University of Groningen

Structure-preserving model reduction of complex physical systems

Schaft, A.J. van der; Polyuga, R.V.

[Link to publication in University of Groningen/UMCG research database](#)

Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

ThB04.4

Structure-preserving model reduction of complex physical systems

A.J. van der Schaft, R.V. Polyuga

Abstract— Port-based network modeling of complex physical systems naturally leads to port-Hamiltonian system models. This motivates the search for structure-preserving model reduction methods, which allow one to replace high-dimensional port-Hamiltonian system components by reduced-order ones. In this paper we treat a family of structure-preserving reduction methods for port-Hamiltonian systems, and discuss their relation with projection-based reduction methods for DAEs.

I. INTRODUCTION

The standard way to model large-scale physical systems is *network modeling*. In this method the overall system is decomposed into (possibly many) interconnected subsystems. Network modeling has many advantages in terms of, e.g., reusability of subsystem models (libraries), flexibility (coarse models of subsystems may be replaced by more refined ones, leaving the rest of the system modeling untouched), hierarchical modeling, and control (by adding new subsystems as control components). In *port-based network modeling* the overall system is decomposed into subsystems which are interconnected to each other through (vector) pairs of variables, whose product is the power exchanged between the subsystems. This approach is especially useful for the systematic modeling of *multi-physics systems*, where the subsystems belong to different physical domains (mechanical, electrical, hydraulic, etc.).

Since the beginning of the 1990s it has been realized [13], [5], [15] that the mathematical models arising from port-based network modeling have a geometric structure, which can be regarded as a generalization of the geometric formulation of analytical mechanics into its Hamiltonian form. These geometric dynamical system models have been called *port-Hamiltonian systems* [13], [15], [6]. Port-Hamiltonian systems are *compositional* in the sense that any (power-conserving) interconnection of port-Hamiltonian systems is again port-Hamiltonian. Furthermore, it has become apparent, see e.g, [14], that port-Hamiltonian systems modeling extends to *distributed-parameter* and *mixed* lumped- and distributed-parameter physical systems, while approaches havr been initiated that deal with the structure-preserving spatial discretization of distributed-parameter port-Hamiltonian systems.

The state space dimension of mathematical models arising from network modeling easily becomes very large; think e.g. of electrical circuits, multi-body systems, or the spatial discretization of distributed-parameter systems. Thus there is

an immediate need for *model reduction* methods. However, since we want the reduced-order models again to be interconnectable to other (sub-)systems, we want to retain the port-Hamiltonian structure of the reduced-order systems. Thus the problem arises of *structure-preserving model reduction* of port-Hamiltonian systems.

Port-Hamiltonian systems are necessarily passive if the Hamiltonian (stored energy) is bounded from below. Hence the structure-preserving model reduction of port-Hamiltonian systems encapsulates passivity-preserving model reduction, which has been a topic of intense research activity over the last few years [1], [12]. On the other hand, port-Hamiltonian system modeling encodes more structural information about the physical system than just passivity (for example, the presence of conservation laws). In fact, port-Hamiltonian systems modeling can be regarded to bridge the gap between passive system models and explicit physical network realizations (such as electrical circuits).

II. PORT-HAMILTONIAN SYSTEMS

First main ingredient in the definition of a port-Hamiltonian system is the notion of a Dirac structure which relates the power variables of the composing elements of the system in a power-conserving manner. The power variables always appear in conjugated pairs (such as voltages and currents, or generalized forces and velocities), and therefore take their values in dual linear spaces.

Definition 2.1: [**?**] Let $\mathcal{F}$ be a linear space with dual space $\mathcal{E} := \mathcal{F}^*$, and duality product denoted as $< e \mid f >= e(f) \in \mathbb{R}$, with $f \in \mathcal{F}$ and $e \in \mathcal{E}$. In vector notation we simply write the duality product as $e^T f$. We call $\mathcal{F}$ the space of *flow* variables, and $\mathcal{E} = \mathcal{F}^*$ the space of *effort* variables. Define on $\mathcal{F} \times \mathcal{E}$ the following indefinite bilinear form

$$\ll (f_1, e_1), (f_2, e_2) \gg = < e_1 \mid f_2 > + < e_2 \mid f_1 >,$$

A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a constant[1] *Dirac structure* if $\mathcal{D} = \mathcal{D}^\perp$, where $\mathcal{D}^\perp$ is the orthogonal complement of $\mathcal{D}$ with respect to the indefinite bilinear form $\ll \cdot, \cdot \gg$.

Remark 2.2: It can be shown [**?**], [6], [5] that in the case of a finite-dimensional linear space $\mathcal{F}$ a Dirac structure $\mathcal{D}$ is equivalently characterized as a subspace such that $e^T f = < e \mid f >= 0$ for all $(f, e) \in \mathcal{D}$, together with $\dim \mathcal{D} = \dim \mathcal{F}$. The property $< e \mid f >= 0$ for all $(f, e) \in \mathcal{D}$ corresponds to power conservation.

A *port-Hamiltonian system* is defined as follows. We start with a Dirac structure $\mathcal{D}$ on the space of all flow and effort

A.J. van der Schaft and R.V. Polyuga are with the Institute for Mathematics and Computing Science, University of Groningen, P.O.Box 407, 9700 AK Groningen, the Netherlands. `A.J.van.der.Schaft@rug.nl`, `R.Polyuga@rug.nl`

[1]For the definition of Dirac structures on *manifolds* we refer to e.g. [5].

variables involved:

$$\mathcal{D} \subset \mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P \qquad (1)$$

The space $\mathcal{F}_x \times \mathcal{E}_x$ is the space of flow and effort variables corresponding to the *energy-storing* elements (to be defined later on), the space $\mathcal{F}_R \times \mathcal{E}_R$ denotes the space of flow and effort variables of the *resistive* elements, while $\mathcal{F}_P \times \mathcal{E}_P$ is the space of flow and effort variables corresponding to the external *ports* (or sources).

The constitutive relations for the *energy-storing* elements are defined as follows. Let the Hamiltonian $H : \mathcal{X} \to \mathbb{R}$ denote the total energy at the energy-storage elements with state variables $x = (x_1, x_2, \cdots, x_n)$; i.e., the total energy is given as $H(x)$. In the sequel we will throughout take $\mathcal{X} = \mathcal{F}_x$, but $\mathcal{X}$ may also denote an n-dimensional manifold (in which case $\mathcal{F}_x$ is the tangent space to this manifold $\mathcal{X}$ at the state x). Then the constitutive relations are given as[2]

$$\dot{x} = -f_x, \quad e_x = \frac{\partial H}{\partial x}(x) \qquad (2)$$

This immediately implies the energy balance

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial x}(x)\dot{x} = -e_x^T f_x, \qquad (3)$$

The constitutive relations for the *resistive elements* are given as

$$f_R = -F(e_R), \qquad (4)$$

for some function F satisfying $e_R^T F(e_R) > 0$ for all $e_R \neq 0$. This implies that

$$e_R^T f_R = -e_R^T F(e_R) < 0, \qquad (5)$$

and that power is always dissipated. For example, linear resistive elements are given as $f_R = -Re_R$, $R = R^T > 0$.

Definition 2.3: Consider a Dirac structure (1), a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$ with constitutive relations (2), and a resistive relation $f_R = -F(e_R)$ as in (5). Then the dynamics of the resulting port-Hamiltonian system is given as

$$\left(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), -F(e_R(t)), e_R(t), f_P(t), e_P(t)\right) \in \mathcal{D} \qquad (6)$$

It follows [15], [6] from the power-conservation property of Dirac structures and (3) and (5) that

$$\frac{d}{dt}H = -e_R^T F(e_R) + e_P^T f_P \leq e_P^T f_P \qquad (7)$$

thus showing *passivity* if H is bounded from below.

A. DAE representations of port-Hamiltonian systems

In general the conditions (6) will define a set of *differential-algebraic equations* (DAEs). Indeed, any Dirac structure $\mathcal{D} \subset \mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P$ can be represented by a linear set of equations involving all the effort and flow variables [13], [5], [15], [6]

$$F_x f_x + E_x e_x + F_R f_R + E_R e_R + F_P f_P + E_P e_P = 0$$

---

[2]The vector $\frac{\partial H}{\partial x}(x)$ of partial derivatives of H will throughout be denoted as a *column* vector.

where the constant[3] matrices $F_x, F_R, F_P, E_x, E_R, E_P$ satisfy

$$E_x F_x^T + F_x E_x^T + E_R F_R^T + F_R E_R^T + E_P F_P^T + F_P E_P^T = 0$$

$$\text{rank} \begin{bmatrix} F_x & F_R & F_P & E_x & E_R & E_P \end{bmatrix} = n_x + n_R + n_p \qquad (8)$$

where $n_x = \dim \mathcal{F}_x, n_R = \dim \mathcal{F}_R, n_P = \dim \mathcal{F}_P$. By substitution of (2) and (5) it follows that any port-Hamiltonian system can be represented as a set of DAEs

$$F_x \dot{x} = E_x \frac{\partial H}{\partial x}(x) - F_R F(e_R) + E_R e_R + F_P f_P + E_P e_P \quad (9)$$

Under general conditions [3] we can solve from these equations for e_R, thus leaving a set of DAEs in the state variables x involving the external port-variables f_P, e_P. Furthermore, there always exists [6] a hybrid partitioning of the port-variables f_P, e_P into input variables $u_i = f_{Pi}, i \in K, u_i = e_{Pi}, i \notin K$, and complementary output variables $y_i = e_{Pi}, i \in K, y_i = f_{Pi}, i \notin K$, for some subset $K \subset \{1, 2, \cdots, n_P\}$. For other useful representations of port-Hamiltonian systems, as well as for the way to transform one representation into another we refer to [5], [15], [6].

Various *pole/zero-dynamics*, which inherit the port-Hamiltonian structure, can be defined for a port-Hamiltonian system. The simplest possibilities are the ones corresponding to constraining either f_P or e_P to zero, while leaving the rest of the external port-variables free. This results in a port-Hamiltonian dynamics without external port variables. For example, if we impose the constraint $e_P = 0$ (while leaving f_P free) then we obtain the port-Hamiltonian system

$$LF_x \dot{x} = LE_x \frac{\partial H}{\partial x}(x) - LF_R F(e_R) + LE_R e_R \qquad (10)$$

where L is any matrix of maximal rank satisfying $LF_P = 0$. Indeed, it can be shown [3] that the equations $LF_x f_x + LE_x e_x + LF_R f_R + LE_R e_R = 0$ define the reduced Dirac structure

$$\mathcal{D}_{\text{red}} \subset \mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_R \times \mathcal{E}_R,$$

which results from interconnection of the original Dirac structure $\mathcal{D}$ with the Dirac structure on the space of external port variables $\mathcal{F}_P \times \mathcal{E}_P$ defined by $e_P = 0$ (see [3] for further information).

The choice $f_P = 0$ is similar, the difference being that L should now satisfy $LE_P = 0$. If the port-Hamiltonian system is linear and f_P is the vector of inputs, then the last case corresponds to the *poles* of the Hamiltonian system, while the first option corresponds to the *zeros* of the system. For a general hybrid partitioning of the port-variables f_P, e_P as above, we may define the reduced Dirac structure corresponding to setting the variables $e_{Pi}, i \in K, f_{Pi}, i \notin K$, equal to zero (while leaving the complementary part free).

B. Linear input-state-output port-Hamiltonian systems

Let us now restrict attention to *linear* port-Hamiltonian systems without algebraic constraints given in *input-state-*

---

[3]In the case of Dirac structures on manifolds these matrices will actually depend on the state variables x.

output form. They take the form [7], [6]

$$\begin{aligned}
\dot{x} &= (J - R)Qx + (G - P)u \\
y &= (G^T + P^T)Qx + (M + S)u
\end{aligned} \tag{11}$$

where J and M are skew-symmetric matrices, and R and S are symmetric matrices satisfying

$$\begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0$$

The energy balance (7) now amounts to

$$\frac{d}{dt}\frac{1}{2}x^T Qx = u^T y - \begin{bmatrix} x^T Q & u \end{bmatrix} \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \begin{bmatrix} Qx \\ u \end{bmatrix} \leq u^T y$$

For a linear input-state-output system the zero-dynamics as introduced above takes the following explicit form. For simplicity let us only consider two typical cases. The first one is where the feedthrough matrix $D := M + S$ is invertible. In this case, constraining y to zero yields the input value

$$u = -D^{-1}(G^T + P^T)Qx$$

which after substitution leads to the following zero dynamics

$$\dot{x} = [J - R - (G - P)D^{-1}(G^T + P^T)]Qx = (\tilde{J} - \tilde{R})Qx$$

where $\tilde{J}$ is obtained by adding the skew-symmetric part of $(G - P)D^{-1}(G^T + P^T)$ to J, and similarly $\tilde{R}$ equals R minus the symmetric part of $(G - P)D^{-1}(G^T + P^T)$. The other typical case is when $M + S = 0$ (no feedthrough), and hence also $P = 0$, in which case the system reduces to

$$\begin{aligned}
\dot{x} &= (J - R)Qx + Gu \\
y &= G^T Qx
\end{aligned} \tag{12}$$

Assuming invertibility of Q we may also write this system into its so-called co-energy variables $e = Qx$ as

$$\begin{aligned}
\dot{e} &= Q(J - R)e + QGu \\
y &= G^T e
\end{aligned} \tag{13}$$

Setting $y = G^T e$ to zero then yields the input

$$u = -(G^T QG)^{-1}G^T Q(J - R)e$$

which after substitution leads to the zero-dynamics

$$\dot{e} = [Q - QG(G^T QG)^{-1}G^T Q](J - R)e$$

More explicitly, taking coordinates $x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$ such that $G = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ the zero-dynamics will be given as

$$\dot{e}_1 = [Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}](J_{11} - R_{11})e_1$$

and thus in corresponding energy variables $x^1 = (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})^{-1}e_1$

$$\dot{x}_1 = (J_{11} - R_{11})[Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}]x_1$$

C. Behavioral properties of linear port-Hamiltonian systems

By the definition of a Dirac structure it follows that for any two vectors of flow and effort variables

$$(f_x^i = -\dot{x}^i, e_x^i = Qx^i, f_R^i = -Re_R^i, e_R^i, u^i, y^i) \in \mathcal{D},$$

for $i = 1, 2$, it holds that

$$\begin{aligned}
&< Qx^1 \mid -\dot{x}^2 > + < Qx^2 \mid -\dot{x}^1 > + < u^1 \mid y^2 > + \\
&+ < u^2 \mid y^1 > + < e_R^1 \mid -Re_R^2 > + < e_R^2 \mid -Re_R^1 >= 0
\end{aligned}$$

By symmetry of R this implies

$$\frac{d}{dt}x^{1T}Qx^2 = < u^1|y^2 > + < u^2|y^1 > -2 < e_R^1|Re_R^2 > \tag{14}$$

By time-integration of (14) we conclude that for any two system trajectories satisfying $x^i(T_1) = x^i(T_2) = 0$, $i = 1, 2$,

$$\begin{aligned}
&\int_{T_1}^{T_2} < u^1(t) \mid y^2(t) > + < u^2(t) \mid y^1(t) > dt \\
&= 2\int_{T_1}^{T_2} < e_R^1(t) \mid Re_R^2(t) > dt
\end{aligned}$$

This has the following consequences. Assume that the linear port-Hamiltonian system is controllable. Define $\mathcal{B}$ to be the external behavior of the port-Hamiltonian system, that is, the set of all its (smooth) input-output trajectories $(u(\cdot), y(\cdot))$: $\mathbb{R} \to U \times Y$. Furthermore, let $\mathcal{B}^c$ denote all trajectories of compact support in $\mathcal{B}$. Define the behavior $\mathcal{B}^\perp$ as the set of all smooth time trajectories $(u^\perp(\cdot), y^\perp(\cdot))$: $\mathbb{R} \to U \times Y$ such that

$$\int_{-\infty}^{\infty} < y(t) \mid u^\perp(t) > + < y^\perp(t) \mid u(t) > dt = 0 \tag{15}$$

for all $(u(\cdot), y(\cdot)) \in \mathcal{B}^c$. It follows that

$$\mathcal{B} \cap \mathcal{B}^\perp = \{(u(\cdot), y(\cdot)) \in \mathcal{B} \mid e_R(t) = 0, \forall t\} \tag{16}$$

Hence $\mathcal{B} \cap \mathcal{B}^\perp$ represents the subbehavior of $\mathcal{B}$ *without internal energy-dissipation*. (For passive systems this subbehavior is equal to the so-called subbehavior of minimal dissipation that was identified in [9] as key to generalizing the passivity-preserving reduction techniques of [1], [12] to behaviors.)

Note that for a port-Hamiltonian system *without* resistive elements (a *conservative* port-Hamiltonian system) it follows from (14) that $\mathcal{B} \subset \mathcal{B}^\perp$. In fact, using the techniques in [4] it can be shown that in the conservative case

$$\mathcal{B} = \mathcal{B}^\perp \tag{17}$$

This has the following appealing interpretation. The external behavior $\mathcal{B}$ of any conservative linear port-Hamiltonian system defines an (infinite-dimensional) Dirac structure with respect to the following indefinite bilinear form on pairs of functions $(u, y) : \mathbb{R} \to U \times Y$ of compact support

$$\ll (u^1, y^1), (u^2, y^2) \gg = \int_{-\infty}^{\infty} u^{1T}(t)y^2(t) + u^{2T}(t)y^1(t)dt$$

This can be seen as the dynamic generalization of the fact that a linear *static* relation between $u \in U$ and $y \in Y = U^*$ is power-conserving if and only if it is a Dirac structure.

III. STRUCTURE-PRESERVING MODEL REDUCTION BASED ON POWER CONSERVATION

Consider a general port-Hamiltonian system, and assume that we have been able (e.g. by some balancing technique) to find a splitting of the state space variables $x = (x^1, x^2)$ having the property that the x^2 coordinates do not much contribute to the input-output behavior of the system, and thus could be omitted. It is easily seen that the usual *truncation* method for obtaining a reduced-order model in the reduced state x^1 in general does *not* preserve the port-Hamiltonian structure (like it will also not preserve the passivity property, see e.g. [1]). The same holds for the so-called singular perturbation reduction method.

In which way is it possible to retain the port-Hamiltonian structure in model reduction ? Recall that in the definition of a port-Hamiltonian system the vector of flow and effort variables is required to be in the Dirac structure

$$(f_x^1, f_x^2, e_x^1, e_x^2, f_R, e_R, f_P, e_P) \in \mathcal{D}, \qquad (18)$$

while the flow and effort variables f_x, e_x are linked to the constitutive relations of the energy-storage by

$$\begin{aligned} \dot{x}^1 &= -f_x^1, & \frac{\partial H}{\partial x^1}(x^1, x^2) &= e_x^1 \\ \dot{x}^2 &= -f_x^2, & \frac{\partial H}{\partial x^2}(x^1, x^2) &= e_x^2, \end{aligned} \qquad (19)$$

The basic idea of structure-preserving model reduction for port-Hamiltonian systems is to 'cut' the interconnection

$$\dot{x}^2 = -f_x^2, \quad \frac{\partial H}{\partial x^2}(x^1, x^2) = e_x^2 \qquad (20)$$

between the energy storage corresponding to x^2 and the Dirac structure, in such a way that no power is transferred. This is done by making both power products $(\frac{\partial H}{\partial x^2})^T \dot{x}^2$ and $(e_x^2)^T f_x^2$ equal to zero. The following main scenario's arise:

1) Set

$$\frac{\partial H}{\partial x^2}(x^1, x^2) = 0, \quad e_x^2 = 0 \qquad (21)$$

The first equation imposes an algebraic constraint on the space variables $x = (x^1, x^2)$. Under general conditions on the Hamiltonian H this constraint allows one to solve x^2 as a function $x^2(x^1)$ of x^1, leading to a *reduced Hamiltonian*

$$H_{\text{red}}^{\text{ec}}(x^1) := H(x^1, x^2(x^1)) \qquad (22)$$

Furthermore, the second equation defines the *reduced Dirac structure*[4]

$$\begin{aligned} \mathcal{D}_{\text{red}}^{\text{ec}} &:= \{(f_x^1, e_x^1, f_R, e_R, f_P, e_P) \mid \exists f_x^2 \text{ such that} \\ & (f_x^1, e_x^1, f_x^2, 0, f_R, e_R, f_P, e_P) \in \mathcal{D}\} \end{aligned} \qquad (23)$$

leading to the reduced port-Hamiltonian system

$$\left(-\dot{x}^1, \frac{\partial H_{\text{red}}^{\text{ec}}}{\partial x^1}(x^1), -F(e_R), e_R, f_P, e_P\right) \in \mathcal{D}_{\text{red}}^{\text{ec}} \qquad (24)$$

[4]$\mathcal{D}_{\text{red}}^{\text{ec}}$ is the composition of the full-order Dirac structure $\mathcal{D}$ with the Dirac structure on the space of flow and effort variables f_x^2, e_x^2 defined by $e_x^2 = 0$. This proves [3] that $\mathcal{D}_{\text{red}}^{\text{ec}}$ is indeed a Dirac structure.

We will call this reduction method the *Effort-constraint* reduction method, since it constrains the efforts e_x^2 and $\frac{\partial H}{\partial x^2}$ to zero.

2) Set

$$\dot{x}^2 = 0, \quad f_x^2 = 0 \qquad (25)$$

The first equation imposes the constraint

$$x^2 = c \text{ (constant)} \qquad (26)$$

and thus defines the reduced Hamiltonian

$$H_{\text{red}}^{\text{fc}}(x^1) := H(x^1, c), \qquad (27)$$

while the second equation leads to the reduced Dirac structure

$$\begin{aligned} \mathcal{D}_{\text{red}}^{\text{fc}} &:= \{(f_x^1, e_x^1, f_R, e_R, f_P, e_P) \mid \exists e_x^2 \text{ such that} \\ & (f_x^1, e_x^1, 0, e_x^2, f_R, e_R, f_P, e_P) \in \mathcal{D}\} \end{aligned} \qquad (28)$$

and the corresponding reduced port-Hamiltonian system

$$\left(-\dot{x}^1, \frac{\partial H_{\text{red}}^{\text{fc}}}{\partial x^1}(x^1), -F(e_R), e_R, f_P, e_P\right) \in \mathcal{D}_{\text{red}}^{\text{fc}} \qquad (29)$$

We call this approach the *Flow-constraint* reduction method.

3) Set

$$\dot{x}^2 = 0, \quad e_x^2 = 0 \qquad (30)$$

This leads to the reduced-order port-Hamiltonian system with reduced Hamiltonian $H_{\text{red}}^{\text{fc}}(x^1)$ and reduced Dirac structure $\mathcal{D}_{\text{red}}^{\text{ec}}$.

4) Set

$$\frac{\partial H}{\partial x^2}(x^1, x^2) = 0, \quad f_x^2 = 0 \qquad (31)$$

This leads to the port-Hamiltonian system with reduced Hamiltonian $H_{\text{red}}^{\text{ec}}(x^1)$ and reduced Dirac structure $\mathcal{D}_{\text{red}}^{\text{fc}}$.

Despite their common basis the above reduction schemes have different physical interpretations and consequences. To illustrate this in a simple context, consider an electrical circuit where x^2 corresponds to the charge Q of a single (linear) capacitor. Application of the Effort-constraint method would correspond to removing the capacitor (and setting its charge equal to zero) and *short-circuiting* the circuit at the location of the capacitor. On the other hand, the Flow-constraint method would correspond to *open-circuiting* the circuit at the location of the capacitor, and keeping the charge of the capacittor constant. Method 3 is in this case very similar to the Effort-constraint method, and corresponds to short-circuiting, with the minor difference of setting the charge of the capacitor equal to a constant. Finally, the method 4 corresponds to open-circuiting while setting the charge of the capacitor equal to zero (and thus is similar to the Flow-constraint method).

A. Equational representations of reduced-order models

We will now provide explicit equational representations of the above four methods for structure-preserving model reduction starting from the general representation by DAEs of the full-order model as in (9):

$$F_x \dot{x} = E_x \frac{\partial H}{\partial x}(x) - F_R F(e_R) + E_R e_R + F_P f_P + E_P e_P$$

(32)

where the matrices $F_x, E_x, F_R, E_R, F_P, E_P$ satisfy (8). Corresponding to the splitting of the state vector x into $x = (x^1, x^2)$ and the splitting of the flow and effort vectors f_x, e_x into f_x^1, f_x^2 and e_x^1, e_x^2 we write

$$F_x = \begin{bmatrix} F_x^1 & F_x^2 \end{bmatrix}, \quad E_x = \begin{bmatrix} E_x^1 & E_x^2 \end{bmatrix}$$

(33)

Now the reduced Dirac structure $\mathcal{D}_{\text{red}}^{\text{ec}}$ corresponding to the effort-constraint $e_x^2 = 0$ is given by the explicit equations (see [3])

$$L^{\text{ec}} F_x^1 f_x^1 + L^{\text{ec}} E_x^1 e_x^1 + L^{\text{ec}} F_R f_R + L^{\text{ec}} E_R e_R +$$
$$L^{\text{ec}} F_P f_P + L^{\text{ec}} E_P e_P = 0$$

(34)

where L^{ec} is any matrix of maximal rank satisfying

$$L^{\text{ec}} F_x^2 = 0$$

(35)

Similarly, the reduced Dirac structure $\mathcal{D}_{\text{red}}^{\text{fc}}$ corresponding to the flow-constraint $f_x^2 = 0$ is given by the equations

$$L^{\text{fc}} F_x^1 f_x^1 + L^{\text{fc}} E_x^1 e_x^1 + L^{\text{fc}} F_R f_R + L^{\text{fc}} E_R e_R +$$
$$L^{\text{fc}} F_P f_P + L^{\text{fc}} E_P e_P = 0$$

(36)

where L^{fc} is any matrix of maximal rank satisfying

$$L^{\text{fc}} E_x^2 = 0$$

(37)

It follows that the reduced-order model resulting from applying the Effort-constraint method is given by

$$L^{\text{ec}} F_x^1 \dot{x}^1 = L^{\text{ec}} E_x^1 \frac{\partial H_{\text{red}}^{\text{ec}}}{\partial x^1}(x^1) - L^{\text{ec}} F_R F(e_R) +$$
$$L^{\text{ec}} E_R e_R + L^{\text{ec}} F_P f_P + L^{\text{ec}} E_P e_P,$$

(38)

whereas the reduced-order model resulting from applying the Flow-constraint method is given by

$$L^{\text{fc}} F_x^1 \dot{x}^1 = L^{\text{fc}} E_x^1 \frac{\partial H_{\text{red}}^{\text{fc}}}{\partial x^1}(x^1) - L^{\text{fc}} F_R F(e_R) +$$
$$L^{\text{fc}} E_R e_R + L^{\text{fc}} F_P f_P + L^{\text{fc}} E_P e_P$$

(39)

Similar expressions follow for the reduced-order models arising from applying Methods 3 and 4.

B. Reduced models for linear input-state-output port-Hamiltonian systems

In the case of linear input-state-output port-Hamiltonian systems (12) (for simplicity *without* feedthrough term) the above reduced-order models take the following form. For clarity of notation denote $K := J - R$ (thus J is the skew-symmetric part and $-R$ the symmetric part of K). Splitting

of the state vector into $x = (x^1, x^2)$ then leads to the following partitioned system description

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} G_1^T & G_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

(40)

Rewriting these equations as DAEs (32), and applying the Effort-constraint reduction method as above, yields (assuming that Q_{22} is invertible) the reduced model

$$\dot{x}^1 = K_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x^1 + G_1 u$$
$$y = G_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x^1$$

(41)

This was already shown by direct methods in [10][5]. The application of the Flow-constraint method is more involved. For simplicity of exposition we will only consider the case $G_2 = 0$. The Flow-constraint method is then seen to lead (assuming that K_{22} is invertible) to the reduced port-Hamiltonian model

$$\dot{x}^1 = (K_{11} - K_{12}K_{22}^{-1}K_{21})Q_{11}x^1 + G_1 u$$
$$y = G_1^T Q_{11}x^1$$

(42)

IV. EFFORT- AND FLOW-CONSTRAINT REDUCTION AS PROJECTION-BASED REDUCTION

The Effort-constraint and Flow-constraint reduction methods for linear port-Hamiltonian systems have a direct interpretation in terms of *projection-based* reduction methods [1], [8]. Consider a port-Hamiltonian system (9) with quadratic Hamiltonian $H(x) = \frac{1}{2}x^T Q x$, $Q = Q^T > 0$, and linear damping $f_R = -Re_R$ given as

$$F_x \dot{x} = E_x Q x - F_R R e_R + E_R e_R + F_P f_P + E_P e_P \quad (43)$$

Let us first consider the Flow-constraint reduction method resulting from setting $f_x^2 =$ and $\dot{x}^2 = x^2 = 0$. This corresponds to an *embedding matrix* $V = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$, with $k = \dim x^1$, with embedded state $x = Vx^1$ given as $x = \begin{bmatrix} x^1 \\ 0 \end{bmatrix}$. The reduced-order port-Hamiltonian system (39) arising from applying the Flow-constraint method is seen to result from substituting the embedded state into the DAEs (43), while *projecting* this dynamics on the reduced vector x^1 of state variables, by premultiplication with the matrix L^{fc}.

For the interpretation of the Effort-constraint reduction method as a projection-based reduction method we will first rewrite the linear port-Hamiltonian system in terms of its *co-energy variables* $e := Qx$ as

$$F_x Q^{-1} \dot{e} = E_x e - F_R R e_R + E_R e_R + F_P f_P + E_P e_P \quad (44)$$

Then the reduced-order port-Hamiltonian system (38) arising from applying the Effort-constraint method is seen to result

---

[5]In [10] it was furthermore shown how the Kalman-decomposition of a non-minimal linear input-state-output port-Hamiltonian system is a combination of Effort- and Flow-constraint reduction.

from substituting the embedded state $e = Ve^1 = \begin{bmatrix} e^1 \\ 0 \end{bmatrix}$ into the dynamics (44), and then *projecting* this dynamics on the reduced state vector e^1 by premultiplication with the matrix L^{ec}. Thus the Flow- and Effort-constraint reduction methods define special projection-based reduction methods which are by construction *atructure-preserving* (and thus, if $Q > 0$, *passivity- and stability-preserving*).

For linear input-state-output port-Hamiltonian systems (12) (again for simplicity of exposition without feedthrough term) the interpretation of the Flow-constraint reduction method as a projection-based reduction method specializes as follows. Denote as above $K := J - R$. Then we rewrite (12) as (assuming invertibility of K)

$$\begin{aligned} K^{-1}\dot{x} &= Qx + K^{-1}Gu \\ y &= G^T Q x \end{aligned} \tag{45}$$

Under the assumption that $\operatorname{im} G \subset \operatorname{im} V$ (corresponding to the previously made assumption $G_2 = 0$), the Flow-constraint reduction now corresponds to the following projection of the dynamics onto a dynamics involving the reduced state vector x_{red}

$$\begin{aligned} \hat{K}^{-1}\dot{x}_{\text{red}} &= \hat{Q}x_{\text{red}} + \hat{K}^{-1}\hat{G}u \\ y &= \hat{G}^T \hat{Q} x_{\text{red}} \end{aligned} \tag{46}$$

where

$$\hat{K}^{-1} = V^T K^{-1} V, \quad \hat{Q} = V^T Q V, \quad \hat{K}^{-1}\hat{G} = V^T K^{-1} G$$

In case of the Effort-constraint reduction method we rewrite the co-energy variable representation (13) as (assuming invertibility of Q)

$$\begin{aligned} Q^{-1}\dot{e} &= Ke + Gu \\ y &= G^T e \end{aligned} \tag{47}$$

Effort-constraint reduction then corresponds to the following projection of the dynamics onto a dynamics involving the reduced state vector e_{red}

$$\begin{aligned} \hat{Q}^{-1}\dot{e}_{\text{red}} &= \hat{K}e_{\text{red}} + \hat{G}u \\ y &= \hat{G}^T e_{\text{red}} \end{aligned} \tag{48}$$

where

$$\hat{Q}^{-1} = V^T Q^{-1} V, \quad \hat{K} = V^T K V, \quad \hat{G} = V^T G$$

V. Conclusions and Outlook

In this paper we have discussed a number of basic properties of port-Hamiltonian systems which are relevant for model reduction, including their DAE representations. We have presented a family of structure-preserving reduction methods which are based on power-conservation. Furthermore, we have discussed the relation of the Effort- and Flow-constraint reduction methods for general port-Hamiltonian

DAEs with projection-based methods, extending the results of [10], [11] on linear input-state-output port-Hamiltonian systems.

Of course, the tight connection with projection-based reduction methods suggests many further research questions, motivated e.g. by [8]. One is the splitting of the state variables $x = (x^1, x^2)$, which may be based on Krylov-type methods or on balancing methods. We refer to [16] for a discussion of various balancing methods for passive systems. The relation with the passivity-preserving model reduction techniques of e.g. [1], [12] also needs further clarification. Another possibility that we want to investigate in further work is to terminate the flows and efforts f_x^2, e_x^2 by a resistive relation $f_x^2 = -De_x^2$ (for some $D = D^T > 0$), instead of the power-conserving terminations $f_x^2 = 0$ or $e_x^2 = 0$.

References

[1] A.C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, SIAM, 2005.

[2] T.J. Courant, "Dirac manifolds", *Trans. Amer. Math. Soc., 319*, pp. 631–661, 1990.

[3] J. Cervera, A.J. van der Schaft, A. Banos, "Interconnection of port-Hamiltonian systems and composition of Dirac structures", *Automatica, vol. 43*, pp. 212–225, 2007.

[4] P.E. Crouch, A.J. van der Schaft, *Variational and Hamiltonian control systems*, Lect. Notes in Control and Information Sciences, Vol. 101, Springer-Verlag, Berlin, 1987, p. 121.

[5] M. Dalsmo, A.J. van der Schaft, "On representations and integrability of mathematical structures in energy-conserving physical systems", *SIAM J. Control and Optimization, vol.37*, pp. 54–91, 1999.

[6] The Geoplex Consortium, *Modeling and Control of Complex Physical Systems; The Port-Hamiltonian Approach*, Springer, to appear.

[7] K. Gerritsen, A.J. van der Schaft, W.P.M.H. Heemels, "On switched Hamiltonian systems", *Proc. 15th International Symposium on Mathematical Theory of networks and Systems* (MTNS2002), Eds. D.S. Gilliam, J. Rosenthal, South Bend, August 12-16, 2002.

[8] S. Gugercin, A.C. Antoulas, C. Beattie, "$\mathcal{H}_2$ model reduction for large-scale linear dynamical systems", *SIAM Journal on Matrix Analysis and Applications*, 30(2): 609–638, 2008.

[9] Ha Bin Minh, *Model Reduction in a Behavioral Framework*, PhD-thesis, University of Groningen, January 2009.

[10] R. Polyuga, A. J. van der Schaft, "Structure preserving model reduction of port-Hamiltonian systems", In 18th International Symposium on Mathematical Theory of Networks and Systems, Blacksburg, Virginia, USA, July 28 - August 1, 2008.

[11] R. Polyuga, A.J. van der Schaft, "Moment matching for linear port-Hamiltonian systems", pp. 4715–4720 in Proceedings 10th European Control Conference (ECC'09), Budapest, 2009.

[12] D.C. Sorensen, "Passivity preserving model reduction via interpolation of spectral zeros", *Systems and Control Letters*, vol.54, pp. 347-360, 2005.

[13] A.J. van der Schaft, B.M. Maschke, "The Hamiltonian formulation of energy conserving physical systems with external ports", *Archiv für Elektronik und Übertragungstechnik, 49*, pp. 362–371, 1995.

[14] A.J. van der Schaft, B.M. Maschke, "Hamiltonian formulation of distributed-parameter systems with boundary energy flow", *Journal of Geometry and Physics, vol. 42*, pp.166-194, 2002.

[15] A.J. van der Schaft, L_2-*Gain and Passivity Techniques in Nonlinear Control*, Lect. Notes in Control and Information Sciences, Vol. 218, Springer-Verlag, Berlin, 1996, p. 168, 2nd revised and enlarged edition, Springer-Verlag, London, 2000 (Springer Communications and Control Engineering series), p. xvi+249.

[16] A.J. van der Schaft, "Balancing of lossless and passive systems', *IEEE Transactions on Automatic Control*, Vol.53(9), pp.2153-2157, 2008.