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Two Identities in the Theory of Polynomials of Binomial Type

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1. INTRODUCTION

Let $\lambda_1,..., \lambda_n, x_1,..., x_n$ be real numbers,

$$\lambda(T) = \sum_{i \in T} \lambda_i, \qquad T \subset [n] = \{1,..., n\} \tag{1}$$

so that $\lambda(\phi) = 0$ and

$$\lambda = \lambda([n]) = \lambda_1 + \cdots + \lambda_n, x = x_1 + \cdots + x_n. \tag{2}$$

Hurwitz [2] by elementary analytical methods proved the identities

$$\sum \prod_{i=1}^{s} x_i(x_i - \lambda(T_i))^{t_i - 1} = x(x - \lambda)^{n-1}, \tag{3}$$

$$\sum (x_s - \lambda(T_s))^{t_s} \prod_{i=1}^{s-1} x_i(x_i - \lambda(T_i))^{t_i - 1} = (x - \lambda)^n, \tag{4}$$

where the sums are over all ordered partitions $(T_1,..., T_s)$ of $[n]$ with $T_i = \phi$ admitted and $t_i = |T_i|$. Françon [1] proved (3) for $s = 2$ by combinatorial methods. For $s = 2$ and $\lambda_i = \lambda$ the relations (3) and (4) reduce to forms of Abel's generalized binomial formula.

We extend these results by applying the theory of polynomials of binomial type developed in [3] and [4] from which we will cite freely. Let $Q = DP$ be a delta operator on the set of all real polynomials with $D = d/dx$ and P an invertible shift-invariant operator that maps any polynomial into a polynomial of the same degree. Shift-invariance means $E^a P = PE^a$, where

$$E^a f(x) = f(x + a). \tag{5}$$

Shift-invariant operators commute. To Q belongs a unique sequence of

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basic polynomials q_n, $n \in \mathbb{N}$, defined by $q_0(x)=1$, $q_n(0)=0$, $Qq_n = nq_{n-1}$, $n \geqslant 1$. The polynomial q_n has degree n and the sequence $\{q_n\}$ is of binomial type. The explicit form of the q_n is given by theorem 4 in [3] and [4]. We choose the representation

$$q_n(x) = xr_n(x), \qquad r_n(x) \equiv r(n; x) = P^{-n}x^{n-1}, \quad n \geqslant 1, \tag{6}$$

which also holds for $n=0$ if we allow $P^0 x^{-1} = Ix^{-1} = x^{-1}$ so that $r_0(x) = x^{-1}$. In Section 2 we prove the following generalization of (3) and (4).

THEOREM 1. *For $r(n; x)$ defined by* (6) *we have*

$$\sum \prod_{i=1}^{s} x_i r(t_i; x_i - \lambda(T_i)) = xr(n; x-\lambda), \tag{7}$$

$$\sum r(t_s+1; x_s - \lambda(T_s)) \prod_{i=1}^{s-1} x_i r(t_i; x_i - \lambda(T_i)) = r(n+1; x-\lambda), \tag{8}$$

where the summations are as in (3) *and* (4) *and* $t_i = |T_i|$.

It is noted that $r_0(x - \lambda(\phi)) = x^{-1}$ since $\lambda(\phi) = 0$ so that both sides of (7) and (8) are polynomials of degree n in $x_1,..., x_n$. For $P = I$ we have $r_n(x) = x^{n-1}$ and (7), (8) reduce to (3), (4). For $Q = I - E^{-1}$ we have $q_n(x) = x^{(n)}$, where

$$x^{(0)} = 1, \; x^{(n)} = x(x+1)\cdots(x+n-1), \qquad n \geqslant 1. \tag{9}$$

Then $r_n(x) = (x+1)^{(n-1)}$, $n \geqslant 0$, if we define $(x+1)^{(-1)} = x^{-1}$ and (7) and (8) become

$$\sum \prod_{i=1}^{s} x_i(x_i + 1 - \lambda(T_i))^{(t_i-1)} = x(x+1-\lambda)^{(n-1)}, \tag{10}$$

$$\sum (x_s + 1 - \lambda(T_s))^{(t_s)} \prod_{i=1}^{s-1} x_i(x_i + 1 - \lambda(T_i))^{(t_i-1)} = (x+1-\lambda)^{(n)}. \tag{11}$$

A combinatorial application of (10) and (11) will be given in Section 2.

2. PROOF AND APPLICATION

First, we prove (7). From (6), since P^{-1} also is shift-invariant,

$$r(t_i; x_i - \lambda(T_i)) = E^{-\lambda(T_i)}P^{-t_i}x_i^{t_i-1} = P^{-t_i}E^{-\lambda(T_i)}x_i^{t_i-1} = P^{-t_i}(x_i - \lambda(T_i))^{t_i-1}. \tag{12}$$

By Theorem 2 in [3] or [4] we have

$$P^{-1} = \sum_{k=0}^{\infty} a_k D^k/k!.$$

Since $(x_i - \lambda(T_i))^{t_i - 1}$ has degree smaller than n, we may replace P^{-1} in (12) by

$$V = \sum_{k=0}^{\infty} b_k D^k/k!,$$

if $b_k = a_k$, $k \leqslant n$. For V we take

$$Vf(x) = \int f(x+u)\, dF(u), \tag{13}$$

where $F\colon \mathbb{R} \to \mathbb{R}$ is of bounded variation. We may choose F so that

$$b_k = \int u^k\, dF(u) = a_k, \qquad k = 0,..., n. \tag{14}$$

It is sufficient to take F constant except for $n+1$ jumps of magnitudes $\eta_0,..., \eta_n$. The coefficient determinant of Eqs. (14) in $\eta_0,..., \eta_n$ then is Vandermonde's. If $T_i = \{j_1,..., j_h\}$ we have from (12) and (13),

$$r(t_i; x_i - \lambda(T_i)) = \int \cdots \int (x_i + u_{j_1} + \cdots + u_{j_h} - \lambda(T_i))^{h-1}\, dF(u_{j_1}) \cdots dF(u_{j_h}). \tag{15}$$

If $T_i = \phi$ we have $r(t_i; x_i - \lambda(T_i)) = x_i^{-1}$. Substituting this into the left-hand side of (7), applying Fubini's theorem to the integrations, interchanging sum and integral and applying (3) with λ_i replaced by $\lambda_i - u_i$ shows that the left-hand side of (7) is equal to

$$\begin{aligned} x \int \cdots \int & (x + u_1 + \cdots + u_n - \lambda)^{n-1}\, dF(u_1) \cdots dF(u_n) \\ &= xV^n(x-\lambda)^{n-1} = xP^{-n}(x-\lambda)^{n-1} = xP^{-n}E^{-\lambda}x^{n-1} \\ &= xE^{-\lambda}P^{-n}x^{n-1} = xr(n; x-\lambda), \end{aligned}$$

which proves (7).

In the proof of (8) we have (15) for $i \leqslant s-1$ and, if $T_s = (i_1, ..., i_k)$, in the same way as above,

$$\begin{aligned} r(t_s+1; x_s - \lambda(T_s)) &= E^{-\lambda(T_s)} P^{-k-1} x_s^k = P^{-k-1}(x_s - \lambda(T_s))^k = V^{k+1}(x_s - \lambda(T_s))^k \\ &= \int \cdots \int (x_s + v + u_{i_1} + \cdots + u_{i_k} - \lambda(T_s))^k \, dF(v) \, dF(u_{i_1}) \cdots dF(u_{i_k}). \end{aligned}$$

Substituting this, together with (15), into the left-hand side of (8), applying Fubini's theorem, interchanging sum and integral and applying (4) with x_s replaced by $x_s + v$ and λ_i by $\lambda_i - u_i$ shows that the left-hand side of (8) is equal to

$$\begin{aligned} &\int \cdots \int (x + v + u_1 + \cdots + u_n - \lambda)^n \, dF(v) \, dF(u_1) \cdots dF(u_n) \\ &\quad = V^{n+1}(x-\lambda)^n = P^{-n-1}(x-\lambda)^n = r(n+1; x-\lambda), \end{aligned}$$

which proves (8).

The relation (10) solves the following combinatorial problem. We want to construct s labeled necklaces of prescribed positive integer lengths $a_1, ..., a_s$ from distinguishable beads with positive integer lengths. There are n beads of lengths $\lambda_i \geqslant 2$, $i \in [n]$ and $a - \lambda$ beads of length 1, where $a = a_1 + \cdots + a_s$, $\lambda = \lambda_1 + \cdots + \lambda_n$, so that the sum of the lengths of the beads fits exactly.

A necklace does not change by circular permutation. In how many ways can this be done? We impose the restrictive condition

$$\lambda \leqslant a_i, \qquad i = 1, ..., s. \tag{16}$$

Without this condition the problem seems to become far more difficult.

First divide the n beads with lengths $\lambda_i \geqslant 2$ into s subsets $T_1, ..., T_s$ and assign T_i to the ith necklace. Distributing the $a - \lambda$ remaining beads over the necklaces and ordering the beads in the necklaces then can be done in

$$\begin{aligned} &(a-\lambda)! \prod_{i=1}^{s} \{(a_i - \lambda(T_i) + t_i - 1)!/(a_i - \lambda(T_i))!\} \\ &\quad = (a-\lambda)! \prod_{i=1}^{s} (a_i + 1 - \lambda(T_i))^{(t_i - 1)} \end{aligned}$$

ways with $t_i = |T_i|$ and $\lambda(T_i)$ given by (1), noting that $(x+1)^{(-1)} = x^{-1}$.

Summing over all ordered partitions $(T_1, \ldots, T_s)$ of $[n]$ shows with (10) that the number of ways is

$$(a-\lambda)!\, a(a+1-\lambda)^{(n-1)}(a_1 \cdots a_s)^{-1} = a(a-\lambda+n-1)!\, (a_1 \cdots a_s)^{-1}.$$

This relation will be used elsewhere to solve a problem on cycles of a random permutation.

If the sth necklace is replaced by a string of beads of length a_s, we find by (11) that, again under (16), the number of ways of distributing the beads is $(a-\lambda+n)!\ (a_1 \cdots a_{s-1})^{-1}$. When two or more necklaces are replaced by strings there does not seem to be a simple answer. This corresponds to the fact that replacing more than one factor $x_i r(t_i; x_i - \lambda(T_i))$ in (7) by $r(t_i+1; x_i-\lambda(T_i))$ does not result in a formula like (8), as was already remarked by Hurwitz [2] for (3) and (4).

References

1. J. Françon, Preuves combinatoires des identités d'Abel, *Discrete Math.* **8** (1974), 331–343.
2. A. Hurwitz, Uber Abel's Verallgemeinerung der binomischen Formel, *Acta Math.* **26** (1902), 199–203.
3. R. Mullin and G.-C. Rota, On the foundations of combinatorial theory: III. Theory of binomial enumeration. *In* "Graph Theory and Its Applications," (B. Harris, Ed.), pp. 167–213, Academic Press, New York, 1970.
4. G.-C. Rota, D. Kahaner, and A. Odlyzko, On the foundations of combinatorial theory. VIII. Finite operator calculus, *J. Math. Anal. Appl.* **42** (1973), 684–760.