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Robust decentralized output regulation with single or multiple reference signals for uncertain heterogeneous systems[‡,§]

Hui Liu[1,2,*,†], Claudio De Persis[2,3] and Ming Cao[2]

[1]*School of Automation, Huazhong University of Science and Technology, Wuhan, China*
[2]*Faculty of Mathematics and Natural Sciences, ITM, University of Groningen, Groningen, the Netherlands*
[3]*Department of Computer, Control and Management Engineering A. Ruberti, Sapienza University of Rome, Rome, Italy*

SUMMARY

We consider the problem in which N coupled heterogeneous uncertain linear systems aim at tracking one or more reference signals generated by given exosystems under the restriction that not all the systems are directly connected to the exosystems. To tackle this problem, the reference signals are reconstructed via local interaction of the systems among themselves and the exosystems in accordance with the given communication graph. Then, decentralized robust controllers using the reconstructed reference signals are designed and shown to result in a closed-loop system whose outputs track the prescribed reference signals. Copyright © 2014 John Wiley & Sons, Ltd.

1. INTRODUCTION

In multi-agent coordination problems, one of the possible tasks, which the agents may have to carry out, is to track an exogenous signal [2–7]. Two possible scenarios can be considered. In the first one, all the agents are assumed to know the reference signal and use this information as well as the relative information coming from the neighboring agents to carry out the control task. The motivation for this scenario stems from the empirical observation that the use of the neighbors' information in the control laws improves the robustness of the overall system. In the second scenario, the reference signal is not available to all the agents, and strategies to overcome this limitation are put in place [7–9].

A related problem has been considered in [10]. Given N heterogeneous linear systems and a communication graph, what are the necessary and sufficient conditions for the systems to achieve output synchronization? Interestingly, the authors have shown that an exosystem [11] that generates the 'reference signal' to which all the systems' outputs converge must necessarily exist. As a result, controllers that guarantee output synchronization are those which solve an output regulation problem associated to that exosystem. In addition, to make sure that the outputs of all the systems converge to a specific reference signal (the same for all the systems) generated by the exosystem, the controllers exchange local information with their neighbors.

Motivated by this result, we turn the attention to the problem in which the systems aim at tracking one or more reference signals generated by exosystems given *in advance* and ask whether there exist controllers that can guarantee the tracking of the reference signals even under the restriction that not all the systems are directly connected to the exosystem. Inspired by [10], we aim at reconstructing the reference signal via local interaction of the systems among themselves and the exosystem in accordance with the given communication graph. Then, controllers which solve the output regulation problem are designed and shown to track the actual reference signal even though they are fed by the local estimate of the signal. Differently from [10], we do not assume that the systems' models are perfectly known and robust regulators have to be designed [11]. The problem in [10] for the case of uncertain systems has been studied in [12] as well but, compared with the latter, the problem formulation in our paper is different and the approach taken in this paper seems to lead to simpler analysis.

Similar approaches have been proposed very recently in the literature. In [13], the systems that do not have direct access to the exosystem exchange information about the local tracking errors. Compared with our contribution, however, the authors require the communication graph to contain no cycles, with the leader (the exosystem) having a directed path to all the other systems (in our paper, we show that the latter condition is enough). Moreover, we assume the uncertainties of the system to range over a (an arbitrarily large) compact set rather that being sufficiently small. The problem of lifting the restrictions on the graph in [13] have been also tackled in the subsequent paper [14]. However, the systems considered in that paper are all assumed to have the same model and no uncertainty is considered.

Other related papers have appeared in the recent literature. To deal with velocity tracking in coordination problems for *passive* systems, [15, Chapter 3] proposes an internal model approach in which the reference trajectory is generated by an exosystem that cannot be accessed by the agents except one. Also, leader-follower problems using the internal model principle have been studied in [16]. In Section III of [7], an internal model approach to (position and) velocity tracking in networks of Euler-Lagrange systems is pursued, but the exosystem is restricted to the trivial one (constant reference velocity). To deal with non-constant reference velocities, the authors rely on a discontinuous control law and require information about one-hop and two-hop neighbors. Related work is also available in [9]. Examples on the use of ideas from output regulation theory and multi-agent systems can be found in the work [17], later developed in for example, [18, 19].

We also look at the problem of tracking multiple reference signals in order to realize clustering in multi-agent systems. Clustering has recently been studied as a coordination task [20–22], and the main challenge is how to have the agents converge to different asymptotic states under the constraints that all the agents are coupled together throughout the system's evolution. Until now, only few of the existing works have considered the situation when the agents are heterogenous. Building upon our results on tracking a single reference signal, we propose a novel robust decentralized output regulation algorithm to track different reference signals for different subgroups of systems and thus realize clustering.

In Section 2, we first formulate the problem of robust decentralized output regulation for uncertain heterogeneous systems along with the standing assumptions, and then state the main results. In Section 3, we extend the results in Section 2 to study clustering output synchronization. The actual design of the controllers is described in Section 4 and then illustrated via two numerical examples in Section 5. Conclusions are drawn in Section 6.

2. OUTPUT REGULATION OF UNCERTAIN HETEROGENEOUS SYSTEMS

2.1. *Problem statement and standing assumptions*

Consider N heterogeneous uncertain linear dynamical systems

$$\mathcal{S}_i : \begin{array}{l} \dot{x}_i = A_i(\mu_i) x_i + B_i(\mu_i) u_i \\ y_i = C_i(\mu_i) x_i, \end{array} \tag{1}$$

with state vector $x_i \in \mathbb{R}^{n_i}$, control input $u_i \in \mathbb{R}^{p_i}$, and output vector $y_i \in \mathbb{R}^q$ for $i = 1, \cdots, N$. Each matrix of the system (1) depends on a vector μ_i of uncertain parameters, which is assumed to range over a given set $\mathcal{P}_i$.

Consider also another system, which we will refer to as the 'leader', whose dynamical behavior is described by the following equation:

$$\begin{aligned} \dot{w}_0 &= S w_0 \\ r &= R w_0, \end{aligned} \tag{2}$$

where $w_0 \in \mathbb{R}^m$, $r \in \mathbb{R}^q$ and matrices $S \in \mathbb{R}^{m \times m}$, $R \in \mathbb{R}^{q \times m}$. These matrices are assumed to satisfy the following:

Assumption 1
The real parts of the eigenvalues of S are zero, i.e., $\sigma(S) \subset \mathbb{C}^0$ and (R, S) is detectable.

The N systems (1) exchange information according to the communication topology described by the directed graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$. Each system is represented by a node in the set $\mathcal{V} = \{1, 2, \ldots, N\}$ and system j sends information to system i if and only if $(j, i) \in \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Associated to the graph $\mathbb{G}$ is the adjacency matrix $A = [a_{ij}]$. The entry $a_{ij} = 1$ if and only if $(j, i) \in \mathcal{E}$ and 0 otherwise. If $a_{ij} = 1$, we say that j is a neighbor of i. We set $a_{ii} = 0$ for each $i = 1, 2, \ldots, N$. The Laplacian L is the matrix $L = D - A$, with $D = \text{diag}(d_1, \ldots, d_N)$ and $d_i = \sum_{j \neq i} a_{ij}$. A directed path from i to j is a sequence of edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ in $\mathcal{E}$ such that $v_0 = i$ and $v_k = j$. In addition to the graph $\mathbb{G}$, we consider the directed graph $\mathbb{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$, obtained as follows. Let system (2) (the leader) be associated with node 0 and set $\mathcal{V}_0 = \mathcal{V} \cup \{0\}$. Moreover, for $i = 1, 2, \ldots, N$, we set $a_{i0} = 1$ if and only if there is an arc from 0 to i and $a_{i0} = 0$ otherwise. Then, we set $\mathcal{E}_0 = \mathcal{E} \cup \{(0, i) : a_{i0} = 1\}$. Compared with $\mathbb{G}$, the graph $\mathbb{G}_0$ additionally describes which followers have direct access to the information of the leader.

In what follows, we exploit the following lemma [7], where we refer to the graphs $\mathbb{G}, \mathbb{G}_0$ and the Laplacian L introduced earlier.

Lemma 1
If in graph $\mathbb{G}_0$, node 0 has directed paths to all the nodes $i = 1, 2, \ldots, N$, then the matrix $L + \text{diag}(a_{10}, \ldots, a_{N0})$ has all the eigenvalues with strictly positive real part.

The objective of the paper is to design the control laws u_i, which guarantee

$$\lim_{t \to \infty} ||y_i(t) - R w_0(t)|| = 0, \quad \text{for all } i = 1, \cdots, N,$$

under the following restrictions on the available measurements:

(i) Only the systems $\mathcal{S}_i$ for which $a_{i0} = 1$ can access the leader and therefore the reference signal r. Hence, only these systems $\mathcal{S}_i$ can measure the *tracking error*

$$\epsilon_i = y_i - R w_0.$$

This restricted access to the leader causes the readability of ϵ_i from y_i not possible for all the systems [13] and makes the problem meaningful.
(ii) The systems $\mathcal{S}_i$'s exchange only relative information.
(iii) For all $i = 1, \cdots, N$, the system $\mathcal{S}_i$ has access to the relative information with respect to $\mathcal{S}_j$ if and only if $\mathcal{S}_j$ is a neighbor of $\mathcal{S}_i$.

Other assumptions are needed in order to state our main result in the next subsection.

Assumption 2
(i) the μ_i-dependent Francis' equations

$$\begin{aligned} \Pi_i(\mu_i)S &= A_i(\mu_i)\Pi_i(\mu_i) + B_i(\mu_i)\Gamma_i(\mu_i) \\ 0 &= C_i(\mu_i)\Pi_i(\mu_i) - R \end{aligned} \tag{3}$$

have a μ_i-dependent solution $\Pi_i(\mu_i), \Gamma_i(\mu_i)$ for each $i = 1, \cdots, N$.

(ii) there exist matrices Φ_i, H_i, $\Sigma_i(\mu_i)$, with Φ_i, H_i independent of μ_i, such that

$$\begin{aligned}
\Sigma_i(\mu_i)S &= \Phi_i\,\Sigma_i(\mu_i) \\
\Gamma_i(\mu_i) &= H_i\,\Sigma_i(\mu_i)
\end{aligned} \tag{4}$$

(iii) there exists a matrix G_i independent of μ_i such that the linear system defined by the triplet

$$\begin{pmatrix} A_i(\mu_i) & B_i(\mu_i)H_i \\ 0 & \Phi_i \end{pmatrix} \quad \begin{pmatrix} B_i(\mu_i) \\ G_i \end{pmatrix} \quad \begin{pmatrix} C_i(\mu_i) & 0 \end{pmatrix}$$

is robustly stabilizable by the dynamic output feedback, i.e. there are matrices K_i, L_i, M_i independent of μ_i, such that the matrix

$$\begin{pmatrix} \begin{pmatrix} A_i(\mu_i) & B_i(\mu_i)H_i \\ 0 & \Phi_i \end{pmatrix} & \begin{pmatrix} B_i(\mu_i) \\ G_i \end{pmatrix} M_i \\ K_i \begin{pmatrix} C_i(\mu_i) & 0 \end{pmatrix} & L_i \end{pmatrix} \tag{5}$$

is Hurwitz.

A few comments on Assumption 2 are in order.
– Fix $i \in \{1, 2, \ldots, N\}$. Suppose that there exists a controller of the form

$$\begin{aligned}
\dot{\xi}_i &= L_i\xi_i + K_i\epsilon_i \\
u_i &= H_i\xi_i + M_i\epsilon_i
\end{aligned} \tag{6}$$

which robustly stabilizes the system $\mathcal{S}_i$. Then, provided that $\sigma(S) \subset \mathbb{C}^0$ (see Assumption 1), Equations (3), (4) are well-known ([11, Proposition 1.4.1]) necessary and sufficient conditions for the controller (6) to solve the tracking problem for the system (1) for each $\mu_i \in \mathcal{P}_i$. Recall that the controller (6) is said to solve the tracking problem for the system (1) for each $\mu_i \in \mathcal{P}_i$ if, for each $\mu_i \in \mathcal{P}_i$, (i) the equilibrium $(x_i, \xi_i) = (\mathbf{0}, \mathbf{0})$ of the unforced closed-loop system (1), (6) is asymptotically stable; (ii) the response of the closed-loop system (1), (6) is such that $\lim_{t\to\infty} \epsilon_i(t) = \mathbf{0}$.
– If in addition condition (iii) in Assumption 2 holds, then one can prove that the dynamic feedback control law

$$\begin{aligned}
\dot{\eta}_i &= \Phi_i\eta_i + G_iM_i\xi_i \\
\dot{\xi}_i &= L_i\xi_i + K_i\epsilon_i \\
u_i &= H_i\eta_i + M_i\xi_i
\end{aligned} \tag{7}$$

solves the tracking problem. Because of the fact that the tracking error ϵ_i may not be available to the controller of system $\mathcal{S}_i$, the previous controller cannot be implemented. In the next section, we overcome this lack of information on ϵ_i with the use of the information collected from the neighbors of system $\mathcal{S}_i$.

2.2. Tracking a single reference

The control strategy we propose to solve the decentralized output regulation problem formulated in the previous section comprises two steps. Because not all the systems $\mathcal{S}_i$ may have access to the reference signal r, we first design systems which aim at asymptotically reconstructing the reference signal using only locally available relative information. As a second step, we use such an asymptotic estimate of the reference signal to feed the tracking controllers and show that they achieve the prescribed control objective.
Motivated by Lemma 1, we introduce the following:

Assumption 3
In the graph G_0, the node 0 has directed paths to all the nodes $i = 1, 2, \ldots, N$.

The assumption implies that there exist $1 \leqslant N_1 \leqslant N$ systems, which has direct access to the leader. Without loss of generality and for the sake of simplicity, we assume that $N_1 = 1$ and that

the system with direct access to the leader is the first one. To reconstruct the reference signal, the systems cooperate to estimate the internal state of the exosystem. For system $\mathcal{S}_1$, the estimation is carried out by

$$
\begin{aligned}
\dot{\hat{w}}_0 &= S\hat{w}_0 + G_0 R(w_0 - \hat{w}_0) \\
\dot{w}_1 &= S w_1 + \sum_{j=1}^{N} a_{1j}(w_j - w_1) + a_{10}(\hat{w}_0 - w_1),
\end{aligned}
\tag{8}
$$

where the matrix G_0 is properly chosen in such a way that $\sigma(S - G_0 R) \subset \mathbb{C}^-$ and $\hat{w}_0$ is an asymptotic estimate of the leader's internal state w_0. For system $\mathcal{S}_i, i \in \{2, \ldots, N\}$, the system which carries out the asymptotic estimation is given by

$$
\dot{w}_i = S w_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i).
\tag{9}
$$

For the system

$$
\begin{aligned}
\dot{w}_0 &= S w_0 \\
\dot{\hat{w}}_0 &= S\hat{w}_0 + G_0 R (w_0 - \hat{w}_0) \\
\dot{w}_1 &= S w_1 + \sum_{j=1}^{N} a_{1j}(w_j - w_1) + a_{10}(\hat{w}_0 - w_1) \\
\dot{w}_i &= S w_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i), \qquad i = 2, \cdots, N,
\end{aligned}
\tag{10}
$$

we have the following result for the convergence of w_i:

Lemma 2
Let Assumptions 1 and 3 hold. Then, $\|w_i(t) - w_0(t)\| \to 0$ exponentially for all $i = 1, \cdots, N$, as $t \to \infty$.

Proof
Let $\tilde{w}_i = w_i - w_0$, for all $i = 1, \cdots, N$. Let $\tilde{w}_0 = \hat{w}_0 - w_0$. Then, we have

$$
\dot{\tilde{w}}_1 = \dot{w}_1 - \dot{w}_0 = S\tilde{w}_1 + \sum_{j=1}^{N} a_{1j}(\tilde{w}_j - \tilde{w}_1) + a_{10}(\tilde{w}_0 - \tilde{w}_1),
$$

and

$$
\dot{\tilde{w}}_i = S\tilde{w}_i + \sum_{j=1}^{N} a_{ij}(\tilde{w}_j - \tilde{w}_i).
$$

Moreover,

$$
\dot{\tilde{w}}_0 = \dot{\hat{w}}_0 - \dot{w}_0 = S(\hat{w}_0 - w_0) + G_0 R(w_0 - \hat{w}_0) = (S - G_0 R)\tilde{w}_0.
$$

Because $\sigma(S - G_0 R) \subset \mathbb{C}^-$, one obtains that $\tilde{w}_0$ converges to the origin exponentially as $t \to \infty$. Following [23], let $\omega_i = e^{-St}\tilde{w}_i$ for all $i = 0, 1, \cdots, N$. Then, we have

$$
\begin{aligned}
\dot{\omega}_1 &= -S e^{-St}\tilde{w}_1 + e^{-St}\left[S\tilde{w}_1 + \sum_{j=1}^{N} a_{1j}(\tilde{w}_j - \tilde{w}_1) + a_{10}(\tilde{w}_0 - \tilde{w}_1) \right] \\
&= \sum_{j=1}^{N} a_{1j}(\omega_j - \omega_1) + a_{10}(\omega_0 - \omega_1),
\end{aligned}
\tag{11}
$$

and

$$\dot{\omega}_i = \sum_{j=1}^{N} a_{ij}(\omega_j - \omega_i), \qquad i = 2, \cdots, N. \tag{12}$$

Let $\omega = \left(\omega_1^T, \omega_2^T, \cdots, \omega_N^T\right)^T \in \mathbb{R}^{Nm}$. We write the aforementioned two equations into the compact form

$$\dot{\omega} = -(L \otimes I_m)\omega - \left(\begin{array}{c|c} a_{10}I_m & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \left(\begin{array}{c} \omega_1 \\ \mathbf{0} \end{array}\right) + \left(\begin{array}{c|c} a_{10}I_m & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \left(\begin{array}{c} \omega_0 \\ \mathbf{0} \end{array}\right)$$

$$= -(\tilde{L} \otimes I_m)\omega + \left(\begin{array}{c|c} a_{10}I_m & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \left(\begin{array}{c} \omega_0 \\ \mathbf{0} \end{array}\right),$$

where $-\tilde{L} = -L - \mathrm{diag}(a_{10}, 0, \cdots, 0)$ and $\otimes$ denotes the Kronecker product. According to Lemma 1, $-\tilde{L}$ is Hurwitz. Thus, $-\tilde{L} \otimes I_m$ is Hurwitz. Moreover, it has been proved that $\tilde{w}_0$ converges to the origin exponentially as $t \to \infty$. Thus, $\omega_0 = e^{-St}\tilde{w}_0$ converges to zero exponentially as $t \to \infty$, because $\sigma(S) \subset \mathbb{C}^0$. Therefore, ω converges to the origin exponentially as $t \to \infty$. Because $\tilde{w}_i = e^{St}\omega_i$ for $i = 1, \cdots, N$ and $\sigma(S) \subset \mathbb{C}^0$, one has that $\tilde{w}_i \to \mathbf{0}$ exponentially as $t \to \infty$. Thus, we arrive at the result $||w_i(t) - w_0(t)|| \to 0$ exponentially for all $i = 1, \cdots, N$, as $t \to \infty$. □

Remark 1
Clearly, the signals $Rw_i(t)$, $i = 1, 2, \ldots, N$, converge exponentially to $r(t)$.

Next, we introduce the controllers for systems (1) as follows. As system $\mathcal{S}_1$ has access to w_0, we design u_1 as

$$\begin{aligned}
\dot{\hat{w}}_0 &= S\hat{w}_0 + G_0 R(w_0 - \hat{w}_0) \\
\dot{w}_1 &= Sw_1 + \sum_{j=1}^{N} a_{1j}(w_j - w_1) + a_{10}(\hat{w}_0 - w_1) \\
\dot{\eta}_1 &= \Phi_1 \eta_1 + G_1 M_1 \xi_1 \\
\dot{\xi}_1 &= L_1 \xi_1 + K_1(y_1 - Rw_1) \\
u_1 &= H_1 \eta_1 + M_1 \xi_1
\end{aligned} \tag{13}$$

For agent $i = 2, \cdots, N$, we design u_i as

$$\begin{aligned}
\dot{w}_i &= Sw_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i) \\
\dot{\eta}_i &= \Phi_i \eta_i + G_i M_i \xi_i \\
\dot{\xi}_i &= L_i \xi_i + K_i(y_i - Rw_i) \\
u_i &= H_i \eta_i + M_i \xi_i
\end{aligned} \tag{14}$$

The matrices $\Phi_i, G_i, M_i, L_i, K_i, H_i$ are those found in Assumption 2.

Theorem 1
Consider N heterogeneous linear systems (1) coupled via the dynamic couplings (13) and (14). Suppose Assumptions 1–3 hold. Then, $||y_i(t) - Rw_0(t)||$ exponentially converges to 0 as $t \to \infty$ for all $i = 1, \cdots, N$.

Proof
Let $\tilde{x}_i = x_i - \Pi_i(\mu_i)w_i$, $\tilde{\eta}_i = \eta_i - \Sigma_i(\mu_i)w_i$. Recall that $a_{i0} > 0$ if and only if $i = 1$ and 0 otherwise. Then, according to Equations (1), (8), and (9), we have

$$\dot{\tilde{x}}_i = A_i(\mu_i)\tilde{x}_i + B_i(\mu_i)H_i\tilde{\eta}_i + B_i(\mu_i)M_i\xi_i - \Pi_i(\mu_i)\left(\sum_{j=1}^{N} a_{ij}(w_j - w_i) + a_{i0}(\hat{w}_0 - w_i)\right),$$

where we have exploited the first equation of (3) and the second equation of (4) in Assumption 2. Furthermore, standard manipulations and the first equation of (4) in Assumption 2 lead to the equation

$$\dot{\tilde{\eta}}_i = \Phi_i \tilde{\eta}_i + G_i M_i \xi_i - \Sigma_i(\mu_i) \cdot \left(\sum_{j=1}^{N} a_{ij}(w_j - w_i) + a_{i0}(\hat{w}_0 - w_i) \right) .$$

One can also observe that

$$
\begin{aligned}
y_i - R w_i &= C_i(\mu_i) x_i - R w_i \\
&= C_i(\mu_i)\tilde{x}_i + (C_i(\mu_i)\Pi_i(\mu_i) - R) w_i \\
&= C_i(\mu_i)\tilde{x}_i ,
\end{aligned}
$$

where we have used the second equation of (3) in Assumption 2. Hence,

$$
\begin{aligned}
\dot{\xi}_i &= L_i \xi_i + K_i(y_i - R w_i) \\
&= L_i \xi_i + K_i \left(C_i(\mu_i)\ \mathbf{0} \right) \begin{pmatrix} \tilde{x}_i \\ \tilde{\eta}_i \end{pmatrix}.
\end{aligned}
\tag{15}
$$

Using the new coordinates $\tilde{x}_i, \tilde{\eta}_i$, and ξ_i, we write the dynamics in the compact form

$$
\begin{aligned}
\begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{\eta}}_i \end{pmatrix} &= \begin{pmatrix} A_i(\mu_i) & B_i(\mu_i)H_i \\ \mathbf{0} & \Phi_i \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{\eta}_i \end{pmatrix} + \begin{pmatrix} B_i(\mu_i) \\ G_i \end{pmatrix} M_i \xi_i \\
&\quad - \begin{pmatrix} \Pi_i(\mu_i) \\ \Sigma_i(\mu_i) \end{pmatrix} \left(\sum_{j=1}^{N} a_{ij}(w_j - w_i) + a_{i0}(\hat{w}_0 - w_i) \right) \\
\dot{\xi}_i &= L_i \xi_i + K_i \left(C_i(\mu_i)\ \mathbf{0} \right) \begin{pmatrix} \tilde{x}_i \\ \tilde{\eta}_i \end{pmatrix}.
\end{aligned}
\tag{16}
$$

The third condition of Assumption 2 shows that the dynamic matrix of the closed loop system (16) is Hurwitz. Because $\sum_{j=1}^{N} a_{ij}(w_j - w_i) + a_{i0}(\hat{w}_0 - w_i)$ converges exponentially to zero, then $\tilde{x}_i \to \mathbf{0}, \tilde{\eta}_i \to \mathbf{0}$ exponentially. Furthermore, $y_i - R w_i = C_i(\mu_i)\tilde{x}_i \to \mathbf{0}$ exponentially. As $w_i - w_0 \to \mathbf{0}$ for all i exponentially, then the latter implies that $y_i(t) \to R w_0(t)$ for all i exponentially. □

Remark 2
Theorem 1 solves the robust decentralized output regulation problem, in which the actual reference signal is tracked relying on local estimates of the signal. The papers [10, 12, 24, 25] have addressed similar problems, with some differences that we are going to discuss in the succeeding text. Differently from [10], we do not assume that the systems' models are perfectly known and robust regulators have to be designed. The problem in [10] has been studied in [12] in the case of uncertain systems. It is worth mentioning that the problem formulation in our paper is different from that in [10, 12]. The work in [10, 12] deals with synchronization problem without any leader and thus they cannot enforce the desired asymptotic regime of individual systems. In contrast, our problem requires that the outputs of individual systems track the prescribed reference signal. More recently, Su and Huang [24, 25] have studied the leader-follower output synchronization problem of heterogeneous linear multi-agent systems. However, the systems considered in the two papers are perfectly known and assumed to have no uncertainty.

The design of robust regulators that fulfill the conditions in Assumption 2 will be discussed in Section 4. For such a design, we will need Corollary 1 in the succeeding text, which deals with the case in which the dynamics of each system (1) is affected by the signals w_i, namely,

$$
\mathcal{S}_i^w : \quad \begin{aligned} \dot{x}_i &= A_i(\mu_i) x_i + B_i(\mu_i) u_i + P_i(\mu_i) w \\ y_i &= C_i(\mu_i) x_i , \end{aligned}
\tag{17}
$$

where $w = \left(w_1^T \ldots w_N^T\right)^T$ is the vector of signals generated by (8), (9) and

$$P_i(\mu_i) = (P_{i1}(\mu_i) \ldots P_{iN}(\mu_i)).$$

The previous theorem can be easily extended provided that Assumption 2 is modified as follows:

Assumption 4
(i) the μ_i-dependent Francis' equations

$$\Pi_i(\mu_i)S = A_i(\mu_i)\Pi_i(\mu_i) + B_i(\mu_i)\Gamma_i(\mu_i) + \sum_{j=1}^{N} P_{ij}(\mu_i) \tag{18}$$

$$0 = C_i(\mu_i)\Pi_i(\mu_i) - R$$

have a μ_i-dependent solution $\Pi_i(\mu_i)$, $\Gamma_i(\mu_i)$ for each $i = 1, \cdots, N$.
(ii) and (iii) are as in Assumption 2.

The result below is used in Section 4 to design the output regulators.

Corollary 1
Consider N heterogeneous linear systems (17). Suppose the systems are coupled via the dynamic couplings (13) and (14). Suppose Assumptions 1, 3 and 4 hold. Then, $\|y_i(t) - Rw_0(t)\|$ exponentially converges to 0 as $t \to \infty$ for all $i = 1, \cdots, N$.

Proof
The result descends from the proof of Theorem 1 after making necessary modifications. In view of (18), the variable $\tilde{x}_i$ of the closed-loop system (17), (13) and (14) satisfies the equation

$$\dot{\tilde{x}}_i = A_i(\mu_i)\tilde{x}_i + B_i(\mu_i)H_i\tilde{\eta}_i + B_i(\mu_i)M_i\xi_i +$$
$$\sum_{j=1}^{N} P_{ij}(\mu_i)(w_j - w_i) - \Pi_i(\mu_i) \cdot \left(\sum_{j=1}^{N} a_{ij}(w_j - w_i) + a_{i0}(\hat{w}_0 - w_1)\right).$$

Repeating the same arguments of Theorem 1, one arrives at the following system:

$$\begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{\eta}}_i \end{pmatrix} = \begin{pmatrix} A_i(\mu_i) & B_i(\mu_i)H_i \\ \mathbf{0} & \Phi_i \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{\eta}_i \end{pmatrix} + \begin{pmatrix} B_i(\mu_i) \\ G_i \end{pmatrix} M_i\xi_i$$

$$+ \begin{pmatrix} \sum_{j=1}^{N} P_{ij}(\mu_i)(w_j - w_i) \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} \Pi_i(\mu_i) \\ \Sigma_i(\mu_i) \end{pmatrix} \left(\sum_{j=1}^{N} a_{ij}(w_j - w_i) + a_{i0}(\hat{w}_0 - w_1)\right)$$

$$\dot{\xi}_i = L_i\xi_i + K_i \left(C_i(\mu_i) \; \mathbf{0} \right) \begin{pmatrix} \tilde{x}_i \\ \tilde{\eta}_i \end{pmatrix}$$

As before, the system earlier is Hurwitz and driven by signals that converge exponentially to zero. The states converge exponentially to zero and the thesis follows. □

In the next section, we further expand the results developed in the section to the case where multiple reference signals have to be tracked.

3. CLUSTERING THROUGH OUTPUT REGULATION

3.1. Problem statement

We again consider N heterogeneous uncertain linear dynamical systems S_i, $1 \leqslant i \leqslant N$, given by (1). In addition, we consider another n systems $\mathcal{L}_j$, $1 \leqslant j \leqslant n$, called 'leaders', with state variables $w_{01}, w_{02}, \ldots, w_{0n}$. Their dynamics are of the same form described by

$$\dot{w}_{0j} = Sw_{0j}$$
$$r_j = Rw_{0j}, \forall j = 1, \ldots, n \tag{19}$$

but with different initial conditions, that is, $w_{01}(0), w_{02}(0), \ldots, w_{0n}(0)$ are different from each other. We will explore topological connections and design decentralized controllers such that, for a given partition of the N heterogeneous systems $\mathcal{S}_i$ with n subsets, the output of each system in the same subset converges to the same reference signal Rw_{0j} for $j \in \{1, \ldots, n\}$ and the outputs of the systems in different subsets converge to different reference signals. The desired behavior is formalized as follows.

Definition 1
Let $\{\mathbb{N}_1, \mathbb{N}_2, \ldots, \mathbb{N}_n\}$ be a partition of the set $\{1, 2, \ldots, N\}$ into n nonempty subsets, which satisfy $\mathbb{N}_i \cap \mathbb{N}_j = \emptyset$ where $i \neq j$ and $\bigcup_{i=1}^{n} \mathbb{N}_i = \{1, 2, \ldots, N\}$. Suppose that $\mathbb{N}_1 = \{1, \ldots, h_1\}, \mathbb{N}_2 = \{h_1+1, \ldots, h_1+h_2\}, \ldots, \mathbb{N}_n = \{h_1+\ldots+h_{n-1}+1, \ldots, h_1+\ldots+h_{n-1}+h_n\}$, where $1 < n < N$, $1 \leq h_i < N$, and $\sum_{i=1}^{n} h_i = N$. A network of N heterogeneous linear systems $\mathcal{S}_i$, partitioned according to $\{\mathbb{N}_1, \mathbb{N}_2, \ldots, \mathbb{N}_n\}$ is said to realize an n-cluster output synchronization, if the outputs y_i of the heterogeneous systems (1) satisfy $\lim_{t \to +\infty} \sum_{j=1}^{n} \sum_{i \in \mathbb{N}_j} ||y_i(t) - Rw_{0j}(t)|| = 0$.

The N systems $\mathcal{S}_i$ exchange information according to the topology described by the directed graph $\mathbb{G}$. Associated to the graph $\mathbb{G}$ is the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$. The entry a_{ij} equals 1 or -1 for $1 \leq i, j \leq N, i \neq j$, if and only if there is a coupling from $\mathcal{S}_j$ to the system $\mathcal{S}_i$; otherwise $a_{ij} = 0$. In this section, we allow couplings among the agents that belong to different subsets to be negative, and as a result $a_{ij} \in \{1, 0, -1\}$. We set $a_{ii} = 0$ for each $i = 1, \ldots, N$. Moreover, the partition $\{\mathbb{N}_1, \mathbb{N}_2, \ldots, \mathbb{N}_n\}$ induces the following block-matrix structure of the matrix A:

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix}.$$

The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ associated with the graph $\mathbb{G}$ is the matrix $L = D - A$, with $D = \text{diag}(d_1, \ldots, d_N)$ where $d_i = \sum_{j=1, j \neq i}^{N} a_{ij}$. Similar to A, the matrix L can be written as

$$L = \begin{pmatrix} L_{11} & L_{12} & \ldots & L_{1n} \\ L_{21} & L_{22} & \ldots & L_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ L_{n1} & L_{n2} & \ldots & L_{nn} \end{pmatrix},$$

with $L_{ij} \in \mathbb{R}^{h_i \times h_j}$ for $i, j = 1, \ldots, n$.

The matrix L is assumed to satisfy the following:

Assumption 5
Suppose that the block-matrices $L_{ij} \in \mathbb{R}^{h_i \times h_j}$, $i, j = 1, \ldots, n$, have zero row sums, namely, $\sum_{\ell=1}^{h_j} l_{k_i+m, k_j+\ell} = 0$ for all $m = 1, \ldots, h_i$, $k_i = h_1 + \ldots h_{i-1}$ and $k_j = h_1 + \ldots h_{j-1}$. Furthermore, the off-diagonal elements of $L_{ii} \in \mathbb{R}^{h_i \times h_i}$ are non-positive.

A few explanations on Assumption 5 are in order. The assumption that all L_{ij} have zero row sums is natural and necessary. It means that the sum of the couplings from the systems in the jth subset to each system in the ith ($i \neq j$) subset is zero. Thus, the effect from the systems in the jth subset to each system in the ith subset will vanish when synchronization in each subset of systems is achieved. This guarantees that clustering synchronization can be realized.
Next, we explain the existence of negative elements in the adjacency matrix A in the framework of clustering synchronization. If $a_{ij} > 0$, a cooperative coupling is enforced, whereas if $a_{ij} < 0$, the coupling is competitive or repulsive. Intuitively, competition or repulsion can affect the synchronization behavior and may result in diverse behaviors in a coupled network. Hence, it is natural to allow negative couplings among different subsets of systems and to have positive couplings among the systems in the same subset.

In what follows, we will also need the following connectivity assumption:

Assumption 6

For each $j = 1, \ldots, n$, there exists at least one system $\mathcal{S}_i, i \in \mathbb{N}_j$ which is connected to the leader $\mathcal{L}_j$.

Moreover, we assume that there is a unique leader for each subset of systems. That is to say, we exclude the possibility that a system $\mathcal{S}_i, i \in \mathbb{N}_j$, is connected to a leader $\mathcal{L}_k$ where $k \neq j$. We use $a_{10}, a_{20}, \ldots, a_{N0}$ to describe the existence of a directed edge from the leader to a system. Namely, for each $j = 1, 2, \ldots, n$, if there is a coupling from the leader $\mathcal{L}_j$ to the system $\mathcal{S}_i, i \in \mathbb{N}_j$, then $a_{i0} = 1$; otherwise $a_{i0} = 0$.

The matrix A only describes the underlying communication topological structure. It does not provide any information about how strong the couplings or connections are. We use the notion of 'coupling strength' to describe the strength of the coupling for an edge. Intuitively, enhancing the couplings among agents inside the same subset will help the whole network to realize clustering synchronization. Hence, we set the coupling strengths among the systems that are inside the set $\mathbb{N}_j$ to be the positive constant $c_j \geqslant 1$. And we set the coupling strengths of the directed edges from the leader $\mathcal{L}_j$ to the systems $\mathcal{S}_i$ where $i \in \mathbb{N}_j$ to be the constant c_j as well. We call the parameters c_j, for $j = 1, \ldots, n$ as the inner coupling strengths.

3.2. Tracking multiple references

The control strategy, which we propose to solve the clustering output synchronization problem formulated in Definition 1, comprises two steps. Because not all the systems $\mathcal{S}_i$ may have access to the leaders, we first design systems which aim at reconstructing the reference signals using only locally available relative information. As a second step, we use such an asymptotic estimate of the reference signal to feed the tracking controllers and show that they achieve the prescribed control objective.

To reconstruct the reference signal, the systems cooperate to estimate the internal state of the exosystems. For system $\mathcal{S}_i, i \in \mathbb{N}_j, j = 1, \ldots, n$, the estimation is carried out by

$$
\begin{aligned}
\dot{\hat{w}}_{0j} &= S\hat{w}_{0j} + G_0 R(w_{0j} - \hat{w}_{0j}), \\
\dot{w}_i &= Sw_i + \sum_{k \in \mathbb{N}_j} c_j a_{ik}(w_k - w_i) + \sum_{k=1, k \notin \mathbb{N}_j}^{N} a_{ik}(w_k - w_i) + c_j a_{i0}(\hat{w}_{0j} - w_i),
\end{aligned}
\tag{20}
$$

where the matrix G_0 is properly chosen in such a way that $\sigma(S - G_0 R) \subset \mathbb{C}^-$ and $\hat{w}_{0j}$ is an asymptotic estimate of the leader's internal state w_{0j}. Let matrix $\Xi_j \in \mathbb{R}^{h_j \times h_j}$ denote the diagonal matrix

$$
\text{diag}\left(a_{k_j+1,0}, \ldots, a_{k_j+h_j,0}\right).
$$

Let $\bar{w}_{h_1} = \left(w_1^T, \ldots, w_{h_1}^T\right)^T, \ldots, \bar{w}_{h_n} = \left(w_{h_1+\cdots+h_{n-1}+1}^T, \ldots, w_{h_1+\cdots+h_n}^T\right)^T$. We now write the dynamics of the estimations as follows:

$$
\begin{pmatrix} \dot{\bar{w}}_{h_1} \\ \dot{\bar{w}}_{h_2} \\ \vdots \\ \dot{\bar{w}}_{h_n} \end{pmatrix} = (I_N \otimes S) \begin{pmatrix} \bar{w}_{h_1} \\ \bar{w}_{h_2} \\ \vdots \\ \bar{w}_{h_n} \end{pmatrix} - \begin{pmatrix} c_1 L_{11} & L_{12} & \ldots & L_{1n} \\ L_{21} & c_2 L_{22} & \ldots & L_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ L_{n1} & L_{n2} & \ldots & c_n L_{nn} \end{pmatrix} \otimes I_m \cdot \begin{pmatrix} \bar{w}_{h_1} \\ \bar{w}_{h_2} \\ \vdots \\ \bar{w}_{h_n} \end{pmatrix}
$$
$$
+ \text{diag}\{c_1 \Xi_1, c_2 \Xi_2, \ldots, c_n \Xi_n\} \otimes I_m \cdot \begin{pmatrix} \mathbf{1}_{h_1} \otimes \hat{w}_{01} - \bar{w}_{h_1} \\ \mathbf{1}_{h_2} \otimes \hat{w}_{02} - \bar{w}_{h_2} \\ \vdots \\ \mathbf{1}_{h_n} \otimes \hat{w}_{0n} - \bar{w}_{h_n} \end{pmatrix},
\tag{21}
$$

where $\mathbf{1}_{h_j} \in \mathbb{R}^{h_j}$ are vectors of all ones, for $j = 1, \ldots, n$.

To show the convergence of w_i, we first introduce some notations of block matrices. Let the matrices

$$L_\Xi \triangleq \begin{pmatrix} c_1\,L_{11} + c_1\,\Xi_1 & L_{12} & \ldots & L_{1n} \\ L_{21} & c_2\,L_{22} + c_2\,\Xi_2 & \ldots & L_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ L_{n1} & L_{n2} & \ldots & c_n\,L_{nn} + c_n\,\Xi_n \end{pmatrix}, \tag{22}$$

$L_\Xi^{(1)} \triangleq \mathrm{diag}\{c_1\,L_{11} + c_1\,\Xi_1, c_2\,L_{22} + c_2\,\Xi_2, \ldots, c_n\,L_{nn} + c_n\,\Xi_n\}$, $L_\Xi^{(2)} \triangleq L_\Xi - L_\Xi^{(1)}$, and $D_\Xi \triangleq \mathrm{diag}\{c_1\,\Xi_1, c_2\,\Xi_2, \ldots, c_n\,\Xi_n\}$. We have the following result for the convergence of w_i:

Lemma 3
Suppose that Assumptions 5 and 6 hold. If the matrix L_Ξ is positive definite, then $w_i, i \in \mathbb{N}_j$ for all $j = 1, \ldots, n$ will asymptotically track the references w_{0j}, that is, it holds that $\lim_{t \to +\infty} \sum_{j=1}^n \sum_{i \in \mathbb{N}_j} \|w_i(t) - w_{0j}(t)\| = 0$.

Proof
Let $\tilde{w}_{h_j} = \bar{w}_{h_j} - \mathbf{1}_{h_j} \otimes w_{0j}$ and $\tilde{w}_{0j} = \hat{w}_{0j} - w_{0j}$ for $j = 1, \ldots, n$. From (21), one has

$$\begin{pmatrix} \dot{\tilde{w}}_{h_1} \\ \dot{\tilde{w}}_{h_2} \\ \vdots \\ \dot{\tilde{w}}_{h_n} \end{pmatrix} = (I_N \otimes S) \begin{pmatrix} \tilde{w}_{h_1} \\ \tilde{w}_{h_2} \\ \vdots \\ \tilde{w}_{h_n} \end{pmatrix} - \begin{pmatrix} c_1\,L_{11} & L_{12} & \ldots & L_{1n} \\ L_{21} & c_2\,L_{22} & \ldots & L_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ L_{n1} & L_{n2} & \ldots & c_n\,L_{nn} \end{pmatrix} \otimes I_m \cdot \begin{pmatrix} \bar{w}_{h_1} \\ \bar{w}_{h_2} \\ \vdots \\ \bar{w}_{h_n} \end{pmatrix}$$
$$- (D_\Xi \otimes I_m) \cdot \begin{pmatrix} \tilde{w}_{h_1} \\ \tilde{w}_{h_2} \\ \vdots \\ \tilde{w}_{h_n} \end{pmatrix} + (D_\Xi \otimes I_m) \cdot \begin{pmatrix} \mathbf{1}_{h_1} \otimes \tilde{w}_{01} \\ \mathbf{1}_{h_2} \otimes \tilde{w}_{02} \\ \vdots \\ \mathbf{1}_{h_n} \otimes \tilde{w}_{0n} \end{pmatrix} \tag{23}$$

Note that L_{ij} are zero-row-sum matrices. It follows that

$$\begin{pmatrix} c_1\,L_{11} & L_{12} & \ldots & L_{1n} \\ L_{21} & c_2\,L_{22} & \ldots & L_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ L_{n1} & L_{n2} & \ldots & c_n\,L_{nn} \end{pmatrix} \otimes I_m \cdot \begin{pmatrix} \mathbf{1}_{h_1} \otimes w_{01} \\ \mathbf{1}_{h_2} \otimes w_{02} \\ \vdots \\ \mathbf{1}_{h_n} \otimes w_{0n} \end{pmatrix} = \mathbf{0}\,.$$

Using the aforementioned equation and the notation L_Ξ, we can rewrite (23) as

$$\begin{pmatrix} \dot{\tilde{w}}_{h_1} \\ \dot{\tilde{w}}_{h_2} \\ \vdots \\ \dot{\tilde{w}}_{h_n} \end{pmatrix} = (I_N \otimes S) \begin{pmatrix} \tilde{w}_{h_1} \\ \tilde{w}_{h_2} \\ \vdots \\ \tilde{w}_{h_n} \end{pmatrix} - (L_\Xi \otimes I_m) \begin{pmatrix} \tilde{w}_{h_1} \\ \tilde{w}_{h_2} \\ \vdots \\ \tilde{w}_{h_n} \end{pmatrix} + (D_\Xi \otimes I_m) \cdot \begin{pmatrix} \mathbf{1}_{h_1} \otimes \tilde{w}_{01} \\ \mathbf{1}_{h_2} \otimes \tilde{w}_{02} \\ \vdots \\ \mathbf{1}_{h_n} \otimes \tilde{w}_{0n} \end{pmatrix} \tag{24}$$

Let $\tilde{w} \triangleq \left(\tilde{w}_{h_1}^T, \tilde{w}_{h_2}^T, \ldots, \tilde{w}_{h_n}^T \right)^T$ and $\tilde{w}_0^* \triangleq \left((\mathbf{1}_{h_1} \otimes \tilde{w}_{01})^T, (\mathbf{1}_{h_2} \otimes \tilde{w}_{02})^T, \ldots, (\mathbf{1}_{h_n} \otimes \tilde{w}_{0n})^T \right)^T$. Then, (24) can be written into the compact form

$$\dot{\tilde{w}} = (I_N \otimes S)\tilde{w} - (L_\Xi \otimes I_m)\tilde{w} + (D_\Xi \otimes I_m)\tilde{w}_0^*. \tag{25}$$

Let $\varpi = (I_N \otimes e^{-St})\tilde{w}$ and $\varphi = (I_N \otimes e^{-St})\tilde{w}_0^*$. Then, one has

$$\begin{aligned} \dot{\varpi} &= -(I_N \otimes e^{-St}S)\tilde{w} + (I_N \otimes e^{-St})\dot{\tilde{w}} \\ &= -(I_N \otimes e^{-St})(L_\Xi \otimes I_m)\tilde{w} + (I_N \otimes e^{-St})(D_\Xi \otimes I_m)\tilde{w}_0^* \\ &= -(L_\Xi \otimes I_m)\varpi + (D_\Xi \otimes I_m)\varphi\,. \end{aligned} \tag{26}$$

According to the condition in Lemma 3, the matrix $-(L_\Xi \otimes I_m)$ is Hurwitz. Moreover, from $\dot{\tilde{w}}_{0j} = \hat{w}_{0j} - \dot{w}_{0j} = (S - G_0\,R)\tilde{w}_{0j}$ and $\sigma(S - G_0\,R) \subset \mathbb{C}^-$, one has that $\tilde{w}_0^*$ converges to zero

exponentially as $t \to \infty$. It further implies $\varphi = \left(I_N \otimes e^{-St}\right) \tilde{w}_0^*$ converges to zero exponentially. Therefore, one has that ϖ converges to the origin exponentially as $t \to \infty$. Furthermore, because $\tilde{w} = (I_N \otimes e^{St}) \varpi$ and $\sigma(S) \subset \mathbb{C}^0$, one has that $\tilde{w} \to \mathbf{0}$ exponentially as $t \to \infty$. Thus, one obtains that $\lim_{t \to +\infty} \sum_{j=1}^{n} \sum_{i \in \mathbb{N}_j} ||w_i(t) - w_{0j}(t)|| = 0.$ $\qquad \square$

Remark 3

Assumption 5 is a trivial condition when clustering synchronization is discussed in multi-agent systems with diffusively coupled dynamical oscillators. For example, Assumption 5 on the Laplacian matrix L is the same as the one in Definition 4 in [20], and similar to that in Proposition 2 in [21]. However, in this subsection, we study different system dynamics and thus a different problem. We have proposed the stability criterion for system (20) in Lemma 3 under suitable assumptions on the communication topologies.

The condition on the matrix L_Ξ is an algebraic condition, which is difficult to check in applications. Now, we specify the connectivity strengths such that the matrix L_Ξ is positive definite. The way to construct the connectivity strengths is motivated by some results in [20, 21]. Because the results in [20, 21] cannot be applied to our problem directly, we carry out the construction as follows:

Lemma 4

[26] Let A and B be $N \times N$ Hermitian matrices, and let the eigenvalues $\lambda_i(A)$, $\lambda_i(B)$, $\lambda_i(A + B)$ be arranged in increasing order as $\lambda_1(\cdot) \leqslant \lambda_2(\cdot) \leqslant \ldots \leqslant \lambda_N(\cdot)$. For each $k = 1, 2, \ldots, N$, we have

$$\lambda_k(A) + \lambda_1(B) \leqslant \lambda_k(A + B) \leqslant \lambda_k(A) + \lambda_N(B).$$

Lemma 5

Suppose that Assumptions 5 and 6 hold. And suppose that the matrix L is symmetric and the matrices L_{jj} for $j = 1, \ldots, n$ are irreducible. If

$$c_j > \max \left\{ -\frac{\lambda_{\min}\left(L_\Xi^{(2)}\right)}{\lambda_{\min}(L_{jj} + \Xi_j)}, 0 \right\}$$

for all $j = 1, \ldots, n$, then the matrix L_Ξ is positive definite.

Proof

We will prove that the matrix L_Ξ is positive definite if the constants c_j for $j = 1, \ldots, n$ are sufficiently large. From Lemma 4, one has

$$\lambda_{\min}(L_\Xi) \geqslant \lambda_{\min}\left(L_\Xi^{(1)}\right) + \lambda_{\min}\left(L_\Xi^{(2)}\right)$$

$$= \min_{1 \leqslant j \leqslant n} \left\{\lambda_{\min}(c_j L_{jj} + c_j \Xi_j)\right\} + \lambda_{\min}\left(L_\Xi^{(2)}\right)$$

$$= \min_{1 \leqslant j \leqslant n} \left\{c_j \lambda_{\min}(L_{jj} + \Xi_j)\right\} + \lambda_{\min}\left(L_\Xi^{(2)}\right).$$

Note that L_{jj} for $j = \{1, \ldots, n\}$ are Laplacian matrices satisfying zero row sums and non-positive off-diagonal elements. Thus, L_{jj} are positive semi-definite. In addition, L_{jj} are irreducible. According to Lemma 1, the matrices $L_{jj} + \Xi_j$ for $j = 1, \ldots, n$ are positive definite. Thus, if $c_j > -\frac{\lambda_{\min}\left(L_\Xi^{(2)}\right)}{\lambda_{\min}(L_{jj}+\Xi_j)}$ for $j = 1, \ldots, n$, then $\lambda_{\min}(L_\Xi) > 0$. We have arrived at the conclusion that the matrix L_Ξ is positive definite if $c_j > \max \left\{ -\frac{\lambda_{\min}\left(L_\Xi^{(2)}\right)}{\lambda_{\min}(L_{jj}+\Xi_j)}, 0 \right\}$ for $j = 1, \ldots, n$. $\qquad \square$

Now, we give some comments on the condition of the inner coupling strengths c_j in Lemma 5.

Remark 4

There might exist other connection patterns such that L_Ξ is positive definite. Lemma 5 provides one way to construct communication topologies for this purpose. It requires lower bounds for c_j to guarantee the positive-definiteness of L_Ξ, which implies that large inner couplings are good for

clustering synchronization in a network. This also can be understood in real-world situations, as clustering is likely to happen in a network if inner connections inside clusters are strong, whereas connections among different clusters are weak. In addition, if more systems in the subset $\mathbb{N}_j$ are connected to their leader $\mathcal{L}_j$, it results in a larger positive value of $\lambda_{\min}(L_{jj} + \Xi_j)$. According to the lower bound for c_j in Lemma 5, a smaller positive c_j may be obtained consequently. This also makes sense in practice.

We have discussed clustering synchronization of the reference trajectories in Lemma 3. The estimations w_i can be treated as a group of reference signals and used to tackle the decentralized n-cluster output synchronization problem for heterogeneous systems. This is pursued in the succeeding text.

We introduce the controllers for systems (1) as follows. For agent $i \in \mathbb{N}_j$, $j = 1, \ldots, n$, we design u_i as

$$
\begin{aligned}
\dot{\hat{w}}_{0j} &= S\hat{w}_{0j} + G_0\,R(w_{0j} - \hat{w}_{0j}) \\
\dot{w}_i &= Sw_i + \sum_{k \in \mathbb{N}_j} c_j\,a_{ik}(w_k - w_i) + \sum_{k=1, k \notin \mathbb{N}_j}^{N} a_{ik}(w_k - w_i) + c_j\,a_{i0}(\hat{w}_{0j} - w_i) \\
\dot{\eta}_i &= \Phi_i\,\eta_i + G_i\,M_i\xi_i \\
\dot{\xi}_i &= L_i\xi_i + K_i(y_i - Rw_i) \\
u_i &= H_i\eta_i + M_i\xi_i
\end{aligned}
\tag{27}
$$

The matrices $\Phi_i, G_i, M_i, L_i, K_i, H_i$ are those found in Assumption 2.

Theorem 2
Consider N heterogeneous linear systems (1) coupled via the dynamic couplings (27). Suppose that Assumptions 1 and 2 hold, and the assumptions in Lemma 3 hold. Then, $\lim_{t \to +\infty} \sum_{j=1}^{n} \sum_{i \in \mathbb{N}_j} \|y_i(t) - Rw_{0j}(t)\| = 0$.

Proof
Let $\tilde{x}_i = x_i - \Pi_i(\mu_i)w_i$, $\tilde{\eta}_i = \eta_i - \Sigma_i(\mu_i)w_i$. Similar manipulations as those in the proof of Theorem 1 lead to

$$
\begin{aligned}
\dot{\tilde{x}}_i = {}& A_i(\mu_i)\tilde{x}_i + B_i(\mu_i)H_i\tilde{\eta}_i + B_i(\mu_i)M_i\xi_i - \\
& \Pi_i(\mu_i)\left(\sum_{k \in \mathbb{N}_j} c_j\,a_{ik}(w_k - w_i) + \sum_{k=1, k \notin \mathbb{N}_j}^{N} a_{ik}(w_k - w_i) + c_j\,a_{i0}(\hat{w}_{0j} - w_i) \right),
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\tilde{\eta}}_i = {}& \Phi_i\tilde{\eta}_i + G_i M_i\xi_i - \Sigma_i(\mu_i)\cdot \\
& \cdot \left(\sum_{k \in \mathbb{N}_j} c_j\,a_{ik}(w_k - w_i) + \sum_{k=1, k \notin \mathbb{N}_j}^{N} a_{ik}(w_k - w_i) + c_j\,a_{i0}(\hat{w}_{0j} - w_i) \right).
\end{aligned}
$$

Furthermore, $y_i - Rw_i = C_i(\mu_i)\tilde{x}_i$. Hence,

$$
\begin{aligned}
\dot{\xi}_i &= L_i\xi_i + K_i(y_i - Rw_i) \\
&= L_i\xi_i + K_i\,\big(\,C_i(\mu_i)\ \mathbf{0}\,\big) \begin{pmatrix} \tilde{x}_i \\ \tilde{\eta}_i \end{pmatrix}.
\end{aligned}
\tag{28}
$$

In the new coordinates $\tilde{x}_i$, $\tilde{\eta}_i$, and ξ_i, the system can be written as

$$
\begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{\eta}}_i \end{pmatrix} = \begin{pmatrix} A_i(\mu_i) & B_i(\mu_i)H_i \\ \mathbf{0} & \Phi_i \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{\eta}_i \end{pmatrix} + \begin{pmatrix} B_i(\mu_i) \\ G_i \end{pmatrix} M_i \xi_i
$$

$$
- \begin{pmatrix} \Pi_i(\mu_i) \\ \Sigma_i(\mu_i) \end{pmatrix} \left(\sum_{k \in \mathbb{N}_j} c_j \, a_{ik}(w_k - w_i) + \sum_{k=1, k \notin \mathbb{N}_j}^{N} a_{ik}(w_k - w_i) + c_j \, a_{i0}(\hat{w}_{0j} - w_i) \right)
$$

$$
\dot{\xi}_i = L_i \xi_i + K_i \, \big(C_i(\mu_i) \ \mathbf{0} \big) \begin{pmatrix} \tilde{x}_i \\ \tilde{\eta}_i \end{pmatrix}.
$$

(29)

By Assumption 2, the dynamic matrix of the closed loop system (29) is Hurwitz. Moreover, all the forcing inputs decay exponentially to zero. As a result, $\lim_{t \to +\infty} \sum_{j=1}^{n} \sum_{i \in \mathbb{N}_j} \|y_i(t) - R w_{0j}(t)\| = 0$. □

4. DESIGN OF THE CONTROLLERS

The actual design of the controllers in the previous Sections 2 and 3 depends on the fulfillment of the conditions in Assumption 2 or 4. In this section, we discuss how this can be achieved. The arguments follow the treatment in [11, Section 1.5]. For the sake of simplicity, we only focus on the design of controllers discussed in Section 2. The controllers discussed in Section 3 can be designed similarly.

We start with condition (ii), namely with the fulfillment of the internal model principle. Let Φ, H and $\Sigma_i(\mu_i)$ be the matrices

$$
\Phi = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_d \end{pmatrix}, H = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}^T
$$

$$
\Sigma_i(\mu_i) = \begin{pmatrix} \Gamma_i(\mu_i) \\ \Gamma_i(\mu_i)S \\ \vdots \\ \Gamma_i(\mu_i)S^{d-2} \\ \Gamma_i(\mu_i)S^{d-1} \end{pmatrix}
$$

where $\lambda^d + a_{d-1}\lambda^{d-1} + a_1 d + a_0$ is the minimal polynomial of S and $\Gamma_i(\mu_i)$ is the matrix that appears in the regulator Equation (3). It is straightforward to check that these matrices satisfy the internal model condition (4).

To the purpose of fulfilling also the robust stability condition (iii), it is convenient to introduce other matrices F_i, G_i, Ψ_i, T_i, which also fulfill the internal model principle. These matrices are detailed in the following lemma ([11, Lemma 1.5.6]):

Lemma 6
Let F_i be any Hurwitz $s \times s$ matrix and let G_i be any $s \times 1$ vector such that the pair (F_i, G_i) is controllable. Let Φ be any $s \times s$ matrix whose eigenvalues are all in $\overline{\mathbb{C}^+}$ and let H be any $1 \times s$ vector such that the pair (H, Φ) is observable.
Then, there exist a $1 \times s$ vector Ψ_i and a nonsingular $s \times s$ matrix T_i such that

$$
(F_i + G_i \Psi_i)T_i = T_i \Phi
$$
$$
\Psi_i T_i = H.
$$

(30)

It is immediate to see that the matrix $\tilde{\Sigma}_i(\mu_i) = T_i \Sigma_i(\mu_i)$ satisfies

$$
\tilde{\Sigma}_i(\mu_i)S = (F_i + G_i \Psi_i)\tilde{\Sigma}_i(\mu_i)
$$
$$
\Gamma_i(\mu_i) = \Psi_i \tilde{\Sigma}_i(\mu_i).
$$

Hence, the internal model principle property (4) is fulfilled by the matrices $F_i + G_i\Psi_i$, Ψ_i, $\tilde{\Sigma}_i(\mu_i)$. The controllers introduced in Section 2.2 can be rewritten as:

$$
\begin{aligned}
\dot{\hat{w}}_0 &= S\hat{w}_0 + G_0 R\,(w_0 - \hat{w}_0) \\
\dot{w}_1 &= S w_1 + \sum_{j=1}^{N} a_{1j}(w_j - w_1) + a_{10}\,(\hat{w}_0 - w_1) \\
\dot{\eta}_1 &= (F_1 + G_1\Psi_1)\,\eta_1 + G_1 M_1\xi_1 \\
\dot{\xi}_1 &= L_1\xi_1 + K_1(y_1 - Rw_1) \\
u_1 &= \Psi_1\eta_1 + M_1\xi_1
\end{aligned}
\tag{31}
$$

and, for agent $i = 2,\cdots,N$,

$$
\begin{aligned}
\dot{w}_i &= S w_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i) \\
\dot{\eta}_i &= (F_i + G_i\Psi_i)\,\eta_i + G_i M_i\xi_i \\
\dot{\xi}_i &= L_i\xi_i + K_i(y_i - Rw_i) \\
u_i &= \Psi_i\eta_i + M_i\xi_i
\end{aligned}
\tag{32}
$$

For the purpose of stabilizing the overall closed-loop system (requirement (iii) in Assumption 2), it is more convenient to work with these controllers rather than with those in (8), (9). In the rest of the section, we turn now to the problem of determining the stabilizing matrices L_i, K_i, M_i, $i = 1, 2, \ldots, N$.

For each i, consider the system (1) with output $\varepsilon_i = y_i - Rw_i$, namely

$$
\begin{aligned}
\dot{x}_i &= A_i(\mu_i)\, x_i + B_i(\mu_i)\, u_i \\
\varepsilon_i &= C_i(\mu_i)\, x_i - Rw_i\,.
\end{aligned}
\tag{33}
$$

As in [11], to reduce the notational burden, we focus on the case in which the inputs u_i and the outputs y_i are scalar, that is, $p_i = 1$ for $i = 1, 2, \ldots, N$ and $q = 1$. Further, assume that $\mathcal{P}_i$ is a compact set and that for each $\mu_i \in \mathcal{P}_i$, the system (33) has the same relative degree r_i from u_i to ε_i. Namely, there exists an integer $r_i \geqslant 1$ such that for each $\mu_i \in \mathcal{P}_i$

$$
\begin{aligned}
C_i(\mu_i) A_i^j(\mu_i) B_i(\mu_i) &= 0\,, \quad j = 0, 1, \ldots, r_i - 2 \\
C_i(\mu_i) A_i^{r_i-1}(\mu_i) B_i(\mu_i) &\neq 0\,.
\end{aligned}
$$

Then, there exists a μ_i-dependent change of coordinates

$$
\begin{pmatrix} z_i \\ e_i \end{pmatrix} =
\left(
\begin{array}{c}
\dfrac{Z_i(\mu_i)}{C_i(\mu_i)} \\
C_i(\mu_i) A_i(\mu_i) \\
\vdots \\
C_i(\mu_i) A_i(\mu_i)^{r_i - 1}
\end{array}
\right) x_i =: \tilde{Z}_i(\mu_i) x_i,
\tag{34}
$$

where $Z_i(\mu_i)$ is a suitable matrix such that $\tilde{Z}_i(\mu_i)$ is nonsingular, such that the system (33) in the new coordinates becomes

$$
\begin{aligned}
\dot{z}_i &= A_i^{(11)}(\mu_i) z_i + A_i^{(12)}(\mu_i) e_i \\
\dot{e}_{i1} &= e_{i2} \\
&\ \ \vdots \\
\dot{e}_{i,r_i-1} &= e_{ir_i} \\
\dot{e}_{i,r_i} &= A_i^{(21)}(\mu_i) z_i + A_i^{(22)}(\mu_i) e_i + b_i(\mu_i) u_i \\
\varepsilon_i &= e_{i1} - Rw_i = \overline{C} e_i - Rw_i,
\end{aligned}
\tag{35}
$$

where in particular $b_i(\mu_i) = C_i(\mu_i) A_i^{r_i - 1}(\mu_i) B_i(\mu_i) \neq 0$.
We further change the coordinates in the following way:

$$\tilde{e}_i = e_i + Q_i w$$

where $Q_i = \left(Q_{i1}^T \ldots Q_{ir}^T \right)^T$, $w = \left(w_1^T \ldots w_N^T \right)^T$,

$$Q_{i1} = \left(\mathbf{0}_{1 \times m} \ldots \mathbf{0}_{1 \times m} \ -R \ \mathbf{0}_{1 \times m} \ldots \mathbf{0}_{1 \times m} \right),$$

$$Q_{i,j+1} = Q_{i,j} \tilde{S}, \quad j = 1, 2, \ldots r_i - 1$$

and $\tilde{S} = (I_N \otimes S - L \otimes I_m)$. Then, we obtain

$$
\begin{aligned}
\dot{z}_i &= A_i^{(11)}(\mu_i) z_i + A_i^{(12)}(\mu_i) \tilde{e}_i + \overline{Q}_i(\mu_i) w \\
\dot{\tilde{e}}_{i1} &= \tilde{e}_{i2} \\
&\;\;\vdots \\
\dot{\tilde{e}}_{i,r_i-1} &= \tilde{e}_{ir_i} \\
\dot{\tilde{e}}_{i,r_i} &= A_i^{(21)}(\mu_i) z_i + A_i^{(22)}(\mu_i) \tilde{e}_i + \tilde{Q}_i(\mu_i) w + b_i(\mu_i) u_i \\
\varepsilon_i &= \tilde{e}_{i1},
\end{aligned}
\tag{36}
$$

with

$$\overline{Q}_i(\mu_i) = -A_i^{(12)}(\mu_i) Q_i, \quad \tilde{Q}_i(\mu_i) = -A_i^{(22)}(\mu_i) Q_i.$$

In the succeeding text, we use the following partition for the two matrices:

$$
\begin{aligned}
\overline{Q}_i(\mu_i) &= \left(\overline{Q}_{i1}(\mu_i) \ldots \overline{Q}_{iN}(\mu_i) \right) \\
\tilde{Q}_i(\mu_i) &= \left(\tilde{Q}_{i1}(\mu_i) \ldots \tilde{Q}_{iN}(\mu_i) \right).
\end{aligned}
$$

Observe that because of the latter change of coordinates, the signal w affects the dynamics of the systems. Hence, (36) falls in the class of systems considered in (17) and Corollary 1 applies. Before doing this, we need an additional assumption. Let the system (33) be minimum-phase, namely

Assumption 7
For each $\mu_i \in \mathcal{P}_i$, all the eigenvalues of $A_i^{(11)}(\mu_i)$ have strictly negative real parts.

As a consequence of this assumption, it is promptly verified (see [11], page 27) that the matrices

$$
\begin{aligned}
\Pi_i(\mu_i) &= \left(\Pi_{i1}(\mu_i)^T \ 0 \ldots 0 \right)^T \\
\Gamma_i(\mu_i) &= -\frac{1}{b_i(\mu_i)} \left[A_i^{(21)}(\mu_i) \Pi_{i1}(\mu_i) - \sum_{j=1}^{N} \tilde{Q}_{ij}(\mu_i) \right],
\end{aligned}
\tag{37}
$$

where $\Pi_{i1}(\mu_i)$ is the unique $r_i \times r_i$ matrix, which solves the Sylvester equation

$$\Pi_{i1}(\mu_i) S = A_i^{(11)}(\mu_i) \Pi_{i1}(\mu_i) + \sum_{j=1}^{N} \overline{Q}_{ij}(\mu_i), \tag{38}$$

satisfy condition (i) in Assumption 4 with

$$
P_i(\mu_i) = \begin{pmatrix} \overline{Q}_i(\mu_i) \\ 0 \\ \vdots \\ 0 \\ \tilde{Q}_i(\mu_i) \end{pmatrix}.
$$

Before carrying out the next step of our robust controllers design, we explain the differences of our design method explored in the aforementioned text, compared with the basic robust regulator design method in [11].

Remark 5

The arguments on the actual design of the decentralized robust controllers follow closely the treatment in [11], but they are not the same. The robust controller design method in [11] only deals with output regulation of a single system. In our case, we design decentralized robust controllers for the cooperative multi-agent systems (1). Consequently, we have to deal with the dynamical coupling terms existing in the controllers (31) and (32). To be specific, in our case only system $\mathcal{S}_1$ has direct access to the exosystem, the other systems $\mathcal{S}_2, \ldots, \mathcal{S}_N$ are fed by the local estimates of the reference signal w_i. As a result, the systems' controllers are coupled with each other through the estimates, and the controllers cannot be designed separately for each system $\mathcal{S}_i$ to track the corresponding reconstructed reference signal w_i. To deal with the difficulties caused by the dynamical couplings and the cooperation framework discussed in this paper, in the calculations to obtain (36), we treat the reconstructed references $w_1, \ldots, w_N$ as a whole, that is generated by an exosystem $\dot{w} = \tilde{S}\, w$. Finally, we have adopted different coordinate changes for $\tilde{e}_i$, compared with those in [11, Section 1.5].

The design of the matrices K_i, L_i, M_i such that condition (iii) is satisfied can be carried out in two steps. Consider the system (36) and write it in the compact form

$$
\begin{aligned}
\dot{z}_i &= A_i^{(11)}(\mu_i)z_i + A_i^{(12)}(\mu_i)\tilde{e}_i + \overline{Q}_i(\mu_i)w \\
\dot{\tilde{e}}_i &= \overline{A}\tilde{e}_i + \overline{B}\left[A_i^{(21)}(\mu_i)z_i + A_i^{(22)}(\mu_i)\tilde{e}_i + \tilde{Q}_i(\mu_i)w + b_i(\mu_i)u_i\right] \\
\varepsilon_i &= \overline{C}\tilde{e}_i,
\end{aligned}
\tag{39}
$$

where $\overline{A}, \overline{B}, \overline{C}$ are understood from the context. Also, consider a controller of the form

$$
\begin{aligned}
\dot{\eta}_i &= F_i\eta_i + G_i u_i \\
u_i &= \Psi_i\eta_i + v_i
\end{aligned}
\tag{40}
$$

where v_i is an additional control input and obtain the closed-loop system

$$
\begin{aligned}
\dot{\eta}_i &= (F_i + G_i\Psi_i)\eta_i + G_i v_i \\
\dot{z}_i &= A_i^{(11)}(\mu_i)z_i + A_i^{(12)}(\mu_i)\tilde{e}_i + \overline{Q}_i(\mu_i)w \\
\dot{\tilde{e}}_i &= \overline{A}\tilde{e}_i + \overline{B}\Big[A_i^{(21)}(\mu_i)z_i + A_i^{(22)}(\mu_i)\tilde{e}_i + \\
&\qquad \tilde{Q}_i(\mu_i)w + b_i(\mu_i)(\Psi_i\eta_i + v_i)\Big] \\
\varepsilon_i &= \overline{C}\tilde{e}_i.
\end{aligned}
\tag{41}
$$

The change of coordinates

$$
\chi_i = \eta_i - \frac{1}{b_i(\mu_i)}G_i\overline{C}\,\overline{A}^{r_i-1}\tilde{e}_i
$$

yields the closed-loop system

$$
\begin{aligned}
\begin{pmatrix} \dot{\chi}_i \\ \dot{z}_i \end{pmatrix} &= \begin{pmatrix} F_i & -\frac{1}{b_i(\mu_i)}G_i A_i^{(21)}(\mu_i) \\ 0 & A_i^{(11)}(\mu_i) \end{pmatrix}\begin{pmatrix} \chi_i \\ z_i \end{pmatrix} + \\
&\quad \begin{pmatrix} \frac{1}{b_i(\mu_i)}\left[F_i G_i \overline{C}\,\overline{A}^{r_i-1} - G_i A_i^{(22)}(\mu_i)\right] \\ A_i^{(12)}(\mu_i) \end{pmatrix}\tilde{e}_i + \begin{pmatrix} -\frac{1}{b_i(\mu_i)}G_i \tilde{Q}_i(\mu_i) \\ \overline{Q}_i(\mu_i) \end{pmatrix}w \\
\dot{\tilde{e}}_i &= \overline{A}\tilde{e}_i + \overline{B}\Big[b_i(\mu_i)\Psi_i\chi_i + A_i^{(21)}(\mu_i)z_i + \\
&\qquad \left(A_i^{(22)}(\mu_i) + \Psi_i G_i\overline{C}\,\overline{A}^{r_i-1}\right)\tilde{e}_i + b_i(\mu_i)v_i + \tilde{Q}_i(\mu_i)w\Big] \\
\varepsilon_i &= \overline{C}\tilde{e}_i.
\end{aligned}
\tag{42}
$$

The zero dynamics of the system is

$$
\begin{pmatrix} \dot{\chi}_i \\ \dot{z}_i \end{pmatrix} = \begin{pmatrix} F_i & -\frac{1}{b_i(\mu_i)}G_i A_i^{(21)}(\mu_i) \\ 0 & A_i^{(11)}(\mu_i) \end{pmatrix}\begin{pmatrix} \chi_i \\ z_i \end{pmatrix}.
$$

This is asymptotically stable for each $\mu_i \in \mathcal{P}_i$ because F_i is Hurwitz by construction and $A_i^{(11)}(\mu_i)$ is Hurwitz by Assumption 7. In view of this property, it is proven in [11, Lemma 1.5.4] that under the assumption that $b_i(\mu_i) \geqslant b_i > 0$ for all $\mu_i \in \mathcal{P}_i$, there exists a positive gain k_i^*, a $1 \times r_i$ vector $\overline{M}_i$ such that for all $k_i \geqslant k_i^*$, the feedback

$$v_i = -k_i \overline{M}_i \tilde{e}_i =: M_i \tilde{e}_i \qquad (43)$$

stabilizes the system (42) for all $\mu_i \in \mathcal{P}_i$. Moreover, the matrix $\overline{M}_i$ is of the form

$$\overline{M}_i = (d_{i0} \ d_{i1} \ \ldots \ d_{i,r_i-2} \ 1)$$

where $\lambda^{r_i-1} + d_{i,r_i-2}\lambda^{r_1-2} + \ldots + d_{i0}$ is any polynomial having all the roots with strictly negative real parts.

The feedback (43) cannot be implemented because it requires the knowledge of $\tilde{e}_i$, which is not available. The second step of the construction consists in the design of a controller, which uses an estimate of $\tilde{e}_i$. This design can be carried out following [11, Lemma 1.5.5]. Consider the dynamic feedback controller

$$\begin{aligned} \dot{\xi}_i &= L_i \xi_i + K_i \varepsilon_i \\ v_i &= M_i \xi_i, \end{aligned} \qquad (44)$$

where

$$L_i = \begin{pmatrix} -g_i c_{i,r_i-1} & 1 & \ldots & 0 \\ -g_i^2 c_{i,r_i-2} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -g_i^{r_i-1} c_{i,1} & 0 & \ldots & 1 \\ -g_i^{r_i} c_{i,0} & 0 & \ldots & 0 \end{pmatrix}, K_i = \begin{pmatrix} g_i c_{i,r_i-1} \\ g_i^2 c_{i,r_i-2} \\ \vdots \\ g_i^{r_i-1} c_{i,1} \\ g_i^{r_i} c_{i,0} \end{pmatrix} \qquad (45)$$

the polynomial $\lambda^{r_i} + c_{i,r_i-1}\lambda^{r_i-1} + \ldots + c_{i,1}\lambda^1 + c_{i,0}$ is any polynomial having all the roots with negative real part, $g_i > 0$ is a design parameter and M_i is as in (43). Under Assumption 7, if $b_i(\mu_i) \geqslant b_i > 0$ for all $\mu_i \in \mathcal{P}_i$, it can be shown that there exists a positive gain $g_i^* > 0$ such that, for all $g_i \geqslant g_i^*$, the controller (44) asymptotically stabilize the system (42) for all $\mu_i \in \mathcal{P}_i$.

The latter statement allows us to summarize as follows:

Proposition 1
Consider the system (39). Let Assumption 7 hold and assume that $b_i(\mu_i) \geqslant b_i > 0$ for all $\mu_i \in \mathcal{P}_i$, with $\mathcal{P}_i$ a compact set. Then, there exists a positive gain $g_i^* > 0$ such that, for all $g_i \geqslant g_i^*$, the matrices L_i, K_i, M_i defined in (45) and (43) are such that the dynamic feedback controller (44) globally asymptotically stabilizes (42) for all $\mu_i \in \mathcal{P}_i$.

Remark 6
The overall controller is given by the interconnection of the internal model (40) and the stabilizer (44). We observe that the design of the two controllers requires local information only. As a matter of fact, the matrices F_i, G_i, Ψ_i of the internal model can be obtained via Lemma 6. On the other hand, the controller (44) is designed to robustly stabilize the system (39). Because the only terms in the system (39) that depend on the Laplacian matrix L are the 'disturbance' vectors $\overline{Q}_i(\mu_i), \tilde{Q}_i(\mu_i)$ which play no role in the stability property of the closed-loop system, one infers that the design of L_i, K_i, M_i is independent of the knowledge of the graph topology.

5. NUMERICAL EXAMPLES

5.1. Tracking a single reference

In this section, we illustrate the design of the robust controllers for decentralized output regulation via a numerical example. The example we consider corresponds to a network of double integrators with different actuator dynamics, namely we consider the case in which the systems (1) are modeled as

$$\dot{x}_i = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & c_i \\ 0 & -d_i & -a_i \end{pmatrix}}_{A_i(\mu_i)} x_i + \underbrace{\begin{pmatrix} 0 \\ 0 \\ b_i \end{pmatrix}}_{B_i(\mu_i)} u_i$$
$$y_i = \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{C_i(\mu_i)} x_i, \quad i = 1, 2, \ldots, N, \tag{46}$$

where $\mu_i = (a_i \, b_i \, c_i \, d_i)^T$ is the vector of uncertain parameters. The example was proposed in [10] where it was assumed that the parameters appearing in the equations are known and used to design the controllers. Here, we consider the case when these parameters are uncertain. Hence, the controllers have to be designed differently. We assume that μ_i is not precisely known and ranges over a compact set $\mathcal{P}_i$, which is contained in $\mathbb{R}^3_{>0} \times \mathbb{R}_{\geqslant 0}$. Observe that the uncertain parameters a_i, b_i, c_i are bounded away from zero. We consider the problem in which the matrices that define the leader's equation (2) are given by

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{47}$$

In other words, the position of the systems (46) has to asymptotically evolve as the ramp reference signal set by the leader.

Following the previous section, we first compute the relative degree r_i of each system. It is easily verified that

$$C_i(\mu_i) B_i(\mu_i) = C_i(\mu_i) A_i(\mu_i) B_i(\mu_i) = 0$$

$$C_i(\mu_i) A_i^2(\mu_i) B_i(\mu_i) = b_i c_i \ .$$

Because $b_i c_i \neq 0$ for each $\mu_i \in \mathcal{P}_i$, the previous equalities show that each system has a relative degree $r_i = 3$. As the relative degree equals the dimension of the systems, the matrix $\tilde{Z}_i(\mu_i)$ in the change of coordinates (34) can be written as

$$\tilde{Z}_i(\mu_i) = \begin{pmatrix} C_i(\mu_i) \\ C_i(\mu_i) A_i(\mu_i) \\ C_i(\mu_i) A_i(\mu_i)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c_i \end{pmatrix}$$

and in the new coordinates the system (35) can be written as

$$\begin{aligned} \dot{e}_{i1} &= e_{i2} \\ \dot{e}_{i2} &= e_{i3} \\ \dot{e}_{i3} &= -c_i d_i e_{i2} - a_i e_{i3} + b_i c_i u_i. \end{aligned} \tag{48}$$

When compared with (35), we observe that the system has no zero dynamics and checking Assumption 7 becomes superfluous. Moreover,

$$A_i^{(21)}(\mu_i) = 0, A_i^{(22)}(\mu_i) = -\begin{pmatrix} 0 & c_i d_i & a_i \end{pmatrix}, b_i(\mu_i) = b_i c_i,$$

from which we conclude that $b_i(\mu_i) \geqslant \bar{b}_i > 0$, for all $\mu_i \in \mathcal{P}_i$, for some $\bar{b}_i$.

Having verified that all the assumptions of Proposition 1 hold, we can determine the controllers.

First of all, we determine the matrices F_i, G_i, Ψ_i in (40). This computation is carried out as in the proof of [11, Lemma 1.5.6]. Because the minimal polynomial of S is λ^2, we have

$$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and let (see Lemma 6 earlier)

$$F_i = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad G_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be a pair of matrices with F_i Hurwitz and (F_i, G_i) controllable. Here, for the sake of simplicity, we take F_i, G_i to be the same for each $i = 1, 2, \ldots, N$. Following the proof of [11, Lemma 1.5.6], one can construct the vector Ψ_i and the nonsingular matrix T_i which satisfy (30) and obtain

$$\Psi_i = \begin{pmatrix} 1 & 2 \end{pmatrix}, \quad T_i = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

This concludes the computation of the matrices F_i, G_i, Ψ_i, which appear in (40).

We turn now to the design of the matrices L_i, K_i, M_i, which appear in (7). Because $r_i = 3$, and letting $\lambda^2 + d_{i1}\lambda + d_{i0} = \lambda^2 + 2\lambda + 1$, $\lambda^3 + c_{i2}\lambda^2 + c_{i1}\lambda + c_{i0} = \lambda^3 + 3\lambda^2 + 3\lambda + 1$ two polynomials with all the roots having strictly negative real parts, the matrices L_i, K_i, M_i are given by

$$L_i = \begin{pmatrix} -3g_i & 1 & 0 \\ -3g_i^2 & 0 & 1 \\ -g_i^3 & 0 & 0 \end{pmatrix}, \quad K_i = \begin{pmatrix} 3g_i \\ 3g_i^2 \\ g_i^3 \end{pmatrix},$$

$$M_i = -k_i \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$

where k_i, g_i are gains to be chosen sufficiently large. Finally, we let $G_0 = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ be such that $S - G_0 R$ is Hurwitz.

We conclude that the controllers (31), (32) with the matrices $F_i, G_i, \Psi_i, L_i, K_i, M_i, G_0$ computed earlier, solve the decentralized output regulation problem for the systems (46), (47).

We have run a simulation for $N = 4$ systems with parameters $\{a_i, b_i, c_i, d_i\}$ chosen to be $\{1 + \mu_1, 1 + \mu_2, 1 + \mu_3, \mu_4\}$, $\{2.5 + \mu_1, 2 + \mu_2, 1 + \mu_3, \mu_4\}$, $\{2 + \mu_1, 1 + \mu_2, 1 + \mu_3, 0.5 + \mu_4\}$, $\{2 + \mu_1, 1 + \mu_2, 1 + \mu_3, 1 + \mu_4\}$, respectively, where $\mu_1, \mu_2, \mu_3, \mu_4$ are $0.3, 0.4, 0.5, 0.7$,respectively. We set the gains $k = 1.1$, $g = 14$. As for the communication graph, we have chosen one with a direct link between the exosystem and the system $\mathcal{S}_1$, that is, there is a directed link $(0, 1)$. The communication graph among the systems $\mathcal{S}_i, i = 1, 2, 3, 4$ is set to be the undirected and static graph with edges $\{(1, 2), (2, 3), (3, 4), (4, 1)\}$. The initial value for the exosystem w_0 is taken to be $\begin{pmatrix} 2 & 1 \end{pmatrix}^T$, whereas all the other initial values are randomly chosen in the interval $[0, 10]$.

Figure 1 shows that the outputs $y_i, i = 1, 2, 3, 4$ of the systems successfully track the exosystem output Rw_0. The simulation result supports the conclusions of Theorem 1 and the controller design method.
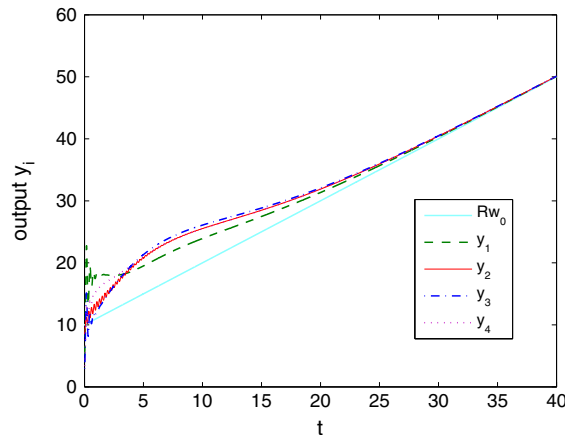

Figure 1. The outputs of the systems track the signal Rw_0.

5.2. Tracking multiple references

In this subsection, we illustrate the design of the robust controllers for decentralized clustering output synchronization through a numerical example. The example we consider corresponds to a network of double integrators with different actuator dynamics, namely we consider the case in which systems (1) are modeled by (46) as well. We consider the problem in which the matrices that define the leader's dynamics (19) are given by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{49}$$

In other words, the trajectory of systems (46) has to converge asymptotically to the sinusoidal reference signals of the leaders.

For systems (46), we have verified that all the assumptions of Proposition 1 hold in the previous subsection. Thus, we can determine the controllers, using the approach given in Section 4.

First of all, we remark that the matrices F_i, G_i in (40) are the same as in the previous subsection. Following again the proof of [11, Lemma 1.5.6], we arrive at the vector Ψ_i and at the nonsingular matrix T_i

$$\Psi_i = \begin{pmatrix} 0 & 2 \end{pmatrix}, \quad T_i = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

such that (30) is satisfied. The matrices L_i, K_i, M_i in (7) can be chosen as in the previous subsection. Therefore, the controllers

$$
\begin{aligned}
\dot{\hat{w}}_{0j} &= S \hat{w}_{0j} + G_0 R(w_{0j} - \hat{w}_{0j}) \\
\dot{w}_i &= S w_i + \sum_{k \in \mathbb{N}_j} c_j \, a_{ik}(w_k - w_i) + \sum_{k=1, k \notin \mathbb{N}_j}^{N} a_{ik}(w_k - w_i) + c_j \, a_{i0}(\hat{w}_{0j} - w_i) \\
\dot{\eta}_i &= (F_i + G_i \Psi_i) \, \eta_i + G_i \, M_i \xi_i \\
\dot{\xi}_i &= L_i \xi_i + K_i(y_i - R w_i) \\
u_i &= \Psi_i \eta_i + M_i \xi_i
\end{aligned}
\tag{50}
$$

for $i \in \mathbb{N}_j$, $j = 1, \ldots, n$, with $G_0 = (2\ 1)^T$ and the matrices $F_i, G_i, \Psi_i, L_i, K_i, M_i$ computed earlier, solve the decentralized output regulation problem for systems (46), (49).

We consider $N = 6$ heterogeneous systems with parameters $\{a_i, b_i, c_i, d_i\}$ for $i = 1, \ldots, 6$ chosen to be $\{1+\mu_1, 1+\mu_2, 1+\mu_3, \mu_4\}$, $\{2.5+\mu_1, 2+\mu_2, 1+\mu_3, \mu_4\}$, $\{2+\mu_1, 1+\mu_2, 1+\mu_3, 0.5+\mu_4\}$, $\{2+\mu_1, 1+\mu_2, 1+\mu_3, 1+\mu_4\}$, $\{2.5+\mu_1, 1.5+\mu_2, 1+\mu_3, 0.5+\mu_4\}$, $\{1+\mu_1, 2+\mu_2, 1+\mu_3, 1+\mu_4\}$ respectively, where the uncertainties $\mu_1, \mu_2, \mu_3, \mu_4$ are $0.3, 0.4, 0.5, 0.7$ respectively. Those systems communicate according to graph $\mathbb{G}$ shown in Figure 2. There are two different leaders $\mathcal{L}_1, \mathcal{L}_2$ as shown in Figure 2. We pick systems 1 and 5 to be connected to the two leaders $\mathcal{L}_1, \mathcal{L}_2$ respectively, such that the systems in the network realize a 2-cluster synchronization and track the two different trajectories of the leaders $\mathcal{L}_1, \mathcal{L}_2$. To be specific, there is a directed edge from leader $\mathcal{L}_1$ to system 1, and a directed edge from leader $\mathcal{L}_2$ to system 5. Leader $\mathcal{L}_1$ has directed paths to all the nodes in $\mathbb{N}_1 = \{1, 2, 3, 4\}$, although $\{1, 2\}$ and $\{3, 4\}$ are connected indirectly through the nodes $\{5, 6\}$. We set the inner coupling strengths $c_1 = c_2 = 2$. Then the matrix L_Ξ in (22) can be described as $L_\Xi = \left(\begin{array}{c|c} c_1 L_{11} & L_{12} \\ \hline L_{21} & c_2 L_{22} \end{array} \right) + \mathrm{diag}\{2, 0, 0, 0, 2, 0\}$, where

$$
\left(\begin{array}{c|c} c_1 L_{11} & L_{12} \\ \hline L_{21} & c_2 L_{22} \end{array} \right) = \left(\begin{array}{cccc|cc} 2 & -2 & 0 & 0 & 1 & -1 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & -1 & 1 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & -1 & -2 & 2 \end{array} \right). \tag{51}
$$

One can check that the matrix L_Ξ is positive definite.
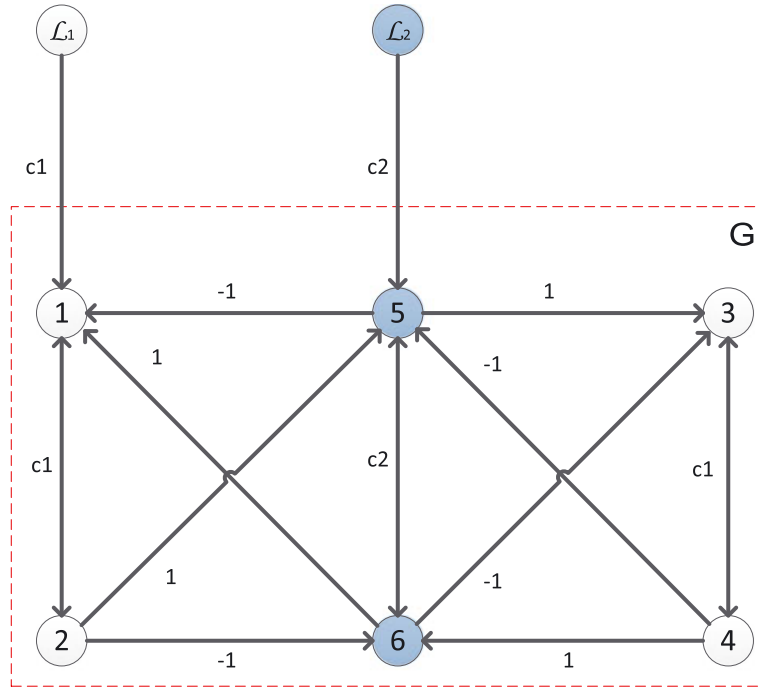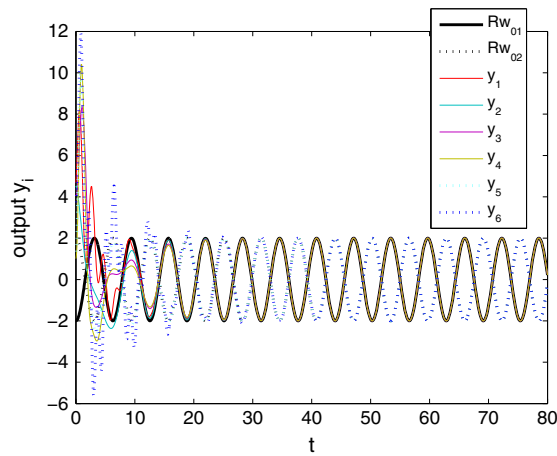
Figure 2. Communication graph.

Figure 3. The outputs of the systems are synchronized in clusters and track the exosystem outputs Rw_{01}, Rw_{02}.

We consider the leader $\mathcal{L}_1$ satisfying $\dot{w}_{01} = Sw_{01}$ with $w_{01}(0) = (-2, 0)^T$, and the leader $\mathcal{L}_2$ satisfying $\dot{w}_{02} = Sw_{02}$ with $w_{02}(0) = (2, 0)^T$, where S has been given in (49). The initial values of w_i for $i = 1, \ldots, 6$ are randomly selected from the interval $[0, 5]$. We ran the simulation for systems (46) and controllers (50) in which the gains k, g are set to be 0.8 and 12, respectively. Figure 3 shows that the outputs $y_i, i = 1, 2, 3, 4$ of the systems asymptotically track the exosystem output Rw_{01}, and the outputs $y_i, i = 5, 6$ asymptotically track the exosystem output Rw_{02}. The simulation result validates the conclusions of Theorem 2 and the controller design method.

6. CONCLUSION

We have tackled the problem of designing decentralized controllers able to track prescribed reference signals generated by exosystems under the restriction that not all the systems can access the information available at the exosystem. Under the assumption that each leader (exosystem) has a directed path to its follower systems, we have shown that there exist decentralized controllers, which achieve the desired regulation task in the presence of arbitrarily large but bounded uncertainties in the systems' models.

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