A state transfer principle for switching port-Hamiltonian systems

Schaft, A.J. van der; Camlibel, M.K.

[Link to publication in University of Groningen/UMCG research database](#)

Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

WeA02.2

A state transfer principle for switching port-Hamiltonian systems

A.J. van der Schaft, M.K. Camlibel

Abstract—**Instantaneous charge/flux transfers may occur in switched electrical circuits when the switch configuration changes. Characterization of such state discontinuities is a classical issue in circuit theory which, typically, is based on the so-called charge and flux conservation principle. This paper proposes a general state transfer principle for arbitrary switching port-Hamiltonian systems. This new principle coincides with the charge and flux conservation principle in the special case of linear RLC circuits, but also covers circuits with nonlinear capacitors and inductors, and of arbitrary topology. Moreover, the new principle is applied to switching mechanical systems.**

I. Introduction

A classical notion in electrical circuit theory concerns the characterization of the discontinuous change in the charges of the capacitors and/or in the magnetic fluxes of the inductors whenever switches are instantaneously closed or opened. This is sometimes referred to as the *charge and flux conservation principle*, and is usually discussed on the basis of examples [14]. Recently, this notion has been articulated in [7], and formulated (for RLC circuits with independent elements) in the following general sense. The discontinuous change in the charges whenever switches are closed corresponds to an impulsive current satisfying Kirchhoff's current laws for the circuit (under the new switch configuration) where the inductors, resistors and external ports have been open-circuited. Dually, the discontinuous change in the fluxes resulting from opening switches corresponds to an instantaneous voltage drop satisfying Kirchhoff's voltage laws for the circuit where the capacitors, resistors and external ports have been short-circuited. In the present paper, we extend this result to arbitrary port-Hamiltonian systems by stating a general state transfer principle whenever switches are being opened or closed. The discontinuous change of the state involved in the transfer principle amounts to an impulsive motion satisfying a set of conservation laws derived from the general conservation laws of the port-Hamiltonian system.

We show how in the case of RLC circuits with independent elements one recovers from this state transfer principle the charge and flux conservation principles as formulated in [7], and how it extends this formulation to RLC-circuits of arbitrary topology and arbitrary constitutive relations for the capacitors and inductors. If the energy function is convex and bounded from below we prove that the switching port-Hamiltonian system satisfying the state transfer principle is *passive*. As a second class of systems we apply the state transfer principle to switching mechanical systems.

Institute of Mathematics and Computing Science University of Groningen, PO Box 407, 9700 AK Groningen, The Netherlands.

II. Switching port-Hamiltonian systems

Underlying the definition of a port-Hamiltonian system is the notion of a Dirac structure, which relates the power variables of the composing elements of the system in a power-conserving manner. The power variables always appear in conjugated pairs (such as voltages and currents, or generalized forces and velocities), and therefore mathematically they are modelled to take their values in dual linear spaces.

Definition 2.1: Let $\mathcal{F}$ be a linear space with dual space $\mathcal{E} := \mathcal{F}^*$, and duality product denoted as $< e \mid f > = e^T f \in \mathbb{R}$ for $f \in \mathcal{F}$ and $e \in \mathcal{E}$. We call $\mathcal{F}$ the space of *flow* variables, and $\mathcal{E} = \mathcal{F}^*$ the space of *effort* variables. Define on $\mathcal{F} \times \mathcal{E}$ the following indefinite bilinear form $\ll (f_1, e_1), (f_2, e_2) \gg = < e_1 \mid f_2 > + < e_2 \mid f_1 >$. A subspace $D \subset \mathcal{F} \times \mathcal{E}$ is a (constant[1]) *Dirac structure* if $D = D^{\mathrm{orth}}$, where D^{orth} is the orthogonal complement of D with respect to this indefinite bilinear form $\ll \cdot, \cdot \gg$.

Remark 2.2: For the case of a finite-dimensional linear space $\mathcal{F}$ (as will be the case throughout this paper) a Dirac structure is equivalently characterized [3], [8], [5] as a subspace such that $< e \mid f > = 0$ for all $(f, e) \in D$ together with $\dim D = \dim \mathcal{F}$. The property $< e \mid f > = 0$ for all $(f, e) \in D$ corresponds to *power conservation*.

For the definition of a *switching port-Hamiltonian system* we need the following ingredients (see [6], [9] for more restricted versions). We start with an overall Dirac structure D on the space of all flow and effort variables involved:

$$D \subset \mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P \times \mathcal{F}_S \times \mathcal{E}_S \quad (1)$$

The space $\mathcal{F}_x \times \mathcal{E}_x$ is the space of flow and effort variables corresponding to the *energy-storing* elements (to be defined later on), the space $\mathcal{F}_R \times \mathcal{E}_R$ denotes the space of flow and effort variables of the *resistive* elements, while $\mathcal{F}_P \times \mathcal{E}_P$ is the space of flow and effort variables corresponding to the external *ports* (or sources). Finally, the linear spaces $\mathcal{F}_S$, respectively $\mathcal{E}_S$, denote the flow and effort spaces of the ideal switches. Let s be the number of switches, then every subset $\pi \subset \{1, 2, \ldots, s\}$ defines a *switch configuration*, according to

$$e_S^i = 0, \quad i \in \pi, \qquad f_S^j = 0, \quad j \notin \pi \quad (2)$$

We will say that in switch configuration π, for all $i \in \pi$ the i-th switch is *closed*, while for $j \notin \pi$ the j-th switch is *open*.

For each fixed switch configuration π this leads to the following subspace D_π of the restricted space of flows and

---

[1]For the definition of Dirac structures on *manifolds* we refer to [3], [5].

efforts $\mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P$:

$$
\begin{aligned}
D_\pi \;=\; & \{(f_x, e_x, f_R, e_R, f_P, e_P) \mid \exists f_S \in \mathcal{F}_S, e_S \in \mathcal{E}_S \\
& \text{such that } e_S^i = 0, i \in \pi, \, f_S^j = 0, j \notin \pi, \text{ and} \\
& ((f_x, e_x, f_R, e_R, f_P, e_P, f_S, e_S) \in D\}
\end{aligned}
$$
(3)

For every π the subspace D_π defines a Dirac structure. Indeed, every switch configuration π given by (2) defines a Dirac structure on the space of flow and effort variables f_S, e_S of the switches, and D_π equals the composition of this Dirac structure with the overall Dirac structure D. Since the composition of any two Dirac stuctures is again a Dirac structure [16], [4] it thus follows that D_π is a Dirac structure.

The dynamics of the switching port-Hamiltonian system is defined by specifying, next to its Dirac structure D, the constitutive relations of the energy-storing and resistive elements. Let the Hamiltonian $H : \mathcal{X} \to \mathbb{R}$ denote the total energy at the energy-storage elements with state variables $x = (x_1, \cdots, x_n)$; i.e., the total energy is given as $H(x)$. In the sequel we will throughout take $\mathcal{X} = \mathcal{F}_x$, but $\mathcal{X}$ may also denote an n-dimensional manifold (in which case $\mathcal{F}_x$ is the tangent space to this manifold $\mathcal{X}$ at the state x). The constitutive relations between the state variables x, and the flow and effort vector of the energy-storing elements are given as[2]

$$
\dot{x} = -f_x, \quad e_x = \frac{\partial H}{\partial x}(x)
$$
(4)

This immediately implies the energy balance

$$
\frac{d}{dt}H = \frac{\partial^T H}{\partial x}(x)\dot{x} = -e_x^T f_x,
$$
(5)

The constitutive relations for the linear resistive elements are given as

$$
f_R = -Re_R, \quad R = R^T > 0,
$$
(6)

implying the power-dissipating property

$$
e_R^T f_R = -e_R^T R e_R < 0, \text{ for all } e_R \in \mathcal{E}_R, e_R \neq 0
$$
(7)

The geometric definition of a switching port-Hamiltonian system is given as follows:

Definition 2.3: Consider a Dirac structure (1), a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, and a resistive relation $f_R = -Re_R$. Then the dynamics of the corresponding switching port-Hamiltonian system is given as

$$
\left(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), -Re_R(t), e_R(t), f_P(t), e_P(t)\right) \in D_\pi
$$
(8)

at all time instants t during which the system is in switch configuration π.

In general the conditions (8) will define a set of *differential-algebraic equations* (DAEs); see below. It follows from the power-conservation property of Dirac structures and (5) that

[2]The vector $\frac{\partial H}{\partial x}(x)$ of partial derivatives throughout denotes a *column* vector.

during the time-interval in which the system is in a fixed switch configuration

$$
\frac{d}{dt}H = -e_R^T Re_R + e_P^T f_P \leq e_P^T f_P,
$$
(9)

thus showing *passivity for each fixed switch configuration* if the Hamiltonian H is bounded from below. For more information regarding port-Hamiltonian systems, their representations, and their properties, we refer to [15], [8], [5].
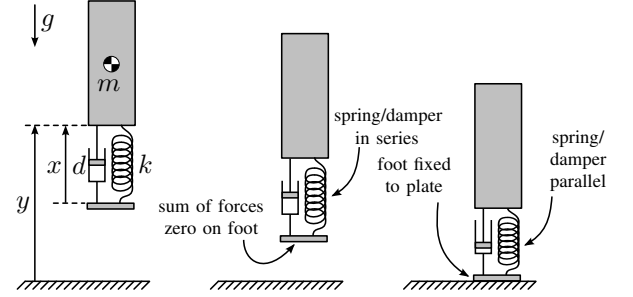

Fig. 1. Bouncing pogo-stick: definition of the variables (left), flying phase (middle), contact phase (right).

Example 2.4: Consider a pogo-stick that bounces on a horizontal plate of variable height (see Figure 1). It consists of a mass m and a mass-less foot, interconnected by a linear spring (with stiffness k and rest length x_0) and a linear damper d. The states of the system are x (length of the spring), y (height of the bottom of the mass), and $z = m\dot{y}$ (momentum of the mass). The total energy is

$$
H(x, y, z) = \frac{1}{2}k(x - x_0)^2 + mg(y + y_0) + \frac{1}{2m}z^2
$$

where y_0 is the distance from the bottom of the mass to the center of mass. The overall Dirac structure of the system is described by the linear equations

$$
\begin{aligned}
& f_y = f_x - f_S, \; f_R = f_x, \; f_z = e_x + e_y + e_R \\
& e_S + e_x + e_R = 0, \; e_z = -f_y
\end{aligned}
$$

whereas the constitutive relations are given as

$$
\begin{aligned}
& f_x = -\dot{x}, \; f_y = -\dot{y}, \; f_z = -\dot{z}, \; f_R = -\frac{1}{d}e_R \\
& e_x = k(x - x_0), \; e_y = mg, \; e_z = \frac{p}{m}
\end{aligned}
$$

In the switch configuration $e_S = 0$ (no external force on the foot) the pogo-stick is in its flying mode, while for $f_S = 0$ the foot is in contact with a horizontal plate.

The conditions (8) for a particular switch configuration π may entail algebraic constraints on the state variables x. These are characterized by the *constraint subspace* defined for each switch configuration π as follows:

$$
\begin{aligned}
C_\pi \;:=\; & \{e_x \in \mathcal{E}_x \mid \exists f_x, f_R, e_R, f_P, e_P, \text{ such that} \\
& (f_x, e_x, f_R, e_R, f_P, e_P) \in D_\pi, \; f_R = -Re_R\}
\end{aligned}
$$
(10)

(Note that C_π may depend on the resistive relation, but *not* on the energy storage constitutive relation.) The subspace C_π determines, together with H, the *algebraic constraints*

in each switch configuration π. Indeed, from (8) it follows that

$$\frac{\partial H}{\partial x}(x(t)) \in C_\pi \qquad (11)$$

for all time instants t during which the system is in switch configuration π. Hence if $C_\pi \neq \mathcal{E}_x$ then in general (depending on the Hamiltonian H) this imposes algebraic constraints on the state vector $x(t)$. (In the above example of the pogo-stick the subspace C_π is *equal* to $\mathcal{E}_x$ for any of the two switch configurations, and hence there are no algebraic constraints. This would change, however, if e.g. the mass of the foot is taken into account.)

Remark 2.5: Under non-degeneracity conditions on the Hamiltonian H, e.g. the Hessian of H being invertible, it can be shown [8] that the algebraic constraints are always of *index one*, implying that every state satisfying the algebraic constraints is a *consistent* state for the set of DAEs, from which a unique solution exists.

Next, we define for each π the *jump space*

$$J_\pi := \{f_x \mid (f_x, 0, 0, 0, 0, 0) \in D_\pi\} \qquad (12)$$

The following crucial relation between the jump space J_π and the constraint subspace C_π holds true. Recall that $J_\pi \subset \mathcal{F}_x$ while $C_\pi \subset \mathcal{E}_x$, where $\mathcal{E}_x = (\mathcal{F}_x)^*$.

Theorem 2.6:

$$J_\pi = C_\pi^\perp \qquad (13)$$

where $\perp$ denotes the orthogonal complement with respect to the duality product between the dual spaces $\mathcal{F}_x$ and $\mathcal{E}_x$.

Proof Let $\tilde{f}_x \in J_\pi$. Furthermore, consider any $e_x \in C_\pi$, that is, there exist f_x, f_R, e_R, f_P, e_P with $f_R = -Re_R$ such that $(f_x, e_x, f_R, e_R, f_P, e_P) \in D_\pi$. Since $(\tilde{f}_x, 0, 0, 0, 0, 0) \in D_\pi$ it follows that

$$0 = \ll (f_x, e_x, f_R, e_R, f_P, e_P), (\tilde{f}_x, 0, 0, 0, 0, 0) \gg = e_x^T \tilde{f}_x,$$

implying that $J_\pi \subset C_\pi^\perp$.

For the converse direction, we take any $\tilde{f}_x \in C_\pi^\perp$. It follows from $e_x^T \tilde{f}_x = 0$ for all $e_x \in C_\pi$ that

$$0 = e_x^T \tilde{f}_x = \ll (f_x, e_x, f_R, e_R, f_P, e_P), (\tilde{f}_x, 0, 0, 0, 0, 0) \gg$$

for all $(f_x, e_x, f_R, e_R, f_P, e_P) \in D_\pi$ with $f_R = -Re_R$. This implies that

$$(\tilde{f}_x, e_x = 0, f_P = 0, e_P = 0) \in (D_\pi \circ \tilde{R})^{\text{orth}}$$

where $\tilde{R}$ denotes the linear relation $\tilde{R} := \{(f_R, e_R) \mid f_R = -Re_R\}$, while $D_\pi \circ \tilde{R}$ denotes the composition of the relations D_π and $\tilde{R}$ via their shared variables f_R, e_R. It has been shown in [11] that

$$(D_\pi \circ \tilde{R})^{\text{orth}} = \{(f_x, e_x, f_P, e_P) \mid \exists \bar{f}_R, \bar{e}_R \text{ with}$$

$$\bar{f}_R = R\bar{e}_R \text{ such that } (f_x, e_x, \bar{f}_R, \bar{e}_R, f_P, e_P) \in D_\pi\}$$

(Note the different sign before the R-matrix !). Hence

$$(\tilde{f}_x, 0, \bar{f}_R, \bar{e}_R, 0, 0) \in D_\pi$$

for some $\bar{f}_R, \bar{e}_R$ with $\bar{f}_R = R\bar{e}_R$. However, any vector $d := (\tilde{f}_x, 0, \bar{f}_R, \bar{e}_R, 0, 0) \in D_\pi$ satisfies

$$0 = \ll d, d \gg = 0 \cdot \tilde{f}_x + \bar{e}_R^T \bar{f}_R + 0 \cdot 0 = \bar{e}_R^T \bar{f}_R$$

Since $\bar{f}_R = R\bar{e}_R$ and R is positive definite this implies $\bar{e}_R = \bar{f}_R = 0$. Hence, $(\tilde{f}_x, 0, \bar{f}_R = 0, \bar{e}_R = 0, 0, 0) \in D_\pi$ showing that $C_\pi^\perp \subset J_\pi$. ∎

The *state transfer principle* for a switch configuration π is now formulated as follows.

Definition 2.7 (State transfer principle): Consider the state x^- of a switching port-Hamiltonian system at a switching time where the switch configuration of the system changes into π. Suppose x^- is *not* satisfying the algebraic constraints corresponding to π, that is

$$\frac{\partial H}{\partial x}(x^-) \notin C_\pi \qquad (14)$$

Then the new state x^+ just after the switching time satisfies

$$x^+ - x^- \in J_\pi, \quad \frac{\partial H}{\partial x}(x^+) \in C_\pi \qquad (15)$$

This means that at this switching time an instantaneous jump from x^- to x^+ with $x_{\text{transfer}} := x^+ - x^- \in J_\pi$ will take place, in such a manner that $\frac{\partial H}{\partial x}(x^+) \in C_\pi$.

The jump space J_π is the space of flows in the state space $\mathcal{X} = \mathcal{F}_x$ that is compatible with zero effort e_x at the energy-storing elements and zero flows f_R, f_P and efforts e_R, e_P at the resistive elements and external ports. Said otherwise, the jump space consists of all flow vectors f_x that may be *added* to the present flow vector corresponding to a certain effort vector at the energy storage and certain flow and effort vectors at the resistive elements and external ports, while remaining in the Dirac structure D_π, *without changing* these other effort and flow vectors. Since D_π captures the full power-conserving interconnection structure of the system while in switch configuration π, reflecting the underlying *conservation laws* of the system, the jump space J_π thus corresponds to a particular subset of conservation laws, and the state transfer principle proclaims that the discontinuous change in the state vector is an impulsive motion satisfying this particular set of conservation laws.

For physical systems one would expect that the value of the Hamiltonian $H(x^+)$ immediately *after* the switching time is less than or equal to the value $H(x^-)$ just before:

Theorem 2.8: Let H be a *convex* function. Then for any x^- and x^+ satisfying the state transfer principle (15)

$$H(x^+) \leq H(x^-) \qquad (16)$$

Proof A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if [12] $f(y) \geq f(x) + < \frac{\partial f}{\partial x}(x) \mid y - x >$ for all x, y. Application to H with $x = x^+$ and $y = x^-$ yields

$$H(x^-) \geq H(x^+) + < \frac{\partial H}{\partial x}(x^+) \mid x^- - x^+ >$$

However, by (15) $< \frac{\partial H}{\partial x}(x^+) \mid x^- - x^+ > = 0$ (since $J_\pi = C_\pi^\perp$), and the result follows. ∎

By combining (9) and Theorem 2.8 we obtain

Corollary 2.9: Consider a switching port-Hamiltonian system satisfying the state transfer principle, with its Hamiltonian H being a convex function. Then for all $t_2 \geq t_1$

$$H(x(t_2)) \leq H(x(t_1)) + \int_{t_1}^{t_2} e_P^T(t) f_P(t) dt$$

47

and thus the system is *passive* [15] if H is bounded from below. Moreover, if H has a strict minimum at some x^* then, whenever $e_P(t)f_P(t)$ is identically zero, the equilibrium x^* is *stable* (and under appropriate conditions on the resistive relation asymptotically stable).

If the Hamiltonian H is a quadratic function $H(x) = \frac{1}{2}x^T K x$ (and thus the port-Hamiltonian system is linear), then the state transfer principle reduces to

$$x_{\text{transfer}} = x^+ - x^- \in J_\pi, \quad Kx^+ \in C_\pi \qquad (17)$$

If $K \geq 0$ then it follows from Theorem 2.8 and Corollary 2.9 that the switching port-Hamiltonian system is passive. Furthermore, for each x^- there exists a x^+ satisfying (17), and moreover if $K > 0$ this x^+ (and the jump x_{transfer}) is *unique*. Indeed, the property $J_\pi = C_\pi^\perp$ implies $\lambda^T K x = 0$ for all $\lambda \in J_\pi$ and all $x \in \mathcal{X}$ with $Kx \in C_\pi$, or equivalently $\lambda^T K x = 0$ for all $x \in C_\pi^K := \{x \in \mathcal{X} \mid Kx \in C_\pi\}$ and all $\lambda \in J_\pi$. Thus, J_π is the orthogonal complement of the subspace C_π^K where the inner product on $\mathcal{X}$ is defined by the positive definite matrix K. Hence it follows that the vector x^+ satisfying (17) is unique.

The state transfer principle in the linear case also allows for a *variational characterization* (see also [7], [9]).

Theorem 2.10: Let $K \geq 0$. A state x^+ satisfying (17) is a solution of the minimization problem (for given x^-)

$$\min_{x, Kx \in C_\pi} \frac{1}{2}(x - x^-)^T K(x - x^-), \qquad (18)$$

and conversely if $K > 0$ then the unique solution of (18) is the unique solution to (17).

Proof By the Lagrange multiplier theory a minimum x^+ of (18) is found by minimizing

$$\frac{1}{2}(x - x^-)^T K(x - x^-) + \lambda^T K x$$

over all $x \in \mathcal{X}$ and $\lambda \in C_\pi^\perp = J_\pi$. A minimizing x^+ is thus found as a solution x of

$$K(x - x^-) + K\lambda = 0, \quad \lambda \in J_\pi, \quad Kx \in C_\pi$$

corresponding to $x^+ = x^- + \lambda, \quad \lambda \in J_\pi, \quad Kx^+ \in C_\pi$. ∎

Furthermore, an application of *Dorn's duality* [10], [2] yields (see also [2], [9])

Theorem 2.11: Let $K > 0$. Then the state transfer $\lambda = x^+ - x^-$ is the unique minimum of

$$\min_{\lambda \in J_\pi} \frac{1}{2}(x^- + \lambda)^T K(x^- + \lambda) \qquad (19)$$

Proof By Lagrange multiplier theory the minimization problem (19) can be rewritten as the minimization

$$\min_{\lambda, e \in C_\pi} \frac{1}{2}(x^- + \lambda)^T K(x^- + \lambda) - \lambda^T e$$

leading to the condition $K(x^- + \lambda) - e = 0, e \in C_\pi$. ∎

III. CHARGE AND FLUX TRANSFER IN SWITCHED RLC CIRCUITS

Consider an RLC-circuit with switches with an *arbitrary topology*. It can be described as a switched port-Hamiltonian system as follows (see also [6]). First consider the oriented graph associated with the circuit. Identify every capacitor, every inductor, every resistor and every switch with an edge. Furthermore, associate with every external port an edge (between the terminals of the port). Denote the incidence matrix [1] of this oriented graph by B. The incidence matrix has as many columns as there are edges, and as many rows as there are vertices in the graph. Each column of B corresponds to an edge, and equals the vector with a 1 at the position of the terminating vertex and a -1 at the position of the starting vertex, and zeros everywhere else. By reordering the edges we partition the incidence matrix as

$$B = [B_C \vdots B_L \vdots B_R \vdots B_S \vdots B_P] \qquad (20)$$

where the submatrices B_C, B_L, B_R, B_S correspond, respectively, to the capacitor, inductor, resistor, and switch edges, and B_P corresponds to the external ports. Then Kirchhoff's current laws are given as

$$B_C I_C + B_L I_L + B_R I_R + B_S I_S + B_P I_P = 0 \qquad (21)$$

with I_C, I_L, I_R, I_S, I_P denoting the currents through, respectively, the capacitors, inductors, resistors, switches, and the external ports.

Correspondingly, Kirchhoff's voltage laws are given as

$$\begin{aligned}
V_C &= B_C^T \psi \\
V_L &= B_L^T \psi \\
V_R &= B_R^T \psi \\
V_S &= B_S^T \psi \\
V_P &= B_P^T \psi
\end{aligned} \qquad (22)$$

with V_C, V_L, V_R, V_S, V_P denoting the voltages across the capacitors, inductors, resistors, switches, and ports, respectively, and ψ being the vector of potentials at the vertices.

Kirchhoff's current and voltage laws define a Dirac structure D on the space of flow and effort variables given as

$$\begin{aligned}
f_x &= (I_C, V_L) \\
e_x &= (V_C, I_L) \\
f_R &= V_R \\
e_R &= I_R \\
f_S &= V_S \\
e_S &= I_S \\
f_P &= V_P \\
e_P &= I_P
\end{aligned} \qquad (23)$$

The constitutive relations for the energy storage are given as

$$\begin{aligned}
(\dot{Q}, \dot{\Phi}) &= -(I_C, V_L) \\
(V_C, I_L) &= \left(\frac{\partial H}{\partial Q}, \frac{\partial H}{\partial \Phi}\right)
\end{aligned} \qquad (24)$$

where Q is the vector of charges at the capacitors, and Φ the vector of fluxes of the inductors. For a linear RLC-circuit

$$H(Q,\Phi) = \frac{1}{2}Q^T C^{-1} Q + \frac{1}{2}\Phi^T L^{-1}\Phi \qquad (25)$$

where the diagonal elements of the diagonal matrices C and L are the capacitances, respectively, inductances, of the capacitors and inductors.

Similarly, the constitutive relations for the linear resistors are given as

$$V_R = -RI_R \qquad (26)$$

with R denoting a diagonal matrix with diagonal elements being the resistances of the resistors.

For every subset $\pi \subset \{1, \cdots, s\}$ (where s is the number of switches) the Dirac structure D_π is defined by the equations

$$B_C I_C + B_L I_L + B_R I_R + B_S I_S + B_P I_P = 0$$

$$
\begin{aligned}
V_C &= B_C^T \psi \\
V_L &= B_L^T \psi \\
V_R &= B_R^T \psi \\
V_S &= B_S^T \psi \\
V_P &= B_P^T \psi \\
V_S^i &= 0,\, i \in \pi, \quad I_S^j = 0,\, j \notin \pi
\end{aligned}
\qquad (27)
$$

(That is, all switches corresponding to the subset π are closed, while the remaining are open.) The constraint subspace C_π for each switch configuration π is given as

$$
\begin{aligned}
C_\pi &= \{(V_C, I_L) \mid \exists I_C, V_L, V_R, I_R, V_S, I_S, V_P, I_P \\
&\qquad \text{such that (26) and (27) is satisfied}\,\}
\end{aligned}
\qquad (28)
$$

Furthermore, the jump space J_π is given as the set of all (I_C, V_L) satisfying for some ψ the equations

$$
\begin{aligned}
B_C I_C + B_S I_S &= 0 \\
0 &= B_C^T \psi \\
V_L &= B_L^T \psi \\
0 &= B_R^T \psi \\
V_S &= B_S^T \psi \\
0 &= B_P^T \psi \\
V_S^i &= 0,\, i \in \pi, \quad I_S^j = 0,\, j \notin \pi
\end{aligned}
\qquad (29)
$$

Hence the jump space can be written as the product of the space

$$\{I_C \mid \exists I_S,\, I_S^j = 0, j \notin \pi,\, B_C I_C + B_S I_S = 0\}$$

with the space

$$
\begin{aligned}
\{V_L \mid \quad &\exists \psi \text{ such that } V_L = B_L^T \psi, 0 = B_C^T \psi, 0 = B_R^T \psi, \\
&0 = B_P^T \psi, V_S = B_S^T \psi, V_S^i = 0, i \in \pi\}
\end{aligned}
$$

Thus the state transfer can be split into a *charge transfer* $Q^+ - Q^- = Q_{\text{transfer}}$ and a *flux transfer* $\Phi^+ - \Phi^- = \Phi_{\text{transfer}}$. The *direction* of the charge transfer Q_{transfer} corresponding to the switch configuration π is specified by

$$B_C Q_{\text{transfer}} + B_S I_S = 0, \quad I_S^j = 0, j \notin \pi \qquad (30)$$

This corresponds to *Kirchhoff's current laws for the circuit with switch configuration π, where the inductors and resistors have been open-circuited, and the currents through the external ports are all zero.*

Furthermore, the *amount* of charge transfer is uniquely determined by the condition

$$C^{-1}(Q^- + Q_{\text{transfer}}) = B_C^T \psi$$

for some ψ satisfying $V_S = B_S^T \psi$, $V_S^i = 0, i \in \pi$

The direction of the flux transfer Φ_{transfer} on the other hand is determined by the equations

$$
\begin{aligned}
0 &= B_C^T \psi \\
\Phi_{\text{transfer}} &= B_L^T \psi \\
0 &= B_R^T \psi \\
V_S &= B_S^T \psi, \quad V_S^i = 0,\, i \in \pi \\
0 &= B_P^T \psi
\end{aligned}
\qquad (31)
$$

These are *Kirchhoff's voltage laws for the circuit corresponding to the switch configuration π, where the capacitors and the resistors have been short-circuited, and the voltages across the external ports are all zero.* Furthermore, the *amount* of flux transfer is uniquely determined by the condition

$$B_C I_C + B_L L^{-1}(\Phi^- + \Phi_{\text{transfer}})$$
$$+ B_R I_R + B_S I_S + B_P I_P = 0,$$

for some I_C, I_R, I_P, I_S with $I_S^j = 0, j \notin \pi$

Since in the case of a linear circuit the Hamiltonian $H(Q,\Phi) = \frac{1}{2}Q^T C^{-1} Q + \frac{1}{2}\Phi^T L^{-1}\Phi$ splits as the sum of a quadratic function of the charge Q and the flux Φ, the variational characterization of the state transfer principle also splits into the variational characterization of the charge transfer principle, given as the minimization of

$$\min_{Q, C^{-1}Q \in C_\pi^V} \frac{1}{2}(Q - Q^-)^T C^{-1}(Q - Q^-) \qquad (32)$$

(where C_π^V denotes the projection of the subspace C_π on the space of voltages V_C) and the variational characterization of the flux transfer principle, given as the minimization of

$$\min_{\Phi, L^{-1}\Phi \in C_\pi^I} \frac{1}{2}(\Phi - \Phi^-)^T L^{-1}(\Phi - \Phi^-) \qquad (33)$$

(where C_π^I denotes the projection of the subspace C_π on the space of currents I_L).

IV. STATE TRANSFER IN SWITCHED MECHANICAL SYSTEMS

Consider a mechanical system subject to linear damping and kinematic constraints, which is written in Hamiltonian form as [15]

$$
\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}(q, p) \\
\dot{p} &= -\frac{\partial H}{\partial q}(q, p) - \bar{R}(q, p)\frac{\partial H}{\partial p}(q, p) + A(q)\lambda + B(q)F \\
0 &= A^T(q)\frac{\partial H}{\partial p}(q, p) \\
v &= B^T(q)\frac{\partial H}{\partial p}(q, p)
\end{aligned}
\qquad (34)
$$

where $q = (q_1, \cdots, q_n)$ denotes the vector of generalized position coordinates, $p = (p_1, \cdots, p_n)$ is the vector of generalized momenta, $F \in \mathbb{R}^m$ the vector of external generalized forces, and $v \in \mathbb{R}^m$ the vector of conjugated generalized velocities. $H(q, p)$ denotes the total energy of the system (which usually can be split into a kinetic and a potential energy contribution). Furthermore, $0 = A^T(q)\frac{\partial H}{\partial p}(q, p) = A^T(q)\dot{q}$ denotes the kinematic constraints (such as rolling without slipping) with corresponding constraint forces $\lambda \in \mathbb{R}^s$, where s is the number of kinematic constraints (equal to the number of rows of the matrix $A^T(q)$).

The damping is characterized by the $n \times n$ matrix $\bar{R}(q, p)$ which is assumed to be symmetric and positive semi-definite, that is, $\bar{R}^T = \bar{R} \geq 0$. This implies the usual energy-balance

$$\frac{dH}{dt}(q, p) = -\frac{\partial H}{\partial p}^T(q, p)\bar{R}(q, p)\frac{\partial H}{\partial p}(q, p) + v^T F \leq v^T F$$

We throughout assume that the matrix $\bar{R}(q, p)$ admits a factorization

$$\bar{R}(q, p) = P^T(q, p)RP(q, p), \quad R = R^T > 0$$

for some $r \times n$ matrix $P(q, p)$ and *constant* $r \times r$ matrix R.

A *switching mechanical system* arises if the kinematic constraints can be turned on and off. Denoting $f_S := \lambda$ and replacing the kinematic constraints in (34) by

$$e_S := A^T(q)\frac{\partial H}{\partial p}(q, p) \tag{35}$$

this defines a switching port-Hamiltonian system as before, where any subset $\pi \subset \{1, \cdots, r\}$ defines as before the switch configuration $e_S^i = 0, i \in \pi, f_S^j = 0, j \notin \pi$. Thus in switch configuration π each i-th kinematic constraint, with $i \in \pi$, is active, while the other kinematic constraints (corresponding to indices not in π) are inactive.

It follows that the constraint subspace C_π in this case is given as

$$C_\pi = \{e_x \mid \exists f_x, f_R, e_R, F, f_S, \text{ with}$$

$$f_S^j = 0, j \notin \Pi, \quad f_R = -Re_R, e_R = P^T(q, p)\frac{\partial H}{\partial p}(q, p)$$

$$-f_x = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_x + P(q, p)f_R + B(q)F + A(q)f_S$$

$$e_S = A^T(q)\frac{\partial H}{\partial p}(q, p), \quad e_S^i = 0, i \in \Pi\}$$

Furthermore, the jump space J_π is given as

$$J_\pi = \{f_x \mid f_x \in \text{im}\begin{bmatrix} 0 \\ A_\pi(q) \end{bmatrix}\}$$

where the matrix $A_\pi(q)$ is obtained from the matrix $A(q)$ by leaving out every j-th column with $j \notin \pi$.

Thus the state transfer principle in this case amounts to a jump in the momentum variables p given as

$$p_{\text{transfer}} = p^+ - p^- \in A_\pi(q), \quad A_\pi^T(q)\frac{\partial H}{\partial p}(q, p^+) = 0$$

If H can be written as the sum of a kinetic and a potential energy $H(q, p) = \frac{1}{2}p^T M^{-1}(q)p + V(q)$, with $M(q) > 0$ denoting the generalized mass matrix, then a variational

characterization of the state transfer principle is given by defining p^+ to be the unique minimum of

$$\min_{p, A_\Pi^T(q)M^{-1}(q)p = 0} \frac{1}{2}(p - p^-)^T M^{-1}(q)(p - p^-) \tag{36}$$

Furthermore, since in this case the kinetic energy is a convex function of the momenta, it follows from Theorem 2.8 and Corollary 2.9 that the switching mechanical system is passive if the potential energy is bounded from below.

V. CONCLUSIONS

Inspired by the charge/flux conservation principle of circuit theory, we presented a state transfer principle for general switching port-Hamiltonian systems. This principle extends the charge/flux conservation principle to RLC circuits of arbitrary topology with nonlinear capacitors and inductors. Also, we applied the principle to switching mechanical systems. A future research line concerns the extension of the state transfer principle to port-Hamiltonian systems that contain state-dependent switching elements, such as diodes in circuits and unilateral constraints for mechanical systems.

REFERENCES

[1] B. Bollobas, *Modern Graph Theory*, Graduate Texts in Mathematics 184, Springer, New York, 1998.
[2] M.K. Camlibel. *Complementarity Methods in the Analysis of Piecewise Linear Dynamical Systems*, PhD thesis, Tilburg University, 2001.
[3] T.J. Courant, "Dirac manifolds", *Trans. Amer. Math. Soc., 319*, pp. 631–661, 1990.
[4] J. Cervera, A.J. van der Schaft, A. Banos, "Interconnection of port-Hamiltonian systems and composition of Dirac structures", *Automatica, vol. 43*, pp. 212–225, 2007.
[5] M. Dalsmo, A.J. van der Schaft, "On representations and integrability of mathematical structures in energy-conserving physical systems", *SIAM J. Control and Optimization, vol.37*, pp. 54–91, 1999.
[6] G. Escobar, A.J. van der Schaft, R. Ortega, "A Hamiltonian viewpoint in the modelling of switching power converters", *Automatica, Special Issue on Hybrid Systems, vol.35*, pp.445–452, 1999.
[7] R. Frasca, M.K. Camlibel I.C. Goknar, L. Iannelli, and F. Vasca, "A system theoretical approach to linear passive networks with ideal switches", *under preparation*, 2009.
[8] The Geoplex Consortium, *Modeling and Control of Complex Physical Systems; The Port-Hamiltonian Approach*, Springer, 2009, to appear.
[9] K. Gerritsen, A.J. van der Schaft, W.P.M.H. Heemels, "On switched Hamiltonian systems", *Proceedings 15th International Symposium on Mathematical Theory of Networks and Systems (MTNS2002)*, Eds. D.S. Gilliam, J. Rosenthal, South Bend, August 12-16, 2002.
[10] O.L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969.
[11] R. Pasumarthy, A.J. van der Schaft, "Achievable Casimirs and its Implications on Control of port-Hamiltonian systems", *Int. J. Control, vol. 80* Issue 9, pp. 1421-1438, 2007.
[12] R.T. Rockafellar and J.-B. Wets, *Variational Analysis*, Series of Comprehensive Studies in Mathematics 317. Springer, 1998.
[13] A.J. van der Schaft, B.M. Maschke, "The Hamiltonian formulation of energy conserving physical systems with external ports", *Archiv für Elektronik und Übertragungstechnik, 49*, pp. 362–371, 1995.
[14] S. Seshu and N. Balabanian, *Linear Network Analysis*, John Wiley & Sons, 1964.
[15] A.J. van der Schaft, L_2-*Gain and Passivity Techniques in Nonlinear Control*, Lect. Notes in Control and Information Sciences, Vol. 218, Springer-Verlag, Berlin, 1996, p. 168, 2nd revised and enlarged edition, Springer-Verlag, London, 2000 (Springer Communications and Control Engineering series), p. xvi+249.
[16] A.J. van der Schaft, "Interconnection and geometry", in *The Mathematics of Systems and Control: from Intelligent Control to Behavioral Systems*, Editors J.W. Polderman, H.L. Trentelman, Groningen, pp. 203-218, 1999.