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INDEPENDENT POISSON PROCESSES GENERATED BY RECORD VALUES AND INTER-RECORD TIMES

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The mth-order upper record values of a sequence of independent random variables with common continuous distribution function, that are kth but not $(k-1)$th-order record values and that precede inter-record times of length j, form a Poisson process, the processes for different (k, j) being independent, $k=1,\ldots,m$, $j=1,2,\ldots$. The records with record epochs after $r>m$, have a similar property if we condition with respect to the mth decreasing order statistic of the sample for times $1,\ldots,r$. These results extend theorems by Ignatov.

records * Poisson process * order statistics * point process * independence

1. Introduction, notations

Let $X(1), X(2), \ldots$ be independent random variables having common continuous distribution function F with

$$(a, b)=\{x: 0<F(x)<1\}, \tag{1.1}$$

where $-\infty \leq a<b \leq \infty$. Since F is continuous, removing a null set gives

$$X(i) \neq X(j), \quad i \neq j. \tag{1.2}$$

Let $X_1(n), \ldots, X_n(n)$ be the decreasing order statistics of $X(1), \ldots, X(n)$. For $n \geq m$, where $m \geq 1$ is fixed throughout this paper unless stated otherwise, we define

$$N_m(1)=m, \tag{1.3}$$

$$N_m(n+1)=\min\{k>N_m(n): X(k)>X_m(N_m(n))\}, n \geq 1, \tag{1.4}$$

$$\nu_m(n)=N_m(n+1)-N_m(n), \tag{1.5}$$

$$Y_h(n)=X_h(N_m(n)), \quad h=1,\ldots,m, n \geq 1, \tag{1.6}$$

where the dependence on m is suppressed. We call the $N_m(n)$, $n=1,2,\ldots$, the mth-order (upper) record epochs, the $\nu_m(n)$ the inter-record times and the $Y_m(n)$ the mth-order record values. By (1.4) the $N_m(n)$ are the jump epochs of the Markov chain $\{X_m(n), n \geq m\}$, see Lamperti [6]. Equivalently, by (1.2), the $N_m(n)$ are the epochs at which the sets $\{X_1(n), \ldots, X_m(n)\}$, $n \geq m$, change. It should be emphasized that we may have $X(N_m(n)) \neq X_m(N_m(n))$. Then $Y_m(n) = X_m(N_m(n)) = X(i)$ with $i < N_m(n)$ and $N_m(n)$ is also a kth-order record epoch and $Y_m(n)$ a kth-order record value for one or more $k < m$. Let $S_0 = \emptyset$ and $S_r = \{X_r(N_r(n)): n \geq 1\}$, $r \geq 1$, so that S_r is the set of rth-order record values. Also $S_r = \{X_r(N_s(n)): n \geq 1\}$, $s > r \geq 1$, since any rth-order record epoch is a sth-order record epoch and $X_r(j)$ with $j \geq r$ is a rth-order record value. We have $X_r(N_r(n)) = X_{r+1}(N_r(n+1))$ and $N_r(n)$ is a $(r+1)$th-order record epoch, since at time $N_r(n+1)$ the set of the highest $r+1$ order statistics of $\{X(i)\}$ changes. So

$$S_r \subset S_{r+1}, \quad r = 1, 2, \ldots \tag{1.7}$$

A countable random point set $S \subset \mathbb{R}^m$ defines a point process as the family of random variables $|S \cap A|$, $A \in \mathscr{B}(\mathbb{R}^m)$, where $\mathscr{B}(E)$ denotes the class of Borel sets of E. Ignatov [7] proved the curious result that the random sets $S_i - S_{i-1}$ define independent point processes on (a, b), each with intensity (or expectation) measure M given by

$$M((a, x)) = -\log(1 - F(x)), \quad a < x < b. \tag{1.8}$$

The set $S_r - S_{r-1}$ consists of those rth-order record values that are not jth-order values, $j < r$. That S_1 defines a Poisson process was proved by Dwass [3] and Shorrock [11]. Resnick [10], Shorrock [12] and Ignatov [8] also proved that the sets of first-order record values with $\nu_1(n) = j$ for different $j = 1, 2, \ldots$ form independent Poisson processes with intensity measures having distribution functions $j^{-1}F^j(x)$.

In Section 3 we extend this result to mth-order records, at the same time combining it with Ignatov's first theorem. We use a different method of proof, referring directly to the record values and derive some new independence properties. The random point set $\{Y_m(i), i = 1, \ldots, n-1, Y_j(n), j = m, \ldots, 1\}$ is the same as $\{X(1), \ldots, X(m), X(N_m(2)), \ldots, X(N_m(n))\}$ and the order in time in which these X-values appear, determines the distribution of them over the sets $S_i - S_{i-1}$, $i = 1, \ldots, m$. It will be shown that this order of appearance and the random vector $(Y_m(1), \ldots, Y_m(n-1), Y_m(n), \ldots, Y_1(n))$ are independent.

The processes of higher-order record values have been the subject of independent research by several authors. After the first version of this paper was submitted, the author received the publication [5] by Goldie and [6] by Goldie and Rogers, giving still different proofs of the combination of Ignatov's results, putting it into a wider context and also containing some results on noncontinuous F. Still another proof of Ignatov's first theorem appeared in Deheuvels [2].

We need some considerations on Poisson processes that may be useful in their own right and are given in Section 2. Section 4 studies record values with record epochs greater than a fixed time r. The point processes then are not Poisson but conditionally Poisson given $X_m(r)$.

Some notation: $\mathbb{N}_1=\{1,2,\ldots\}$, $E_m=\{1,\ldots,m\}$,

$$G_n=\{x\in\mathbb{R}^n\colon a<x_1<\cdots<x_n<b\}. \tag{1.9}$$

For intervals $A_i=[c_i,d_i]\subset(a,b)$ we have $A_1\times\cdots\times A_n\subset G_n$ if and only if $a<c_1<d_1<\cdots<c_n<d_n<b$ and then write

$$A_1<A_2<\cdots<A_n. \tag{1.10}$$

For measures Q and R on $\mathbb{R}^n$ the notation $Q(\mathrm{d}x_1\cdots\mathrm{d}x_n)=f(x_1,\ldots,x_n)R(\mathrm{d}x_1\cdots\mathrm{d}x_n)$ or $\mathrm{d}Q(x)=f(x)\,\mathrm{d}R(x)$ means

$$Q(A)=\int_A f(x)\,\mathrm{d}R(x),\quad A\in\mathcal{B}(\mathbb{R}^n). \tag{1.11}$$

If Q is the joint probability distribution of random variables $Z_1,\ldots,Z_n$ we write $P(Z_i\in\mathrm{d}x_i,\ i\in E_n)$ for $Q(\mathrm{d}x_1\cdots\mathrm{d}x_n)$ in the above convention.

2. Poisson processes

Consider a Poisson process with intensity measure Λ on $T\times D$, where $T=(a,b)$ with $-\infty\leqslant a<b\leqslant\infty$ and $D\in\mathcal{B}(\mathbb{R}^k)$.

Putting

$$\Lambda_1(A)=\Lambda(A\times D),\quad A\in\mathcal{B}(T), \tag{2.1}$$

we assume

$$\Lambda(T\times D)=\infty, \tag{2.2}$$

$$\Lambda_1((a,x))<\infty,\quad a<x<b, \tag{2.3}$$

$$\Lambda_1(\{c\})=0,\quad a<c<b. \tag{2.4}$$

It follows that the random vectors $(Z(i),V(i))$, $i=1,2,\ldots$ are defined a.s. uniquely by taking them to be the coordinates of the points of the Poisson process so that

$$Z(i)\in T,\qquad V(i)\in D,\qquad Z(i)<Z(i+1),\qquad i\in\mathbb{N}_1. \tag{2.5}$$

We see that the joint probability distribution of $(Z(i),V(i))$, $i\in E_n$ is restricted to $G_n\times D^n$, see (1.9), and that for $z\in G_n$, $v\in D^n$

$$P(Z(i)\in\mathrm{d}z_i,\ V(i)\in\mathrm{d}v_i,\ i\in E_n)$$
$$=\exp\{-\Lambda_1((a,z_n))\}\,\mathrm{d}\Lambda(z_1,v_1)\cdots\mathrm{d}\Lambda(z_n,v_n). \tag{2.6}$$

This gives, for $z\in T$, $v\in D$,

$$P(Z(1)\in\mathrm{d}z,\ V(1)\in\mathrm{d}v)=\exp\{-\Lambda_1((a,z))\}\,\mathrm{d}\Lambda(z,v), \tag{2.7}$$

$$P(Z(1) > z) = \exp\{-\Lambda_1((a, z))\}, \tag{2.8}$$

$$P(Z(i) \in \mathrm{d}z_i,\ V(i) \in \mathrm{d}v_i,\ i \in E_n) = \prod_{i=1}^{n} P(Z(1) \in \mathrm{d}z_i,\ V(1) \in \mathrm{d}v_i) \prod_{i=1}^{n-1} P(Z(1) > z_i). \tag{2.9}$$

The following lemma is a simple special case of Theorem 5.13 in Çinlar [1].

Lemma 1. *Let* $(Z(i), V(i))$, $i \in \mathbb{N}_1$ *be a stochastic process with* $Z(i) \in T$, $V(i) \in D$, $Z(i) < Z(i+1)$, $i \in \mathbb{N}_1$ *and* $P(Z(1) = x) = 0$, $P(Z(1) > x) > 0$, $x \in T$. *The points of this process form a Poisson process on* $T \times D$ *with intensity measure* Λ *satisfying* (2.2)-(2.4) *if and only if* (2.9) *holds,* $n \in \mathbb{N}_1$, *and then* Λ *is given by*

$$\mathrm{d}\Lambda(z, v) = P(Z(1) \in \mathrm{d}z,\ V(1) \in \mathrm{d}v)/P(Z(1) > z), \quad z \in T,\ v \in D. \tag{2.10}$$

Proof. Necessity follows from the computations above. For sufficiency define the measures Λ on $T \times D$ by (2.10) and Λ_1 by (2.1). Then

$$\exp\{-\Lambda_1((a, z))\} = P(Z(1) > z), \quad z \in T. \tag{2.11}$$

It follows that Λ satisfies (2.2)-(2.4) and that (2.6) holds. Since the probability law of the point process and the probability law of the process $(Z(i), V(i))$ determine each other, the assertion of the lemma follows.

Remark. Since the restrictions of a Poisson process to disjoint sets are independent Poisson processes, the points $Z(i)$ in T with $V(i) \in D_k$ form independent Poisson processes in T if the D_k are disjoint subsets of D. The $V(i)$ are independent with common probability distribution P_V and independent of $\{Z(j), j \in \mathbb{N}_1\}$ if and only if Λ is the product measure of Λ_1 and P_V, as is seen from (2.10) and (2.9). The process $(Z(i), V(i))$ then is called a marked Poisson process on T, i.e. the $Z(i)$ are marked independently with $V(i)$. This shows that Ignatov's theorems may be formulated in terms of independent Poisson processes but also in terms of a single Poisson process on a product space. The first theorem states that $(Y_m(n), \sigma_m(n))$, $n \in \mathbb{N}_1$, is a marked Poisson process, with $\sigma_m(n) = k$ if $Y_m(n) \in S_k - S_{k-1}$.

Taking $D = \{0\}$ we see from Lemma 1 that the $Z(i)$ form a Poisson process if and only if

$$P(Z(i) \in \mathrm{d}z_i,\ i \in E_n) = \prod_{i=1}^{n} P(Z(1) \in \mathrm{d}z_i) \Big/ \prod_{i=1}^{n-1} P(Z(1) > z_i).$$

This means that the $Z(i)$ have the distribution of a process of first order record values. The proof given in Shorrock [8] used a similar reasoning. Lemma 1 shows that a process $\{(Z(i), V(i)), i \in \mathbb{N}_1\}$ defines a Poisson process on $T \times D$ if and only if $\{Z(i)\}$ has the distribution of an upper record value process and the dependence between $\{Z_i\}$ and $\{V_j\}$ has the special form following from (2.9).

Remark. The mth-order record values defined in (1.6) have the same joint distribution as the first order record values of independent $X^*(i)$ with $P(X^*(i)<x)=1-(1-F(x))^m$. This was proved by Dwass [4], Stam [13] and it will follow from results in Section 3.

3. The *m*th-order record process

Let $\{a_1,\ldots,a_h\}$ be a set of h different real numbers. Then a_k has rank i in this set if a_k is the ith smallest of $a_1,\ldots,a_h$. By (1.2), (1.4) and (1.6) the random variables $\tau_k(n)$, $k\in E_m$, $n\in\mathbb{N}_1$ are defined uniquely by

$$Y_k(n)=X(\tau_k(n)). \tag{3.1}$$

For fixed m we define $\theta^{(n)}(i)$ as the rank of $\tau_m(i)$ and $\theta_j^{(n)}$ as the rank of $\tau_j(n)$ in the set $\{\tau_m(1),\tau_m(2),\ldots,\tau_m(n-1),\tau_m(n),\tau_{m-1}(n),\ldots,\tau_1(n)\}$, $i\in E_{n-1}$, $j\in E_m$. If the value of n is understood, the superscript is omitted. The random vector

$$\theta=(\theta(1),\ldots,\theta(n-1),\theta_m,\ldots,\theta_1) \tag{3.2}$$

specifies the order of appearance in time of $Y_m(1),\ldots, Y_m(n-1), Y_m(n),\ldots, Y_1(n)$. We have $\theta\in\Theta$ where Θ is the set of all sequences (3.2) of $m+n-1$ different positive integers satisfying

$$1\leqslant\theta(i)\leqslant m+i-1,\quad i\in E_{n-1}, \tag{3.3}$$

$$1\leqslant\theta_j\leqslant m+n-1,\quad j\in E_m, \tag{3.4}$$

so that $\theta_j=m+n-1$ for some j. All $m!\,m^{n-1}$ elements of Θ are possible. This is seen from the identities between point sets

$$\{Y_m(1),\ldots, Y_1(1)\}=\{X(1),\ldots,X(m)\}, \tag{3.5}$$

$$\{Y_m(1), Y_m(2), Y_{m-1}(2),\ldots, Y_1(2)\}=\{X(1),\ldots,X(m),X(N_m(2))\}, \tag{3.6}$$

$$\begin{aligned}\{Y_m(1),\ldots, Y_m(n-1), Y_m(n), Y_{m-1}(n),\ldots, Y_1(n)\}\\ =\{X(1),\ldots,X(m),X(N_m(2)),\ldots,X(N_m(n))\}.\end{aligned} \tag{3.7}$$

Theorem 1. *The random vectors θ and $(Y, X(N_m(n+1)), \nu_m(1),\ldots,\nu_m(n))$, with*

$$Y=(Y_m(1),\ldots, Y_m(n-1), Y_m(n),\ldots, Y_1(n))\in G_{n+m-1}$$

are independent with $P(\theta=\alpha)=(m!m^{n-1})^{-1}$, $\alpha\in\Theta$ *and*

$$P(Y_m(i)\in \mathrm{d}y_i,\ \nu_m(i)=h_i,\ i\in E_n,\ Y_j(n)\in \mathrm{d}z_j, j\in E_{m-1})$$

$$=m!m^{n-1}\left\{\prod_{i=1}^{n-1} F^{h_i-1}(y_i)\,\mathrm{d}F(y_i)\right\}$$

$$\times(1-F(y_n))F^{h_n-1}(y_n)\,\mathrm{d}F(y_n)\prod_{j=m-1}^{1}\mathrm{d}F(z_j),$$

$$(y_1,\ldots,y_n,z_{m-1},\ldots,z_1)\in G_{n+m-1}. \tag{3.8}$$

Proof. From (3.7) we see that Y and θ specify the values of $X(1),\ldots,X(m)$, $X(N_m(2)),\ldots,X(N_m(n))$. Prescribing $\nu_m(1),\ldots,\nu_m(n)$ fixes the values of $N_m(1),\ldots,N_m(n+1)$. We must have $X(k)<X_m(N_m(i))$ for $N_m(i)<k<N_m(i+1)$. This gives, for intervals $A_1<\cdots<A_n<B_{m-1}<\cdots<B_1$ and $C>A_n$, see (1.10),

$$P(\theta=\alpha,\ Y_m(i)\in A_i,\ \nu_m(i)=h_i,\ i\in E_n,\ Y_j(n)\in B_j,$$

$$j\in E_{m-1}, X(N_m(n+1))\in C)$$

$$=\left\{\prod_{i=1}^{n}\int_{A_i} F^{h_i-1}(y_i)\,\mathrm{d}F(y_i)\right\}\left\{\prod_{j=1}^{m-1} P(X(1)\in B_j)\right\}P(X(1)\in C). \tag{3.9}$$

So, for $(y_1,\ldots,y_n,z_{m-1},\ldots,z_1)\in G_{n+m-1}$ and $x>y_n$,

$$P(\theta=\alpha,\ Y_m(i)\in \mathrm{d}y_i,\ \nu_m(i)=h_i,\ i\in E_n,\ Y_j(n)\in \mathrm{d}z_j,$$

$$j\in E_{m-1}, X(N_m(n+1))\in \mathrm{d}x)$$

$$=\prod_{i=1}^{n} F^{h_i-1}(y_i)\,\mathrm{d}F(y_i)\cdot\prod_{j=m-1}^{1}\mathrm{d}F(z_j)\cdot\mathrm{d}F(x). \tag{3.10}$$

The right-hand side of (3.10) does not depend on α, so θ and $(Y, X(N_m(n+1)), \nu_m(1),\ldots,\nu_m(n))$ are independent and $P(\theta=\alpha)=(m!m^{n-1})^{-1}$. Summation with respect to α and integration with respect to x over (y_n, b) in (3.10) give (3.8).

Remark. By integration in (3.8) with respect to $z_1,\ldots,z_{m-1}$ and then summing over the h_i we find

$$P(Y_m(i)\in \mathrm{d}y_i,\ \nu_m(i)=h_i,\ i\in E_n)$$

$$=m^n(1-F(y_n))^m\prod_{i=1}^{n} F^{h_i-1}(y_i)\,\mathrm{d}F(y_i),\quad y\in G_n, \tag{3.11}$$

$$P(Y_m(i)\in \mathrm{d}y_i,\ i\in E_n)$$

$$=\mathrm{d}F_m(y_1)\cdots\mathrm{d}F_m(y_n)\Big/\prod_{i=1}^{n-1}(1-F_m(y_i)),\quad y\in G_n. \tag{3.12}$$

with $F_m(x)=1-(1-F(x))^m$. This proves the last remark in Section 2.

For m fixed define the random variables $\sigma(n)$, $n \in \mathbb{N}_1$ by

$$\sigma(n) = k \text{ if } Y_m(n) \in S_k - S_{k-1}, \quad k \in E_m, \tag{3.13}$$

i.e. if $Y_m(n)$ is a kth-order but not a jth-order record value, $j < k$, see (1.7). The next theorem shows that the process $\{\sigma(n)\}$ is not only independent of the mth-order record values but also of the mth-order record epochs.

Theorem 2. *The random vectors* $(\sigma(1), \ldots, \sigma(n))$ *and* $(Y_m(1), \ldots, Y_m(n), \nu_m(1), \ldots, \nu_m(n))$ *are independent, the latter with distribution given by* (3.12). *The* $\sigma(i)$ *are independent with* $P(\sigma(i) = k) = m^{-1}$, $k \in E_m$, $i \in E_n$.

Proof. Let $\xi_h(n)$ be the rank of $\theta_h^{(n)}$ in the set $\{\theta_1^{(n)}, \ldots, \theta_m^{(n)}\}$ so that the vector $\xi(n) = (\xi_1(n), \ldots, \xi_m(n))$ specifies the order of appearance of $Y_1(n), \ldots, Y_m(n)$. Since the latter are the first m order statistics of $X(i)$, $1 \leq i \leq N_m(n)$ we have

$$\sigma(n) = \xi_m(n). \tag{3.14}$$

The vector $\theta^{(n)}$ determines $\theta^{(n-1)}$, in fact the latter is derived from $\theta^{(n)}$ by deleting the component that has the value $n + m - 1$ and corresponds to the 'new' value $X(N_m(n))$ at time $N_m(n)$. It follows that $\theta^{(n)}$ determines $\theta^{(j)}$, $j = 1, \ldots, n-1$, and (3.14) then shows that $(\sigma(1), \ldots, \sigma(n))$ is a function of $\theta^{(n)}$. The first assertion of Theorem 2 now follows from Theorem 1.

In Theorem 1 it was proved that the distribution of θ is uniform on Θ defined in (3.3) and (3.4). This implies

$$P(\xi_k(n) = i_k, k \in E_m) = 1/m! \tag{3.15}$$

for any permutation $(i_1, \ldots, i_m)$ of $(1, \ldots, m)$. This proves with (3.14) that $P(\sigma(n) = k) = m^{-1}$, $k \in E_m$.

Now let $\varepsilon_k(n)$, $k \in E_{m-1}$, be the rank of $\xi_k(n)$ in the set $\{\xi_1(n), \ldots, \xi_{m-1}(n)\}$ or, equivalently, the rank of $\theta_k^{(n)}$ in $\{\theta_1^{(n)}, \ldots, \theta_{m-1}^{(n)}\}$, so that the vector $\varepsilon(n) = (\varepsilon_1(n), \ldots, \varepsilon_{m-1}(n))$ specifies the relative order of appearance of $Y_1(n), \ldots, Y_{m-1}(n)$. We show by induction on n that $\xi_m(n)$, $\varepsilon(n)$ and $(\sigma(1), \ldots, \sigma(n-1))$ are independent, which by (3.14) implies the independence of the $\sigma(i)$. From (3.15) it follows that $\xi_m(1)$ and $\varepsilon(1)$ are independent. For the step $n \to n+1$ we note that $\xi(n+1)$ is a function of $\varepsilon(n)$ and $Z = (Y_{m-1}(n), \ldots, Y_1(n), X(N_m(n+1)))$, viz.

$$\xi(n+1) = (\varepsilon_1(n), \ldots, \varepsilon_{m-1}(n), m) \text{ if } X(N_m(n+1) < Y_{m-1}(n), \tag{3.16a}$$

$$\xi(n+1) = (\varepsilon_1(n), \ldots, \varepsilon_i(n), m, \varepsilon_{i+1}(n), \ldots, \varepsilon_{m-1}(n))$$

$$\text{if } Y_{i+1}(n) < X(N_m(n+1)) < Y_i(n), \quad 1 \leq i \leq m-2, \tag{3.16b}$$

$$\xi(n+1) = (m, \varepsilon_1(n), \ldots, \varepsilon_{m-1}(n)) \text{ if } X(N_m(n+1)) > Y_1(n). \tag{3.16c}$$

From Theorem 1, noting that $(\varepsilon(n), \sigma(1), \ldots, \sigma(n))$ is a function of $\theta^{(n)}$, we see that Z and $(\varepsilon(n), \sigma(1), \ldots, \sigma(n))$ are independent. By (3.14) and the induction assumption $\varepsilon(n)$ and $(\sigma(1), \ldots, \sigma(n))$ are independent. So Z, $\varepsilon(n)$ and $(\sigma(1), \ldots, \sigma(n))$ are independent and (3.16) shows that $\xi(n+1)$ and $(\sigma(1), \ldots, \sigma(n))$ are independent. But $\xi(n+1)$ and $(\xi_m(n+1), \varepsilon(n+1))$ determine each other completely, whereas $\xi_m(n+1)$ and $\varepsilon(n+1)$ are independent by (3.15). This proves the assertion for $n+1$.

Theorem 3. *The points $(Y_m(n), \nu_m(n), \sigma(n))$, $n \in \mathbb{N}_1$ form a Poisson process in $(a, b) \times \mathbb{N}_1 \times E_m$, with intensity measure Λ given by*

$$\Lambda((a, x) \times \{j\} \times \{k\}) = j^{-1} F^j(x), \quad a < x < b, \; j \in \mathbb{N}_1, \; k \in E_m. \tag{3.17}$$

Remark. From (3.17) and the remark following Lemma 1 we see that the $Y_m(n)$, $n \in \mathbb{N}_1$, with $\nu_m(n) = j$, $\sigma(n) = k$ for different (j, k) form independent Poisson processes on (a, b) with intensity measures Λ_{jk} given by $\Lambda_{jk}((a, x)) = j^{-1} F^j(x)$, $a < x < b$. Since this holds for any m, it implies the independence of the point processes $S_i - S_{i-1}$, $i \geq 1$. It also follows that the numbers of the mth-order inter-record times of length j are independent Poisson random variables with parameters mj^{-1}, $j = 1, 2, \ldots$.

Proof of Theorem 3. We apply Lemma 1 with $D = \mathbb{N}_1 \times E_m$. From Theorem 2 and (3.11)

$$P(Y_m(1) \in \mathrm{d}y, \nu_m(1) = h, \sigma(1) = k) = F^{h-1}(y)(1 - F(y))^m \, \mathrm{d}F(y), \tag{3.18}$$

$$P(Y_m(1) \in \mathrm{d}y) = m(1 - F(y))^{m-1} \, \mathrm{d}F(y),$$

$$P(Y_m(1) > y) = (1 - F(y))^m. \tag{3.19}$$

The relation (2.9) now becomes, by Theorem 2,

$$m^{-n} P(Y_m(i) \in \mathrm{d}y_i, \nu_m(i) = h_i, i \in E_n)$$
$$= \prod_{i=1}^{n} P(Y_m(1) \in \mathrm{d}y_i, \nu_m(1) = h_i, \sigma(1) = k_i) \Big/ \prod_{i=1}^{n} P(Y_m(1) > y_i),$$

and that it is satisfied, follows from (3.11), (3.18) and (3.19). From (2.10) (3.18) and (3.19)

$$\Lambda(\mathrm{d}y \times \{j\} \times \{k\}) = F^{j-1}(y) \, \mathrm{d}F(y)$$

and this gives (3.17).

4. The *m*th-order records after time *r*

For fixed $r \geq m$ let us denote the mth-order record epochs $N_m(n)$ with $N_m(n) > r$ as $N'_m(1) < N'_m(2) < \cdots$, and let us put $Y'_h(n) = X_h(N'_m(n)$, $\nu'_m(n) = N'_m(n+1) - N'_m(n)$ and $\sigma'(n) = k$ if $Y'_m(n) \in S_k - S_{k-1}$, $k \in E_m$.

For fixed m and n we define the random vector $\theta'=(\theta'(1),\dots,\theta'(n-1),\theta'_m,\dots,\theta'_1)$ with respect to $Y'_m(1),\dots,Y'_m(n),\dots,Y'_1(n)$ in the same way as θ in (3.2) with respect to $Y_m(1),\dots,Y_m(n),\dots,Y_1(n)$, so that θ' determines the order of appearance of $Y'_m(1),\dots,Y'_m(n),\dots,Y'_1(n)$. We may have $Y'_m(h)=X(i)$ or $Y'_k(n)=X(j)$ with $i\leq r$ or $j\leq r$. We have $\theta'\in\Theta$, with Θ defined by (3.3) and (3.4), all $\theta'\in\Theta$ being possible. This is seen from the identities between point sets, cf. (3.5)-(3.6):

$$\{Y'_m(1),\dots,Y'_1(1)\}=\{X_1(r),\dots,X_{m-1}(r),X(N'_m(1))\},$$

$$\begin{aligned}&\{Y'_m(1),Y'_m(2),Y'_{m-1}(2),\dots,Y'_1(2)\}\\&\quad=\{X_1(r),\dots,X_{m-1}(r),X(N'_m(1)),X(N'_m(2))\},\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\{Y'_m(1),Y'_m(2),\dots,Y'_m(n),Y'_{m-1}(n),Y'_{m-2}(n),\dots,Y'_1(n)\}\\&\quad=\{X_1(r),\dots,X_{m-1}(r),X(N'_m(1),\dots,X(N'_m(n))\}.\end{aligned}$$

We need the distribution function F_{mr} of $X_m(r)$. It is given by

$$\mathrm{d}F_{mr}(u)=r\binom{r-1}{m-1}(1-F(u))^{m-1}F^{r-m}(u)\,\mathrm{d}F(u).\tag{4.2}$$

Theorem 4. *Under the condition $X_m(r)=u$, $a<u<b$, the points $Y'_m(n)$, $\nu'_m(n)$, $\sigma'(n)$, $n\in\mathbb{N}_1$, form a Poisson process on $(u,b)\times\mathbb{N}_1\times E_m$ with intensity measure Λ_u given by*

$$\Lambda_u((u,x)\times\{j\}\times\{k\})=j^{-1}(F^j(x)-F^j(u)).\tag{4.3}$$

Remark. The theorem implies that either under $X_m(r)=u$ or unconditionally the points with $\sigma'(n)=k$ for different $k\in E_m$ form independent point processes, the points $(Y'_m(n),\nu'_m(n))$ being marked independently with $\sigma'(n)$ from the uniform distribution on E_m.

Proof of Theorem 4. Noting that $X_m(r)=X_m(N_m(h))=Y_m(h)$ where $N_m(h)$ is the last mth-order record epoch in $\{m,\dots,r\}$ and $Y'_m(i)=Y_m(h+i)$, $\sigma'(i)=\sigma(h+i)$, the statement in the remark follows from Theorem 2 by specifying h. Therefore we restrict our attention to the points $(Y'_m(n),\nu'_m(n))$ in $(a,b)\times\mathbb{N}_1$.

Still we begin the proof making use of the vector θ'. We have $(X_m(r),Y')\in G_{n+m}$, with $Y'=(Y'_m(1),\dots,Y'_m(n),Y'_{m-1}(n),\dots,Y'_1(n))$. From (4.1) we see that Y' and θ' specify the values of $X_1(r),\dots,X_{m-1}(r)$, $X(N'_m(1)),\dots,X(N'_m(n))$ and the relative order of appearance of $X_1(r),\dots,X_{m-1}(r)$. For $\tau_1,\dots,\tau_m$ with $X_i(r)=X(\tau_i)$, $i\in E_m$, then there are $r\binom{r-1}{m-1}$ possibilities. So for $A_0<A_1<\cdots<A_n<B_{m-1}<$

$\cdots < B_1$ and $\alpha' \in \Theta$ we have

$$P(X_m(r) \in A_0,\ N'_m(1) = r+k,\ Y'_m(i) \in A_i,\ \nu'_m(i) = h_i,$$
$$i \in E_n,\ Y'_j(n) \in B_j, j \in E_{m-1},\ \theta' = \alpha')$$
$$= r\binom{r-1}{m-1} \int_{A_0} F^{r-m+k-1}(u)\,\mathrm{d}F(u) \int_{A_n} F^{h_n-1}(y_n)(1-F(y_n))\,\mathrm{d}F(y_n)$$
$$\cdot \prod_{i=1}^{n-1} \int_{A_i} F^{h_i-1}(y_i)\,\mathrm{d}F(y_i) \prod_{j=1}^{m-1} P(X(1) \in B_j).$$

Summing over k and the $m!m^{n-1}$ values of α' we see that for $(u, y_1, \ldots, y_n, z_{m-1}, \ldots, z_1) \in G_{n+m}$

$$P(X_m(r) \in \mathrm{d}u,\ Y'_m(i) \in \mathrm{d}y_i,\ \nu'_m(i) = h_i,\ i = 1, \ldots, n,$$
$$Y'_j(n) \in \mathrm{d}z_j, j = m-1, \ldots, 1)$$
$$= m!m^{n-1} r\binom{r-1}{m-1} F^{r-m}(u)(1-F(u))^{-1}(1-F(y_n)) \prod_{i=1}^{n} F^{h_i-1}(y_i)$$
$$\cdot \mathrm{d}F(u)\,\mathrm{d}F(y_1)\cdots\mathrm{d}F(y_n)\,\mathrm{d}F(z_{m-1})\cdots\mathrm{d}F(z_1).$$

By integration with respect to $z_1, \ldots, z_{m-1}$ we find, for $a < u < y_1 \cdots < y_n < b$,

$$P(X_m(r) \in \mathrm{d}u,\ Y'_m(i) \in \mathrm{d}y_i,\ \nu'_m(i) = h_i,\ i \in E_n)$$
$$= m^n r\binom{r-1}{m-1} F^{r-m}(u)(1-F(u))^{-1}(1-F(y_n))^m$$
$$\cdot \prod_{i=1}^{n} F^{h_i-1}(y_i)\,\mathrm{d}F(u)\,\mathrm{d}F(y_1)\cdots\mathrm{d}F(y_n). \tag{4.4}$$

Denoting conditional probability given $X_m(r) = u$ by P_u, we see from (4.2) and (4.4) that for $u < y_1 < \cdots < y_n < b$

$$P_u(Y'_m(i) \in \mathrm{d}y_i,\ \nu'_m(i) = h_i,\ i \in E_n)$$
$$= m^n(1-F(u))^{-m}(1-F(y_n))^m \prod_{i=1}^{n} F^{h_i-1}(y_i)\,\mathrm{d}F(y_i). \tag{4.5}$$

We apply Lemma 1 with $T = (u, b)$, $D = E_m$. From (4.5), for $u < y < b$

$$P_u(Y'_m(1) \in \mathrm{d}y,\ \nu'_m(1) = j) = m(1-F(u))^{-m}(1-F(y))^m F^{j-1}(y)\,\mathrm{d}F(y), \tag{4.6}$$

$$P_u(Y'_m(1) > y) = (1-F(y))^m(1-F(u))^{-m}. \tag{4.7}$$

From (4.5), (4.6) and (4.7)

$$P_u(Y'_m(i) \in \mathrm{d}y_i,\ \nu'_m(i) = h_i,\ i \in E_n)$$
$$= \prod_{i=1}^{n} P_u(Y'_m(1) \in \mathrm{d}y_i,\ \nu'_m(1) = h_i) \Big/ \prod_{i=1}^{n-1} P_u(Y'_m(1) > y_i),$$

and this is the relation (2.9) for the process $(Y'_m(n), \nu'_m(n))$, $n \in \mathbb{N}$ under $X_m(r) = u$, so that it is a Poisson process on $(u, b) \times \mathbb{N}_1$ with intensity measure Λ_u given by (2.10) as

$$\begin{aligned} \Lambda_u(\mathrm{d}y \times \{j\}) &= mF^{j-1}(y)\,\mathrm{d}F(y), \\ \Lambda_u((u, x) \times \{j\}) &= mj^{-1}(F^j(x) - F^j(u)). \end{aligned} \tag{4.8}$$

Remark. It follows from Theorem 4 that the points $(Y'_m(n), \nu'_m(n))$, $n \in \mathbb{N}_1$, form a weighted Poisson process. In particular by (4.2) and (4.8) the numbers M_j, $j \in \mathbb{N}_1$, of mth-order record epochs $N_m(n) > r$ with $\nu_m(n) = j$, are weighted independent Poisson with

$$\begin{aligned} Ez_1^{M_1} \cdots z_k^{M_k} &= \int \mathrm{d}F_{mr}(u) \exp\left\{ \sum_{j=1}^{k} (z_j - 1) mj^{-1}(1 - F^j(u)) \right\} \\ &= r\binom{r-1}{m-1} \int_0^1 (1-t)^{m-1} t^{r-m} \exp\left\{ \sum_{j=1}^{k} (z_j - 1) mj^{-1}(1 - t^j) \right\} \mathrm{d}t. \end{aligned}$$

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