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Abstract—In this paper a general approach is taken to yield a characterization of the class of stable plant controller pairs which is a generalization of the Youla parameterization for linear systems. This is based on the idea of representing the input–output pairs of the plant and controller as elements of the kernel of some related operator which is denoted the kernel representation of the system. It is demonstrated that in some sense the kernel representation is a generalization of the left coprime factorization of a general nonlinear system. Results giving one method of deriving a kernel representation for a nonlinear plant with a general state-space description are presented.

I. INTRODUCTION

MOST controller synthesis problems may be formulated as follows: Given a plant G, design a controller K such that the closed-loop system is stable and satisfies given (optimal) performance criteria. It is thus convenient to have a parameterization of the class of controllers which stabilize the plant and to optimize the performance criteria within this class. This approach has been quite successful in the linear case, the result being the Youla–Kucera parameterization, yielding for example an approach to the solution of the $\mathcal{H}_\infty$-control problem [3]. By providing such a parameterization for nonlinear feedback systems, it is hoped that nonlinear controller synthesis problems may also become more tractable. In particular, one would like to tackle the nonlinear $\mathcal{H}_\infty$-optimal control problem in this way.

The object of this paper is to reproduce the results which gave the Youla parameterization for nonlinear systems, as obtained using left coprime factorizations, of [13] in a more general framework. By taking this more general approach some implicit assumptions are avoided, and it is shown that state-space versions of these results are immediately available. This overcomes a major weakness in the left factorization theory, where state-space descriptions have only been found for two cases [9], [17].

A method of representing nonlinear systems is presented which we denote the kernel representation of the system. The input–output pairs of a system may be found in the kernel of this related operator which maps from the combined input and output spaces to some other space. This has obvious links to the behavioral approach to control developed by Willems; see, for example, [24] and [25] and the references therein. In

this paper the authors do not explore these links but develop a framework in which kernel representations may be used in the definition of such concepts as well posedness and stability of a closed-loop system and investigate their role as a generalization of left coprime factorizations of nonlinear operators.

It is demonstrated that with the characterizations of well posedness and stability presented, the class of plants stabilized by a given controller, the class of controllers stabilizing a given plant, and the class of all stable plant controller pairs may be easily parameterized. This mimics the results of [13], where such results were obtained using left coprime factorizations of the plant and controller, and the linear results of [19] which uses the Youla parameterization. The results presented in this paper are, however, more general. First, as they are applicable to a wider class of systems, and second as they are derived without distinguishing between the input and output spaces of the plant and controller. The development of the relationship between the kernel representation and the input–output representation of the system is delayed until after the presentation of the main results to emphasize the latter fact.

Once it has been demonstrated how a kernel representation may be specialized to yield an input–output representation for the system, a version of the Youla parameterization for nonlinear systems is presented in terms of kernel representations. The links to factorization of nonlinear operators are then explored, and it is suggested that stable kernel representations give a more appropriate generalization of left coprime factorizations of linear systems than the nonlinear left factorizations which have been investigated in the literature to date; see for example [1], [5], [6], [9], [11], [12], [14], and [17]. A further specialization shows how state-space realizations of kernel representations for nonlinear systems may be derived.

This work continues a series of investigations into the use of coprime factorizations in nonlinear systems analysis. Specifically, the motivation for these results is due to the use of left coprime factorizations of nonlinear systems, where a version of the Youla parameterization, giving the class of all linear controllers stabilizing a linear plant, has been derived; see [13] or [12] for details. This line of investigation was initiated by Hammer [5], [6], where it was demonstrated that if a right coprime factorization of a nonlinear plant satisfies a Bezout identity, then there exists a stabilizing pre- and post-compensator pair for the plant. Tay [18] showed that this leads to a class of such compensator pairs. This was then generalized by Paice and Moore [11], [12], [14] to derive a nonlinear version of the Youla parameterization.

Recent work by Hammer [7] appears at first glance to be similar to the results presented in this paper; however, the overall approach is different. The idea of using a stabilizing pre- and post-compensator pair has been generalized in [7], and although it leads to a class of stabilizing compensators, the results obtained are of a different form to those derived here.

Other work has been conducted attempting to derive a right factorization based approach to these results, notably the work by Verma [20]–[22]; see also the work by Chen [1], [2]. A general Youla parameterization using this approach has not been derived. However, based on the work of Sontag [16], a state-space formulation has been derived for right coprime factorizations. To date, state-space formulations for nonlinear left coprime factorizations have been derived for a special case in Moore and Irlicht [9] and more generally in [17]. In the latter work, sufficient conditions are derived for left factorizations to exist; however, these are very restrictive, and it is not clear how they should be checked.

The paper is organized as follows. In Section II a general framework for using stable kernel representations is presented. The concepts of well posedness and stability of feedback systems are developed for use within this framework. The main results of the paper are presented in Section III, giving the class of stable closed-loop systems which are representable within this framework. This class is parameterized in a way which specializes to the Youla parameterization in the linear case. The relationship of the stable kernel representation (*skr*) of a system to its input–output representation is then developed. It is shown that the existence of an input–output representation is equivalent to the well posedness of the system when it is in closed loop with the zero operator. Using this specialization, more direct versions of the Youla parameterization are derived. In Section IV it is demonstrated that the results which are currently available via nonlinear left coprime factorizations may also be derived using stable kernel representations. It is suggested that stable kernel representations are a more useful nonlinear generalization of linear left coprime factorizations than those currently available in the literature. This is further supported in Section V, where a state-space approach to deriving stable kernel representations due to Scherpen and van der Schaft [15] is presented. It may be seen that the state-space representation is a specialization of the more general framework developed in Section II. Conclusions are drawn in Section VI.

II. KERNEL REPRESENTATIONS

In this section the notion of representing a general system, Σ, as being represented by the kernel of a family of operators parameterized by the initial conditions of the system, is introduced. This is extended to give kernel representations of feedback systems by simply joining the kernel representation of the system and its compensator. Definitions of well posedness and stability of feedback systems are then presented for use within this framework.

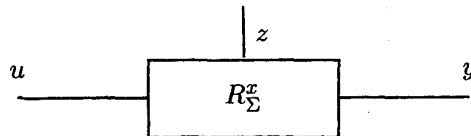In the sequel the term system will be taken to denote a general (dynamical) system, and the terms feedback system or

Fig. 1. The kernel representation of Σ for initial condition $x \in \mathcal{X}$.

closed-loop system will be used to indicate an interconnection of such systems.

A. Representing a General System

Consider the system Σ, with input and output spaces $\mathcal{U}$ and $\mathcal{Y}$ respectively, and initial condition space $\mathcal{X}_\Sigma$. Note that $\mathcal{U}$ and $\mathcal{Y}$ are taken to be signal spaces, that is sets of functions from a given time domain to a given signal set, which may be for example a discrete set, manifold, or vector space, whereas the initial condition space $\mathcal{X}_\Sigma$ is not a function space. It is assumed that every such system under consideration may be described by a family of maps

$$R_\Sigma^x : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}, \qquad \forall x \in \mathcal{X}_\Sigma \tag{1}$$

where $\mathcal{Z}$ is a signal space, known as the kernel representation of Σ, such that all possible input–output pairs u, y for the system Σ with initial conditions $x \in \mathcal{X}_\Sigma$ satisfy

$$R_\Sigma^x(y, u) = 0. \tag{2}$$

Note that it is not assumed that $\mathcal{Z}$ is a vector space; our approach does not use any properties of $\mathcal{Z}$. We only assume that there is a distinguished element of $\mathcal{Z}$ which we denote by zero.

Remark 2.1: For the subsequent developments, until Section III-B, it is in fact not necessary to distinguish *a priori* between inputs and outputs. Indeed, if we group u and y into a single vector w, then the entire framework and all results will remain valid for systems described as $R_\Sigma^x(w) = 0$, where R_Σ^x is an operator from $\mathcal{W}$ (the space of external signals) to $\mathcal{Z}$. This is clearly related to the behavioral approach to control, see, e.g., [24] and [25], although in this latter context one does not usually consider kernel representations of this type. (Instead one normally considers kernel representations of the form $\tilde{R}_\Sigma(w) = 0$, where $\tilde{R}_\Sigma$ is some (possibly nonlinear) differential operator. Furthermore, in this context the state-space $\mathcal{X}$ is usually not *a priori* given, but must be derived, e.g., from the above description of the behavior as a set of (higher-order) differential equations.)

In general it is not possible to describe a kernel representation by a single map $R_\Sigma : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}$; however for brevity, we shall refer to the kernel representation R_Σ. The key to the development of the following results is to examine the solutions to

$$R_\Sigma^x(y, u) = z \tag{3}$$

where z is not necessarily equal to zero. This may be visualized as in Fig. 1.

For z arbitrary, the input–output map induced by the solution pairs to (3) for a given initial condition $x \in \mathcal{X}$ will
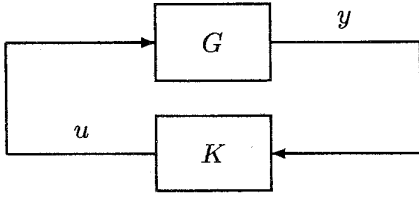
Fig. 2.   The system $\{G, K\}$ without external inputs.

be denoted by $\Sigma_z(x): \mathcal{U} \mapsto \mathcal{Y}$; although as noted above, the following development is not dependent on the existence of this map. The input–output map $\Sigma_0(x): \mathcal{U} \mapsto \mathcal{Y}$ will be simply denoted by $\Sigma(x)$, the input–output map of Σ for initial condition x.

Note that the kernel representation for a given system will not be unique, for example any input–output map $\Sigma(x): \mathcal{U} \mapsto \mathcal{Y}$ may be represented in this form by $R_\Sigma^x(y, u) = y - \Sigma(x)u$, although, this is not a useful form for the purposes of this paper. Later, additional assumptions will be assumed to hold for the map $R_\Sigma(\cdot, \cdot)$.

B. Feedback Systems

In this section the notion of interconnecting two systems, the plant and the controller, to form a closed-loop or feedback system is introduced and developed for use within this framework. Note that it is common to allow for the introduction of external signals between the plant and controller to account for reference signals or noise signals corrupting the control or measured signal; see, for instance, [13]. For simplicity, only the case where these external signals are zero will be considered; this is referred to as the noise-free case.

Remark 2.2: In general the problem becomes much more difficult when considering a feedback system with external inputs, so this is left to another study. In the linear case, any external disturbances may be directly accounted for by perturbations to z_G and z_K. In the case that all external disturbances can be represented by stable perturbations to z_G and z_K, it is expected that the loop may be reposed as in Fig. 3, yielding stability results for the closed loop.

Consider a plant, $G: \mathcal{U} \to \mathcal{Y}$, and controller, $K: \mathcal{Y} \to \mathcal{U}$, with kernel representations

$$R_G: \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_G \tag{4}$$

$$R_K: \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_K \tag{5}$$

which are interconnected to form the system $\{G, K\}$ as in Fig. 2. The closed loop then has a kernel representation

$$R_{\{G,K\}}: \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_G \times \mathcal{Z}_K$$
$$\begin{pmatrix} u \\ y \end{pmatrix} \mapsto \begin{pmatrix} z_G \\ z_K \end{pmatrix} = \begin{pmatrix} R_G(y, u) \\ R_K(u, y) \end{pmatrix} \tag{6}$$

as in Fig. 3.

The existence of a solution pair (u, y) for a given (z_G, z_K) is not guaranteed. Thus, to work with feedback systems within this framework, we will need to assume that for each (z_G, z_K) pair, a solution exists and is unique. This property is known as well posedness.
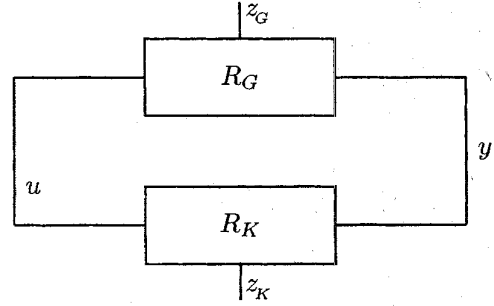

Fig. 3.   The kernel representation of $\{G, K\}$.

Definition 2.3 (Well Posedness): The system $\{G, K\}$ is well posed iff for all initial conditions, $(x_G, x_K) \in \mathcal{X}_G \times \mathcal{X}_K$, and for all $(z_G, z_K) \in \mathcal{Z}_G \times \mathcal{Z}_K$, a solution (u, y) to (6) exists and is unique. That is, for all $(x_G, x_K) \in \mathcal{X}_G \times \mathcal{X}_K$

$$[R_{\{G,K\}}^{x_G, x_K}]^{-1}: \mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{U} \times \mathcal{Y} \qquad \text{exists.} \tag{7}$$

$\square$

Remark 2.4: The above definition of well posedness of a feedback system, when specialized to linear systems, is very similar to the notion of regular feedback interconnection as proposed in [26]. Note that the requirement of existence of unique solutions $w = \begin{pmatrix} y \\ u \end{pmatrix}$ for every $x \in \mathcal{X}$ excludes the possibility of singular feedback [26].

In the sequel, the well posedness of such feedback systems will be considered over increasingly large cross products of signal and initial condition spaces. Thus the following notational convenience is adopted:

$$\mathcal{Z}_{GK} = \mathcal{Z}_G \times \mathcal{Z}_K, \qquad z_{GK} = (z_G, z_K) \in \mathcal{Z}_{GK}$$
$$\mathcal{X}_{GK} = \mathcal{X}_G \times \mathcal{X}_K, \qquad x_{GK} = (x_G, x_K) \in \mathcal{X}_{GK}.$$

C. Stability

We now define the concept of stability for general nonlinear operators and feedback systems. This is defined implicitly via the notion of stability on the various input and output spaces of these operators. A signal space $\mathcal{Z}$ is divided into two disjoint subsets as follows:

$$\mathcal{Z} = \mathcal{Z}^s \cup \mathcal{Z}^u, \qquad \mathcal{Z}^s \cap \mathcal{Z}^u = \emptyset$$

where $\mathcal{Z}^s$ denotes the set of all stable signals and $\mathcal{Z}^u$ the set of all unstable signals. For the space $\mathcal{Z}_{GK}$, $\mathcal{Z}_{GK}^s$ is defined to be $\mathcal{Z}_G^s \times \mathcal{Z}_K^s$, and $\mathcal{Z}_{GK}^u$ is the remainder of the space. $\square$

Note that $\mathcal{Z}$ may be partitioned in many ways. It is not assumed that $\mathcal{Z}^s$ is open, closed, or a vector space, although it is assumed that the distinguished element $0 \in \mathcal{Z}^s$. Commonly these sets are formed by defining a norm on the space $\mathcal{Z}$ and then defining a signal to be stable iff it has finite norm.

Definition 2.5 (Operator Stability): An input–output map $\Sigma: \mathcal{U} \to \mathcal{Y}$ is said to be stable if the image of $\mathcal{U}^s$ under Σ is a subset of $\mathcal{Y}^s$. $\square$

Note that for a given system, Σ, there are many possible kernel representations. In the sequel only those kernel representations which are stable will be considered.

Definition 2.6 (Stable Kernel Representations): A kernel representation $R_\Sigma: \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}$ of Σ is called a stable kernel representation (*skr*) of Σ iff for all initial conditions $x \in \mathcal{X}_\Sigma$, $R_\Sigma^x(\cdot, \cdot)$ is a stable operator. That is, if $y \in \mathcal{Y}^s$, $u \in \mathcal{U}^s$, then $z = R_\Sigma^x(y, u) \in \mathcal{Z}^s$. □

Remark 2.7: Note that in most cases there will exist a stable kernel representation for Σ. Consider $\Sigma: \mathcal{U} \to \mathcal{Y}$ and assume that there exists a mapping $F: \mathcal{Y} \to \mathcal{Y}$ which is stable and that $\forall y \in \mathcal{Y}$, $Fy \neq 0$. Then a stable kernel representation R_Σ may be defined as follows:

$$R_\Sigma(y, u) = \begin{cases} y - \Sigma u, & y \in \mathcal{Y}^u \text{ or } u \in \mathcal{U}^u \\ y - \Sigma u, & y \in \mathcal{Y}^s, u \in \mathcal{U}^s, (y - \Sigma u) \in \mathcal{Y}^s \\ Fy, & \text{otherwise.} \end{cases}$$
(8)

The signal space $\mathcal{Z}$ is then $\mathcal{Y}$, and $\mathcal{Z}^s = \mathcal{Y}^s$.

Remark 2.8: In general, the definition of the signal space $\mathcal{Z}$ will depend on the definitions of $\mathcal{U}$ and $\mathcal{Y}$, as in the construction (8). Although it would be possible to trivially define $\mathcal{Z}^s$ as the image of $\mathcal{Y}^s \times \mathcal{U}^s$ under $R_\Sigma(y, u) = y - \Sigma u$, this would lead to trivial results in the sense that the stability characterizations thus obtained would only hold for that definition of $\mathcal{Z}^s$. Such results would not specialize to the linear results. The results developed in this paper are applicable for systems where the stable signals $\mathcal{Z}^s$ are *a priori* defined.

Unless otherwise stated, all kernel representations used in the sequel will be *skr*'s.

The definition of stability is now extended to include closed-loop systems.

Definition 2.9 (Closed-Loop Stability): The closed-loop system $\{G, K\}$ with *skr* $R_{\{G,K\}}$ as in (6) is stable over $\mathcal{B}_{GK} \subset \mathcal{Z}_{GK}^s \times \mathcal{X}_{GK}$ if it is well posed, and for all pairs $(z_{GK}, x_{GK}) \in \mathcal{Z}_{GK}^s \times \mathcal{X}_{GK}$ the solution (y, u) to (6) is stable iff $(z_{GK}, x_{GK}) \in \mathcal{B}_{GK}$.

A section of $\mathcal{B}_{GK}$ corresponding to the initial condition x is defined as follows:

$$\mathcal{B}_{GK}^x = \{z: (z, x) \in \mathcal{B}_{GK}\}.$$

□

The system $\{G, K\}$ is said to be generally stable, or simply stable, if it is stable over $\mathcal{Z}_{GK}^s \times \mathcal{X}_{GK}$. The system signals (u, y) must be unstable for $z_{GK} \in \mathcal{Z}_{GK}^u$, otherwise the stability of the kernel representations R_G and R_K would be contradicted.

Lemma 2.10: The system $\{G, K\}$ is well posed and stable over $\mathcal{B} \subset \mathcal{Z}_{GK}^s \times \mathcal{X}_{GK}$ iff for all $x_{GK} \in \mathcal{X}_{GK}$ the map

$$[R_{\{G,K\}}^{x_{GK}}]^{-1}: \mathcal{Z} \to \mathcal{Y} \times \mathcal{U} \quad \text{exists}$$
(9)

and

$$[R_{\{G,K\}}^{x_{GK}}]^{-1}\mathcal{B}^{x_{GK}} \subset \mathcal{Y}^s \times \mathcal{U}^s.$$
(10)

□

The proof arises out of the definitions and is left to the reader.

III. MAIN RESULTS

In this section the results of [12] and [14] giving nonlinear versions of the Youla parameterization are generalized to use the framework presented in the previous section.

The construction of a general well posed and stable class of plant-controller pairs from a given well posed and stable feedback system $\{G, K\}$ with *skr* (6) is first presented. It is shown that this generates the class of all well posed and stable feedback systems which are expressible within this framework. A specialization of our framework which admits a well-defined input–output operator for each kernel representation is then developed. Results giving the class of stabilizing controllers for a given plant and the class of plants stabilized by a given controller are then stated.

A. Class of Stabilizing Plant Controller Pairs

Consider the systems $S: \mathcal{Z}_K \to \mathcal{Z}_G$ and $Q: \mathcal{Z}_G \to \mathcal{Z}_K$ with *skr*'s

$$R_S: \mathcal{Z}_G \times \mathcal{Z}_K \to \mathcal{Z}_S$$
(11)

$$R_Q: \mathcal{Z}_K \times \mathcal{Z}_G \to \mathcal{Z}_Q$$
(12)

and initial condition spaces $\mathcal{X}_S$, $\mathcal{X}_Q$, respectively.

The systems G_S and K_Q are defined via their stable kernel representations R_{G_S} and R_{K_Q}

$$R_{G_S}: \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_S$$
$$\begin{pmatrix} y \\ u \end{pmatrix} \mapsto z_s = R_S(R_G(y, u), R_K(u, y)) \quad (13)$$

and

$$R_{K_Q}: \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_Q$$
$$\begin{pmatrix} u \\ y \end{pmatrix} \mapsto z_Q = R_Q(R_K(u, y), R_G(y, u)). \quad (14)$$

Note that the initial condition spaces of G_S and K_Q are $\mathcal{X}_S \times \mathcal{X}_{GK}$ and $\mathcal{X}_Q \times \mathcal{X}_{GK}$, respectively.

The properties of the feedback loop $\{G_S, K_Q\}$, as shown in Fig. 4, are now investigated. This investigation yields the main results of the paper and follows in a straightforward fashion from the definitions of the previous section.

The following results become quite complex but may be readily understood when the equivalence of the two system representations, Figs. 4 and 5, is recognized.

Theorem 3.1: Consider a well-posed system $\{G, K\}$ with *skr* (6), and systems S, Q, with *skr*'s (11) and (12), respectively, giving G_S and K_Q with *skr*'s (13) and (14), respectively. Then the closed-loop system $\{G_S, K_Q\}$ is well posed iff the closed-loop system $\{S, Q\}$ is well posed.

Further, given a well-posed system $\{G^*, K^*\}$ with *skr* $R_{\{G^*,K^*\}}$, there exist kernel representations R_{S^*}, R_{Q^*}, for the systems S^* and Q^* such that $G_{S^*} = G^*$ and $K_{Q^*} = K^*$, and the system $\{S^*, Q^*\}$ is well posed. If $\{G, K\}$ is stable, then R_{S^*} and R_{Q^*} will be stable kernel representations; otherwise, they will be stable over a set of inputs and initial conditions which includes $\mathcal{B}_{GK} \times \mathcal{X}_{G^*K^*}$. □
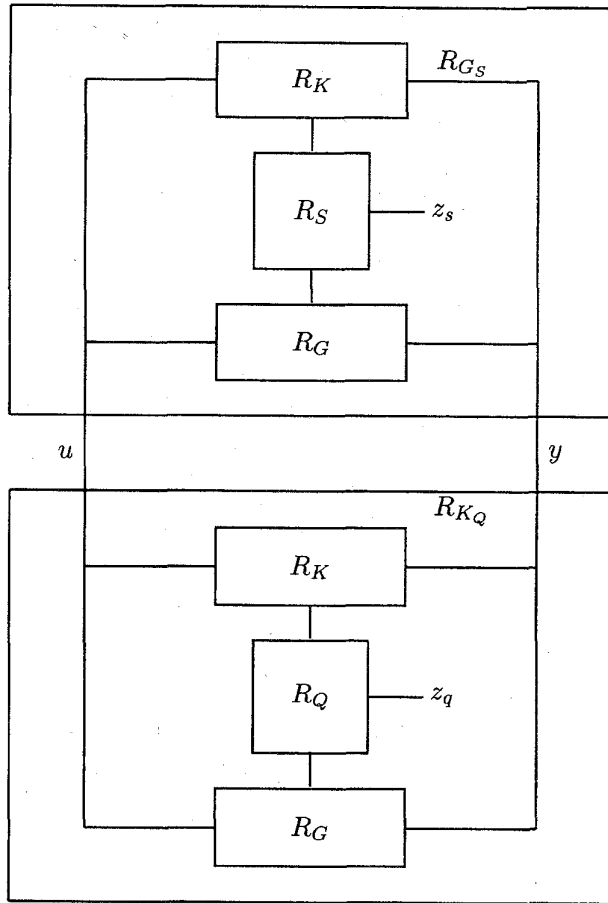
Fig. 4. The kernel representation of $\{G_S, K_Q\}$.
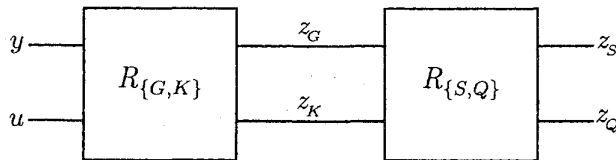
Fig. 5. An equivalent representation of $\{G_S, K_Q\}$.

Proof: By well posedness of $\{G, K\}$, for all $x_{GK} \in \mathcal{X}_{GK}$ the map $[R_{\{G,K\}}^{x_{GK}}]^{-1}$ exists. From (11) and (12), the kernel representation of $\{S, Q\}$ is given by

$$R_{\{S,Q\}} : \mathcal{Z}_G \times \mathcal{Z}_K \rightarrow \mathcal{Z}_S \times \mathcal{Z}_Q$$

$$\begin{pmatrix} z_G \\ z_K \end{pmatrix} \mapsto \begin{pmatrix} z_S \\ z_Q \end{pmatrix} = \begin{pmatrix} R_S(z_G, z_K) \\ R_Q(z_K, z_G) \end{pmatrix} \quad (15)$$

and via (13) and (14), the kernel representation of $\{G_S, K_Q\}$ is given by

$$R_{\{G_S,K_Q\}} : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}_S \times \mathcal{Z}_Q$$

$$\begin{pmatrix} y \\ u \end{pmatrix} \mapsto \begin{pmatrix} z_S \\ z_Q \end{pmatrix}$$

$$= \begin{pmatrix} R_S(R_G(y, u), R_K(u, y)) \\ R_Q(R_K(u, y), R_G(y, u)) \end{pmatrix}. \quad (16)$$

Thus, for arbitrary initial conditions $x_{GK} \in \mathcal{X}_{GK}, x_{SQ} \in \mathcal{X}_{SQ}$

$$R_{\{G_S,K_Q\}}^{x_{GK},x_{SQ}} = R_{\{S,Q\}}^{x_{SQ}} R_{\{G,K\}}^{x_{GK}} \quad (17)$$

where juxtaposition of kernel representations indicates composition. For each set of initial conditions (x_{GK}, x_{SQ}), the map $[R_{\{G,K\}}^{x_{GK}}]^{-1}$ exists; therefore, $[R_{\{G_S,K_Q\}}^{x_{GK},x_{SQ}}]^{-1}$ will exist if and only if $[R_{\{S,Q\}}^{x_{SQ}}]^{-1}$ exists. Thus, by the definition of well posedness, $\{G_S, K_Q\}$ is well posed iff $\{S, Q\}$ is well posed, proving the first part of the theorem.

Suppose $\{G^*, K^*\}$ is well posed, with initial condition space $\mathcal{X}_{G^*K^*}$, and *skr*

$$R_{\{G^*,K^*\}} : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}_{G^*} \times \mathcal{Z}_{K^*}. \quad (18)$$

The systems S^* and Q^*, with initial condition spaces $\mathcal{X}_{S^*} = \mathcal{X}_{G^*} \times \mathcal{X}_{GK}$, $\mathcal{X}_{Q^*} = \mathcal{X}_{K^*} \times \mathcal{X}_{GK}$, respectively, are defined via the kernel representation for $\{S^*, Q^*\}$ as follows:

$$R_{\{S^*,Q^*\}}^{x_{G^*K^*},x_{GK}} = R_{\{G^*,K^*\}}^{x_{G^*K^*}} [R_{\{G,K\}}^{x_{GK}}]^{-1} : \mathcal{Z}_{GK} \rightarrow \mathcal{Z}_{G^*K^*}. \quad (19)$$

Thus

$$R_{S^*}^{x_{S^*}} = R_{G^*}^{x_{G^*}} [R_{\{G,K\}}^{x_{GK}}]^{-1} : \mathcal{Z}_{GK} \rightarrow \mathcal{Z}_{G^*} \quad (20)$$

$$R_{Q^*}^{x_{Q^*}} = R_{K^*}^{x_{K^*}} [R_{\{G,K\}}^{x_{GK}}]^{-1} : \mathcal{Z}_{GK} \rightarrow \mathcal{Z}_{K^*}. \quad (21)$$

As $\{G^*, K^*\}$ and $\{G, K\}$ are well posed, (19) gives well posedness of the system $\{S^*, Q^*\}$. The initial condition space of G_S is given as

$$\mathcal{X}_{G_{S^*}} = \mathcal{X}_{S^*} \times \mathcal{X}_{GK} = \mathcal{X}_{G^*} \times \mathcal{X}_{GK} \times \mathcal{X}_{GK} \quad (22)$$

and thus covers $\mathcal{X}_{G^*}$. By the definitions given above

$$R_{G_{S^*}}^{(x_{G^*},x_{GK}^{(1)},x_{GK}^{(2)})} = R_{G^*}^{x_G^*} [R_{\{G,K\}}^{x_{GK}^{(1)}}]^{-1} R_{\{G,K\}}^{x_{GK}^{(2)}} \quad (23)$$

which is equal to $R_G^{x_G^*}$ provided $x_{GK}^{(1)} = x_{GK}^{(2)}$. Thus G_{S^*} with the restricted initial condition space $\mathcal{X}_{G^*} \times \text{diag}\, (\mathcal{X}_{GK} \times \mathcal{X}_{GK})$ equals G^*. Strictly speaking, this is a nonminimal realization of G^*. The same argument applies, giving equality of K_{Q^*} and K^*.

If $\{G, K\}$ is generally stable, then it is immediate from (20) that R_{S^*} is a *skr* for S^* iff R_{G^*} is a *skr* for G^*, and from (21), R_{Q^*} is a *skr* for Q^* iff R_{K^*} is a *skr* for K^*. If $\{G, K\}$ is not generally stable, then for the set of inputs and initial conditions for which $(z_{GK}, x_{GK}, x_{G^*K^*}) \in \mathcal{B}_{GK} \times \mathcal{X}_{G^*K^*}$, $[R_{\{G,K\}}^{x_{GK}}]^{-1} z_{GK}$ is stable, and thus $R_{S^*}^{x_{GK},x_{G^*}} z_{GK}$ and $R_{Q^*}^{x_{GK},x_{K^*}} z_{GK}$ will be stable, completing the proof. $\square$

Remark 3.2: Note that for $\{G, K\}$ not generally stable, the actual set of input, initial condition sets over which R_{S^*} and R_{Q^*} are stable may be wider than that given by the theorem. The additional inputs for which R_{S^*} and R_{Q^*} give stable outputs are given, for each set of initial conditions $(x_{GK}, x_{G^*K^*})$, by the sets

$$\mathcal{B}_{S^*}^* = R_{G^*}^{x_{G^*}} ([R_{\{G,K\}}^{x_{GK}}]^{-1} \mathcal{Z}_{GK}^s \bigcap [R_{G^*}^{x_{G^*}}]^{-1} \mathcal{Z}_{G^*}^s \bigcap (\mathcal{Y} \times \mathcal{U})^u)$$

and

$$\mathcal{B}_{Q^*}^* = R_{K^*}^{x_{K^*}} ([R_{\{G,K\}}^{x_{GK}}]^{-1} \mathcal{Z}_{GK}^s \bigcap [R_{K^*}^{x_{K^*}}]^{-1} \mathcal{Z}_{K^*}^s \bigcap (\mathcal{Y} \times \mathcal{U})^u)$$

respectively.

Remark 3.3: Note that the initial conditions of the plant and controller must be available when constructing the new system $\{G_S, K_Q\}$; this means that K_Q uses the exact initial conditions of the plant G. In practice this may not be feasible which limits the applicability of these results; however, this is a feature of the input–output approach which seems to be unavoidable. It is hoped that the notation developed here, where initial conditions must be explicitly accounted for, makes this point explicit. Note that the effects of different plant initial conditions may be modeled within the parameter Q, and thus stability of the closed loop may be examined using the techniques of the following theorem. In this way the extent of the stability set $\mathcal{B}_{GK}$ may be explored.

The previous theorem is now extended to give the stability properties of the closed-loop $\{G_S, K_Q\}$.

Theorem 3.4: Consider a system $\{G, K\}$ with *skr* (6) which is well posed and stable over $\mathcal{B}_{GK}$ and systems S, Q, with *skr*'s (11) and (12), respectively, giving G_S and K_Q with *skr*'s (13) and (14), respectively. Then the closed-loop system $\{G_S, K_Q\}$ of (16) will be well posed and stable over $\mathcal{B}_{G_S K_Q}$ iff the closed-loop system $\{S, Q\}$ of (15) is well posed and stable over $\mathcal{B}_{SQ}$, where

$$(z_{SQ}, x_{SQ}, x_{GK}) \in \mathcal{B}_{G_S K_Q} \Leftrightarrow (z_{SQ}, x_{SQ}) \in \mathcal{B}_{SQ}$$
$$\text{and } ([R_{\{S,Q\}}^{x_{SQ}}]^{-1} z_{SQ}, x_{GK}) \in \mathcal{B}_{GK}. \quad (24)$$

Further, given a system $\{G^*, K^*\}$ with *skr* $R_{\{G^*,K^*\}}: (u, y) \mapsto z_{G^*K^*}$ which is well posed and stable over $\mathcal{B}_{G^*K^*}$, there exist "stable" kernel representations for the systems S^* and Q^* given by (20) and (21), respectively, such that $G_{S^*} = G^*$ and $K_{Q^*} = K^*$, and the system $\{S^*, Q^*\}$ is well posed and stable over

$$\mathcal{B}_{SQ}^* = \mathcal{B}_{G^*K^*} \times \mathcal{X}_{GK} \bigcup \Big\{ (z_{G^*K^*}, x_{G^*K^*}, x_{GK}):$$
$$(z_{G^*K^*}, x_{G^*K^*})$$
$$\notin \mathcal{B}_{G^*K^*}, [R_{\{S^*,Q^*\}}^{x_{G^*K^*},x_{GK}}]^{-1} z_{G^*K^*} \in \mathcal{Z}_{GK}^s \Big\}. \quad (25)$$

If $\{G, K\}$ is generally stable, then (20) and (21) give *skr*'s for S^* and Q^*, and the set in parenthesis in (25) will be empty.   □

Remark 3.5: Note that the second set in the right-hand side of (25) may be given the form

$$\{(z_{G^*K^*}, x_{G^*K^*}, x_{GK}): z_{G^*K^*} \in R_{\{G^*,K^*\}}^{x_{G^*K^*}}$$
$$\cdot ([R_{\{G,K\}}^{x_{GK}}]^{-1} \mathcal{Z}_{GK}^s \bigcap [R_{\{G^*,K^*\}}^{x_{G^*K^*}}]^{-1} \mathcal{Z}_{G^*K^*}^s \bigcap (\mathcal{Y} \times \mathcal{U})^u)\}$$

as may be easily seen from Remark 3.2.

Proof: By Theorem 3.1, $\{G_S, K_Q\}$ is well posed iff $\{S, Q\}$ is well posed and (17) holds. This equation may be inverted. Thus, for all initial conditions $x_{GK} \in \mathcal{X}_{GK}$ and $x_{SQ} \in \mathcal{X}_{SQ}$

$$[R_{\{G_S,K_Q\}}^{x_{GK},x_{SQ}}]^{-1} = [R_{\{G,K\}}^{x_{GK}}]^{-1}[R_{\{S,Q\}}^{x_{SQ}}]^{-1} \quad (26)$$

$$[R_{\{S,Q\}}^{x_{SQ}}]^{-1} = R_{\{G,K\}}^{x_{GK}}[R_{\{G_S,K_Q\}}^{x_{GK},x_{SQ}}]^{-1}. \quad (27)$$

Necessity of (24) is now proven. Suppose that $\{S, Q\}$ is stable over $\mathcal{B}_{SQ}$, and consider $(z_{SQ}, x_{SQ}) \in \mathcal{B}_{SQ}$, then $z_{GK} = [R_{\{S,Q\}}^{x_{SQ}}]^{-1} z_{SQ}$ is stable. If $(z_{GK}, x_{GK}) \in \mathcal{B}_{GK}$,

then $(y, u) = [R_{\{G,K\}}^{x_{GK}}]^{-1} z_{GK}$ is stable. By (26), $(y, u) = [R_{\{G_S,K_Q\}}^{x_{SQ},x_{GK}}]^{-1} z_{SQ}$, and thus $(z_{SQ}, x_{SQ}, x_{GK}) \in \mathcal{B}_{G_S K_Q}$, as required.

To prove sufficiency of (24), assume $\{G_S, K_Q\}$ stable over $\mathcal{B}_{G_S K_Q}$, and consider the action of (27) on $(z_{SQ}, x_{SQ}, x_{GK}) \in \mathcal{B}_{G_S K_Q}$. Then $(y, u) = [R_{\{G_S,K_Q\}}^{x_{GK},x_{SQ}}]^{-1} z_{SQ}$ is stable, and as $R_{\{G,K\}}$ is a *skr* for $\{G, K\}$, $[R_{\{S,Q\}}^{x_{SQ}}]^{-1} z_{SQ} = R_{\{G,K\}}^{x_{GK}}(y, u)$ is stable, giving $(z_{SQ}, x_{SQ}) \in \mathcal{B}_{SQ}$. Further, due to the stability of (y, u) and well posedness of $\{G, K\}$, $(z_{GK}, x_{GK}) \in \mathcal{B}_{GK}$. Thus $\{G_S, K_Q\}$ is stable over $\mathcal{B}_{G_S K_Q}$ iff $\{S, Q\}$ is stable over $\mathcal{B}_{SQ}$ as in (24).

Suppose that the system $\{G^*, K^*\}$ is well posed and stable over $\mathcal{B}_{G^*K^*}$. By Theorem 3.1, there exist systems S^*, Q^*, defined via the kernel representations (20) and (21), respectively, such that $G^* = G_{S^*}$ and $K^* = K_{Q^*}$, and $\{S^*, Q^*\}$ is well posed.

The stability of $\{S^*, Q^*\}$ is now investigated by constructing $\mathcal{B}_{S^*,Q^*}$. By Definition 2.9

$$\mathcal{B}_{SQ}^* = \{(z_{G^*K^*}, x_{G^*K^*}, x_{GK})$$
$$: [R_{\{S^*,Q^*\}}^{x_{G^*K^*},x_{GK}}]^{-1} z_{G^*K^*} \in \mathcal{Z}_{GK}^s\}. \quad (28)$$

Note that if $(z_{G^*K^*}, x_{G^*K^*}) \in \mathcal{B}_{G^*K^*}$, then $(y, u) = [R_{\{G^*,K^*\}}^{x_{G^*K^*}}]^{-1} z_{G^*,K^*}$ is stable, and by stability of the kernel representation $R_{\{G,K\}}$ it is evident that

$$[R_{\{S^*,Q^*\}}^{x_{G^*K^*},x_{GK}}]^{-1} z_{G^*K^*}$$
$$= R_{\{G,K\}}^{x_{GK}}[R_{\{S^*,Q^*\}}^{x_{G^*K^*},x_{GK}}]^{-1} z_{G^*K^*} \in \mathcal{Z}_{GK}^s.$$

Thus $\mathcal{B}_{G^*K^*} \times \mathcal{X}_{GK} \subset \mathcal{B}_{S^*Q^*}$. The expression (25) for $\mathcal{B}_{SQ}^*$ follows immediately.

If $\{G, K\}$ is generally stable, then by Theorem 3.1, (20) and (21) give *skr*'s for S^* and Q^*. Further, for all triples $(z_{G^*K^*}, x_{G^*K^*}, x_{GK})$ such that $[R_{\{S^*,Q^*\}}^{x_{G^*K^*},x_{GK}}]^{-1} z_{G^*K^*} \in \mathcal{Z}_{GK}^s$

$$[R_{\{G,K\}}^{x_{GK}}]^{-1}[R_{\{S^*,Q^*\}}^{x_{G^*K^*},x_{GK}}]^{-1} z_{G^*K^*}$$
$$= [R_{\{G^*K^*\}}^{x_{G^*K^*}}]^{-1} z_{G^*K^*} \in \mathcal{U}^s \times \mathcal{Y}^s$$

and thus $(z_{G^*K^*}, x_{G^*K^*}) \in \mathcal{B}_{G^*K^*}$. Thus the second set in the right-hand side of (25) is empty.   □

Remark 3.6: Given a closed-loop system $\{G, K\}$ with a stable kernel representation which is well posed and generally stable, it is possible to parameterize the class of all well-posed and stable systems which have *skr*'s. Thus these theorems give a generalization of the linear results of Tay *et al.* [19] and the nonlinear results of Paice and Moore [12] when these are restricted to the noise-free case.

Remark 3.7: Note that the apparent discrepancy between (24) and (25) is due to the fact that (20) and (21) do not necessarily give stable kernel representations for S^* and Q^*.

Remark 3.8: That *skr*'s lead to a parameterization of stable closed-loop systems suggests a link to the theory of coprime factorizations. This is explored in Section IV.

Remark 3.9: By considering the system $0: u \mapsto 0$ and defining how a given *skr* should relate to an input–output operator, it is possible to derive more explicit analogues of the Youla parameterization. This is explored in the following section.

B. Kernel Representations and Input–Output Operators

In this section the definitions required to specialize the framework presented in Section II to an input–output framework are presented. It is seen that the key to these results is to apply the definitions of well posedness and stability for a closed loop to the system when in closed loop with the zero operator.

As noted previously, since we have not distinguished between the input and output spaces, the previous results may be considered from a behavioral point of view, Remark 2.1. In the case that we wish to move to an input–output or state-space point of view, it becomes necessary to assume that it is possible to identify inputs and outputs, and that once the inputs are specified, the outputs are determined. This is equivalent to assuming that given a set of initial conditions $x \in \mathcal{X}_\Sigma$, each $z \in \mathcal{Z}$, yields an input–output map

$$\Sigma_z(x) : \mathcal{U} \to \mathcal{Y} \qquad (29)$$

such that $y = \Sigma_z(x)u$ satisfies (3) for all $u \in \mathcal{U}$. This property is denoted well definedness of the skr.

Definition 3.10 (Well Definedness): A kernel representation (1) is said to be well-defined if for each $z \in \mathcal{Z}$, and initial conditions $x \in \mathcal{X}_\Sigma$, (29) exists, so that for all $u \in \mathcal{U}$, $y = \Sigma_z(x)u$ iff $R_\Sigma^x(u, y) = z$. ☐

Note that R_Σ can be well-defined for $x \in \mathcal{X}_\Sigma$ only if the map

$$R_\Sigma^x(\cdot, u) : \mathcal{Y} \to \mathcal{Z} \qquad (30)$$

is one to one and onto, i.e., invertible. We denote this inverse

$$[R_\Sigma^x]^{-1}(\cdot, u) : \mathcal{Z} \to \mathcal{Y}. \qquad (31)$$

This is summarized in the following result.

Proposition 3.11: A given kernel representation (1) of Σ is well defined iff for all $x \in \mathcal{X}$ and all $u \in \mathcal{U}$, the map $[R_\Sigma^x]^{-1}(\cdot, u)$ of (31) exists. ☐

The proof is trivial and is left to the reader.

We will also need to discuss the stability of an input–output operator. This is defined as follows.

Definition 3.12 (System Stability): A system Σ with stable kernel representation $R_\Sigma(\cdot, \cdot)$, as in (1), is stable over the set $\mathcal{B} \subset \mathcal{Z}^s \times \mathcal{X}_\Sigma$ if for all $(z, x) \in \mathcal{Z}^s \times \mathcal{X}_\Sigma$ the input–output map $\Sigma_z(x)$ is stable iff $(z, x) \in \mathcal{B}$. ☐

The system Σ with skr (1) is called generally stable, or simply stable, if it is stable over $\mathcal{Z}^s \times \mathcal{X}$.

By considering the zero operator in closed loop with another system, it is possible to relate well definedness with well posedness and system stability with the previous definition of closed-loop stability. This property is presented in Lemma 3.13. We first define a well-defined skr for the zero operator, 0, defined by

$$0 : \mathcal{U} \to \mathcal{Y}, \qquad \forall u \in \mathcal{U}, \, 0(u) = 0 \qquad (32)$$

as being given by

$$R_0(y, u) = y. \qquad (33)$$

Lemma 3.13: Consider a system Σ with skr $R_\Sigma : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_\Sigma$ which is placed in closed-loop with the system $0 : \mathcal{Y} \to \mathcal{U}$ with skr $R_0(u, y) = u$ [note that this is the reverse case to (33)]. Then

1) R_Σ is well-defined iff the closed-loop $\{\Sigma, 0\}$ is well posed.
2) The operator $\Sigma_z : \mathcal{U} \to \mathcal{Y}$ is stable over $\mathcal{B}_\Sigma \subset \mathcal{Z}_\Sigma^s \times \mathcal{X}_\Sigma$ iff the feedback system $\{\Sigma, 0\}$ is stable over $\mathcal{B}_\Sigma \times \mathcal{Z}_0^s$. ☐

Note that the zero operator has no state space, and thus no initial condition space, so $\mathcal{B}_{\Sigma 0} = \mathcal{B}_\Sigma \times \mathcal{Z}_0^s$ which is consistent with Definition 2.9.

Proof: The system $\{\Sigma, 0\}$ is well posed iff for all initial conditions $x_\Sigma \in \mathcal{X}_\Sigma$, and all z_Σ, z_0 the solutions to

$$R_\Sigma^{x_\Sigma}(y, u) = z_\Sigma$$
$$R_0(u, y) = z_0 \qquad (34)$$

exist and are unique. By (33), $z_0 = u$, thus $\{\Sigma, 0\}$ is well posed iff for all z_Σ, u the solution to

$$R_\Sigma^{x_\Sigma}(y, u) = z_\Sigma$$

exists and is unique. This will hold iff the function $[R_\Sigma^{x_\Sigma}]^{-1}(\cdot, u)$ exists and by Proposition 3.11 R_Σ is well defined. Thus $\{\Sigma, 0\}$ is well posed iff R_Σ is well defined.

We assume now that the closed loop is well defined and consider its stability. The closed-loop $\{\Sigma, 0\}$ is stable over $\mathcal{B}_\Sigma \times \mathcal{Z}_0^s$ iff for all initial conditions $x_\Sigma \in \mathcal{X}_\Sigma$, $z_\Sigma \in \mathcal{B}_\Sigma^{x_\Sigma} z_0 \in \mathcal{Z}_0^s$, the unique solution to (34) is stable. By the preceding argument, this solution is given by $u = z_0$, $y = [R_\Sigma^{x_\Sigma}]^{-1}(\cdot, u)$. Thus the closed loop is stable over $\mathcal{B}_\Sigma \times \mathcal{Z}_0^s$ iff for all stable u, $y = [R_\Sigma^{x_\Sigma}]^{-1}(\cdot, u) = \Sigma_{z_\Sigma}^{x_\Sigma}(u)$ is stable, giving stability of the system over $\mathcal{B}_\Sigma$. ☐

The corollaries to Theorem 3.4 derived by considering alternately $S = 0$ and $Q = 0$ are now expressible in a form more easily seen to be generalizations of the existing results giving the Youla parameterization.

C. The Youla Parameterization via Stable Kernel Representations

The following corollaries to Theorem 3.4 give the class of all controllers which stabilize a given plant and the class of all plants stabilized by a given controller, respectively. They are generalizations of the results given in [11]–[13] which were the first results giving Youla parameterizations for general nonlinear systems.

Corollary 3.14: Consider a system $\{G, K\}$ with skr (6) which is well posed and stable over $\mathcal{B}_{GK}$ and the system Q with skr (12) such that K_Q is given by the skr (14). Then the closed-loop system $\{G, K_Q\}$ will be well posed iff the skr for Q is well defined. Further, $\{G, K_Q\}$ will be stable over $\mathcal{B}_{GK_Q}$ iff Q is well defined and stable over $\mathcal{B}_Q$, where[1]

$$(z_G, z_Q, x_Q, x_{GK}) \in \mathcal{B}_{GK_Q} \Leftrightarrow (z_Q, x_Q) \in \mathcal{B}_Q$$
$$\text{and } (z_G, Q_{z_Q}(x_Q)z_G, x_{GK}) \in \mathcal{B}_{GK}. \qquad (35)$$

---

[1] Recall that $Q_{z_Q}(x_Q)$ denotes an input–output in the family of input–output maps associated with Q via its stable kernel representation, so $y_Q = Q_{z_Q}(x_Q)z_G$ is the unique solution to $R_Q^{x_Q}(y_Q, z_G) = z_Q$.

Further, given a K^* with skr $R_{K^*}: (u, y) \mapsto z_{K^*}$, the closed-loop system $\{G, K^*\}$ is well posed iff the kernel representation for the system Q^* given by (21) is well defined. If the system $\{G, K^*\}$ is stable over $\mathcal{B}_{GK^*}$, then the stability properties of Q^* are defined as follows:

$$Q_{z_{K^*}}(x_{K^*}, x_{GK})z_G = z_K \in \mathcal{Z}_K^s \Leftrightarrow (z_{K^*}, z_G, x_{K^*}, x_G)$$
$$\in \mathcal{B}_{GK^*} \bigcup \mathcal{B}_{GK^*}^* \quad (36)$$

where

$$\mathcal{B}_{GK^*}^* = R_{\{G,K^*\}}^{x_G, x_{K^*}}([R_{\{G,K\}}^{x_G, x_K}]^{-1}\mathcal{Z}_{GK}^s$$
$$\bigcap [R_{\{G,K^*\}}^{x_G, x_{K^*}}]^{-1}\mathcal{Z}_{GK^*}^s \bigcap (\mathcal{Y} \times \mathcal{U})^u) \times \{x_{K^*}\}.$$

If $\{G, K\}$ is generally stable, the set $\mathcal{B}_{GK^*}^*$ is empty. $\square$

Corollary 3.15: Consider a system $\{G, K\}$ with skr (6) which is well posed and stable over $\mathcal{B}_{GK}$ and the system S with skr (11) such that G_S is given by the skr (13). Then the closed-loop system $\{G_S, K\}$ will be well posed iff the system S is well defined. Further, $\{G_S, K\}$ will be stable over $\mathcal{B}_{G_S K}$ iff S is stable over $\mathcal{B}_S$, where

$$(z_S, z_K, x_S, x_{GK}) \in \mathcal{B}_{G_S K} \Leftrightarrow (z_S, x_S) \in \mathcal{B}_S$$
$$\text{and } (S_{z_S}(x_S)z_K, z_K, x_{GK}) \in \mathcal{B}_{GK}. \quad (37)$$

Further, given a G^* with skr $R_{G^*}: (u, y) \mapsto z_{G^*}$, the closed-loop system $\{G^*, K\}$ is well posed iff the kernel representation for the system S^* given by (20) is well defined. If the system $\{G^*, K\}$ is stable over $\mathcal{B}_{G^* K}$, then the stability properties of S^* are defined as follows:

$$S_{z_{G^*}}(x_{G^*}, x_{GK})z_K = z_G \in \mathcal{Z}_G^s$$
$$\Leftrightarrow (z_{G^*}, z_K, x_{G^*}, x_K) \in \mathcal{B}_{G^* K} \bigcup \mathcal{B}_{G^* K}^* \quad (38)$$

where

$$\mathcal{B}_{G^* K}^* = R_{\{G^*,K\}}^{x_K, x_{G^*}}([R_{\{G,K\}}^{x_G, x_K}]^{-1}\mathcal{Z}_{GK}^s$$
$$\bigcap [R_{\{G^*,K\}}^{x_{G^*}, x_K}]^{-1}\mathcal{Z}_{G^* K}^s \bigcap (\mathcal{Y} \times \mathcal{U})^u) \times \{x_{G^*}\}.$$

If $\{G, K\}$ is generally stable, the set $\mathcal{B}_{G^* K}^*$ is empty. $\square$

Remark 3.16: These corollaries give generalizations of the results presented in [12] to the stable kernel representation framework. They give explicit versions of the Youla parameterization for linear systems. Further, as seen in Section V, it is possible to derive skr's for nonlinear systems with general state-space representations. By applying these corollaries to this special case, a state-space characterization of the Youla parameterization for nonlinear systems may be derived. We believe that these are the first such results presented in the literature.

Remark 3.17: That these results generalize the existing linear results may be easily seen as follows. Consider a linear system given by transfer matrix $G(s)$ and a stabilizing controller given by the transfer matrix $K(s)$. Write the left coprime factorizations as $G(s) = M^{-1}(s)N(s)$, $K(s) = V^{-1}(s)U(s)$ for stable rational matrices $M(s)$, $N(s)$, $U(s)$, $V(s)$. Note that we can equivalently associate $G(s)$, $K(s)$, with the kernels of the matrices $R_G = [M(s) \mid -N(s)]$, respectively, $R_K = [-U(s) \mid V(s)]$. Following Corollary
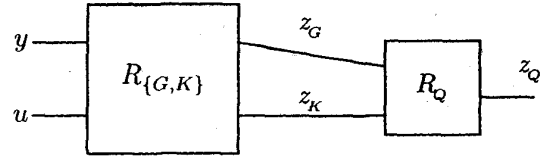

Fig. 6. The Youla–Kucera parameterization.

3.14, see also Fig. 5, the input–output behavior of any stabilizing controller is generated as the set of (u, y) resulting in $z_Q = 0$ in Fig. 6 for some stable system $Q(s)$. Restricting to zero initial conditions and linear stabilizing controllers, it follows that the input–output behavior of all linear stabilizing controllers is given by the kernels of

$$[I \mid -Q(s)]\begin{bmatrix} V(s) & -U(s) \\ -N(s) & M(s) \end{bmatrix}$$
$$= [V(s) + Q(s)N(s) \mid -U(s) - Q(s)M(s)] \quad (39)$$

for any stable rational matrix $Q(s)$. Note that the transfer matrix corresponding to (39) is

$$K_Q(s) = [V(s) + Q(s)N(s)]^{-1}[U(s) + Q(s)M(s)]$$

and we recover the classical Youla–Kucera parameterization of linear stabilizing controllers in an insightful manner.

IV. RELATIONSHIP TO COPRIME FACTORIZATIONS

In the previous section it was seen that the main results obtained in nonlinear factorization theory using left factorizations are duplicable using stable kernel representations. We now further explore the relationship between skr's and coprime factorizations and demonstrate that the skr of a general operator is a generalization of its left coprime factorization. It is shown that any operator with a left coprime factorization has a stable kernel representation and that the results derived linking nonlinear left and right coprime factorizations may also be obtained using skr's.

In the sequel, all statements will be assumed to hold for arbitrary initial conditions, and so for notational convenience, the superscripts denoting the initial conditions have been suppressed. However, it should be noted that attention must be paid to initial conditions, as the validity of the factorization of an operator is initial condition dependent.

We first define stable factorizations of a general nonlinear operator.

Definition 4.1 (Stable Factorizations): The system $\Sigma: \mathcal{U} \to \mathcal{Y}$ has a stable right factorization if there exist stable operators

$$D: \mathcal{Z}_r \to \mathcal{U} \quad \text{invertible,}$$
$$N: \mathcal{Z}_r \to \mathcal{Y} \quad (40)$$

such that $\Sigma = ND^{-1}$.

The system $\Sigma: \mathcal{U} \to \mathcal{Y}$ has a stable left factorization if there exist stable operators

$$\tilde{D}: \mathcal{Y} \to \mathcal{Z}_l \quad \text{invertible,}$$
$$\tilde{N}: \mathcal{U} \to \mathcal{Z}_l \quad (41)$$

such that $\Sigma = \tilde{D}^{-1}\tilde{N}$. $\square$

The following result establishes that the *skr* of a system is a generalization of the left factorization of a system.

Proposition 4.2: A system $\Sigma : \mathcal{U} \to \mathcal{Y}$ will have a stable left factorization (41) iff there exists a stable kernel representation R_Σ for Σ (1) which is well defined and separable in the sense that

$$R_\Sigma(y, u) = R_y(y) - R_u(u). \tag{42}$$

The stable factorization will be given by

$$\tilde{D} = R_y, \qquad \tilde{N} = R_u. \tag{43}$$

$\square$

Proof: Suppose that $\Sigma = \tilde{D}^{-1}\tilde{N}$ is a stable left factorization, then it is straightforward to see that

$$R_\Sigma : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}, \qquad R(y, u) = \tilde{D}y - \tilde{N}u \tag{44}$$

is a stable kernel representation for Σ. Further, as $\tilde{D}$ is invertible, the operator $[R_\Sigma]^{-1}(\cdot, u)$ exists and by Proposition 3.11 Σ is well defined for this *skr*.

Conversely, suppose that $R_\Sigma : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}$ is a stable kernel representation for Σ which is well defined and separable in the sense of (42). The operators $\tilde{D}$ and $\tilde{N}$ of (43) will be stable, and to prove that this is a stable left factorization of Σ it only remains to be shown that $\tilde{D}$ is invertible. By Proposition 3.11, the operator $[R_\Sigma]^{-1}(\cdot, u)$ exists, that is, once u is fixed, there exists a one-to-one and onto mapping between z and y. It is straightforward to see that this implies that $\tilde{D}$ is invertible, and the proof is complete. $\square$

Note that in the linear case, all *skr*'s are separable and thus equivalent to left factorizations.

Remark 4.3: A similar result to that presented in this proposition was proven by Hammer [4, Theorem 5.8]. Note, however, that the result is concerned with a recursive representation for a discrete time system Σ rather than a stable kernel representation as is the case here.

Coprime factorizations of a given system are now defined in terms of Bezout identities. Right and left factorizations have been previously defined both in terms of Bezout identities; see for example [20], [1], and [2], and from a set theoretic point of view, see [13], [5], and [9]. In the linear case these definitions are equivalent, and in the nonlinear case these definitions may be seen to be equivalent for right coprime factorizations. However, for left coprime factorizations, the connection is not well established, and further, there are weaknesses in the set-theoretic approach to the definition of left coprimeness which lead us to taking a Bezout-based approach.

Definition 4.4 (Coprime Factorizations): Consider a system $\Sigma : \mathcal{U} \to \mathcal{Y}$ which has stable right and left factorizations as in (40) and (41). Then $\Sigma = ND^{-1}$ is a right coprime factorization iff there exists a stable operator $L : \mathcal{Y} \times \mathcal{U} \mapsto \mathcal{Z}_r$ such that

$$L \begin{bmatrix} D \\ N \end{bmatrix} = I_{\mathcal{Z}_r}. \tag{45}$$

Similarly, $\Sigma = \tilde{D}^{-1}\tilde{N}$ is a left coprime factorization iff there exists a stable operator $T : \mathcal{Z}_l \to \mathcal{Y} \times \mathcal{U}$ such that

$$[\tilde{D} \ \ \tilde{N}]T = I_{\mathcal{Z}_l}. \tag{46}$$

$\square$

Remark 4.5: The identity operators of (45) and (46) are sometimes replaced by unimodular operators, i.e., stable operators with stable inverses; however, it may be seen that identity operators may be used without loss of generality.

Remark 4.6: Note that the operator T of (46) may be written as $\begin{bmatrix} T_y \\ T_u \end{bmatrix}$, where $T_y : \mathcal{Z}_l \to \mathcal{Y}$, $T_u : \mathcal{Z}_l \to \mathcal{U}$. Then if T_y is invertible, (46) implies that $T_u T_y^{-1}$ gives a right coprime factorization of some other operator.

Thus the existence of a left coprime factorization implies the existence of a right factorization for some other operator. This is exploited in [13] to show that if the plant and controller of a stable and well-posed system have left factorizations, then they will also have right factorizations.

This result may also be proven for a well posed and stable system $\{G, K\}$ with *skr* $R_{\{G,K\}}$.

Proposition 4.7: Consider a system $\{G, K\}$ with *skr* (6) which is well posed and generally stable, and that G and K are well defined. Then there exist right coprime factorizations for G and K

$$G = NM^{-1}, \qquad \begin{matrix} N : \mathcal{Z}_K \mapsto \mathcal{Y} \\ M : \mathcal{Z}_K \mapsto \mathcal{U} \end{matrix} \tag{47}$$

$$K = UV^{-1}, \qquad \begin{matrix} U : \mathcal{Z}_G \mapsto \mathcal{U} \\ V : \mathcal{Z}_G \mapsto \mathcal{Y} \end{matrix} \tag{48}$$

which satisfy the generalized Bezout identities

$$R_G \begin{bmatrix} V \\ U \end{bmatrix} = I_{\mathcal{Z}_G}, \qquad R_G \begin{bmatrix} N \\ M \end{bmatrix} = 0$$

$$R_K \begin{bmatrix} V \\ U \end{bmatrix} = 0, \qquad R_K \begin{bmatrix} N \\ M \end{bmatrix} = I_{\mathcal{Z}_K}. \tag{49}$$

$\square$

Proof: Consider a feedback system $\{G, K\}$ with *skr* $R_{\{G,K\}}$ which is well posed and generally stable. By Definition 2.3, the operator $[R_{\{G,K\}}]^{-1}$ exists and is stable, and thus the operators $T_G : \mathcal{Z}_G \to \mathcal{Y} \times \mathcal{U}$ and $T_K : \mathcal{Z}_K \to \mathcal{Y} \times \mathcal{U}$ defined by

$$T_G z_G = [R_{\{G,K\}}]^{-1} \begin{bmatrix} z_G \\ 0 \end{bmatrix}, \qquad T_K z_K = [R_{\{G,K\}}]^{-1} \begin{bmatrix} 0 \\ z_K \end{bmatrix} \tag{50}$$

are also be stable. The stable operators

$$M : \mathcal{Z}_K \to \mathcal{U}, \qquad N : \mathcal{Z}_K \to \mathcal{Y}$$
$$V : \mathcal{Z}_G \to \mathcal{Y}, \qquad U : \mathcal{Z}_G \to \mathcal{U}$$

are now defined by

$$\begin{bmatrix} N \\ M \end{bmatrix} = T_K, \qquad \begin{bmatrix} V \\ U \end{bmatrix} = T_G. \tag{51}$$

By the definitions of T_G and T_K, the following identities hold:

$$R_G T_G = R_G \begin{bmatrix} V \\ U \end{bmatrix} = I_{\mathcal{Z}_G}, \qquad R_G T_K = R_G \begin{bmatrix} N \\ M \end{bmatrix} = 0$$

$$R_K T_G = R_K \begin{bmatrix} V \\ U \end{bmatrix} = 0, \qquad R_K T_K = R_K \begin{bmatrix} N \\ M \end{bmatrix} = I_{\mathcal{Z}_K}. \tag{52}$$

To prove the proposition, it only remains to prove that M and V are invertible and thus give right factorizations for G and K.

We first prove that M is invertible. Suppose that M were not injective, then there exist z_1, $z_2 \in \mathcal{Z}$, $z_1 \neq z_2$ such that $Mz_1 = Mz_2 = u$. By well definedness of G, there exists a unique $y \in \mathcal{Y}$ such that $R_G(y, u) = 0$. Thus

$$[R_{\{G,K\}}]^{-1}\begin{pmatrix} 0 \\ z_1 \end{pmatrix} = [R_{\{G,K\}}]^{-1}\begin{pmatrix} 0 \\ z_2 \end{pmatrix} = \begin{pmatrix} y \\ u \end{pmatrix}.$$

However, by well posedness of the system, this implies that $z_1 = z_2$. Thus M is injective. Given any $u \in \mathcal{U}$, there exists a $y \in \mathcal{Y}$ such that $R_G(y, u) = 0$, and thus a $z_k = R_K(u, y)$. Note that for this z_K, $u = Mz_k$. Thus M is surjective and is thus invertible.

Further for all $u \in \mathcal{U}$, $y = NM^{-1}u$ satisfies $R_G(y, u) = 0$, and thus $y = Gu$, and so $G = NM^{-1}$ is a stable factorization. As M and N also satisfy the Bezout identity (49) this is a coprime factorization.

A dual argument exploiting the well definedness of K shows that V is also invertible and that $K = UV^{-1}$ is a right coprime factorization, and the proof is complete. $\square$

Remark 4.8: The dual result, showing that if the plant and controller of a well posed and stable system have right factorizations then there will exist left factorizations, holds in the linear case where

$$\begin{bmatrix} V & N \\ U & M \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{bmatrix}.$$

However, in the nonlinear case, the lack of a separability property, as in Proposition 4.7, means that this dual result is not available for nonlinear systems described by left coprime factorizations.

Thus the results obtained previously for nonlinear left coprime factorizations may be duplicated, at least in the noise free case, by stable kernel representations. Thus skr's give a generalization of nonlinear left coprime factorizations. Furthermore, they may be more easily linked to the state-space literature as is explored in the following section.

V. STATE-SPACE RESULTS

In this section we present some state-space results which were recently obtained by Scherpen and van der Schaft [15]. An skr is derived for a general nonlinear system with a state-space description. This overcomes a major weakness of the nonlinear left factorization theory, where, except in special cases such as [9] and [17], a method for deriving left factorizations from a state-space realization of a nonlinear operator has not been derived.

Note that only the essentials of the development are presented here, Definition 5.1 and Theorem 5.3 are based on, but not directly taken from, the results presented in [15].

Consider a nonlinear system $G: \mathcal{U} \to \mathcal{Y}$ which has state-space description

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{53}$$

where $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and $x = (x_1, \cdots, x_n)$ are local coordinates for a smooth state-space manifold, M. G defines an input–output map $G(x_0)$ when the initial condition $x(0) = x_0$ is specified. It is assumed that the system has an equilibrium, without loss of generality this taken is to be zero, i.e., $f(0) = 0$, and $h(0) = 0$.

The equation $z = h(x) - y$ is considered to derive a stable kernel representation. This is motivated by the linear theory, where transforming the state equations such that the map $(u, y) \mapsto z$ is input to state stable, and $z = 0$ for $y = Gu$ yields a stable left factorization of the original system; see [10].

A particular form of left coprime factorization, a normalized left coprime factorization, is dealt with in [15]. To define this, it is necessary to define the notion of a co-inner nonlinear system. A detailed consideration of these conditions is beyond the scope of this paper, so we work with the following definition of left coprimeness.

Definition 5.1 (Left Coprimeness): A left coprime factorization of a nonlinear system (53) is represented by a system of the form

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\begin{bmatrix} u \\ y \end{bmatrix} \\ z &= \tilde{h}(\tilde{x}) - y \end{aligned} \tag{54}$$

where $\tilde{f}$ is Lyapunov stable, the input–output map for every initial condition is L_2-stable, the dynamics resulting from the constraint $z = h(\tilde{x}) - y = 0$, i.e.,

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}) + \tilde{g}_1(\tilde{x})u + \tilde{g}_2(\tilde{x})\tilde{h}(\tilde{x}) \\ y &= \tilde{h}(\tilde{x}) \end{aligned} \tag{55}$$

equals (53), and there exists a right-inverse for (54)

$$\begin{aligned} \dot{p} &= \overline{f}(p) + \overline{g}(p)s \\ \begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} \overline{h}_1(p) \\ \overline{h}(p) - s \end{bmatrix} \end{aligned} \tag{56}$$

with $\overline{f}$ Lyapunov stable. $\square$

Remark 5.2: Note that this definition of left coprimeness may not be consistent with Definition 4.4 as it is not guaranteed that this realization be separable in the sense of Proposition 4.2. However, it does give connections to Proposition 4.7, as the map from s to y is invertible, so that this right-inverse to (54) gives the right coprime factorization for some other system.

The following two Hamilton–Jacobi–Bellman equations are introduced in relation to (53):

$$\frac{\partial V}{\partial x}(x)f(x) - \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x)g(x)^T\frac{\partial^T V}{\partial x}(x)$$
$$+ \frac{1}{2}h(x)^T h(x) = 0, \qquad V(0) = 0 \tag{57}$$

$$\frac{\partial W}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial W}{\partial x}(x)g(x)g(x)^T\frac{\partial^T W}{\partial x}(x)$$
$$- \frac{1}{2}h(x)^T h(x) = 0, \qquad W(0) = 0. \tag{58}$$

It is assumed that (57) and (58) have smooth nonnegative definite solutions, W and V, respectively, at least on a neighborhood of 0. (See also Remark 5.6.)

Based on the solutions to these Hamilton–Jacobi–Bellman equations, a left coprime factorization, in the sense of Definition 5.1, may be derived for (53).

Theorem 5.3: Let V and W be smooth positive definite solutions (i.e., $V(x) > 0$, $W(x) > 0$, $x \neq 0$) to the Hamilton–Jacobi–Bellman equations (57) and (58), respectively. Since $\frac{\partial W}{\partial x}(0) = 0$ and $h(0) = 0$, there exist smooth matrices $M(x)$ and $C(x)$, such that

$$\frac{\partial W}{\partial x}(x) = x^T M(x), \qquad h(x) = C(x)x. \tag{59}$$

Assume that $M(x)$ is invertible for all x. Then a left coprime factorization of the system (53) is given by

$$\begin{aligned} \dot{x} &= [f(x) - M^{-1}(x)C^T(x)h(x)] \\ &\quad + [g(x)M^{-1}(x)C^T(x)]\begin{bmatrix} u \\ y \end{bmatrix} \\ z &= h(x) - y \end{aligned} \tag{60}$$

where $f(x) - M^{-1}(x)C(x)^T h(x)$ is Lyapunov stable with Lyapunov function W. Furthermore, an internally stable right inverse of (60) is given by

$$\begin{aligned} \dot{p} &= f(p) - g(p)g^T(p)\frac{\partial^T V}{\partial p}(p) - M^{-1}(p)C^T(p)s \\ \begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} -g^T(p)\frac{\partial^T V}{\partial p}(p) \\ h(p) - s \end{bmatrix}. \end{aligned} \tag{61}$$

Remark 5.4: Furthermore, if V and W are proper (i.e., for each $c > 0$ the set $\{x \in M \mid 0 \leq V(x) \leq c\}$ is compact, and similarly for W) and $\dot{x} = f(x)$, $y = h(x)$ is zero-state detectable (i.e., $y(t) \equiv 0$, implies that $\lim_{t \to \infty} x(t) = 0$), then $f(x) - M^{-1}(x)C(x)^T h(x)$ and $f(p) - g(p)g^T(p)\frac{\partial^T V}{\partial p}(p)$ are globally asymptotically stable.

Remark 5.5: Note that local invertibility of $M^{-1}(x)$ can be ensured by the usual minimality assumptions on the linearization of the system at the equilibrium $x = 0$. In fact, this will ensure that $M^{-1}(0) > 0$ and thus $M^{-1}(x) > 0$ for x close to 0. Another, less restrictive, alternative is to replace $M^{-1}(x)C(x)$ by $k(x)$, where $k(x)$ is the solution to $W_x(x)k(x) = h^T(x)$.

Proof: From (60) and (61) we deduce

$$\begin{aligned} &\frac{\partial W}{\partial x}[[f - M^{-1}C^T h] + gu + M^{-1}C^T y] \\ &= -\frac{1}{2}\left\| u - g^T\frac{\partial^T W}{\partial x} \right\|^2 - \frac{1}{2}\|z\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|y\|^2. \end{aligned} \tag{62}$$

By integration and using $W \geq 0$ it follows:

$$\int_0^T \|z\|^2 \, dt \leq \int_0^T (\|u\|^2 + \|y\|^2) \, dt + 2W(x(0)) \tag{63}$$

for all $T > 0$ and all $x(0)$, implying L_2-stability (in fact, L_2-gain ≤ 1). Furthermore, taking $u = y = 0$, we immediately obtain

$$\frac{\partial W}{\partial x}[f - M^{-1}C^T h] \leq -\frac{1}{2}h^T h \leq 0$$

implying Lyapunov stability of $f - M^{-1}C^T h$.

Direct calculation shows that the input–output map of (61) followed by (60) for $x(0) = p(0)$ is the identity map. Furthermore, by using (59) we obtain

$$\begin{aligned} &\frac{\partial V}{\partial p}(p)\left[f(p) - g(p)g^T(p)\frac{\partial^T V}{\partial p}(p) \right] \\ &= -\frac{1}{2}h^T(p)h(p) - \frac{1}{2}\frac{\partial V}{\partial p}(p)g(p)g^T(p)\frac{\partial^T V}{\partial p}(p) \leq 0 \end{aligned}$$

implying Lyapunov stability of $f(p) - g(p)g^T(p)\frac{\partial^T V}{\partial p}(p)$, completing the proof. □

Remark 5.6: If the linearization of (53) at $x = 0$ is controllable and observable, then at least on a neighborhood of $x = 0$ there exist smooth positive definite solutions V, W to (57) and (58), respectively.

Remark 5.7: In the case of a linear system (53), the left coprime factorization (60) reduces to the state-space representation of a normalized left coprime factorization; see Meyer and Franklin [8] and Vidyasagar [23].

Remark 5.8: In the notation of the previous sections, this will be a stable kernel representation for the operator with state-space realization (53).

Thus, at least in a local setting, there exists a procedure for deriving a stable kernel representation for a general nonlinear system. This may be applied to the results of the previous sections, giving state-space version of the Youla parameterization for nonlinear systems.

VI. CONCLUSION

In this paper we have developed the theory of stable kernel representations for nonlinear systems and demonstrated that they are a generalization of left coprime factorizations for linear systems. The results presented in Section IV and Section III demonstrate that in the noise-free case it is possible to duplicate all of the currently available results available for left factorizations in nonlinear factorization theory simply by replacing the left factorizations by stable kernel representations. Specifically, the Youla parameterization of all stabilizing plant-controller pairs has been shown to result from this approach, and the links between right factorizations and left factorizations for linear systems seem to be duplicated by the links between stable kernel representations and right coprime factorizations for nonlinear systems.

As further support for this approach to nonlinear control, a derivation of a stable kernel representation for a general nonlinear plant was presented from Scherpen and van der Schaft [15].

It is expected that results in nonlinear robust control may be derived from the results presented in this paper, as was the case with the results for nonlinear left coprime factorizations,

and that these results may now be translated into a state-space domain. It is hoped that the many useful techniques which result in the linear theory due to the use of coprime factorization analysis may now be derived in a nonlinear form.

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