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An Intrinsic Hamiltonian Formulation of Network Dynamics : Non-standard Poisson Structures and Gyrators

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ABSTRACT: *The aim of this paper is to provide an intrinsic Hamiltonian formulation of the equations of motion of network models of non-resistive physical systems. A recently developed extension of the classical Hamiltonian equations of motion considers systems with state space given by Poisson manifolds endowed with degenerate Poisson structures, examples of which naturally appear in the reduction of systems with symmetry. The link with network representations of non-resistive physical systems is established using the generalized bond graph formalism which has the essential feature of symmetrizing all the energetic network elements into a single class and introducing a coupling unit gyrator. The relation between the Hamiltonian formalism and network dynamics is then investigated through the representation of the invariants of the system, either captured in the degeneracy of the Poisson structure or in the topological constraints at the ports of the gyrative type network structure. This provides a Hamiltonian formulation of dimension equal to the order of the physical system, in particular, for odd dimensional systems. A striking example is the direct Hamiltonian formulation of electrical LC networks.*

1. Introduction

The modern mathematical theory of analytical mechanics evolved from the Newtonian and variational formulations to a differential-geometric frame where the Hamiltonian formalism is recognized as the fundamental axiom (**1**–**4**). Underlying the dynamic systems of analytical mechanics is a certain geometric structure, called symplectic or Poisson structure of full-rank (**5**). Using this structure one

gains crucial qualitative information, in particular on stability (**2, 4**). Apart from applications in mechanics, as in celestial mechanics (**2**) or mechanisms (**6**), this geometric approach found applications in electrical engineering, for instance in electrical circuit theory (**7**), as well as in control theory (**8–10**).

Arising from a different tradition based on network representations, a unified formalization of physical systems emerged (**11–13**), covering domains as different as chemical reactions (**14, 15**), thermodynamics (**13**) and mechanisms (**16–18**). It displays an additional network structure on the fundamental physical concepts of energy and conservation principles (**13, 19–22**), where the constitutive assumptions on the elements are based on a classification of physical variables and energies derived in accordance with the fundaments of thermodynamics (**23–25**).

This paper is intended to enlighten some aspects of physical systems theory in the double context of analytical mechanics and network theory. The relation between these two formalisms will be investigated through the different ways in which the dynamical invariants of physical systems are expressed. The authors regard this paper as the continuation of previous works on the structure of the dynamics of network representations of electrical circuits (**7, 26**), of chemical reactions (**14, 15**), general non-dissipative systems (**27**) and control systems (**8, 9**).

In Section II, the differential-geometric definition of Hamiltonian systems is recalled in terms of a Poisson structure defined on the state-space manifold (**5**). The main departure from a classical presentation is to allow for general Poisson structures, including degenerate ones (**28–32**). The degeneracy of the Poisson structure is discussed with regard to invariance: a set of smooth real-valued functions (different from the energy function) is left invariant by any Hamiltonian dynamics defined with regard to the Poisson structure. Thus these invariants are fully captured in the geometric structure of the state manifold. It is recalled that in contrast, for standard Hamiltonian systems defined w.r.t. non-degenerate Poisson structures (i.e. symplectic structures), the invariance properties are expressed as symmetries of the Hamiltonian function. Finally, the Hamiltonian system defined on a state manifold with degenerate Poisson structure will be regarded as the result of an intermediate step of a reduction procedure from an embedding standard Hamiltonian system with symmetries to a reduced standard Hamiltonian system.

Section III recalls the physical concepts underlying the modelling of physical systems in network terms using the bond graph notation (**11, 12**). More precisely the generalized bond graph formalism (**13**) is used: its main feature is to symmetrize all energetic elements to one class and to introduce a unit gyrator called "symplectic" gyrator (**25**), which enables the relation with the Hamiltonian formalism. Indeed the main objects of the Hamiltonian formalism (Hamiltonian function, Poisson structure) will be related to a class of constitutive relations of some network elements. Also the dynamic equations associated with a network model will be shown to be Hamiltonian by allowing for degenerate Poisson structures on the state-space manifold (this solves the problem of odd dimensional systems discussed in (**27, 33, 34**)). The last part is devoted to the bond graph realization of the reduced and embedding standard Hamiltonian systems in order to compare the different representations of physical invariants in network as well as in differential geometric terms.

II. Poisson Structures, Symmetries and Reduction

The Hamiltonian formulation of analytical mechanics puts the dynamical equations of motion of conservative mechanical systems into the following form (known as Hamiltonian equations):

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q_1, \ldots, q_n, p_1, \ldots, p_n) \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q_1, \ldots, q_n, p_1, \ldots, p_n) \end{aligned} \qquad i = 1, \ldots, n, \tag{1}$$

where q_i, $i = 1, \ldots, n$, are the generalized configuration variables, p_i, $i = 1, \ldots, n$, are the generalized momentum variables and $H(q, p)$ is the Hamiltonian function which usually can be equated with the total energy of the system. The Hamiltonian equations of motion can be derived from and are equivalent to the Euler–Lagrange equations of motion (derived from a variational point of view, i.e. Hamilton's principle which in turn rests upon d'Alembert's principle of virtual work) in case the Hessian matrix of $H(q, p)$ with respect to p is invertible (the so-called hyperregular case which is mostly satisfied). The generalized momentum variables p_i can be defined through the generalized velocities in the Lagrangian function; however the real power of the Hamiltonian formalism (which is at the heart of theoretical physics) lies in treating the variables q and p at the same level.

An important mathematical achievement of the last century has been the *coordinate free* definition of the Hamiltonian equations (1) in terms of *symplectic* manifolds (**34, 35**); for a complete modern account of this topic see (**1, 2**). Here $q_1, \ldots, q_n, p_1, \ldots, p_n)$ are seen as *special* local coordinates for a manifold M (the phase space) which is endowed with a particular geometric structure called "symplectic structure" (defined by a non-degenerate closed two-form ω on M). In fact, these coordinates are adapted to ω in the sense that ω takes a particularly simple form in these coordinates: it can be equated with the constant bilinear form defined by the matrix:

$$J^{\text{symp}} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{2}$$

The special coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ are called "canonical coordinates". Subsequently it was realized that an equivalent geometric definition of the Hamiltonian equations can be given in terms of a non-degenerate *Poisson structure* (which mathematically is dual to a symplectic structure (**5**)). The expression in local coordinates of such a non-degenerate Poisson structure is nothing other than the classical *Poisson bracket*. The geometric coordinate-free formulation of the Hamiltonian equations (1) is now performed by identifying a phase space M endowed with a symplectic form ω (or equivalently with a non-degenerate Poisson structure), together with a Hamiltonian function $H: M \to \mathbb{R}$. The triplet (M, ω, H) determines the dynamics (1) uniquely by the fact that the symplectic structure defines a mapping from functions on M, in particular from H, to vector fields on M.

Recently it was recognized that an important generalization of the standard Hamiltonian equations (1) can be obtained by *not* being restricted to a non-degenerate Poisson structure, but instead to allow for general ones. General Poisson structures have been already introduced by Lie (**28**) mainly for studying systems of partial differential equations of first order, e.g. Hamilton–Jacobi equations (see also (**29**)), but were rediscovered only recently (**30–32**). The relevance of such general Poisson structures has already become evident in the study of reduced-order systems arising from systems having symmetries, such as the Euler equations for the motion of the rigid body or more complex systems with symmetries (**5, 37**). The basic connection of the Euler equations with the canonical Poisson structure on the dual of the space of 3×3 skew-symmetric matrices (see Examples 3 and 7) was identified in (**38**). Also, for infinite dimensional systems the Poisson structure point of view has proven to be very valuable, see e.g. (**35**).

A main purpose of this paper is to show that Poisson structures appear naturally in the dynamic equations associated with a network representation of physical systems (here restricted to systems without dissipative phenomena).

2.1. *Poisson structures and Hamiltonian systems*

We will now briefly recapitulate the basic mathematical ingredients of general Poisson structures and the resulting generalized Hamiltonian equations of motion as alluded to in the introduction to this section. Because of space limitations we can only sketch the outline of the theory. For a more elaborate treatment we refer the reader to (**5, 30, 35**), where detailed proofs can also be found.

We start with the definition of general Poisson structures, which constitutes the basis of the definition of generalized Hamiltonian systems.

Let M be a smooth (i.e. C^∞) manifold and let $C^\infty(M)$ denote the smooth real functions on M. A Poisson structure on M is a bilinear map from $C^\infty(M) \times C^\infty(M)$ into $C^\infty(M)$, called the *Poisson bracket* and denoted as:

$$(F, G) \mapsto \{F, G\} \in C^\infty(M), \quad F, G \in C^\infty(M),$$

which satisfies for every F, G, $H \in C^\infty(M)$ the following properties:

skew-symmetry:

$$\{F, G\} = -\{G, F\}, \tag{3}$$

Jacobi identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0, \tag{4}$$

Leibniz rule:

$$\{F, G \cdot H\} = \{F, G\} \cdot H + G \cdot \{F, H\}. \tag{5}$$

M together with the Poisson structure is called a *Poisson manifold.*

Now let M be a Poisson manifold with Poisson bracket $\{,\}$. Then for any $H \in C^\infty(M)$ and arbitrary $x \in M$, we can define the mapping:

$$X_H(x): C^\infty(M) \to \mathbb{R}$$

as

$$X_H(x)(F) = \{F, H\}(x), F \in C^\infty(M). \tag{6}$$

It follows from the bilinearity of the Poisson bracket and (5) that $X_H(x) \in T_xM$ for every $x \in M$ (where T_xM denotes the *tangent space* to M at the point $x \in M$). Consequently X_H defines a smooth vector field on M, called the *Hamiltonian vector field* corresponding to the Hamiltonian function H (and with respect to the Poisson bracket $\{,\}$). Thus (6) can be seen as the (Lie) derivative of the function F along the vector field X_H.

Note that this definition implies that the Hamiltonian H is necessarily a *conserved quantity* for the Hamiltonian vector field X_H. Indeed by using skew-symmetry (3), we have by (6):

$$X_H(x)(H) = \{H, H\}(x) = 0. \tag{7}$$

Furthermore, we note that since

$$\begin{aligned} \{F, G\} &= X_G(F)(x) = \mathrm{d}F(x)(X_G(x)), \\ \{F, G\} &= -\{G, F\}(x) = -X_F(G)(x) = -\mathrm{d}G(x)(X_F(x)) \end{aligned} \tag{8}$$

the value of the Poisson bracket $\{F, G\}$ in any point $x \in M$ only depends on the differentials of F and G in that same point. Therefore the mapping $H \in C^\infty(M)$ to $X_H(x) \in T_xM$ for any $x \in M$, as defined by (6), can be also seen as a mapping from $\mathrm{d}H(x) \in T_x^*M$ (the *cotangent space* to M at $x \in M$) to $X_H(x) \in T_xM$. Hence the Poisson structure defines for any $x \in M$ a (linear) mapping from T_x^*M to T_xM.

A more concrete representation of this mapping is obtained by taking local coordinates $x_1, \ldots, x_n$ for M (defined on a neighborhood of some point $x_0 \in M$). Since the Poisson bracket $\{F, G\}(x)$ only depends on the differentials of F and G in x, we obtain the following coordinate expression for the Poisson bracket:

$$\{F, G\}(x) = \sum_{k,l=1}^{n} \frac{\partial F}{\partial x_k}(x) J_{kl}(x) \frac{\partial G}{\partial x_l}(x) \tag{9}$$

for some functions $J_{kl} \in C^\infty(M)$, $k, l = 1, \ldots, m$, which in fact are determined by:

$$J_{kl}(x) = \{x_k, x_l\}(x), \quad k, l = 1, \ldots, m. \tag{10}$$

(Here x_k, x_l are seen as coordinate functions on M, i.e. as elements of $C^\infty(M)$.) From (3), we obtain

$$J_{kl}(x) = -J_{lk}(x), \quad k, l = 1, \ldots, m \tag{11}$$

while (4) yields

$$\sum_{l=1}^{m} \left(J_{lj} \frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x) \frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk} \frac{\partial J_{ji}}{\partial x_l} \right)(x) = 0, \quad i, j, k = 1, \ldots, m. \tag{12}$$

Conversely, if the smooth functions $J_{kl}, k, l = 1, \ldots, m$, defined locally on M, satisfy (11) and (12) then one can define locally the Poisson bracket $\{,\}$ as in (9) and it

may be verified that the Poisson bracket verifies (3), (4), (5). In conclusion, locally the Poisson bracket is uniquely determined by its *structure matrix*:

$$J(x) = (J_{kl}(x))_{k,l=1,\ldots,m} \tag{13}$$

with J_{kl} satisfying (11), (12). Furthermore it follows from (6), that the Hamiltonian vector field X_H expressed in local coordinates $x_1, \ldots, x_m$ as the vector $(X_H^1, \ldots, X_H^m)^t$ is given as:

$$\begin{pmatrix} X_H^1(x) \\ \cdots \\ \cdots \\ X_H^m(x) \end{pmatrix} = J(x) \begin{pmatrix} \dfrac{\partial H}{\partial x_1}(x) \\ \cdots \\ \cdots \\ \dfrac{\partial H}{\partial x_m}(x) \end{pmatrix} \tag{14}$$

and thus the dynamical equations of motion determined by X_H read in local coordinates as:

$$\begin{pmatrix} \dot{x}_1 \\ \cdots \\ \cdots \\ \dot{x}_m \end{pmatrix} = J(x) \begin{pmatrix} \dfrac{\partial H}{\partial x_1}(x) \\ \cdots \\ \cdots \\ \dfrac{\partial H}{\partial x_m}(x) \end{pmatrix}. \tag{15}$$

The above also implies that the map:

$$J(x): T_x^*M \mapsto T_xM \tag{16}$$

can be seen as the local matrix representation of a bundle map from the cotangent bundle T^*M to the tangent bundle TM.

Let us remark at this point that for the definition of $X_H(x) \in T_xM$ in (6) or the coordinate representation $J(x): T_x^*M \mapsto T_xM$ in (13), we did not yet use the Jacobi identity (4) or equivalently (12); i.e. we could have defined everything without this assumption on the bracket $\{,\}$. The Jacobi identity does however play a crucial role in the following property of any Hamiltonian vector field X_H. Let $F, G \in C^\infty(M)$, then by (6), the Jacobi identity (4) and (3):

$$X_H(\{F,G\}) = \{H,\{F,G\}\} = \{\{H,F\},G\} + \{F,\{H,G\}\} \tag{17}$$

and thus:

$$X_H(\{F,G\}) = \{X_H(F),G\} + \{F,X_H(G)\}. \tag{18}$$

Property (18) means that X_H *leaves* the Poisson bracket *invariant*. Indeed, if we denote by X_H^t the time t integral of X_H (i.e. $X_H^t(x_0)$ is the solution at time t of $\dot{x} = X_H(x)$ with $x(0) = x_0$) then (18) is equivalent to:

$$\{F \circ X_H^t, G \circ X_H^t\}(x) = \{F, G\}(x) \quad \forall F, G \in C^\infty(M), \quad \forall t \geqslant 0. \tag{19}$$

Conversely, given a bracket $\{\,,\}$, satisfying (3) and (5), one can define for each real valued function H on $\mathbb{R}$, a vector field X_H as in (6). Then the bracket satisfies the Jacobi identity (4) (and thus is Poisson bracket) if and only if for each H the Hamiltonian vector field X_H leaves the bracket $\{\,,\}$ invariant (i.e. satisfies (18), (19)).

The *rank* of the Poisson bracket $\{\,,\}$ in any point $x \in M$, is defined as the rank of the structure matrix $J(x)$ in this point. (This can be shown to be independent of the choice of the local coordinates.) By (11) necessarily the rank at any point is *even.* A Poisson manifold M having the property that the rank of the Poisson bracket is equal everywhere to the dimension of M is called a *symplectic* manifold. (Thus necessarily the dimension of a symplectic manifold is even.) On the other hand a general Poisson manifold M can be seen as a union of symplectic manifolds which fit together in a smooth way in the following sense: through each point of M there passes a unique submanifold with the property that the rank of the Poisson bracket in every point of this submanifold is equal to the dimension of the submanifold (**5, 30**). In fact (see (**5**)), this submanifold is the maximal integral manifold (through x_0) of the distribution:

$$\text{span}\,\{X_F(x) | F \in C^\infty(M)\}, \quad x \in M. \tag{20}$$

Example 1. (Standard Hamiltonian equations.) Let $M = \mathbb{R}^{2n}$ with natural coordinates $(x_1, \ldots, x_{2n}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and define the *standard* Poisson bracket:

$$\{F, G\}(q, p) = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)(q, p). \tag{21}$$

Notice that

$$J(x) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad x \in \mathbb{R}^{2n},$$

and thus the rank equals $2n$ everywhere. For any Hamiltonian $H \in C^\infty(\mathbb{R}^{2n})$, the Hamiltonian vector field X_H is given by the familiar equations of motion (*cf.* (15)):

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q, p), \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q, p), \quad i = 1, \ldots, n \tag{22}$$

called the *standard* Hamiltonian equations. $M = \mathbb{R}^{2n}$ with bracket (21) is a symplectic manifold and $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$ are called the generalized configuration coordinates, respectively generalized momenta. Using the (inverse) Legendre transform:

$$H(q, p) \mapsto L(q, \dot{q}) = \sum_{i=1}^{n} p_i \dot{q}_i - H(q, p)$$

with p determined by $\dot{q} = \partial H/\partial p$ one may arrive at the equivalent Euler–Lagrange equations.

Example 2. Consider a simple linear electrical circuit consisting of two inductances L_1, L_2 and one capacitor C (see Fig. 1).

The total energy stored in the three elements is given as

$$H(Q, \phi_1, \phi_2) = \frac{Q^2}{2C} + \frac{\phi_1^2}{2L_1} + \frac{\phi_2^2}{2L_2} \tag{23}$$

where ϕ_1, ϕ_2 denote the fluxes of inductances L_1 and L_2, respectively, and Q is the charge of the capacitor C. Using the constitutive relations of L_1, L_2 and C together with Kirchhoff's laws, we obtain the equation of motion:

$$\dot{Q} = -\frac{\phi_1}{L_1} + \frac{\phi_2}{L_2}, \quad \dot{\phi}_1 = \frac{Q}{C}, \quad \dot{\phi}_2 = -\frac{Q}{C} \tag{24}$$

which can be rewritten as:

$$\begin{pmatrix} \dot{Q} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial Q} \\ \dfrac{\partial H}{\partial \phi_1} \\ \dfrac{\partial H}{\partial \phi_2} \end{pmatrix}. \tag{25}$$

This defines a Hamiltonian vector field on $M = \mathbb{R}^3$ corresponding to the Hamiltonian $H(Q, \phi_1, \phi_2)$ and the Poisson bracket on $\mathbb{R}^3$ given in natural coordinates for $\mathbb{R}^3$ by the constant structure matrix:

$$J(Q, \phi_1, \phi_2) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{26}$$

Notice that J is independent from the parameters L_1, L_2 and C, and in fact is solely

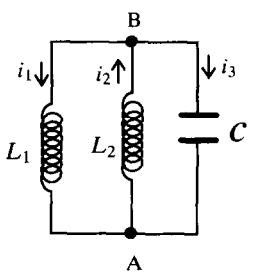


FIG. 1. An electrical circuit.

determined by the topological structure of the network. In particular, (25) remains valid if the elements are nonlinear and thus define a non-quadratic Hamiltonian $H(Q, \phi_1, \phi_2)$. Clearly the rank of J is everywhere equal to 2, and $M = \mathbb{R}^3$ is the union of all the two-dimensional symplectic manifolds:

$$\{(Q, \phi_1, \phi_2) | \phi_1 + \phi_2 = k\}, \tag{27}$$

with k denoting a constant.

Example 3. (The rigid body.) Consider $\mathbb{R}^3$ with natural coordinates denoted as (p_x, p_y, p_z). It can be verified that the matrix:

$$J(p) = \begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix}, \quad p = (p_x, p_y, p_z)' \in \mathbb{R}^3 \tag{28}$$

satisfies (11) and (12) and thus is the structure matrix of a Poisson structure on M. Consider furthermore the Hamiltonian function:

$$H(p_x, p_y, p_z) = \frac{p_x^2}{2J_x} + \frac{p_y^2}{2J_y} + \frac{p_z^2}{2J_z}. \tag{29}$$

Then by (14) and (15), we obtain the following equations of motion:

$$\dot{p}_x = \frac{J_y - J_z}{J_y J_z} p_y p_z, \quad \dot{p}_y = \frac{J_z - J_x}{J_z J_x} p_z p_x, \quad \dot{p}_z = \frac{J_x - J_y}{J_x J_y} p_x p_y \tag{30}$$

which are the well-known *Euler equations* for the motion of a rigid body spinning around its centre of mass (p_x, p_y, p_z are the body angular momenta, see (**35**)). Note that rank $J(p) = 2$ at every point $(p_x, p_y, p_z) \neq (0, 0, 0)$, and that $M = \mathbb{R}^3$ can be seen as the union of the symplectic submanifolds of the form:

$$\{(p_x, p_y, p_z) | p_x + p_y^2 + p_z^2 = k\} \tag{31}$$

with K being a constant.

The Poisson bracket introduced in Example 3 is an example of a much more general construction. Indeed, let V be a matrix Lie algebra, i.e. a linear space of $n \times n$ matrices which is closed under the matrix commutator

$$[A, B] = AB - BA. \tag{32}$$

Then the dual space V^* has a natural Poisson structure, called Lie–Poisson structure. Indeed, let F and G be real functions on V^*. Then the differentials $\mathrm{d}F(x)$, $\mathrm{d}G(x)$, $x \in V^*$ (i.e. the row vectors of partial derivatives) are elements of $(V^*)^*$, and thus can be *identified* with elements in V. Using this identification, one defines

$$\{F, G\}(x) = \langle x, [\mathrm{d}F(x), \mathrm{d}G(x)] \rangle, \quad x \in V^*, \tag{33}$$

where $\langle , \rangle$ denotes the pairing between V and V^* and $[,]$ is the matrix commutator (32). It can be verified that $\{ , \}$ satisfies (3), (4) and (5) and thus defines a Poisson

bracket on V^*. In more concrete terms, let $v_1, \ldots, v_m$ be a basis of V. Then $v_i \in V$ can be identified with an element in $(V^*)^*$, i.e. with a linear coordinate function x_i on V^*, $i = 1, \ldots, m$. Now define

$$J_{ij}(x) = \{x_i, x_j\} = [v_i, v_j], \quad x \in V^*, \tag{34}$$

then $J(x) = [J_{ij}(x)]_{i,j=1,\ldots,m}$ is the structure matrix of the Poisson bracket defined in (33) and satisfies (11) and (12).

Example 3 fits into this theory as follows: the matrix Lie group underlying to rigid body motion is $SO(3)$, the space of 3×3 orthonormal matrices with matrix algebra $so(3)$, the space of skew-symmetric 3×3 matrices. Thus let us consider $V = so(3)$, with the standard basis:

$$v_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{35}$$

and commutation relations:

$$[v_1, v_2] = v_3, \quad [v_2, v_3] = v_1, \quad [v_3, v_1] = v_2. \tag{36}$$

Using (34), one sees that the Lie–Poisson bracket on $V^* = so^*(3)$ is given by the structure matrix:

$$J(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \quad x = (x_1, x_2, x_3)^t \in so^*(3) \tag{37}$$

and thus by identifying M in Example 3 with $so^*(3)$, i.e. identifying $(p_x, p_y, p_z)^t$ with $(x_1, x_2, x_3)^t$, one recovers the structure matrix (28).

2.2. *Symmetries and Hamiltonian reduction*

We start with the following basic theorem on general Poisson brackets (essentially due to Lie (**28**)), which generalizes the definition of canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ for symplectic manifolds to general Poisson manifolds.

Theorem 1

Let M be an m-dimensional Poisson manifold. Suppose the Poisson bracket has constant rank $2n$ in a neighborhood of a point $x_0 \in M$. Then locally around x_0 we can find coordinates $(q, p, r) = (q_1, \ldots, q_n, p_1, \ldots, p_n, r_1, \ldots, r_l)$ for M of dimension $(2n+l)$ satisfying:

$$\begin{aligned} &\{q_i, p_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0, \quad i,j = 1, \ldots, n \\ &\{q_i, r_{i'}\} = \{p_i, r_{i'}\} = \{r_{j'}, r_{i'}\} = 0, \quad i = 1, \ldots, n, \quad i', j' = 1, \ldots, l \end{aligned} \tag{38}$$

or equivalently, the $m \times m$ structure matrix $J(q, p, r)$ is given as:

$$J(q,p,r) = \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{39}$$

The coordinates (q,p,r) satisfying (38) are called (generalized) canonical coordinates.

Remark 1. If M is a symplectic manifold then $l = 0$, $m = 2n$ and r is void, thus leading to the structure matrix:

$$J(q,p) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{40}$$

Hence Theorem I implies in particular that on every symplectic manifold we can take local coordinates $q_1,\ldots,q_n,p_1,\ldots,p_n$ in which the Poisson bracket takes the form of the *standard* Poisson bracket, see Example 1.

Let M be an m-dimensional Poisson manifold with Poisson bracket of constant rank $2n$ in a neighborhood of a point $x_0 \in M$. Then in view of (14) and (15) in the local canonical coordinates of Theorem I any Hamiltonian vector field X_H, $H: M \mapsto \mathbb{R}$, takes the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q,p,r), \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q,p,r), \quad \dot{r}_j = 0, \quad i = 1,\ldots,n, \quad j = 1,\ldots,l, \tag{41}$$

which generalizes the standard Hamiltonian equations (1).

Example 4. Let M be a symplectic manifold with Hamiltonian vector field X_H, $H \in C^\infty(M)$. Then by Remark 1 and (41) one can find local canonical coordinates $q_1,\ldots,q_n,p_1,\ldots,p_n$ such that the equations of motion corresponding to X_H are locally given by the standard Hamiltonian equations (22).

For any Poisson manifold we define the *distinguished* or *Casimir* functions as those smooth functions $F: M \mapsto \mathbb{R}$, such that:

$$\{F,G\} = 0, \quad \forall G \in C^\infty(M) \tag{42}$$

or equivalently, since $\{F,G\} = -X_F(G)$ (see (6)): $X_F = 0$. Hence the Casimir functions correspond to the *kernel* of the map: $F \mapsto X_F$ from $C^\infty(M)$ modulo $\mathbb{R}$ to the vector fields on M given by (6). In local terms F is a Casimir function if $\mathrm{d}F(x) \in \ker J(x)$, i.e. if $\mathrm{d}F(x)$ is in the kernel of the map $J(x): T_x^*M \mapsto T_xM$ for every x. Under the assumptions of Theorem I, the Casimir functions are locally given by the coordinate functions $r_1,\ldots,r_l$ together with all functions on M depending only on $r_1,\ldots,r_l$.

Theorem I has some interesting consequences concerning the *local* embedding of a Poisson manifold into a higher-dimensional *symplectic* manifold, as well as the local *reduction* of a Poisson manifold to a lower-dimensional *symplectic*

manifold. First, for any two Poisson manifolds M_1 and M_2, one defines a smooth map: $\phi: M_1 \mapsto M_2$ to be a *Poisson* (or *canonical*) mapping if:

$$\{F \circ \phi, G \circ \phi\}_1 = \{F, G\}_2 \circ \phi, \forall F, G \in C^\infty(M_2) \tag{43}$$

where $\{\,,\}_1$ and $\{\,,\}_2$ denote the Poisson brackets on M_1 and M_2, respectively. Now consider the local coordinates (q, p, r) of Theorem I. Then the map $\pi: \mathbb{R}^{2n+l} \mapsto \mathbb{R}^{2n}$ given by:

$$(q_1, \ldots, q_n, p_1, \ldots, p_n, r_1, \ldots, r_l) \overset{\pi}{\mapsto} (q_1, \ldots, q_n, p_1, \ldots, p_n) \tag{44}$$

is a *Poisson mapping* from the Euclidean space $\mathbb{R}^{2n+l}$ with Poisson bracket given by (39) to the lower dimensional space $\mathbb{R}^{2n}$ endowed with the standard Poisson bracket (40). On the other hand, consider the *extended* space $\mathbb{R}^{2n+2l}$ with natural coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n, s_1, \ldots, s_l, r_1, \ldots, r_l)$ and standard Poisson bracket given by (38), together with the additional relations:

$$\begin{aligned} &\{s_i, r_j\} = \delta_{ij}, \quad i', j' = 1, \ldots, l \\ &\{q_i, s_{i'}\} = \{p_i, s_{i'}\} = 0, \quad i = 1, \ldots, n, \quad i', j' = 1, \ldots, l. \end{aligned} \tag{45}$$

Then the map $\pi: \mathbb{R}^{2n+2l} \mapsto \mathbb{R}^{2n+l}$ given by:

$$(q_1, \ldots, q_n, p_1, \ldots, p_n, s_1, \ldots, s_l, r_1, \ldots, r_l) \overset{\pi}{\mapsto} (q_1, \ldots, q_n, p_1, \ldots, p_n, r_1, \ldots, r_l) \tag{46}$$

is also a *Poisson mapping*. Hence in view of Theorem I, we have the following picture in a neighborhood of any point x_0 in the $(2n+l)$-dimensional Poisson manifold M where the Poisson bracket has constant rank $2n$:

$$U^e \subset \mathbb{R}^{2n+2l} \overset{\Pi}{\mapsto} U \subset \mathbb{R}^{2n+l} \overset{\pi}{\mapsto} U^r \subset \mathbb{R}^{2n}, \tag{47}$$

where U, U^e, U^r are open parts of $\mathbb{R}^{2n+l}$, $\mathbb{R}^{2n+2l}$, $\mathbb{R}^{2n}$, respectively. Hence the coordinate neighborhood U can be reduced (via π) to a symplectic space U^r and can be embedded (via Π) into a symplectic space U^e. (Furthermore it can be shown that Π and π are *unique* in the sense that if (47) holds for *different* Poisson mappings π' and Π' then there exist Poisson mappings $\psi: \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}$, $\Psi: \mathbb{R}^{2n+2l} \mapsto \mathbb{R}^{2n+2l}$ of maximal rank such that $\Psi \circ \pi' = \pi$, $\Psi \circ \Pi' = \Pi$ (see **(30)**.)

The above discussion on local normal forms of Poisson structures with constant rank has the following implications for the structure of any Hamiltonian vector field on the Poisson manifold M in the neighborhood of a point $x_0 \in M$ where the Poisson bracket has constant rank. Let $H: M \mapsto \mathbb{R}$ be a Hamiltonian function, then locally about x_0 the Hamiltonian vector field X_H is given in the local coordinates (q, p, r) of Theorem I by (41), i.e.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q, p, r), \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q, p, r), \quad \dot{r}_j = 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, l. \tag{48}$$

Following (47) these equations can be *restricted* for every constant $(r_1, \ldots, r_l)$ to the standard Hamiltonian equations on $\mathbb{R}^{2n}$ (*cf.* (1) or (22))

$$\dot{q}_i = \frac{\partial H^r}{\partial p_i}(q,p), \quad \dot{p}_i = -\frac{\partial H^r}{\partial q_i}(q,p), \quad i = 1, \ldots, n, \tag{49}$$

where $H^r: \mathbb{R}^{2n} \mapsto \mathbb{R}$ is given by $H^r(q,p) = H(q,p,r)$, and can be embedded into the standard Hamiltonian equations on $\mathbb{R}^{2n+2l}$:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q,p,r), \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q,p,r), \quad \dot{s}_j = \frac{\partial H}{\partial r_j}(q,p,r), \quad \dot{r}_j = -\frac{\partial H}{\partial s_j}(q,p,r) = 0,$$
$$i = 1, \ldots, n, \quad j = 1, \ldots, l. \tag{50}$$

Note that in (50) the Hamiltonian H seen as a function on $\mathbb{R}^{2n+2l}$ does *not* depend on s, while in (49) the Hamiltonian $H^r(q,p) = H(q,p,r)$ is seen as a function on $\mathbb{R}^{2n}$ *parametrized* by r.

The transition from the standard Hamiltonian equations (50) on $\mathbb{R}^{2n+2l}$ into the standard Hamiltonian equations on (49) on $\mathbb{R}^{2n}$, via the non-standard Hamiltonian equations (48) can be interpreted as a *canonical reduction of order* caused by symmetry of the Hamiltonian function, which is a vast subject in analytical mechanics (see e.g. (**2**, **39**–**41**)). Indeed the crucial point is that H in (50) does not depend on the canonical coordinates $s_1, \ldots, s_l$, or equivalently that H is *invariant* under the infinitesimal canonical mappings $\partial/\partial s_1, \ldots, \partial/\partial s_l$. It is well known in the theory of Hamiltonian systems with symmetry (see (**35**)), that this implies that $\mathbb{R}^{2n+2l}$ may be projected to $\mathbb{R}^{2n+l}$ with Poisson structure given by (38), and that the Hamiltonian dynamics (50) *project* to (48). Furthermore the Casimir functions $r_1, \ldots, r_l$ for (48) are conserved quantities and thus (48) can be *restricted* (for any constant $r_1, \ldots, r_l$) to (49). Thus alternatively the Poisson manifold M, locally given as $\mathbb{R}^{2n+l}$ with Poisson structure (38), and the Hamiltonian dynamics (41) defined on it, may be considered as an intermediate step of the canonical reduction of order of the standard Hamiltonian equations (50) to the standard Hamiltonian equations (49).

Example 5. Consider the electrical circuit of Example 2, with Poisson structure determined by (26). Canonical coordinates are given by

$$q = Q, \quad p = \tfrac{1}{2}(\phi_2 - \phi_1), \quad r = \phi_1 + \phi_2 \tag{51}$$

and the total energy (23) expressed in these coordinates is

$$H(q,p,r) = \frac{Q^2}{2C} + \frac{\left(\frac{r}{2} - p\right)^2}{2L_1} + \frac{\left(\frac{r}{2} + p\right)^2}{2L_2} \tag{52}$$

resulting in the dynamical equations (*cf.* (52)):

$$\dot{q} = \frac{\partial H}{\partial p}(q,p,r) = -\frac{\frac{r}{2} - p}{L_1} + \frac{\frac{r}{2} + p}{L_2}, \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p,r) = -\frac{Q}{C}, \quad \dot{r} = 0. \tag{53}$$

The last equation reflects the fact that $r = \phi_1 + \phi_2$ (the total flux) is a Casimir function, and thus a conserved quantity.

Example 6. Consider the mass-spring system described in Fig. 2 which constitutes the mechanical analog of the electrical circuit described in Example 2. Its total energy is given by

$$H(x_{12}, p_1, p_2) = \frac{kx_{12}^2}{2} + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \tag{54}$$

with p_1, p_2 denoting the momenta of mass m_1 and m_2, respectively, and x_{12} the distance between them and k denoting the spring constant. The Poisson structure is again given by (23), resulting in the equations of motion (compare with (22)):

$$f = \begin{pmatrix} \dot{x}_{12} \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} kx_{12} \\ \dfrac{p_1}{m_1} \\ \dfrac{p_2}{m_2} \end{pmatrix}. \tag{55}$$

Canonical coordinates are given, analogously to (51), as

$$q = x_{12}, \quad p = \tfrac{1}{2}(p_2 - p_1), \quad r = p_1 + p_2, \tag{56}$$

where the Casimir function $r = p_1 + p_2$ is the total momentum of the mechanical system. The system can be embedded into standard Hamiltonian equations on $\mathbb{R}^4$ (*cf.* (50) for $n = 2$) by adding a variable s satisfying (*cf.* (45)):

$$\{s, r\} = 1, \quad \{q, s\} = \{p, s\} = 0. \tag{57}$$

The resulting standard equations are

$$\dot{q} = -\frac{\frac{r}{2} - p}{m_1} + \frac{\frac{r}{2} + p}{m_2}, \quad \dot{p} = -kq, \dot{s} = \frac{\partial H}{\partial r}(q, p, r) = -\frac{\frac{r}{2} - p}{m_1} + \frac{\frac{r}{2} + p}{m_2}, \quad \dot{r} = 0 \tag{58}$$

which is isomorphic, using the canonical mapping implicitly defined by (56) and $s = \frac{1}{2}(x_1 + x_2)$, $x_{12} = x_2 - x_1$, to

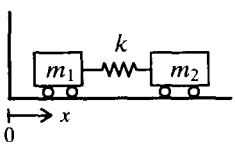


FIG. 2. A one-dimensional mechanical system.

$$\dot{x}_1 = \frac{p_1}{m_1}, \quad \dot{x}_2 = \frac{p_2}{m_2}, \quad \dot{p}_1 = k(x_2 - x_1), \quad \dot{p}_2 = -k(x_2 - x_1), \tag{59}$$

which are the standard Hamiltonian equations of motion for the mechanical system, where x_1 and x_2 denote the configuration coordinates of masses m_1 and m_2, respectively, in an inertial frame.

Example 7. Consider the Euler equations for the motion of a rigid body as dealt with in Example 3, with p_x, p_y, p_z the body angular momentum variables. Canonical coordinates (outside the origin) are, for instance, (see e.g. **(35)**):

$$q = p_z, \quad p = \arctan \frac{p_y}{p_x}, \quad r = p_x^2 + p_y^2 + p_z^2. \tag{60}$$

Notice that r is the *total angular momentum*, which indeed is a conserved quantity. The embedding standard Hamiltonian system (50) in this case does not seem to have an easy physical interpretation, and will not be specified. (Notice that in **(39)**, example 3.2) it is briefly indicated how alternatively a 4-dimensional embedding Hamiltonian system may be obtained by using the fact that $so^*(3)$ is isomorphic to the complex matrix Lie algebra $su(2)$, which acts symplectically on the symplectic space $\mathbb{C}^2$.)

Of course the rank of the Lie–Poisson structure is zero at the origin. Thus a truly globally defined embedding Hamiltonian system has to be 6-dimensional and, in fact, may be obtained in the following way (see **(2)** for details). The configuration space for the dynamics of a rigid body around its centre of mass is $SO(3)$, the matrix group of 3×3 orthogonal matrices. Hence the phase space of the standard Hamiltonian equations for the rigid body equals $T^*SO(3)$. Now $T^*SO(3)$ can be identified with $SO(3) \times so^*(3)$ in the following way (see e.g. **(2)**). Define for any $g \in SO(3)$ the left translation $L_g : SO(3) \mapsto SO(3)$ as:

$$L_g h = gh, \quad h \in SO(3). \tag{61}$$

Denoting the 3×3 identity matrix $I_3 \in SO(3)$ by e, we obtain the diffeomorphism:

$$L_g^* : T_g^*SO(3) \mapsto T_e^*SO(3) =: so^*(3) \tag{62}$$

and the resulting diffeomorphism $\lambda : T_g^*SO(3) \mapsto SO(3) \times so^*(3)$ defined as

$$\lambda(g, \alpha) = (g, L_g^* \alpha), \quad g \in SO(3), \quad \alpha \in T_g^*SO(3). \tag{63}$$

Identifying in this way $T^*SO(3)$ *with* $SO(3) \times so^*(3)$ we see that the Hamiltonian function $H : SO(3) \times so^*(3) \mapsto \mathbb{R}$, being the kinetic energy, only depends on the natural coordinates p_x, p_y, p_z for $so^*(3)$, and in fact is given as in Example 3. Hence the embedding standard Hamiltonian equations are in local coordinates of the form (50), and thus project to the Euler equations (30) on the Poisson manifold $SO(3) \times so^*(3)/SO(3) = so^*(3)$.

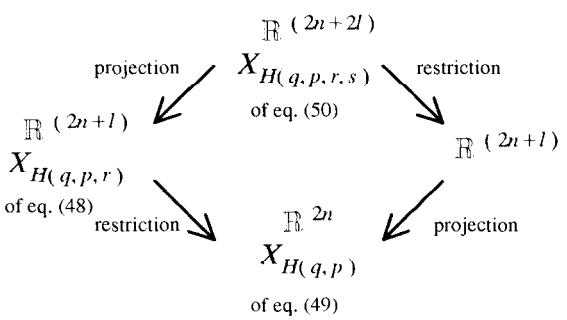
Given the standard Hamiltonian equations (50), where H does not depend on the canonical variables $s = (s_1, \ldots, s_l)$, there exists also an *alternative* reduction

procedure which leads eventually to the same reduced standard Hamiltonian system (49), but *not* via the non-standard Hamiltonian equations (48). Instead, one uses the fact that $r = (r_1, \ldots, r_l)$ are *conserved* quantities for (50), from which it follows that (50) can be *restricted* for every constant vector r to dynamics $\mathbb{R}^{2n+l}$, and subsequently may be projected to the same standard Hamiltonian dynamics (48). Pictorially both reduction schemes can be summarized by the commutative Diag. 1.

In the present context of network representation of physical systems, however, the left-hand side reduction procedure appears to be the most natural one. Since this reduction procedure *starts* with the invariance (or symmetry) of the Hamiltonian (internal energy), the conserved quantities become fully captured in the Poisson structure, i.e. the *geometry* underlying the general Hamiltonian equations (48). Indeed the conserved quantities are determined by the Casimir functions for this Poisson structure, *independently* from the Hamiltonian of (48).

Remark 2. Let us mention that we have restricted ourselves to a purely *local* description of the reduction procedure from equations (50) to (49), via (48). For a global and coordinate free treatment, we refer to e.g. (**35**), see also (**2**). Of particular interest is the case of general Hamiltonian dynamics (48) defined on a Poisson manifold which is the dual of some Lie algebra V (endowed with the Lie–Poisson bracket, *cf.* the discussion after example 3), such as $so^*(3)$. Then an embedding standard Hamiltonian system is to be found by looking for a symplectic action of the corresponding (simply connected covering) Lie group on some symplectic manifold M (see (**30, 41**)). In this case the *momentum map* $M \to V^*$ (see (**2**)) provides the projection map from (50) to (48), and the embedding standard Hamiltonian system is living on this manifold M.

Remark 3. In (**41**) it is shown how the dynamics of incompressible fluids, described as general Hamiltonian equations of motion on the dual of the Lie algebra corresponding to the group of volume-preserving diffeomorphisms, can be embedded into a standard Hamiltonian system with canonical coordinates given by the classical Clebsch variables.



DIAG. 1.

III. Bond Graph Models and their Dynamics

3.1. *Physical systems in terms of bond graphs*

The modern graphical description of physical systems emerged from the description of electrical circuits. Kirchhoff introduced, from the inspection of its material organization, the first graphical type abstraction into constitutive relations associated with electrical elements (R, L, C) and the topology defining the circuit (Kirchhoff's current and voltage laws). Although mechanical systems provided the framework for a general analytical formalism of physical systems models, leading to analytical mechanics, it was also soon recognized that circuit theory is not restricted to electrical circuits. Indeed Maxwell already stated Newton's second law in a circuit-theoretical way, in the form of a balance equation.

The network formalism as a unifying conceptual framework for physical systems was developed mainly in engineering (for historical developments see (**21**)), where it revealed an extraordinary efficiency not only for the design of electrical circuits, but also for industrial devices pertaining to different physical domains. For the latter systems a particular graphical notation, called bond graphs, was developed originally by Paynter (**11**). Its network structure called "junction structure" displays the invariants of the physical system according to Tellegen's and Kirchhoff's theorems on networks, but corresponds to an additional abstraction level. Indeed it represents abstract energy flows between elements and no more material interconnections. In the same way, the bond graph formalism is based on a classification of physical variables, rooted in thermodynamics (**23–25**). This led to the generalized bond graph formalism (**13**) based on an abstraction of the elements of a network (their phenomenological properties) and the systematic use of a unit gyrator (the gyrator element was invented by Tellegen). This element, called "symplectic gyrator" (**25**), will be a fundamental link relating analytical mechanics with the network formalism.

3.1.1. *Energy and the capacitor element.* The elemental components of a bond graph model are elemental systems endowed with energy, called (multiport) energy storage elements and denoted by C (**11**, chap. 4; **13**, chap. 5; **25**).

Such an elemental system concentrates some physical properties inside abstract boundaries defining it with respect to its environment. Its internal state is defined by a vector x in $\mathbb{R}^n$ of energy variables and a real-valued smooth energy function $H(x)$ describing its static characteristics. The existence of an energy-function corresponds physically to the first law of thermodynamics. Then the variation of the energy and the energy variables are related by Gibbs fundamental equation:

$$\mathrm{d}H = \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \mathrm{d}x_i. \tag{64}$$

The interactions with the environment through the boundaries of the elemental system are defined by the time-variation of the energy at the ports of the elemental systems. The time-variation of the energy may be expressed by two vectors of so-called conjugated power variables called "effort" e and "flow" f:

$$f = \dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)', \quad e = \mathrm{d}H = \left(\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}\right). \tag{65}$$

The variation of the energy function is then the inner product of these two vectors:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_{i=1}^{n} e_i \cdot f_i. \tag{66}$$

It may be noticed that as a consequence of the existence of an energy function, the effort as a function of the energy variables verifies the Maxwell reciprocity relations:

$$\frac{\partial e_i}{\partial q_j} = \frac{\partial e_j}{\partial q_i}, \quad i,j \in 1, \ldots, n \tag{67}$$

(and by Poincaré's lemma the converse also holds).

In summary the energy storage elements C are represented in bond graph terms as in Fig. 3.

Example 8. (Point-mass.) A point-mass may be represented as a 1-port energy storage element (in the kinetic domain) with the translational momentum p as energy variable and the kinetic energy $H_{\mathrm{kin}}(p) = p^2/2m$ as energy function, where m denotes the mass. The power variables at its port are the forces acting on the mass (i.e. the rate of change of the momentum) as flow variable and the translational velocity of the point-mass as effort variable.

Example 9. (Electrical capacitor.) An electrical capacitor may also be represented as a 1-port energy storage element with the electrical charge q as energy variable and the electrical energy as energy function (in the linear case: $q^2/2C$, where C denotes the linear capacitance). The flow variable is the current through the capacitor (i.e. the rate of change of the charge) and the effort variable is the voltage at its port.

Example 10. (Simple thermodynamical system.) A simple thermodynamical system is characterized by an internal energy function $U(S, V, N)$ (**13**, chap. 5). Such a system may be represented as a 3-port energy storage element with the number of moles N, the volume V and the entropy S as energy variables and the

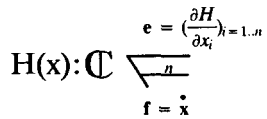


FIG. 3. An energy storage element.

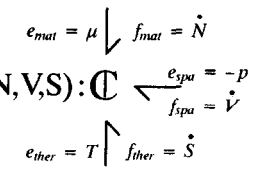


FIG. 4. A simple thermodynamical system.

internal energy as energy function (see Fig. 4). The conjugated effort variables are the material potential $\mu = \partial U/\partial N$, the pressure $P = -(\partial U/\partial V)$ and the absolute temperature $T = \partial U/\partial S$ (where μ is depending on P and T due to the first-order homogeneous nature of the energy $U(S, V, N)$ (**13**, chap. 5)). Such a 3-port energy storage element can be used, for instance, to model an open container with piston and containing a gas.

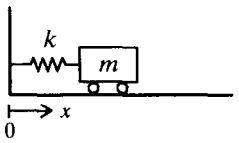
3.1.2. *Dynamical interactions and the symplectic gyrator.* In the previous section we have defined elemental systems endowed with energy, but which cannot undergo any dynamic evolution *per se*. Any dynamics arises from the interaction between elemental systems. These interactions are described as relations between the effort and flow variables, called phenomenological laws like Newton's or Ohm's laws. In bond graph terms these interactions are described by elements whose constitutive relation represents a phenomenological or constitutive law and are power continuous, i.e. the total power flow at their ports is equal to zero (as any energy storage, i.e. any energy function, should be associated with an energy storage element C). There are two types of such elements.

The resistive irreversible transducer relates some physical domain with the thermal domain and describes resistive laws (**13–15**). This element will not be considered in the present paper, as we have restricted ourselves to the so-called "conservative" systems without any resistive or, rather, irreversible phenomena. The other element relates two different physical domains: either the kinetic with the elastic (potential) domain, or the electrical with the magnetic domain. The kinetic and elastic domains include the special forms which are present in hydraulics or acoustics. This second element is called "symplectic gyrator" (**25**). It is a $2n$-port element whose constitutive relation is, if f and e denote the ($2n$-dimensional) flow and effort vectors at its ports and J^{symp} the symplectic matrix of order $2n$:

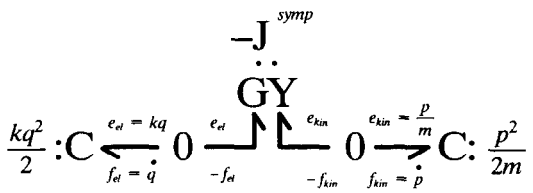
$$f = -J^{\text{symp}} e, \quad J^{\text{symp}} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{68}$$

where I_n is the identity matrix of order n.

Example 11. (Mass-spring system.) Consider the mass-spring system and its bond graph representation depicted in Fig. 5. With the spring a one-port energy storage element is associated: it defines the elastic potential energy of the system as a function $H_{\text{el}}(q)$ of the displacement q (the energy variable of the elastic domain). For a linear element the energy function is:



(a)



(b)

FIG. 5. Mass–spring interaction and bond graph model.

$$H_{\mathrm{el}}(q) = \tfrac{1}{2}kq^2, \tag{69}$$

with k the stiffness of the spring. The point mass is represented by an energy storage element with the momentum p of the mass as energy variable and the kinetic energy $H_{\mathrm{kin}}(p)$, which is in the linear case:

$$H_{\mathrm{kin}}(p) = \frac{1}{2}\frac{p^2}{m} \tag{70}$$

with m the mass of the point mass. The complete bond graph model is obtained by relating the two energy storage elements via so-called bonds through a symplectic gyrator. The bonds may be considered as identities on the power variables, i.e. efforts and flow variables of the related ports of the elements. (The next section will give the proper definition of the bonds). The symplectic gyrator represents the following two relations. First the attachment of the end of the spring to the mass in globally Eulerian but locally Lagrangian coordinates with respect to an inertial frame (**13**, chap. 6):

$$\frac{\mathrm{d}q}{\mathrm{d}t} = f_{\mathrm{el}} = e_{\mathrm{kin}} = v_{\mathrm{mass}}. \tag{71}$$

The variable e_{kin} is the effort in the kinetic domain, i.e. the velocity of the mass v_{mass} and f_{el} is the flow of the electric-potential domain. The second relation represents Newton's second law in an inertial reference frame:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f_{\mathrm{kin}} = -e_{\mathrm{el}} = -F_{\mathrm{el}}. \tag{72}$$

The variable e_{el} is the effort in the elastic potential domain, i.e. the force induced by the spring, and f_{kin} is the flow of the kinetic domain, i.e. the time variation of the momentum of the mass.

Example 12. (Inductor and capacitance circuit.) Consider the electrical circuit of Fig. 6 and its bond graph model. The electrical domain is represented by an energy storage element C with energy variable Q (the electrical charge of the capacitor) and the energy function $H_{el}(Q)$ which is, for instance, with the assumption of the linearity of the constitutive relation:

$$H_{el}(Q) = \frac{1}{2}\frac{Q^2}{C} \tag{73}$$

with C the capacitance.

The magnetic domain is represented by an energy storage element C, with the magnetic flux linkage ϕ as energy variable and energy function $H_{mg}(\phi)$ which is, for instance, with the assumption of linearity of the constitutive relation:

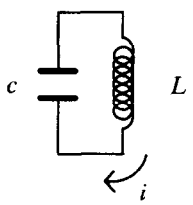
$$H_{mg}(\phi) = \frac{1}{2}\frac{\phi^2}{L} \tag{74}$$

with L the inductance.

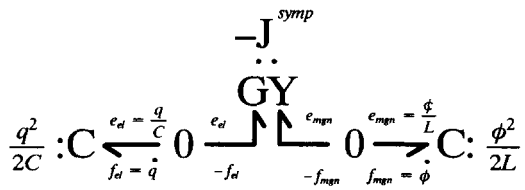
The symplectic gyrator represents Maxwell's second and fourth law coupling the electric and magnetic fields in the quasi-stationary approximation of electrical circuits (**25**) according to

$$\frac{dQ}{dt} = f_{el} = i = e_{mg}, \quad \frac{d\phi}{dt} = f_{mg} = -u = -e_{el}. \tag{75}$$

The generalized bond graph formalism recalled here makes a gyrative coupling between different physical domains appear explicitly, which would not appear in the "conventional" bond graph approach (as well as in electrical networks). In the



(a)



(b)

FIG. 6. Inductor capacitance interaction and its bond graph model.

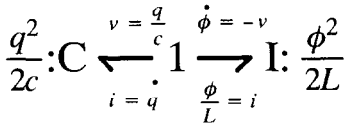


FIG. 7. Conventional model of the mass–spring interaction.

conventional bond graph approach, for both examples, there would be two kinds of storage elements in interaction as represented in Fig. 7. The bond still represents a coupling through power exchange, but using an operational identification (through measurement) of across-variables (like velocities or potentials) and through-variables (like forces or currents). The definition of effort and flow variables are in this sense less general than in the generalized bond graph formalism and are linked to the concrete network. The two kinds of storage elements I and C have a dual definition with respect to the power variables. In generalized bond graph terms the I element is the aggregation of a C element with a symplectic gyrator: this aggregation is called "dualization" of the C element (**13, 25**). Indeed the generalized bond graph formalism prefers to give an asymmetric definition to flows and efforts and introduces explicitly the gyrative couplings, based on the fact that in the thermal domain there is no "inertance" and that in the conventional approach some phenomena may be overlooked (**25**).

3.1.3. *Power continuity as an abstract network postulate.* Complex autonomous physical systems may now be defined as a set of interacting elemental systems (i.e. energy storage elements denoted by C) exchanging energy in such a way that it is conserved over the whole system. The postulate of power continuity sets forth that one may define the conservation of energy in the form of an abstract topology of power flows between elemental systems (**11, 22**). This topology of power flows is called "junction structure". Consequently the edges of this network transport power flows and are called "power bonds" or simply "bonds". In order to represent the interaction between subsystems, for instance elemental systems, they satisfy the "power postulate": there exist two dual variables, called power variables, such that their pairing is equal to the power flow. The two variables are called a flow f (usually a vector in $\mathbb{R}^n$) and effort e (usually also identified with a vector in $\mathbb{R}^n$ although it is a dual quantity to the flow f). In summary a power bond is associated with three variables (flow f, effort e and instantaneous power P) which verify:

$$\langle e,f \rangle = P. \tag{76}$$

In Fig. 8, the half arrow on the power bond indicates the positive orientation for

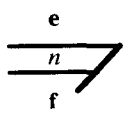


FIG. 8. Power (multi-) bond with sign orientation.

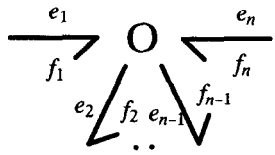


FIG. 9. A 0-junction.

the power P and the flow variable f. The nodes of the junction structure are necessarily power continuous elements and will be presented in the next paragraph.

3.1.4. *Invariants displayed by generalized junction structures.* The elements of a generalized junction structure extend the representation of invariants of the physical system from the energy (represented as energy storage elements) to structural invariants induced by structural constraints (the topology of an electrical circuit or kinematic constraints) or by interdomain couplings. Thereby the generalization from quasi-statics to dynamics may be realized.

Two basic elements are the simple junctions, denoted by 0 and 1. Using scattering variables it can be proven that they are the only possible realization of port symmetric, power continuous elements (**11, 42**). They represent invariants on the dual power variables efforts and flows (**11**, chap. 6; **22**). A 0-junction corresponds to "effort potentiality", i.e. defines a common effort variable to all bonds connected to it (see Fig. 9). The symmetry of the junction and its power continuity imply the following constitutive relations:

$$e_1 = \cdots = e_n, \quad \sum_{i=1}^{n} \varepsilon_i \cdot f_i = 0, \tag{77}$$

where ε_i are equal to plus or minus one depending on the orientation of the power bonds (half arrows in Fig. 9).

A 1-junction represents "flow continuity", i.e. common flow variables to all bonds connected to it (see Fig. 10). It has the following constitutive relations (dual to that of a 0-junction):

$$f_1 = \cdots = f_n, \quad \sum_{i=1}^{n} \varepsilon_i \cdot e_i = 0, \tag{78}$$

where ε_i are the sign functions depending on the orientation of the power bonds.

Simple junction structures are networks of power bonds and junctions which

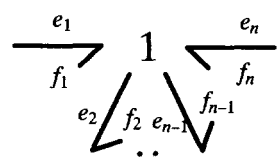


FIG. 10. A 1-junction.

describe sign-weighted invariants (i.e. constraints on the power variables). Such local invariants may arise directly from the material organization of the system, abstracted as a directed graph, for instance in the case of electrical circuits or one-dimensional mechanical systems. In the case of electrical circuits the invariants generated by the directed graph are called Kirchhoff's voltage and current laws. For such invariants, arising from graphical interconnection constraints, several systematic procedures exist of the realization of the equivalent simple junction structures (**42–46**).

Example 13. (Simple junction structure of an electrical circuit.) Let us consider the electrical circuit of Example 2 (see Fig. 1). The underlying directed graph (with the elements associated with the branches indicated in brackets) and the corresponding simple junction structure are represented in Fig. 11. The constitutive relations of the two simple junction structures represent, in the case of electrical circuits, Kirchhoff's current and voltage laws with the dual definition of currents as either flow or effort variables.

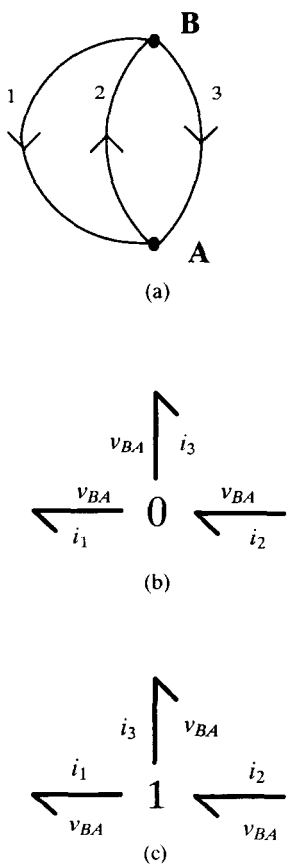


FIG. 11. Directed graph underlying an electrical circuit and the two dual junction structures.

FIG. 12. A causally augmented bond.

Hence a simple junction structure is constructed out of junctions representing local invariants (in the sense that they are represented by distinguished nodes of a bond graph), but it defines itself global invariants on the efforts and flows at its ports (**40**, **41**, **43**). They are graphically displayed as a causal augmentation of the bonds at the ports of the junction structure: a causal stroke is added at one end of the bond (see Fig. 12). A causal stroke directed towards (or outwards, respectively) a simple junction structure indicates an independent effort variable and dependent flow variable (or independent flow and dependent effort, respectively). The consistency of the invariants at the ports and at a junction in a simple junction structure is ensured by causality restrictions at the junction (see Fig. 13) and for particular bond loop structures by some global causality restrictions (i.e. restrictions on some part of the simple junction structure (**40**, **41**)). For simple junction structures representing electrical circuits, the causal augmentation at the ports corresponds to the choice of a spanning tree or cotree of the underlying directed graph (**41**, **43**, **44**).

Example 14. (Causal restriction in an electrical circuit.) Consider the electrical circuit of Example 2 and its directed graph in Fig. 11. Then a consistent causal augmentation and choice of the spanning tree (here a single branch) is shown in Fig. 14.

One may note that simple junction structures allow one to represent very general power continuity relations arising from topological constraints in different kinds of physical domains. For instance, the complex topological constraints of planar

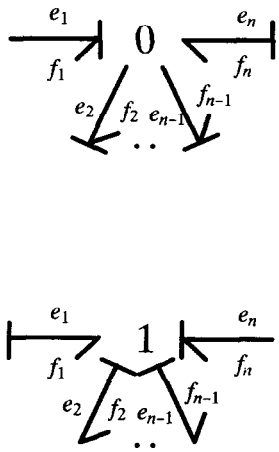


FIG. 13. Causal restrictions at junctions.

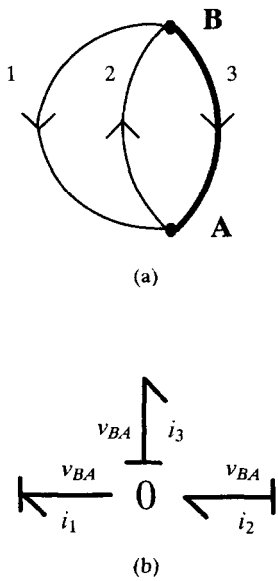


FIG. 14. Choice of causality and associated spanning tree.

and spatial mechanical linkages described in skeleton diagrams or interchange graphs may be represented in kinematic junction structures using generalized junctions (**47, 48**).

An additional power continuous element, called "transformer" and denoted by TF, generalizes the sign-weighted relations expressed in simple junction structures to linear relations. A junction structure containing transformers is called *weighted junction structure*. A multiport transformer imposes linear relations between the power variables (f_2, e_2) of port 1 with dimension n_1 and the power variables (f_1, e_1) of port 2 with dimension n_2. Its constitutive relation is characterized by an $(n_1 \times n_2)$ matrix M with coefficients in some field defining a vector space on the power variables; it is defined by Eq. (79) according to the representation in Fig. 15:

$$f_2 = M_x f_1, \quad e_1 = M_x^t e_2. \tag{79}$$

If the coefficients of the matrix M_x are functions, the transformer is said to be "modulated" and is denoted by MTF. The coefficients of the transformer are, for instance, integers representing stoichiometric coefficients for chemical reactions' models (**14, 15**), real numbers representing geometric parameters or real valued functions of some energy and configuration variables for multibody systems (**18**).

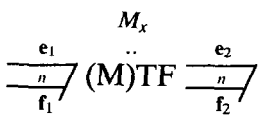


FIG. 15. A multiport transformer.

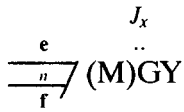


FIG. 16. A multiport gyrator.

The causal constraints at the ports of a multiport transformer depend on the rank properties of M_x. But for a single bond transformer, they are analogous to the constraints on a single bond which may be considered as a unit transformer. They provide constraints at the ports of a weighted junction structure which are analogous to the constraints on the ports of a simple junction structure.

Finally, the complete network interconnecting capacitive elements, i.e. elemental systems of different physical domains, may contain additionally some gyrative element, denoted by GY and represented in Fig. 16. The complete network is called: *generalized junction structure*. The constitutive relation of a gyrator is defined by a skew symmetric matrix J_x according to Eq. (80):

$$f = J_x e, \tag{80}$$

where the order of J_x is equal to the dimension of the port of the multiport gyrator. Again the causal constraints at its ports depend on the rank properties of J, but they may be precised in the case of a two-port gyrator (necessarily of rank 2): the two causality assignments are depicted in Fig. 17. The causal constraints at the port of a generalized junction structure reflect the invariant algebraic relations on the power variables at the ports in the same way as for weighted junction structures. Also, it has an additional interpretation if one connects the ports of a capacitor of appropriate dimension to its ports, thus obtaining a complete (autonomous) bond graph model. Then a causal restriction on the efforts at the ports implies the restriction of the state space to a proper subset of $\mathbb{R}^n$, where n is the number of energy variables. Indeed each causal constraint corresponds to some linear relation on the efforts

$$\sum_{i=1}^{n} \alpha_i(x) e_i = 0, \tag{81}$$

and according to the definition of the effort variables at the ports of an energy storage element this relation also induces a constraint on the energy variables:

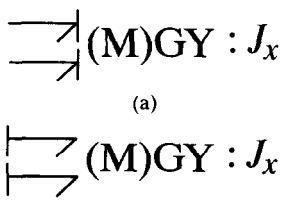


FIG. 17. Causal constraints on a two-port gyrator.

$$\sum_{i=1}^{n} \alpha_i(x) \frac{\partial H}{\partial x_i}(x) = 0, \tag{82}$$

where $H(x)$ denotes the energy function of the n-port energy storage element.

In an analogous way, causal restrictions on the flows at the ports connecting the generalized junction structure and the energy storage elements induce a restriction on the time variation of the energy variables and hence on the dynamics of the system which will be addressed in the following section.

Example 15. (Bond graph model of an electrical circuit.) Consider again the electrical circuit of Example 2 and represented in Fig. 1. A generalized bond graph model of this circuit is depicted in Fig. 18a. It contains three one-port energy storage elements: two of them represent the inductances L_1 and L_2, and the third one represents the capacitor C. The topology of the electrical circuit is realized by a simple junction structure constituted by a 1-junction with the bonds connected according to Fig. 11. The symplectic gyrator indicates the coupling between the magnetic and electrical domains (as in Example 12). The simple junction structure together with the symplectic gyrator forms the generalized junction structure connecting the three energy storage elements in the complete generalized bond graph model of the electrical circuit. The causally augmented bond graph depicted in Fig. 18a shows that it is possible to assign integral causality to all energy storage elements: the efforts at their ports are independent. As the energy variables are related one-to-one to the effort variables through the constitutive relation of the energy storage elements (here even linear relations), the dimension of the state-space is three. Considering now the flows at the ports of the generalized junction structure, the causally augmented bond graph of Fig. 18b shows the maximum number of independent flows: at most two causal strokes may be directed outwards. This corresponds to a constraint on the flows which is here quite simple: the common flow expressed by the 1-junction induces the following relation on the generalized flows:

$$\dot{\phi}_1 = v_1 = -v_2 = -\dot{\phi}_2. \tag{83}$$

This relation indeed restricts the time variation of the energy variable to remain within a plane defined by Eq. (84)

$$\phi_1 + \phi_2 = K, \tag{84}$$

where K is some real number.

Example 16. (Bond graph model of a one-dimensional mechanical system.) Consider now the one-dimensional mechanical system of Example 6 and represented in Fig. 2. Its generalized bond graph model is analogous to Fig. 18 by substituting the momenta p_1 and p_2 for the magnetic flux linkages, the displacement q_{12} of the spring for the electrical charge, the masses m_1 and m_2 for the inductances and finally the compliance C for the capacitance. The simple junction structure represents the force balance (and dually the kinematic constraints) in the system. The symplectic gyrator indicates the coupling between the elastic and the kinetic

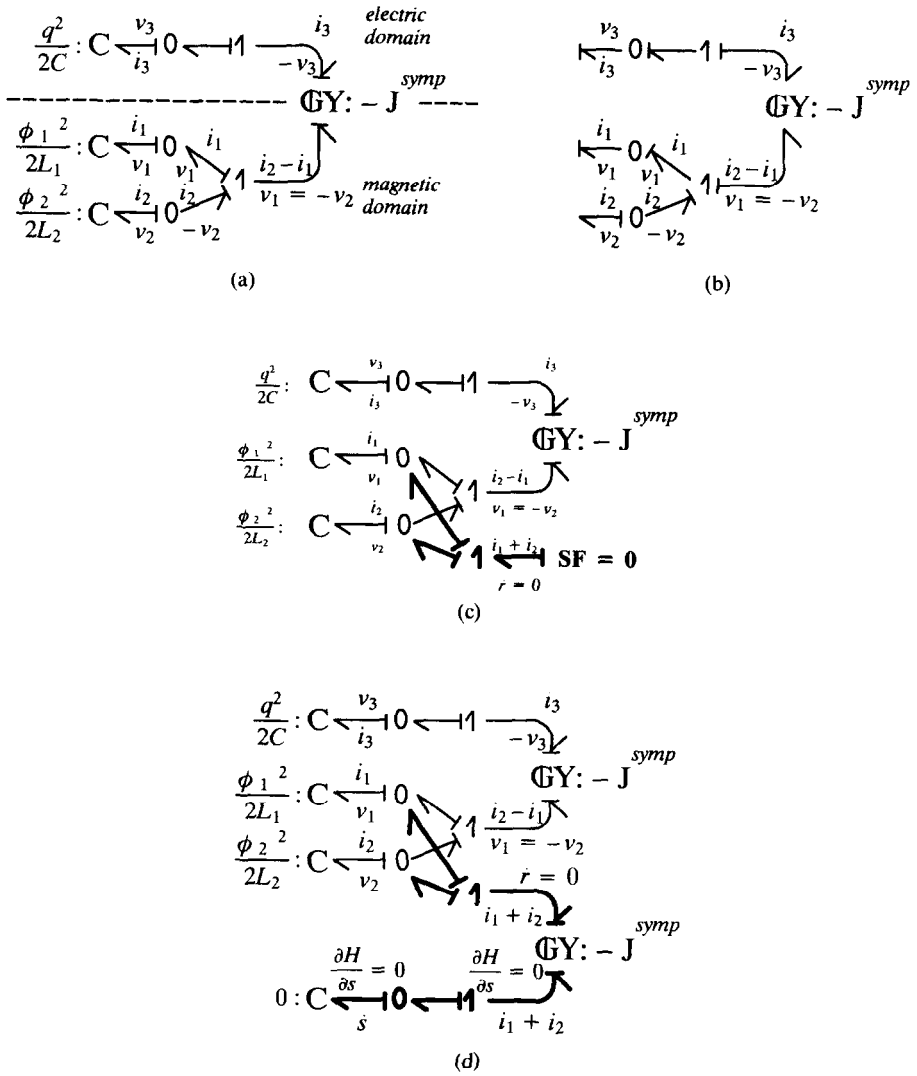


FIG. 18. Bond graph models of the dynamics of an electrical circuit.

domains. The causal strokes in Fig. 18 indicate in the same way that the dimension of the state space is three, but that the time variation of the state variables is restricted to the force balance relation:

$$\dot{p}_1 = F_1 = -F_2 = -\dot{p}_2, \tag{85}$$

where F_1 and F_2 denote the forces exerted on the masses.

Example 17. (Bond graph model of a rigid body in free motion.) The model of

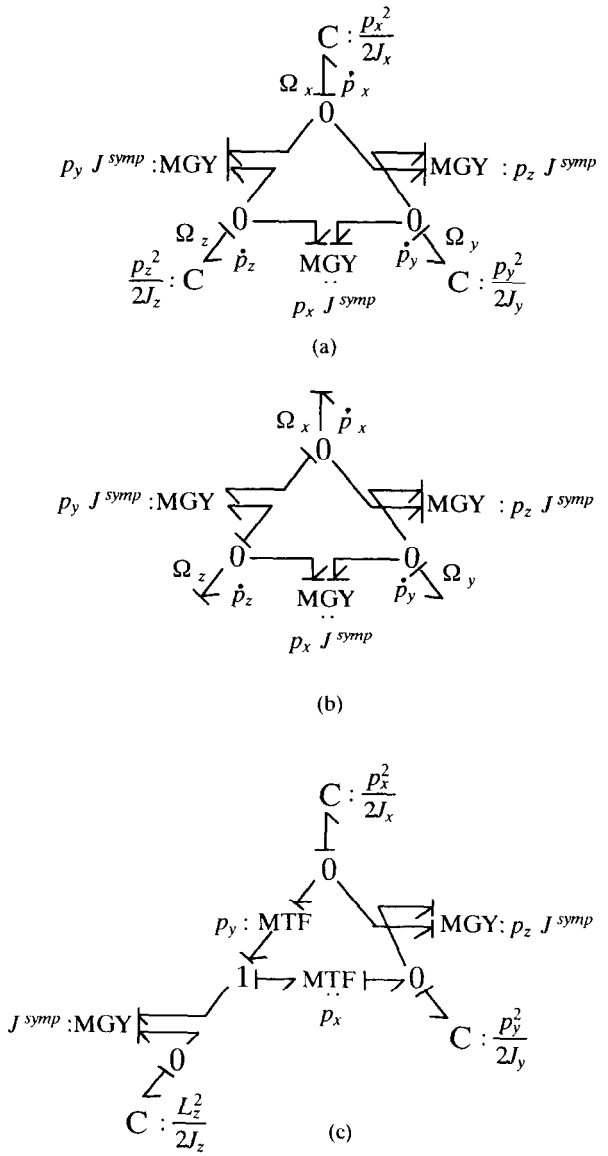


FIG. 19. Bond graph models of rigid body dynamics.

a rigid body in free motion is depicted in Fig. 19. The three one-port energy storage elements represent the three components of the rotational kinetic energy of the body along its principal axis with the projections p_x, p_y, p_z of the total angular momentum on the principal axes as energy variables. The generalized junction structure of Fig. 19 connected to the three capacitors provides the bond graph realization of Euler's equations (30). It is different from the "Eulerian junction

structure" proposed by Karnopp (**49**) which is a nonlinear element in the velocities, whereas here it is modulated by the momenta. The causality assignment of Fig. 19a shows that the three efforts at the ports of the generalized junction structure are independent and hence the state space has dimension three. Figure 19b gives at most two independent flows at the ports of the energy storage elements; thus the variations of the energy variables are constrained on the so-called "invariant plane" (**2**) defined by Eq. (86):

$$p_x\dot{p}_x + p_y\dot{p}_y + p_z\dot{p}_z = 0. \tag{86}$$

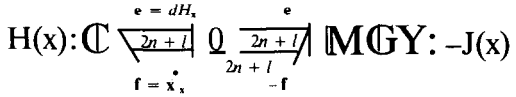
3.2. *The dynamics of the energy variables as a Hamiltonian system*

Here we are concerned with the analytical formulation of the dynamics associated with the energy variables of a complete bond graph model: an n-port energy storage element connected to a generalized junction structure. For the sake of simplicity the following assumption is made on the causality at the ports of the generalized junction structure: it admits flow causality at all its ports. According to the preceding section, this means that the efforts at the ports are independent. Hence the generalized junction structure is equivalent at its ports to a modulated gyrator (with constitutive matrix equal to minus the junction structure matrix J_x due to the power bond orientation according to Fig. 20a) with constitutive equation (80). Furthermore the assumption on the independency of the efforts at the ports of the generalized junction structure means that the energy variables of the connected energy storage element are independent, which implies that the state space is simply $\mathbb{R}^n$.

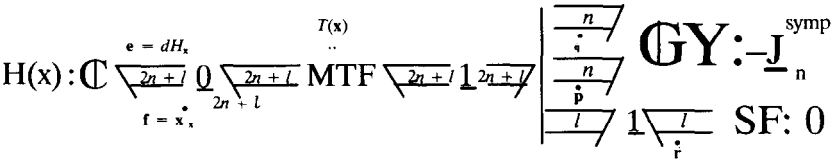
If the causal assumption is not satisfied, then the space of the energy variables (i.e. the state space) is restricted to a proper subset of $\mathbb{R}^n$, but the results presented in this section remain valid provided that one restrict the maps defined hereafter to this proper subset of $\mathbb{R}^n$ induced by the causal constraints on the efforts at the ports of the generalized junction structure (**27**).

Let us first interpret the constitutive relations of an energy storage element in differential geometric terms. The energy function $H(x)$ of the energy storage element defines a smooth real-valued function on $\mathbb{R}^n$, the space of the energy variables x. Identifying the tangent bundles $T\mathbb{R}^n$ of $\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$ and the cotangent bundle $T^*\mathbb{R}^n$ with $\mathbb{R}^n \times (\mathbb{R}^n)^*$, the power variables at the ports of the energy storage element may be interpreted as follows. The flow variable f, describing the time variation $\dot{x}$ of the energy variable is an element of the tangent space $T_x\mathbb{R}^n$ at the state x. The effort variable e is the differential $dH(x)$ of the energy function $H(x)$ at x, thus is a cotangent vector belonging to $T_x^*\mathbb{R}^n$. Then the constitutive relation of an energy storage element defined in Eq. (65) may be defined as the local definition of the application given in Diag. 2.

In the same way the relations on the power variables at the ports of the generalized junction structure may be interpreted in differential geometric terms. Taking into account the assumption on the causality restriction at its ports, the generalized junction structure induces algebraic relations on the power variables at its ports equivalent to the constitutive relations of a gyrator which may be modulated (see Eq. 80). One may again identify the flow variables at its ports with an element of



(a)



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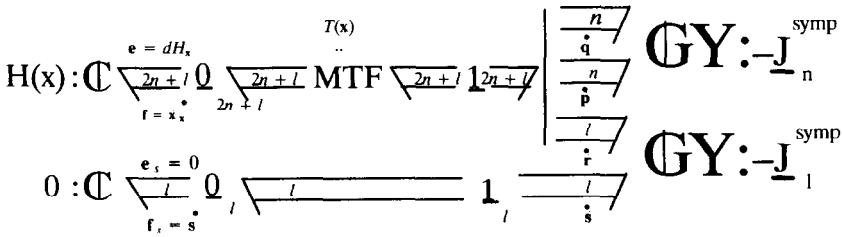

(c)

FIG. 20. Bond graph interpretation of the Hamiltonian reduction procedure.

the tangent space $T_x\mathbb{R}^n$ at x and the effort variables as an element of the cotangent space $T_x^*\mathbb{R}^n$ at x. Then the causal relations at the ports of the generalized junction structure correspond to Diag. 3, where J_x denotes the junction structure matrix. This diagram would correspond to the mapping from one-forms to Hamiltonian vector fields defined by a Poisson bracket if the junction structure matrix J_x satisfies the constitutive assumptions of structure matrices of Poisson structures. These

$$e : \mathbf{R}^n \xrightarrow{dH} T^*(\mathbf{R}^n)$$
$$x \longmapsto e_x = dH_x$$

$$Id : T(\mathbf{R}^n) \xrightarrow{Id} T(\mathbf{R}^n)$$
$$f_x \longmapsto f_x = \dot{x}_x$$

DIAG. 2.

$$\begin{array}{ccc} (\mathbf{R}^n)^* & \xrightarrow{J_x} & \mathbf{R}^n \\ \mathbf{e}_x & \longmapsto & \mathbf{f}_x \end{array}$$

DIAG. 3.

constitutive assumptions are given by Eqs (11) and (12), and will now be discussed with respect to the constitutive assumptions on junction structure matrices (or gyrators' constitutive relations). Firstly in order to ensure the power continuity of the generalized junction structure, the matrix J_x has to be skew-symmetric (i.e. antisymmetric) (**13**); this is exactly the condition (11). On the other hand, the Jacobi conditions (12) on a structure matrix (13) are usually not required for a (generalized) junction structure matrix or the constitutive matrix of a gyrator. For Hamiltonian systems, they play an essential role (see Section II) since they are necessary and sufficient conditions in order to ensure that an arbitrary choice of a Hamiltonian function induces a vector field which leaves the Poisson structure, i.e. the geometric structure, of the system invariant. Hence in an analogous way for networks they are necessary and sufficient conditions to ensure that one may choose arbitrary energy functions for the energy storage elements independently of the generalized junction structure. It is interesting to note moreover that as a consequence of the Jacobi conditions, the invariants on the flow variables at the ports of the generalized junction structure may be related one-to-one to invariants on the energy variables whatever the actual energy function is. Physically this may be explained by the very general derivation of power continuity from the conservation of physical quantities like the electrical charge or the kinetic momentum (**11**, chaps 4 and 5). In terms of Hamiltonian systems, the Jacobi conditions allow to express the degeneracy of the Poisson structure (corresponding to invariants on the flow variables) as symmetries on the Hamiltonian function (defined on the state variables) of some embedding standard Hamiltonian systems as recalled in Section II. Now we may consider again the complete bond graph model with causality assigned according to the assumptions on the generalized junction structure and represented on Fig. 20a. Assembling the diagrams representing the maps defined above, one obtains the map represented in Diag. 4. This map is the differential geometric interpretation of the usual construction of the dynamics of bond graph models using causal paths in a generalized junction structure. Moreover, if the junction structure matrix verifies the skew-symmetry and the Jacobi conditions (see Eq. (12)), then the dynamics corresponds to a Hamiltonian system with Hamiltonian function equal to the energy function $H(x)$ on the Poisson manifold admitting the junction structure matrix as structure matrix of its Poisson structure.

$$\begin{array}{ccccc} \mathbf{R}^n & \xrightarrow{dH} & T^*(\mathbf{R}^n) & \xrightarrow{J_x} & T(\mathbf{R}^n) \\ x & \longmapsto & dH_x & \longmapsto & X_{H_x} \end{array}$$

DIAG. 4.

Example 18. (Dynamics of an electrical circuit.) Consider the circuit of Example 2 and its bond graph representation in Fig. 18a. The array of three one-port energy storage elements may be interpreted as one three-port energy storage element with energy function:

$$H(Q, \phi_1, \phi_2) = \frac{Q^2}{2C} + \frac{\phi_1^2}{2L_1} + \frac{\phi_2^2}{2L_2} \tag{87}$$

with Q the charge of the capacitor and ϕ_1, ϕ_2 the magnetic flux linkages of the inductors. The energy function is the sum of the energy associated with each one-port energy storage element. The constitutive relation of the three-port capacitor is then linear and diagonal:

$$e = \begin{pmatrix} v_3 \\ i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 1/C & 0 & 0 \\ 0 & 1/L_1 & 0 \\ 0 & 0 & 1/L_2 \end{pmatrix} \begin{pmatrix} Q \\ \phi_1 \\ \phi_2 \end{pmatrix}. \tag{88}$$

The causal relations at the ports of the junction structure in Fig. 18a induces the following relation:

$$f = \begin{pmatrix} i_3 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_3 \\ \phi_1 \\ \phi_2 \end{pmatrix}. \tag{89}$$

The matrix defining the linear input–output relation at the ports of the junction structure is the junction structure matrix. The relation (89) corresponds to the relations induced by the fundamental loop and cutset matrices associated with the tree in Fig. 14. Furthermore the junction structure matrix is skew-symmetric and linear (and thus it satisfies necessarily the Jacobi identities). Thus it defines a Poisson structure on $\mathbb{R}^3$. The combination of Eqs (87), (88) and (89), according to Diag. 4, defines the dynamics of the electrical circuit as Hamiltonian dynamics on $\mathbb{R}^3$ endowed with the Poisson structure defined by the junction structure matrix, with the sum of the electrical and magnetic energy functions as Hamiltonian function:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} Q \\ \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial Q} \\ \dfrac{\partial H}{\partial \phi_1} \\ \dfrac{\partial H}{\partial \phi_2} \end{pmatrix}. \tag{90}$$

One recovers the dynamics presented in Example 2; however it was now deduced using the causal procedure on the causally augmented bond graph model (**12**, **42**).

Example 19. (Dynamics of a mechanical system.) The one-dimensional mech-

anical system of Example 6 has a bond graph model analogous to Fig. 18a (see Example 16). Hence its dynamics may also be expressed as a Hamiltonian system. Its Hamiltonian function is the sum of the elastic potential energy of the spring and the kinetic energies of the two masses:

$$H(x_{12}, p_1, p_2) = \frac{kx_{12}^2}{2} + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}. \tag{91}$$

This energy function implies the following constitutive relation at the ports of the three-port energy storage element:

$$e = \begin{pmatrix} F_{el} \\ v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} \dfrac{\partial H}{\partial x_{12}} \\ \dfrac{\partial H}{\partial p_1} \\ \dfrac{\partial H}{\partial p_2} \end{bmatrix} = \begin{bmatrix} kx_{12} \\ \dfrac{p_1}{m_1} \\ \dfrac{p_2}{m_2} \end{bmatrix}. \tag{92}$$

The generalized junction structure induces the following causal relations at the ports of the junction structure:

$$f = \begin{pmatrix} \dot{x}_{12} \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} e. \tag{93}$$

Combining Eqs (92) and (93) one recovers the Hamiltonian dynamics:

$$f = \begin{pmatrix} \dot{x}_{12} \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial x_{12}} \\ \dfrac{\partial H}{\partial p_1} \\ \dfrac{\partial H}{\partial p_2} \end{bmatrix}. \tag{94}$$

Example 20. (Rigid body dynamics.) Consider again the rigid body in free motion (Example 3) and its bond graph model in Fig. 19a. The array of the three one-port energy storage elements may be grouped into one three-port energy storage element. Its energy variable is simply the angular momentum p whose projections p_x, p_y, p_z on the three principal axes of the rigid body are the energy variables of the one-port energy storage elements. The energy function is the rotational kinetic energy of the rigid body and is equal to the sum of its components on the three principal axes:

$$H(p_x, p_y, p_z) = \frac{p_x^2}{2J_x} + \frac{p_y^2}{2J_y} + \frac{p_z^2}{2J_z}. \tag{95}$$

The causal relations at the ports of the generalized junction structure are:

$$f = \begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix} e. \tag{96}$$

The junction structure matrix of the bond graph model in Fig. 19a corresponds exactly to the structure matrix of the Poisson structure of the Euler equations (see Example 3).

The constitutive relations of the three-port energy storage element with energy function $H(p)$ give:

$$e = \mathrm{d}H_p = \begin{pmatrix} \frac{p_x}{J_x} \\ \frac{p_y}{J_y} \\ \frac{p_z}{J_z} \end{pmatrix}, \quad f = \begin{pmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{pmatrix}. \tag{97}$$

Combining Eqs (96) and (97) according to Diag. 4, i.e. following the causal strokes in the bond graph model, one recovers the Hamiltonian dynamics defined by the Euler equations:

$$\begin{pmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{pmatrix} = \begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix} \begin{pmatrix} \frac{p_x}{J_x} \\ \frac{p_y}{J_y} \\ \frac{p_z}{J_z} \end{pmatrix}. \tag{98}$$

3.3. *Relation with standard Hamiltonian systems and their bond graph realization*

Now we shall discuss the relation of the dynamics of the energy variables of a bond graph model with associated reduced and embedding standard Hamiltonian systems. The discussion focuses on their bond graph realization, and special attention will be paid to the choice of coordinates of each representation.

First, the dynamics of the energy variables is discussed with respect to the reduced standard Hamiltonian system obtained through the reduction by the dynamical invariants. Its bond graph realization is based upon the analytical decomposition of the generalized junction structure (**13**): it makes appear explicitly its structure and identifies the canonical variables in the decomposed bond graph. Secondly, the bond graph realization of an embedding standard Hamiltonian system will be obtained by the addition of some energy storage element with an energy function which is identically equal to zero. Again, the associated symplectic structure and canonical coordinates will be identified with some elements and flow variables of the bond graph realization.

3.3.1. *The reduced standard Hamiltonian system.* Let us consider the Hamiltonian system describing the dynamics of the general bond graph model of Fig. 20a. In order to reduce this system through its invariants, let us first consider its expression in canonical coordinates following Section II. According to Theorem II, one knows that there exists (locally) a change of coordinates denoted by ϕ which provides canonical coordinates:

$$\phi(x) = (q, p, r), \tag{99}$$

where q and p are in $\mathbb{R}^n$ and r is in $\mathbb{R}^l$ if $2n$ is the rank of the Poisson structure and l is the dimension of its kernel.

In these coordinates, the structure matrix J_x of the Poisson structure takes the form:

$$J(q,p,r) = \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{100}$$

Consequently the structure matrix is split into a symplectic matrix of rank $2n$ and a zero matrix corresponding to its kernel.

Moreover the change of coordinates ϕ locally induces the two following adjoint maps: the tangent map $(\phi^{-1})_{*x}$ and the cotangent map $(\phi^{-1})_x^*$ on the tangent and cotangent bundles of $\mathbb{R}^{2n+l}$. In coordinates, $(\phi^{-1})_{*x}$ may be represented by a matrix T_x (the subscript indicates that this matrix in general is a function of x); while the adjoint map $(\phi^{-1})_x^*$ is represented by the transpose T_x^t of the real-valued matrix T_x. The matrix T_x defines the analytical decomposition of the generalized junction structure (**13**). The corresponding bond graph model is shown in Fig. 20b: the generalized junction structure is decomposed into a modulated multiport transformer with constitutive matrix T_x relating the energy storage elements to a $2n$-port (symplectic) gyrator with constitutive matrix J_n^{symp}:

$$J_n^{\text{symp}} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{101}$$

and a zero-flow source of dimension l.

This decomposition of the bond graph model was obtained in an analytical way and may or may not coincide with the junction structure obtained from physical considerations. It makes explicit the causal constraints on the flows at the ports of the generalized junction structure in the form of the zero-flow source of dimension l. According to the definition of the transformation matrix T_x, the generalized flows at the ports of the zero-flow source are the time-variation of the redundant variables, which obviously obey the dynamic equation:

$$\dot{r} = 0. \tag{102}$$

In the same way the generalized flow variables at the ports of the $2n$-port symplectic gyrator correspond to the time-variations $\dot{q}$ and $\dot{p}$ of the canonical coordinates of the reduced system. Thus they obey the constitutive relation:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J_n^{\text{symp}} \begin{pmatrix} \dfrac{\partial H}{\partial q} \\ \dfrac{\partial H}{\partial p} \end{pmatrix}. \tag{103}$$

Finally, one may observe that considering the energy flows, only the power bonds at the ports of the symplectic gyrator contribute to the variation of the energy in the energy storage elements. Indeed the power bonds with the flow variables being the variation $\dot{r}$ of the redundant variables convey a power equal to zero.

Example 21. Reconsider the electrical circuit of Fig. 1 and its bond graph model in Fig. 18a. The analytically decomposed generalized junction structure, defined by the change of variables (51), is shown in Fig. 18c. Compared to the physical junction structure in Fig. 18a, a zero-flow source was added and a junction structure relating it to the energy storage elements (they are denoted by the bold bonds in Fig. 18c). The flow variable at the port of the zero flow source is

$$0 = \dot{r} = \tfrac{1}{2}(v_1 + v_2) = \tfrac{1}{2}(\dot{\phi}_1 + \dot{\phi}_2). \tag{104}$$

The zero-flow source shows explicitly the conservation of the total magnetic flux. The flow variables at the ports of the symplectic gyrator are

$$\dot{q} = i_3 = \dot{Q}, \quad p = \tfrac{1}{2}(v_2 - v_1) = \tfrac{1}{2}(\dot{\phi}_2 + \dot{\phi}_1). \tag{105}$$

The symplectic gyrator induces the dynamics of the reduced Hamiltonian system: here it is a linear harmonic oscillator, This reduced Hamiltonian system at the ports of the symplectic gyrator is exactly the system obtained by Bernstein and Lieberman (7). The change of coordinates performed in this example is the alternative change of coordinates presented in (7) and leaving the capacitors' charges invariant; they call the redundant variables r "latent" variables and the canonical coordinates of the reduced system "active variables". Summarizing, the analytical decomposition of the generalized junction structure corresponds, for the electrical circuit, to the addition of a simple junction structure and a zero-flow source to the original physical junction structure. This additional junction structure is necessary in order to define a regular change of coordinates, but is redundant in the sense that its power bonds convey a power identically equal to zero. Moreover, the generalized bond graph formalism uses the symplectic gyrator as a physical coupling element of the network: at its ports one reads directly the reduced Hamiltonian dynamics. This explains why one may define a topological reduction procedure of the dynamics of electrical circuits as in (7).

Example 22. Consider now the dynamics of the rigid body treated in Examples 3 and 17 and its bond graph representation in Fig. 19a. The change of variables (60) would lead in the same way as for the electrical circuit, to the decomposition of the junction structure into a two-port symplectic gyrator and a zero-flow source. At the ports of the symplectic gyrator one recovers the reduced standard Hamiltonian dynamics and the zero-flow source represents explicitly the constraint on the generalized flows:

$$p_x\dot{p}_x + p_y\dot{p}_y + p_z\dot{p}_z = 0. \tag{106}$$

The junction structure matrix verifies the Jacobi conditions, hence the constraints on the flow variables imply the existence of some invariant on the energy variables which is here the conservation of the total momentum:

$$p_x^2 + p_y^2 + p_z^2 = k, \tag{107}$$

where k is some constant.

In contrast to the electrical circuit models, the analytical decomposition destroys the original Eulerian junction structure realization. Furthermore, the analytical decomposition of the generalized junction structure implies the use of varying junction structures according to the singularity of the chosen coordinate transformation. For instance, the coordinate transformation defined in (60) gives the following singularity on the transformers moduli:

$$p_y^2 + p_z^2 = 0. \tag{108}$$

Three different decomposed junction structures, obtained by circular permutation of p_x, p_y, p_z are necessary to cover the entire state space. This disadvantage of the analytical canonical decomposition of generalized junction structure may be avoided using a graphical transformation. This transformation is based on the "partial dualization" of the junction structure (**50**). It consists in making symplectic gyrators appear at some ports of the junction structure by dualization, i.e. exchanging the flows and effort variables at these ports, and minimizing the number of the gyrators remaining in the junction structure. In the case of the Eulerian junction structure (Fig. 19a), a partial dualization leads to the junction structure in Fig. 19c. It is obtained by dualization of the port attached to the energy storage element representing the kinetic energy along the x-axis of the rigid body. Then the zero junction connected to this port and the gyrators attached to this junction are dualized into a 1-junction and two transformers, leaving one gyrator in the junction structure. In general the minimal sets of gyrators remaining in the junction structure after partial dualization at its ports are called "essential" (**50**). At the ports of the symplectic gyrator one may again read the reduced standard Hamiltonian dynamics. But the main advantage of the dualization procedure with respect to the analytical decomposition is that the reduced Hamiltonian system is explicit graphically without generating any singularity on the remaining dualized junction structure.

3.3.2. *The embedding standard Hamiltonian system.* Reconsider the general bond graph model in Fig. 20a and its dynamics described by (15). In the previous paragraph we have seen that the regular change of coordinates into canonical coordinates leads to the analytical decomposition of the junction structure into a symplectic gyrator (corresponding to the reduced dynamics) and makes appear a zero-flow source representing the singularity of the junction structure. However, according to Section II, one may further require the regularity of the geometric structure of the dynamical system considering the embedding standard Hamiltonian system (50).

Its bond graph realization arises from the decomposed junction structure in Fig. 20b; the embedding system is defined in canonical coordinates (q, p, r) associated with the Poisson structure. In order to regularize the geometric structure of the system, the additional coordinate s is introduced, conjugated to variable r (see Eq. 45). In bond graph terms, the generalized junction structure of the embedding system is realized by introducing a bond carrying the flow variable $\dot{s}$ and a symplectic gyrator of dimension $2l$ representing the conjugacy of $\dot{s}$ with $\dot{r}$ (see Fig. 20c). Hence the augmented generalized junction structure is now regular. On the extended state space the Hamiltonian function of the embedding system (42) is:

$$\hat{H}(q, p, s, r) = H(q, p, r) + 0(s), \tag{109}$$

where $0(s)$ represents the function of s equal to zero.

Equation (109) simply expresses that the Hamiltonian function H is independent of s, the vector of the conserved quantities of the embedding system. Thus the energy associated with s is equal to zero: this is represented by an energy storage element attached to the bond carrying $\dot{s}$ with an energy function equal to zero (see Fig. 20c). Now the dynamics of the canonical coordinates (q, p, s, r) reads at the ports of the two symplectic gyrators. At the ports of the symplectic gyrator of dimension $2n$ one reads the dynamics of the reduced Hamiltonian system considering (109):

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J_n^{\text{symp}} \begin{pmatrix} \dfrac{\partial \hat{H}}{\partial q} \\ \dfrac{\partial H}{\partial p} \end{pmatrix} = J_n^{\text{symp}} \begin{pmatrix} \dfrac{\partial H}{\partial q} \\ \dfrac{\partial H}{\partial p} \end{pmatrix}. \tag{110}$$

In the same way, at the ports of the symplectic gyrator of dimension $2l$, one reads the dynamics of the canonical variables s and r. The time variation of the redundant variables r_i becomes:

$$\dot{r}_j = \frac{\partial 0(s)}{\partial s_j} = 0, \quad j \in 1, \ldots, l. \tag{111}$$

The invariance of the vector of redundant variable r is no longer expressed by a zero-flow source (as in Fig. 20b) but by the degeneracy of an energy storage element (Fig. 20c) which has an energy function equal to zero and thus constrains the efforts at its ports to remain equal to zero. In network terms this means that this storage element is equivalent at its port to a zero-effort source; combined with the symplectic gyrator this results again in an equivalent zero-flow source. The flow variables at the ports of the energy storage elements are not causally constrained by the extended generalized junction structure which is regular, but by the non-invertibility of the constitutive relation (65) of the energy storage with energy function equal to zero. The dynamics of the conserved quantities s is also read at the ports of the symplectic gyrator of dimension $2l$:

$$\dot{s}_j = -\frac{\partial \hat{H}(q, p, s, r)}{\partial r_j} = \frac{\partial H(x)}{\partial r_j}, \quad j \in 1, \ldots, l. \tag{112}$$

Thereby the efforts $\partial H(x)/\partial r_j$ conjugated to the zero-flow appearing with the analytical decomposition of a generalized junction structure, may be interpreted as the time variation $\dot{s}_j$ of the conserved quantities of the embedding standard Hamiltonian system.

Example 23. The bond graph representation of the embedding system associated with the electrical circuit in Fig. 18a is shown in Fig. 18d. It was deduced from the decomposed junction structure in Fig. 18c by replacing the flow source by a gyrator and an energy storage element with an energy function equal to zero. At the ports of the symplectic gyrator one reads:

$$\dot{s}_j = -(i_1+i_2), \quad (v_1+v_2) = \dot{r} = e_s = 0, \tag{113}$$

where e_s denotes the effort at the port of the degenerate energy storage element.

Thus the degeneracy of the additional energy storage element ensures the constraint on the voltages. The current type variable $-(i_1+i_2)$ conjugated to the redundant variable may be interpreted as the time variation of the conserved quantity of the embedding standard Hamiltonian system. It may further be noticed that the embedding system does *not* have a realization in electrical circuit terms as the simple junction structure in Fig. 18c, d is not graphical (**45**).

Example 24. Take again the one-dimensional mechanical system of Example 6, which has a bond graph representation analogous to the electrical circuit, and consider the bond graph representation of its embedding standard Hamiltonian system in Fig. 18b. Then at the ports of the additional symplectic gyrator, one reads again the dynamics of the redundant variable $r = (p_1+p_2)$, i.e. the total momentum of the system and of its conjugated conserved quantity:

$$\dot{s} = -(v_1+v_2), \quad (\dot{p}_1+\dot{p}_2) = \dot{r} = e_s = 0. \tag{114}$$

The conserved quantity may be interpreted as the position (up to some multiplicative constant) of the masses.

IV. Conclusion

The structural relations between network representations of systems not containing resistive (or irreversible) phenomena (such as LC electrical circuits or mechanical systems without friction) and the Hamiltonian formalism of analytical mechanics were investigated.

To this aim, generalized Hamiltonian systems were considered, i.e. systems defined on degenerate Poisson manifolds which appear in the context of reduction of standard Hamiltonian systems with symmetry. This generalization allows in particular the handling of state spaces of odd dimension and the explication of the invariants of the system in terms of the geometric structure, i.e. the Poisson structure. One interesting consequence of the use of Poisson structures is that it implies the use of dual sets of variables.

The bond graph notation was used for the network representation of systems and the generalized bond graph formalism allowed to make the relation between

the network and Hamiltonian formalisms. Indeed its main feature is to symmetrize all energetic network elements in a single energy storage element and accordingly to explicate the coupling between different physical domains by a unit gyrator element, thus endowing the network including the gyrators (and called generalized junction structure) with a gyrator-type constitutive relation at its ports.

Then the relation between the network and the Hamiltonian formalism is made by matching the energy function associated with the energy storage elements of the network with the Hamiltonian function and the generalized junction structure relations at its ports with the local definition of a Poisson structure. It was proved that the dynamic equations associated with the energy variables (of the energy storage element) is Hamiltonian and that the topological (or graphical) construction of these dynamics matches exactly the analytical definition of Hamiltonian dynamics.

The crucial point of the relation of the network and Hamiltonian formalisms is the representation of the dynamic invariants. Indeed the (general) Poisson structure explicates the invariants of the system as the elements of its kernel. These may be related with the algebraic and topologic constraints induced by the generalized network (the generalized junction structure) at its port variables.

Furthermore, a network interpretation of the relation with the embedding and reduced standard Hamiltonian systems was given. The reduced Hamiltonian system was shown to correspond to the generalized flows at the ports of the symplectic gyrators, appearing as coupling elements between the different physical domains or by the analytical decomposition of the generalized junction structure. The embedding system was constructed using a degenerate energy storage element with energy function equal to zero and the symmetry variables of the embedding system as energy variable.

In summary, the consideration of generalized Hamiltonian systems defined with respect to degenerate Poisson structure is essential for an intrinsic analytical representation of the main features of network representations: the duality of port variables and the invariants intrinsically given by the network topology.

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