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Stam, A.J.

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## Two Identities in the Theory of Polynomials of Binomial Type

A. J. STAM

*Mathematisch Instituut, Rijksuniversiteit Groningen,  
Groningen, The Netherlands*

*Submitted by G.-C. Rota*

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### 1. INTRODUCTION

Let  $\lambda_1, \dots, \lambda_n, x_1, \dots, x_n$  be real numbers,

$$\lambda(T) = \sum_{i \in T} \lambda_i, \quad T \subset [n] = \{1, \dots, n\} \tag{1}$$

so that  $\lambda(\emptyset) = 0$  and

$$\lambda = \lambda([n]) = \lambda_1 + \dots + \lambda_n, \quad x = x_1 + \dots + x_n. \tag{2}$$

Hurwitz [2] by elementary analytical methods proved the identities

$$\sum \prod_{i=1}^s x_i (x_i - \lambda(T_i))^{t_i - 1} = x(x - \lambda)^{n-1}, \tag{3}$$

$$\sum (x_s - \lambda(T_s))^{t_s} \prod_{i=1}^{s-1} x_i (x_i - \lambda(T_i))^{t_i - 1} = (x - \lambda)^n, \tag{4}$$

where the sums are over all ordered partitions  $(T_1, \dots, T_s)$  of  $[n]$  with  $T_i = \emptyset$  admitted and  $t_i = |T_i|$ . Françon [1] proved (3) for  $s=2$  by combinatorial methods. For  $s=2$  and  $\lambda_i = \lambda$  the relations (3) and (4) reduce to forms of Abel's generalized binomial formula.

We extend these results by applying the theory of polynomials of binomial type developed in [3] and [4] from which we will cite freely. Let  $Q = DP$  be a delta operator on the set of all real polynomials with  $D = d/dx$  and  $P$  an invertible shift-invariant operator that maps any polynomial into a polynomial of the same degree. Shift-invariance means  $E^a P = P E^a$ , where

$$E^a f(x) = f(x + a). \tag{5}$$

Shift-invariant operators commute. To  $Q$  belongs a unique sequence of

basic polynomials  $q_n, n \in \mathbb{N}$ , defined by  $q_0(x) = 1, q_n(0) = 0, Qq_n = nq_{n-1}, n \geq 1$ . The polynomial  $q_n$  has degree  $n$  and the sequence  $\{q_n\}$  is of binomial type. The explicit form of the  $q_n$  is given by theorem 4 in [3] and [4]. We choose the representation

$$q_n(x) = xr_n(x), \quad r_n(x) \equiv r(n; x) = P^{-n}x^{n-1}, \quad n \geq 1, \quad (6)$$

which also holds for  $n=0$  if we allow  $P^0x^{-1} = Ix^{-1} = x^{-1}$  so that  $r_0(x) = x^{-1}$ . In Section 2 we prove the following generalization of (3) and (4).

**THEOREM 1.** *For  $r(n; x)$  defined by (6) we have*

$$\sum \prod_{i=1}^s x_i r(t_i; x_i - \lambda(T_i)) = xr(n; x - \lambda), \quad (7)$$

$$\sum r(t_s + 1; x_s - \lambda(T_s)) \prod_{i=1}^{s-1} x_i r(t_i; x_i - \lambda(T_i)) = r(n + 1; x - \lambda), \quad (8)$$

where the summations are as in (3) and (4) and  $t_i = |T_i|$ .

It is noted that  $r_0(x - \lambda(\phi)) = x^{-1}$  since  $\lambda(\phi) = 0$  so that both sides of (7) and (8) are polynomials of degree  $n$  in  $x_1, \dots, x_n$ . For  $P = I$  we have  $r_n(x) = x^{n-1}$  and (7), (8) reduce to (3), (4). For  $Q = I - E^{-1}$  we have  $q_n(x) = x^{(n)}$ , where

$$x^{(0)} = 1, x^{(n)} = x(x + 1) \cdots (x + n - 1), \quad n \geq 1. \quad (9)$$

Then  $r_n(x) = (x + 1)^{(n-1)}, n \geq 0$ , if we define  $(x + 1)^{(-1)} = x^{-1}$  and (7) and (8) become

$$\sum \prod_{i=1}^s x_i (x_i + 1 - \lambda(T_i))^{(t_i-1)} = x(x + 1 - \lambda)^{(n-1)}, \quad (10)$$

$$\sum (x_s + 1 - \lambda(T_s))^{(t_s)} \prod_{i=1}^{s-1} x_i (x_i + 1 - \lambda(T_i))^{(t_i-1)} = (x + 1 - \lambda)^{(n)}. \quad (11)$$

A combinatorial application of (10) and (11) will be given in Section 2.

### 2. PROOF AND APPLICATION

First, we prove (7). From (6), since  $P^{-1}$  also is shift-invariant,

$$r(t_i; x_i - \lambda(T_i)) = E^{-\lambda(T_i)} P^{-t_i} x_i^{t_i-1} = P^{-t_i} E^{-\lambda(T_i)} x_i^{t_i-1} = P^{-t_i} (x_i - \lambda(T_i))^{t_i-1}. \quad (12)$$

By Theorem 2 in [3] or [4] we have

$$P^{-1} = \sum_{k=0}^{\infty} a_k D^k/k!.$$

Since  $(x_i - \lambda(T_i))^{h-1}$  has degree smaller than  $n$ , we may replace  $P^{-1}$  in (12) by

$$V = \sum_{k=0}^{\infty} b_k D^k/k!,$$

if  $b_k = a_k, k \leq n$ . For  $V$  we take

$$Vf(x) = \int f(x+u) dF(u), \tag{13}$$

where  $F: \mathbb{R} \rightarrow \mathbb{R}$  is of bounded variation. We may choose  $F$  so that

$$b_k = \int u^k dF(u) = a_k, \quad k = 0, \dots, n. \tag{14}$$

It is sufficient to take  $F$  constant except for  $n+1$  jumps of magnitudes  $\eta_0, \dots, \eta_n$ . The coefficient determinant of Eqs. (14) in  $\eta_0, \dots, \eta_n$  then is Vandermonde's. If  $T_i = \{j_1, \dots, j_h\}$  we have from (12) and (13),

$$r(t_i; x_i - \lambda(T_i)) = \int \cdots \int (x_i + u_{j_1} + \cdots + u_{j_h} - \lambda(T_i))^{h-1} dF(u_{j_1}) \cdots dF(u_{j_h}). \tag{15}$$

If  $T_i = \phi$  we have  $r(t_i; x_i - \lambda(T_i)) = x_i^{-1}$ . Substituting this into the left-hand side of (7), applying Fubini's theorem to the integrations, interchanging sum and integral and applying (3) with  $\lambda_i$  replaced by  $\lambda_i - u_i$  shows that the left-hand side of (7) is equal to

$$\begin{aligned} &x \int \cdots \int (x + u_1 + \cdots + u_n - \lambda)^{n-1} dF(u_1) \cdots dF(u_n) \\ &= xV^n(x - \lambda)^{n-1} = xP^{-n}(x - \lambda)^{n-1} = xP^{-n}E^{-\lambda}x^{n-1} \\ &= xE^{-\lambda}P^{-n}x^{n-1} = xr(n; x - \lambda), \end{aligned}$$

which proves (7).

In the proof of (8) we have (15) for  $i \leq s - 1$  and, if  $T_s = (i_1, \dots, i_k)$ , in the same way as above,

$$\begin{aligned} r(t_s + 1; x_s - \lambda(T_s)) &= E^{-\lambda(T_s)} P^{-k-1} x_s^k = P^{-k-1} (x_s - \lambda(T_s))^k = V^{k+1} (x_s - \lambda(T_s))^k \\ &= \int \cdots \int (x_s + v + u_{i_1} + \cdots + u_{i_k} - \lambda(T_s))^k dF(v) dF(u_{i_1}) \cdots dF(u_{i_k}). \end{aligned}$$

Substituting this, together with (15), into the left-hand side of (8), applying Fubini's theorem, interchanging sum and integral and applying (4) with  $x_s$  replaced by  $x_s + v$  and  $\lambda_i$  by  $\lambda_i - u_i$  shows that the left-hand side of (8) is equal to

$$\begin{aligned} \int \cdots \int (x + v + u_1 + \cdots + u_n - \lambda)^n dF(v) dF(u_1) \cdots dF(u_n) \\ = V^{n+1} (x - \lambda)^n = P^{-n-1} (x - \lambda)^n = r(n + 1; x - \lambda), \end{aligned}$$

which proves (8).

The relation (10) solves the following combinatorial problem. We want to construct  $s$  labeled necklaces of prescribed positive integer lengths  $a_1, \dots, a_s$  from distinguishable beads with positive integer lengths. There are  $n$  beads of lengths  $\lambda_i \geq 2, i \in [n]$  and  $a - \lambda$  beads of length 1, where  $a = a_1 + \cdots + a_s, \lambda = \lambda_1 + \cdots + \lambda_n$ , so that the sum of the lengths of the beads fits exactly.

A necklace does not change by circular permutation. In how many ways can this be done? We impose the restrictive condition

$$\lambda \leq a_i, \quad i = 1, \dots, s. \tag{16}$$

Without this condition the problem seems to become far more difficult.

First divide the  $n$  beads with lengths  $\lambda_i \geq 2$  into  $s$  subsets  $T_1, \dots, T_s$  and assign  $T_i$  to the  $i$ th necklace. Distributing the  $a - \lambda$  remaining beads over the necklaces and ordering the beads in the necklaces then can be done in

$$\begin{aligned} (a - \lambda)! \prod_{i=1}^s \{ (a_i - \lambda(T_i) + t_i - 1)! / (a_i - \lambda(T_i))! \} \\ = (a - \lambda)! \prod_{i=1}^s (a_i + 1 - \lambda(T_i))^{(t_i - 1)} \end{aligned}$$

ways with  $t_i = |T_i|$  and  $\lambda(T_i)$  given by (1), noting that  $(x + 1)^{(-1)} = x^{-1}$ .

Summing over all ordered partitions  $(T_1, \dots, T_s)$  of  $[n]$  shows with (10) that the number of ways is

$$(a - \lambda)! a(a + 1 - \lambda)^{(n-1)} (a_1 \cdots a_s)^{-1} = a(a - \lambda + n - 1)! (a_1 \cdots a_s)^{-1}.$$

This relation will be used elsewhere to solve a problem on cycles of a random permutation.

If the  $s$ th necklace is replaced by a string of beads of length  $a_s$ , we find by (11) that, again under (16), the number of ways of distributing the beads is  $(a - \lambda + n)! (a_1 \cdots a_{s-1})^{-1}$ . When two or more necklaces are replaced by strings there does not seem to be a simple answer. This corresponds to the fact that replacing more than one factor  $x_i r(t_i; x_i - \lambda(T_i))$  in (7) by  $r(t_i + 1; x_i - \lambda(T_i))$  does not result in a formula like (8), as was already remarked by Hurwitz [2] for (3) and (4).

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