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Stable n-pointed trees of projective lines

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ABSTRACT

Stable n-pointed trees arise in a natural way if one tries to find moduli for totally degenerate curves:

Let C be a totally degenerate stable curve of genus $g \geq 2$ over a field k. This means that C is a connected projective curve of arithmetic genus g satisfying

(a) every irreducible component of C is a rational curve over k.
(b) every singular point of C is a k-rational ordinary double point.
(c) every nonsingular component L of C meets $\overline{C-L}$ in at least three points.

It is always possible to find g singular points $P_1, \ldots, P_g$ on C such that the blow up $\tilde{C}$ of C at $P_1, \ldots, P_g$ is a connected projective curve with the following properties:

(i) every irreducible component of $\tilde{C}$ is isomorphic to $\mathbb{P}_k^1$
(ii) the components of $\tilde{C}$ intersect in ordinary k-rational double points
(iii) the intersection graph of $\tilde{C}$ is a tree.

The morphism $\phi : \tilde{C} \to C$ is an isomorphism outside $2g$ regular points $Q_1, Q_1', \ldots, Q_g, Q_g'$ and identifies Q_i with Q_i'. ϕ is uniquely determined by the g pairs of regular k-rational points (Q_i, Q_i'). A curve C satisfying (i)-(iii) together with n k-rational regular points on it is called a n-pointed tree of projective lines. C is stable if on every component there are at least three points which are either singular or marked. The object of this paper is the classification of stable n-pointed trees. We prove in particular the existence of a fine moduli space B_n of stable n-pointed trees. The discussion above shows that there is a surjective map $\pi : B_{2g} \to D_g$ of B_{2g} onto the closed subscheme D_g of the coarse moduli scheme $\bar{M}_g$ of stable curves of genus g corresponding to the totally degenerate curves. By the universal property of $\bar{M}_g$, π is a (finite) morphism. π factors through $\bar{B}_{2g} = B_{2g}$ mod the action of the group of pair preserving permutations of $2g$ elements (a group of order $2^g g!$, isomorphic to a wreath product of S_g and $\mathbb{Z}/2\mathbb{Z}$).

The induced morphism $\bar{\pi} : \bar{B}_{2g} \to D_g$ is an isomorphism on the open subscheme of irreducible curves in D_g, but in general there may be nonequivalent choices of g singular points on a totally degenerated curve for the above construction, so $\bar{\pi}$ has nontrivial fibres. In particular, π is not the

quotient map for a group action on B_{2g}. This leads to the idea of constructing a Teichmüller space for totally degenerate curves whose irreducible components are isomorphic to B_{2g} and on which a discontinuous group acts such that the quotient is precisely D_g; π will then be the restriction of this quotient map to a single irreducible component. This theory will be developped in a subsequent paper.

In this paper we only consider stable n-pointed trees and their moduli theory. In § 1 we introduce the abstract cross ratio of four points (not necessarily on the same projective line) and show that for a field k the k-valued points in the projective variety B_n of cross ratios are in $1-1$ correspondence with the isomorphy classes of stable n-pointed trees of projective lines over k. We also describe the structure of the subvarieties $B(T, \psi)$ of stable n-pointed trees with fixed combinatorial type.

We generalize our notion in § 2 to stable n-pointed trees of projective lines over an arbitrary noetherian base scheme S and show how the cross ratios for the fibres fit together to morphisms on S. This section is closely related to [*Kn*], but it is more elementary since we deal with a special case.

§ 3 contains the main result of the paper: the canonical projection $B_{n+1} \rightarrow B_n$ is the universal family of stable n-pointed trees. As a by-product of the proof we find that B_n is a smooth projective scheme of relative dimension $2n-3$ over $\mathbb{Z}$. We also compare B_n to the fibre product $B_{n-1} \times_{B_{n-2}} B_{n-1}$ and investigate the singularities of the latter.

In § 4 we prove that the Picard group of B_n is free of rank

$$2^{n-1}-(n+1)-\frac{n(n-3)}{2}.$$

We also give a method to compute the Betti numbers of the complex manifold $B_n(\mathbb{C})$.

In § 5 we compare B_n to the quotient $Q_n := \mathbb{P}^n_{ss}/PGL_2$ of semi-stable points in $\mathbb{P}^n_1$ for the action of fractional linear transformations in every component. This orbit space has been studied in greater detail by several authors, see [GIT], [MS], [G]. It turns out that B_n is a blow-up of Q_n, and we describe the blow-up in several steps where at each stage the obtained space is interpreted as a solution to a certain moduli problem.

1. STABLE n-POINTED TREES OF PROJECTIVE LINES OVER A FIELD

(1.1) Let C be a connected projective variety over a field k and $\phi = (\phi_1, \ldots, \phi_n)$ be a n-tupel of distinct k-rational point of C.

DEFINITION. *The pair (C, ϕ) is called a stable n-pointed tree of projective lines over k if*

(1) *every component of C is isomorphic to the projective line over k*

(2) *every singular point of C is k-rational and an ordinary double point*

(3) *The intersection graph of the components of C is a tree*

(4) *The set*

$$\{\phi_1, \ldots, \phi_n\} \cup \{\textit{singular points of } C\}$$

has at least 3 *points on every component of C*

(5) *$\phi_1, \ldots, \phi_n$ are regular points on C.*

We call ϕ the marking of (C, ϕ).

(C, ϕ) and (C', ϕ') are isomorphic if there exists an isomorphism $\alpha: C \rightarrow C'$ such that $\alpha(\phi_i) = \phi_i'$ for all i. If (C, ϕ) and (C', ϕ') are isomorphic, this isomorphism is unique. Indeed let β be an automorphism of (C, ϕ). Then we have to show that β is the identity. Let L be an end component of C; this means that L meets only one other component or $L = C$. Now β must be the identity on L because it fixes at least three points. In the same way one shows that β is the identity on any component of C.

Let L be a component of C. There is a unique projection $\pi_L : C \to L$; π_L maps the components different from L to k-rational points of L.

Let $d = (d_1, d_2, d_3)$ be a triple of three different indices of $\underline{n} = \{1, 2, \ldots, n\}$ and let $D = D_n$ denote the set of all these triples. Then there is a unique component L_d of C such that $\pi_{L_d}(\phi_{d_1})$, $\pi_{L_d}(\phi_{d_2})$, $\pi_{L_d}(\phi_{d_3})$ are distinct. Thus one gets a unique morphism

$$\lambda_d : C \to \mathbb{P}_k^1$$

with $\lambda_d(\phi_{d_1}) = 0$, $\lambda_d(\phi_{d_2}) = \infty$, $\lambda_d(\phi_{d_3}) = 1$ which is an isomorphism on L_d and constant on all the other components of C. The component L_d is called the median component relative to the tripel d.

(1.2) Let T be a finite tree in the sense of graph theory. We will denote by T_0 the set of vertices of T and by T_1 the set of edges of T.

Let ψ be a mapping $\underline{n} \to T_0$.

DEFINITION. *The pair (T, ψ) is called a stable n-marked tree if for every $t \in T_0$ the number*

$$\text{val } t : = \# \phi^{-1}(t) + \# \{\textit{edges of } T \textit{ with end point } t\}$$

is ≥ 3.

Let T be the intersection graph of the components of a n-pointed tree of projective lines (C, ϕ). Then the marking ϕ defines a mapping $\psi : \underline{n} \to T_0$ by letting $\psi(i)$ be the component of C on which $\phi(i)$ is sitting. We will call (T, ψ) the combinatorial type of (C, ϕ).

The median component L_d of a tripel $d \in D$ is the median of the subtree of T generated by the vertices $\psi(d_1)$, $\psi(d_2)$, $\psi(d_3)$.

(1.3) Let $V = V_n$ be the set of quadruples $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4)$ of distinct indices of $\underline{n}$ and $\mathbb{P}^V$ the product of $\# V$ copies of the projective line $\mathbb{P}$ over $\mathbb{Z}$. Thus $\mathbb{P}^V = \prod_{\upsilon \in V} \mathbb{P}_\upsilon$ and any $\mathbb{P}_\upsilon$ is a copy of $\mathbb{P}$.

Let (C, ϕ) be a b-pointed tree of projective lines over k. Then

$$\lambda_{\upsilon_1 \upsilon_2 \upsilon_3 \upsilon_4} : = \lambda_{\upsilon_1 \upsilon_2 \upsilon_3}(\phi_{\upsilon_4})$$

is a k-valued point of $\mathbb{P}$ for any $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4) \in V$ and

$$\lambda(C, \phi) : = (\lambda_\upsilon)_{\upsilon \in V}$$

is a k-valued point of $\mathbb{P}^V$. It will be called the system of cross-ratios for (C, ϕ).

One gets the following relations

$$(1) \qquad \lambda_{\upsilon_1 \upsilon_2 \upsilon_3 \upsilon_4} = \frac{1}{\lambda_{\upsilon_2 \upsilon_1 \upsilon_3 \upsilon_4}}$$

(2) $\lambda_{v_2v_3v_4v_1} = 1 - \lambda_{v_1v_2v_3v_4}$

(3) $\lambda_{v_1v_2v_4v_5} \cdot \lambda_{v_1v_2v_3v_4} = \lambda_{v_1v_2v_3v_5}$.

PROOF. (1) is obvious as $\lambda_{v_1v_2v_3} = \dfrac{1}{\lambda_{v_2v_1v_3}}$.

Ad (2): Let L be the median component of C relative to (v_1, v_2, v_3). If $\pi_L(\phi_{v_4}) \neq \pi_L(\phi_{v_i})$ for $i=2$ and $i=3$, then L is also the median component of (v_2, v_3, v_4). Then $\lambda_{v_1v_2v_3v_4}$ is the cross ratio of the points $\pi_L(\phi_{v_1})$, $\pi_L(\phi_{v_2})$, $\pi_L(\phi_{v_3})$, $\pi_L(\phi_{v_4})$ while $\lambda_{v_2v_3v_4v_1}$ is the cross ratio of the points $\pi_L(\phi_{v_2})$, $\pi_L(\phi_{v_3})$, $\pi_L(\phi_{v_4})$, $\pi_L(\phi_{v_1})$.

This shows that (2) is a well-known formula for cross-ratios on a line.

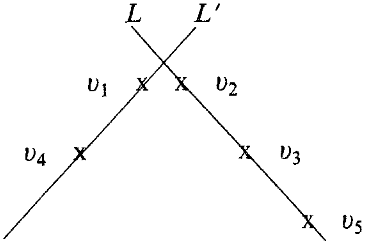
If $\pi_L(\phi_{v_4}) = \pi_L(\phi_{v_2})$, then $\lambda_{v_1v_2v_3v_4} = \infty$. If L' is the median component of (v_2, v_3, v_4), then $\pi_{L'}(\phi_{v_1}) = \pi_{L'}(\phi_{v_3})$ and one gets that the cross-ratio of $\pi_{L'}(\phi_{v_2})$, $\pi_{L'}(\phi_{v_3})$, $\pi_{L'}(\phi_{v_4})$, $\pi_{L'}(\phi_{v_1})$ which is $\lambda_{v_2v_3v_4v_1}$ is ∞.

If $\pi_L(\phi_{v_4}) = \pi_L(\phi_{v_3})$, then $\lambda_{v_1v_2v_3v_4} = 1$. If L' is the median component of (v_2, v_3, v_4) then $\pi_{L'}(\phi_{v_1}) = \pi_{L'}(\phi_{v_2})$ and thus $\lambda_{v_2v_3v_4v_1}$ = cross-ratio of $\pi_{L'}(\phi_{v_2})$, $\pi_{L'}(\phi_{v_3})$, $\pi_{L'}(\phi_{v_4})$, $\pi_{L'}(\phi_{v_1}) = 0$.

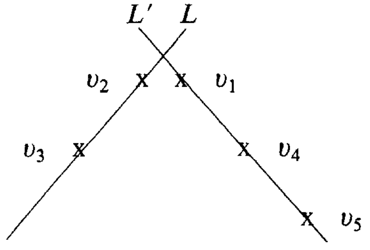
Ad (3): Let L be the median component of (v_1, v_2, v_3). If $\pi_L(\phi_{v_4}) \neq \pi_L(\phi_{v_1})$ and $\neq \pi_L(\phi_{v_2})$ then L is also the median component of (v_1, v_2, v_4). Then (3) is another well-known formula for cross-ratios on the line L. If $\pi_L(\phi_{v_4}) = = \pi_L(\phi_{v_1})$ then $\lambda_{v_1v_2v_3v_4} = 0$ and if L' is the median component relative to (v_1, v_2, v_4) then $\pi_{L'}(\phi_{v_2}) = \pi_{L'}(\phi_{v_3}) = \pi_{L'}(L)$.

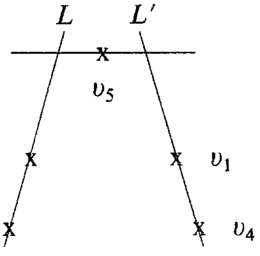
If $\pi_L(\phi_{v_5}) \neq \pi_L(\phi_{v_1})$, then $\lambda_{v_1v_2v_3v_5} \neq 0$ and $\pi_{L'}(\phi_{v_5}) = \pi_{L'}(L) = \pi_{L'}(\phi_{v_2})$ and thus $\lambda_{v_1v_2v_4v_5} = \infty$. The formula thus reads $\infty \cdot 0 = \lambda_{v_1v_2v_3v_4}$ which is correct.

An intuitive picture for this situation is:



If $\pi_L(\phi_{v_5}) = \pi_L(\phi_{v_1})$, then $\lambda_{v_1v_2v_3v_5} = 0$ and $\pi_{L'}(\phi_{v_2}) = \pi_{L'}(\phi_{v_3}) \neq \pi_{L'}(\phi_{v_5})$. Then $\lambda_{v_1v_2v_3v_4} = 0$ and the formula is correct as $\lambda_{v_1v_2v_4v_5} \neq \infty$. Intuitive pictures for these situation are





Similar reasoning shows that (3) is correct also if $\pi_L(\phi_{v_4}) = \pi_L(\phi_{v_2})$.

(1.4) Let $\mathbb{P}_\upsilon$ be provided with homogenous coordinates a_υ, b_υ such that Proj $\mathbb{Z}[a_\upsilon, b_\upsilon] = \mathbb{P}_\upsilon$.

B_n is the closed subscheme of $\mathbb{P}^V$ given by the multihomogeneous ideal in the multigraded ring $\mathbb{Z}[a_\upsilon, b_\upsilon : \upsilon \in V]$ generated by the equations

(1) $a_{\upsilon_2\upsilon_1\upsilon_3\upsilon_4} \cdot a_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4} = b_{\upsilon_2\upsilon_1\upsilon_3\upsilon_4} b_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}$

(2) $a_{\upsilon_2\upsilon_3\upsilon_4\upsilon_1} \cdot b_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4} = b_{\upsilon_2\upsilon_3\upsilon_4\upsilon_1} \cdot b_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4} - a_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4} \cdot b_{\upsilon_2\upsilon_3\upsilon_4\upsilon_1}$

(3) $\mathrm{a}_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4} \cdot a_{\upsilon_1\upsilon_2\upsilon_4\upsilon_5} \cdot b_{\upsilon_1\upsilon_2\upsilon_3\upsilon_5} = a_{\upsilon_1\upsilon_2\upsilon_3\upsilon_5} \cdot b_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4} \cdot b_{\upsilon_1\upsilon_2\upsilon_4\upsilon_5}$

where $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5$ is any system of 5 distinct elements of $\underline{n}$. With respect to the inhomogeneous coordinates a_υ/b_υ one gets formally the relations deduced for the system of cross-ratios of an n-pointed tree of projectives lines in the above subsection.

PROPOSITION 1. *Let $q = (q_\upsilon)_{\upsilon \in V}$, $q_\upsilon := a_\upsilon/b_\upsilon(q)$, be a k-valued point of B_n. Then there exists a stable n-pointed tree of projective lines (C, ϕ) over k such that*

$$\lambda(C, \phi) = q.$$

The curve (C, ϕ) is unique up to isomorphisms.

PROOF. 1) Let $d = (d_1, d_2, d_3)$ a tripel of distinct elements in $\underline{n}$. For any $i \in \underline{n}$, $i \neq d_j$, $(d, i) = di$ is a quadrupel in V. If $i = d_1$ (resp. $i = d_2$, resp. $i = d_3$) we define q_{di} as 0 (resp. ∞, resp. 1).

We define an equivalence relation $\underset{d}{\sim}$ on $\underline{n}$:

$$i \underset{d}{\sim} j \text{ iff } q_{di} = q_{dj}.$$

The following properties hold:

a) If $D' = (d'_1, d'_2, d'_3)$ is a permutation of d, then $\underset{d}{\sim} = \underset{d'}{\sim}$

b) If $d_j \underset{d}{\sim} d'_j$ for $1 \leq j \leq 3$, then $\underset{d}{\sim} = \underset{d'}{\sim}$

c) If d'_3 is not $\underset{d}{\sim}$-equivalent to d_1 and d_2 then $\underset{d}{\sim} = \underset{d'}{\sim}$ where $d' = (d_1, d_2, d'_3)$

d) If d'_3 is in the $\underset{d}{\sim}$-equivalence class of d_1 (resp. d_2), then the union of all the equivalence classes relative to $\underset{d}{\sim}$ not containing d_1 (resp. d_2) is in one equivalence class with respect to $\underset{d'}{\sim}$ where $d' = (d_1, d_2, d'_3)$, where $d'_3 \neq d_1$, resp. $d'_3 \neq d_2$.

We now prove these properties:

Ad a) Let $\lambda_\upsilon := a_\upsilon/b_\upsilon$. From the set of relations (1), (2) for B_n one can easily deduce the following relations:

$$\lambda_{\upsilon_2\upsilon_3\upsilon_1\upsilon_4} = \frac{1}{1 - \lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}}$$

$$\lambda_{\upsilon_3\upsilon_2\upsilon_1\upsilon_4} = 1 - \lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}$$

$$\lambda_{\upsilon_1\upsilon_3\upsilon_2\upsilon_4} = \frac{\lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}}{1 - \lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}}$$

$$\lambda_{\upsilon_3\upsilon_1\upsilon_2\upsilon_4} = \frac{1-\lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}}{\lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}}$$

This shows that $q_{di}=q_{dj}$ if and only if $q_{d'i}=q_{d'j}$.

Ad b) Assume first that $d_1'=d_1$, $d_2'=d_2$. Then $q_{d_1d_2d_3d_3'}=1$. From relation (3) we get

$$q_{d_1d_2d_3'i} \cdot q_{d_1d_2d_3d_3'} = q_{d_1d_2d_3i}.$$

This shows that $q_{d_1d_2d_3'i}=q_{d_1d_2d_3'j}$ if and only if $q_{di}=q_{dj}$. Thus

$$\underset{d}{\sim} = \underset{d_1d_2d_3'}{\sim} = \underset{d_1d_3'd_2}{\sim} = \underset{d_1d_3'd_2'}{\sim} = \underset{d_3'd_2'd_1}{\sim} = \underset{d_3'd_2'd_1'}{\sim} = \underset{d'}{\sim}$$

and b) proved.

Ad c) The proof is similar to the one for b). $q_{dd_3'}\neq 0$ and $\neq\infty$ and $q_{d_1d_2d_3'i}\cdot q_{di}=q_{dd_3'}$ which shows that $q_{di}=q_{dj}$ if and only if $q_{d_1d_2d_3'i}=q_{d_1d_2d_3'j}$.

Ad d) Let d_3' be in the $\underset{d}{\sim}$-equivalence class of d_1. Then $q_{dd_3'}=0$. Again

$$q_{d_1d_2d_3'i}\cdot q_{di}=q_{dd_3'}=0.$$

For any i such that $q_{di}\neq 0$ we get $q_{d_1d_2d_3'i}=0$ which shows that all i not $\underset{d}{\sim}$ equivalent to d_1 are in one equivalence class with respect to $\underset{d_1d_2d_3'}{\sim}$.

In the same way one proves the result if $q_{dd_3}=\infty$.

2) As a corollary to the properties a)–d) one gets: there are tripels $d\in D_n$ such that all the equivalence classes with respect to $\underset{d}{\sim}$ except one class contain just one element.

3) We now prove the proposition by induction on n. The induction starts with $n=4$. This case is quite simple.

We pick a triple d such that all equivalence classes except one with respect to $\underset{d}{\sim}$ consist of one element only.

We may assume that one class is $\{n\}$ because if σ is a permutation of $\underline{n}$, then

$$q' := (q_\upsilon')_{\upsilon\in V},\ q_\upsilon' := q_{\sigma(\upsilon_1)\sigma(\upsilon_2)\sigma(\upsilon_3)\sigma(\upsilon_4)},$$

is also a k-valued point of B_n. If $\sigma^{-1}(i)=n$ then $\{n\}$ is an $\sigma^{-1}d_1$, $\sigma^{-1}d_2$, $\sigma^{-1}d_3$-equivalence class relative to the point q'. The curves for q and q' will be the same and the markings are transformed through σ.

We consider two cases.

Case 1: We assume that there are ≥ 4 equivalence classes relative to $\underset{d}{\sim}$. Then we may assume that $d_i\neq n$ by property 1d). We consider $q' := (q_\upsilon)_{\upsilon\in V_{n-1}}$ which is obviously a k-valued point of B_{n-1}. By the induction hypothesis we obtain a $(n-1)$-pointed curve (C',ϕ') with $\lambda(C',\phi')=q'$. Let L be the median component of C' with respect to d and let ϕ_n be that point of L such that the cross-ratio of the sequence of points $\pi_L(\phi_{d_1})$, $\pi_L(\phi_{d_2})$, $\pi_L(\phi_{d_3})$, ϕ_n is equal to $q_{d_1d_2d_3n}$ which is different from 0, ∞, 1 and also different from q_{di} for any $i \neq n$ because there is no $i \neq n$ equivalent to n relative to $\underset{d}{\sim}$. Thus $\phi_n \neq \pi_L(\phi_i)$ for $i<n$ and $\phi_n \neq \phi_i$ for $i<n$.

Let $C := C'$, $\phi = (\phi'_1, \ldots, \phi'_{n-1}, \phi_n)$. Then (C, ϕ) is an n-pointed tree of projective lines. One checks easily that $\lambda(C, \phi) = q$.

Case 2: Assume now that there are just three equivalence classes relative to $\underset{d}{\sim}$. Then one of the d_i must be equal to n. Let $d_3 = n$. We assume that $\{d_2\}$ is also an equivalence class relative to $\underset{d}{\sim}$ and that $d_2 = n-1$. Let again (C', ϕ') be a $(n-1)$-pointed curve such that $\lambda(C', \phi') = q' := (q_\upsilon)_{\upsilon \in V_{n-1}}$ and let L be the component of C' on which $\phi'(d_{n-1})$ is sitting. Let $C := C' \cup L'$ where L' is an extra projective line over k which meets C' only in the point ϕ'_{n-1} and such that $L' \cap C' = \{\phi'_{n-1}\}$ is an ordinary double point of C. Let ϕ_{n-1}, ϕ_n be two distinct k-rational ploints on L' different from ϕ'_{n-1}. Then C is a tree of projective lines over k and $\phi := (\phi'_1, \ldots, \phi'_{n-2}, \phi_{n-1}, \phi_n)$ is a marking of C which makes (C, ϕ) into a stable n-pointed tree of lines. One checks easily that $\lambda(C, \phi) = q$.

4) Uniqueness follows because in the construction of (C, ϕ) in 3) the $(n-1)$-pointed curve (C', ϕ') is unique and there is no freedom in the choice of ϕ_n in case 1 while in case 2 there is a unique isomorphism α on C which is the identity on $C' \subset C$ and which sends ϕ_{n-1}, ϕ_n to any pair of distinct k-rational points of $L' - \{\phi'_{n-1}\}$.

(1.5) Let (T, ψ) be a n-marked stable tree and t a vertex of T. We define an equivalence relation $\underset{t}{\sim}$ on $\underline{n}$:

$$i \underset{t}{\sim} j \text{ iff } i=j \text{ or } \psi(i) \text{ can be connected to } \psi(j) \text{ by a path in } T \text{ not passing through } t.$$

If (C, ϕ) is a n-pointed tree of projective lines and (T, ψ) the combinatorial type of (C, ϕ), then the system of crossratios $q = (q_\upsilon) = \lambda(C, \phi)$ satisfies the following equations:

$$q_\upsilon = 0 \text{ for all } \upsilon \in V_T$$

where $V_T := \{(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4) \in V_n : \upsilon_1 \underset{t}{\sim} \upsilon_4,\ \upsilon_2 \text{ and } \upsilon_3 \text{ are not } \underset{t}{\sim}\text{-equivalent to } \upsilon_1 \text{ for some } t \in T_0\}$.

This is easily proved because $\underset{t}{\sim} = \underset{d}{\sim}$ for $d = (\upsilon_1, \upsilon_2, \upsilon_3) \in D_n$. The median component L_d is just the vertex t, if $\upsilon_2 \underset{t}{\not\sim} \upsilon_3$.

In order to formulate a converse statement we need the concept of contractions.

DEFINITION. *Let (T, ψ), (T', ψ') be n-marked stable trees and $\varepsilon : T' \to T$ a mapping. ε is called a contraction if*

(1) $\varepsilon(T_0')=T_0$ *and* $\varepsilon\circ\psi'=\psi$

(2) $\varepsilon^{-1}(t)$ *is a subtree of* T' *for any vertex* t *of* T

(3) *If* t *is an endpoint of* $k\in T_1'$, *then* $\varepsilon(t)$ *is an endpoint of* $\varepsilon(k)$ *if* $\varepsilon(k)$ *is an edge or* $\varepsilon(t)=\varepsilon(k)$.

It is easy to show that if $\varepsilon:(T',\psi')\to(T,\psi)$ is a contraction of stable n-marked trees, then ε is uniquely determined by (T',ψ'), (T,ψ).

Let now $B(T,\psi)$ be the closed subscheme of B_n given by the equations

$$\lambda_v=0$$

for all $v\in V_T$.

If $\lambda(C,\phi)=q$ is a k-valued point of $B(T,\psi)$ and if (T',ψ') is the combinatorial type of (C,ϕ), then (T',ψ') contracts to (T,ψ). In general (T',ψ') will be different from (T,ψ). It is easy to see that $B(T',\psi')$ is a closed subscheme of $B(T,\psi)$ if (T',ψ') contracts to (T,ψ).

Let $B(T,\psi)^*$ be the open subscheme of $B(T,\psi)$ which is the complement of the union of all the $B(T',\psi')$ for which (T',ψ') contract to (T,ψ) and $(T',\psi')\neq(T,\psi)$.

PROPOSITION 2. *$B(T,\psi)$ is canonically isomorphic to* $\prod_{t\in T_0}B_{\mathrm{val}\,t}$. *Moreover $B(T,\psi)^*$ is isomorphic to* $\prod_{t\in T_0}B^*_{\mathrm{val}\,t}$ *where* $B_n^*=B(T^0,\psi^0)^*$ *where* (T^0,ψ^0) *is the unique n-marked tree possessing just one vertex and* val $t:=\#\{$*edges of* T *adjacent to* $t\}+\#\{\psi^{-1}(t)\}$.

PROOF. We start the proof of the first statement by examining the special case where $\#T_0=2$, say $T_0=\{t_0,t_1\}$, where $\psi^{-1}(t_0)=\{1,\dots,k\}$ and $\psi^{-1}(t_1)=\{k+1,\dots,n\}$.

Let $\mathcal{I}$ denote the sheaf of ideals defining $B=B(\psi,T)$. As a scheme B equals $(B,(\mathcal{O}_{B_n}/\mathcal{I})|B)$. We will construct morphisms $g:B\to B_{k+1}\times B_{n-k+1}$ and $f:B_{k+1}\times B_{n-k+1}\to B$ such that $f\circ g$ is the identity.

The first map g is obtained from $h=(h_1,h_2):B_n\to B_{k+1}\times B_{n-k+1}$ by restriction to B. Here h_1 is the projection induced by the natural injection $\underline{k+1}\to\underline{n}$ and h_2 is induced by the injection $\underline{n-k+1}\to\underline{n}$ given by $i\to i+k-1$. The second morphism is obtained from a morphism $e:B_{k+1}\times B_{n-k+1}\to\prod_{v\in V_n}\mathbb{P}_v$ given in coordinates e_v by the following formulas:

$$v=(v_1,v_2,v_3,v_4),\text{ with } v_1<v_2<v_3<v_4$$

if $v_1<v_2<v_3<v_4\leq k+1$ then e_v is the projection on the factor $\mathbb{P}_v$ of B_{k+1}.

if $v_1<v_2<v_3<k+1\leq v_4$ then e_v is the projection of B_{k+1} on its factor $\mathbb{P}_{(v_1,v_2,v_3,k+1)}$.

if $v_1<v_2<k+1\leq v_3<v_4$ then e_v is the constant map with image 1.

if $v_1<k+1\leq v_2<v_3<v_4$ then e_v is the projection of B_{n-k+1} on its factor $\mathbb{P}_w$ with $w=(1,v_2-k+1,v_3-k+1,v_4-k+1)$.

if $k+1\leq v_1<v_2<v_3<v_4$ then e_v is the projection of B_{n-k+1} on its factor $\mathbb{P}_w$ with $w=(v_1-k+1,v_2-k+1,v_3-k+1,v_4-k+1)$.

The group S_4 acts on $\mathbb{P}_1$ in the well known way:

$S_4 \rightarrow S_4/K \cong S_3$ where K is the group of Klein and S_3 acts on $\mathbb{P}$ and permutes 0, ∞, 1. For $\sigma \in S_4$ we write $\tilde{\sigma}$ for the corresponding automorphism of $\mathbb{P}$.

The definition of the e_υ's is now completed by:

$$e_{(\upsilon_{\sigma(1)}, \upsilon_{\sigma(2)}, \upsilon_{\sigma(3)}, \upsilon_{\sigma(4)})} = \tilde{\sigma} \circ e_{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4}$$

where $\upsilon_1 < \upsilon_2 < \upsilon_3 < \upsilon_4$ and $\sigma \in S_4$.

A straightforward verification shows that the image of e lies in $B_n \subset \subset \prod_{\upsilon \in V_n} \mathbb{P}_\upsilon$. The equation $e_\upsilon = 1$ for $\upsilon_1 < \upsilon_2 < k+1 \leq \upsilon_3 < \upsilon_4$, yields that the image of e lies in $(B, \mathcal{O}_{B_n}/\mathcal{I})$. Another trivial verification shows that f and g are each others inverses.

We consider now $B(T, \psi)$ where $\# T_0 \geq 3$. Let t_0 be an end vertex of the tree and put $S = \psi^{-1}(t_0)$. Consider the marked tree (T', ψ') such that T' has two vertics t_0' and t_1' and such that $\psi'(S) = t_0'$ and $\psi'(S^*) = t_1'$, where $S^* = \underline{n} - S$.

Then we have $B(T, \psi) \subseteq B(T', \psi')$ (meaning the opposite inclusion of the sheaves of ideals). The isomorphism

$$B(T', \psi') \xrightarrow{\sim} B_{\mathrm{val}\, t_0'} \times B_{n - \mathrm{val}\, t_0' + 1}$$

induces an isomorphism of $B(T, \psi)$ with the closed subscheme $B_{\mathrm{val}\, t_0'} \times \times B(T'', \psi'')$. This (T'', ψ'') is constructed from (T, ψ) as follows. For convenience we suppose that $S = \{k+1, \ldots, n\}$; put (t_0, t_1) for the only edge in T with endpoint t_0. Then T'' is obtained from T by deleting t_0. Further $\psi'' : \underline{k+1} \rightarrow (T'')_0$ is defined by $\psi''(i) = \psi(i)$ for $i \leq k$ and $\psi''(k+1) = t_1$.

Induction now finishes the proof.

The proof of the second statement proceeds in the same way.

COROLLARY. *Let k be a field with q elements and $B(T, \psi)(k)$ the set of k-valued points of the scheme $B(T, \psi)$. Then*

$$\# B(T, \psi)(k) = \sum_{(T', \psi')} (q-2)^{r_4(T', \psi')} (q-3)^{r_5(T', \psi')} \cdot \ldots \cdot (q-n+2)^{r_n(T', \psi')}$$

where $r_i(T', \psi')$ is the number of vertices of (T', ψ') of valence $\geq i$ and where the summation has to be extended over the set of isomorphism classes of stable n-marked trees contracting to (T, ψ).

PROOF. $\# B_n^*(k) = (q-2)(q-3) \cdot \ldots \cdot (q-n+2)$ because a point in $B_n^*(k)$ is given by a projective line C over k and an injective mapping $\phi : \underline{n} \rightarrow C(k)$. But (C, ϕ) and (C, ϕ') are isomorphic if and only if there is a fractional linear transformation $\alpha : C \rightarrow C$ such that $\alpha \circ \phi = \phi'$. If $\phi(i) = \phi'(i)$ for $1 \leq i \leq 3$, then (C, ϕ) is isomorphic to (C, ϕ') only if $\phi = \phi'$. The number of possibilities to pick the points $\phi(4), \ldots, \phi(n)$ is therefore

$$(q-2)(q-3) \cdot \ldots \cdot (q-n+2).$$

Now we use the fact that $B(T, \psi) = \cup B(T', \psi')^*$ where the union is over the set of n-marked stable trees contracting to (T, ψ) and that

$$B(T', \psi')^* \cong \prod_{t \in T_0'} B^*_{\mathrm{val}\, t}.$$

Thus

$$\begin{cases} \# B(T', \psi')(k) = \prod\limits_{t \in T_0'} (q-2)(q-3) \cdot \ldots \cdot ((q - (\mathrm{val}\ t) + 2)) = \\ \qquad = (q-2)^{r_4(T', \psi')} \cdot \ldots \cdot (q-n+2)^{r_n(T', \psi')}. \end{cases}$$

2. STABLE n-POINTED TREES

(2.1) Let $\pi : X \rightarrow S$ be a proper and flat morphism of noetherian schemes and $\phi = (\phi_1, \ldots, \phi_n)$ be a n-tupel of morphisms $S \rightarrow X$.

DEFINITION. The pair (π, ϕ) is called a stable n-pointed tree of projective lines over S, if

(1) $\pi \circ \phi_i = id_S$ for all i

(2) for every point $s \in S$ the fibre X_s with the points $\phi_1(s), \ldots, \phi_n(s)$ on it is a stable n-pointed tree of projective lines over the field $k(s)$ of values at s.

We will show that the system of cross-ratios $\lambda(X_s, \phi)$ is a morphism on S.

PROPOSITION 3. *There is a morphism* $u : S \rightarrow \mathbb{P}^V$ *such that* $u(s)$ *is the system of cross-ratios of the n-pointed tree* (X_s, ϕ) *of projective lines over* $k(s)$.

The proof is achieved with the help of the dualizing sheaf $\omega_{X/S}$, see [DM], [Kn], p. 163 and will be given in some detail at the end of (2.3). It parallels the proofs in [Kn] about the properties of the contraction; it is however more elementary as we only treat a special case compared to the setting in [Kn].

(2.2) In this subsection we give the construction of the dualizing sheaf for trees of projective lines.

LEMMA 1. *Let* $X \xrightarrow{\pi} S$ *be a n-pointed tree of projective lines. Then* $\phi_i(S)$ *is a divisor in X.*

PROOF. Put $\phi = \phi_i$. The set $\phi(S) \subset X$ is closed because $\phi(S) = \phi \circ \pi(X)$ and $\phi \circ \pi : X \rightarrow X$ is an S-morphism. Apply now [H] p. 104, ex. 4.4. We have to find for every point $x = \phi(s)$ with $s \in S$ a neighbourhood U of x and a non-zero-divisor t on U such that the ideal of $U \cap \phi(S)$ equals (t). One knows:

$$\mathscr{O}_{S,s} \xrightarrow{\pi^*} \mathscr{O}_{X,x} \xrightarrow{\phi^*} \mathscr{O}_{S,s} = id \text{ and } \mathscr{O}_{X_s,x} = \mathscr{O}_{X,x} / \pi^*(\underline{m}_{S,s}) \mathscr{O}_{X,x}.$$

The local ring $\mathscr{O}_{X_s,x}$ has coefficient field $k(s)$ and is regular of dimension 1. Choose a parameter of $\mathscr{O}_{X_s,x}$ and a pre-image $t \in \underline{m}_{X,x}$ with $\phi^*(t) = 0$. Then $\pi^*(\underline{m}_{S,s}) \cup \{t\}$ generates the maximal ideal of $\mathscr{O}_{X,x}$. The ringhomomorphism $\sigma : \hat{\mathscr{O}}_{S,s}[\![T]\!] \rightarrow \hat{\mathscr{O}}_{X,x}$ given by $\sum a_n T^n \rightarrow \sum \pi^*(a_n) t^n$ is then surjective. Hence $\ker \hat{\phi}^* = t \hat{\mathscr{O}}_{X,x}$ and by flatness $\ker \phi^* = t \mathscr{O}_{X,x}$. We will show that σ is an isomorphism and so in particular t is a non-zero-divisor.

Let $I = \ker \sigma$ and suppose that $I \neq 0$. Take $b \geq 1$ minimal such that I contains an element $f = \sum_{i \geq b} a_i T^i$ with $a_b \neq 0$. Let I_b denote the ideal in $\hat{\mathscr{O}}_{S,s}$ consisting of the b^{th}-coefficients of the elements in I.

The exact sequence $0 \to I \to \hat{\mathcal{O}}_{S,s}[\![T]\!] \to \hat{\mathcal{O}}_{X,x} \to 0$ remains exact after $- \otimes_{\hat{\mathcal{O}}_{S,s}} k(s)$ because $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{S,s}$. This implies $I \otimes k(s) = 0$ and so $I = \hat{\underline{m}}_{S,s} I$. Then also $I_b = \hat{\underline{m}}_{S,s} I_b$ and Nakayama's lemma implies $I_b = 0$. This is a contradiction and so σ is an isomorphism.

Choose a neighbourhood W of x in X, put $U = \phi^{-1} W$ and $V = W \cap \pi^{-1} U$. Then $V \supseteq \phi(U)$ and it follows that $\pi(V) = U$ since $\pi\phi = id$. One finds ringhomomorphisms

$$\mathcal{O}_S(U) \xrightarrow[\pi_V^*]{} \mathcal{O}_X(V) \xrightarrow[\phi_V^*]{} \mathcal{O}_S(U) = id.$$

Taking $\varinjlim_{V}$, one finds ker $(\varinjlim_{V} \phi_V^*) = (t)$. For a suitable V one has already ker $\phi_V^* = (t)$ and t is a non-zero-divisor on V. It follows that $\phi(U) = \{ v \in V \mid t(v) = 0 \}$ and $(t) =$ the ideal of $\phi(U) \subset V$. The proves the lemma.

LEMMA 2. *Let $x \in X$ be a singular point of the fibre X_s, with $s = \pi(x)$. Then $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{S,s}[\![A, B]\!]/_{(AB-m)}$ for some element $m \in \hat{\underline{m}}_{S,s}$.*

PROOF. $\hat{\mathcal{O}}_{X_s,x} \cong k(s)[\![\bar{a}, \bar{b}]\!]/_{(\bar{a}\bar{b})}$ and from this one finds a surjective ringhomomorphism $\hat{\mathcal{O}}_{S,s}[\![A, B]\!]/_{(AB-m)} \xrightarrow{\sigma} \hat{\mathcal{O}}_{X,x}$ with $\sigma(A) = a$, $\sigma(B) = b$; σ is $\hat{\mathcal{O}}_{S,s}$-linear; $m \in \hat{\underline{m}}_{S,s}[\![A, B]\!]$ and a, b map to $\bar{a}, \bar{b}$ in $\hat{\mathcal{O}}_{X_s,x}$. After changing the formal variables A and B one can arrange that $m \in \hat{\underline{m}}_{S,s}$. Put $I = \ker \sigma$ and represent each $f \in I$ as

$$f = f_0 + \sum_{n \geq 1} a_n A^n + \sum_{m \geq 1} b_m B^m \text{ with } f_0, a_n, b_m \in \hat{\mathcal{O}}_{S,s}.$$

The collection of all coefficients f_0 (for $f \in I$) form an ideal $I_0 \subset \hat{\mathcal{O}}_{S,s}$. Let ${}_nI$ denote the ideal of the coefficients a_n (for $f \in I$) and let I_m denote the ideal of the coefficients b_m.

As in lemma 1 flatness implies $I = \hat{\underline{m}}_{S,s} I$. Let J denote any of the ideals I_0, ${}_nI$ or I_m. Then $J = \hat{\underline{m}}_{S,s} J$ and by Nakayama $J = 0$. Hence $I = 0$ and the lemma is proved.

LEMMA 3. *Let $\pi : X \to S$ be a n-pointed tree of projective lines. There exists an invertible sheaf $\omega = \omega_{X/S}$ on X such that for every fibre $\alpha : X_s \to X$ one has $\alpha^* \omega \cong \omega_{X_s|k(s)} =$ the dualizing sheaf on X_s over $k(s)$.*

PROOF. The sheaf of differentials $\Omega_{X/S}$ satisfies: $\Omega_{X/S,x}$ is a free module of rank 1 if x is a regular point of its fibre and $\Omega_{X/S,s}$ has two generators and one relation if x is a singular point of its fibre. The last statement follows from lemma 2, namely using the above notation: $\Omega_{X/S,x}$ is generated by da, db and has the one relation $bda + adb = 0$.

This implies that $\Omega_{X/S}$ has locally on X a 2-step resolution

$$0 \to \mathscr{E}_1 \xrightarrow{\beta} \mathscr{E}_0 \xrightarrow{\alpha} \Omega_{X/S} \to 0$$

with $\mathscr{E}_1$, $\mathscr{E}_0$ free $\mathcal{O}_X$-modules of rank 1 and 2. One forms locally $\omega = \Lambda^2 \mathscr{E}_0 \otimes \otimes \mathscr{E}_1^{\vee}$, and a morphism $\Omega_{X/S} \to \omega$. The morphism is defined as follows: let v_1

generated $\mathscr{E}_1$ and let $v_1^\vee$ denote the dual element, generating $\mathscr{E}_1^\vee$, let $a \in \Omega_{X/S}$ have preimage $b \in \mathscr{E}_0$. Then the image of a in ω is given by $(a \wedge v_1) \otimes v_1^\vee$. One can show that the local construction above glues over all of X and that the construction permutes with base-change. The construction above carried out for X_s yields the dualizing sheaf $\omega_{X_s/k(s)}$. This proves the lemma.

(2.3) We study now the contraction morphism of a n-pointed tree of lines; it comes into the game if one forgets some of the marking sections ϕ_i.

LEMMA 4. *Let $X \to S$ be a n-pointed tree and let $\mathscr{L}$ denote the line bundle $\omega_{X/S}(\phi_1(S) + \dots + \phi_n(S))$. Then:*
1) $R^i\pi_*\mathscr{L} = 0$ *for* $i \geq 1$.
2) $\pi_*\mathscr{L}$ *is a vector bundle on S of rank* $n-1$.
3) $X \xrightarrow{\sim} Proj.\ (\bigoplus_{m \geq 0} \pi_*(\mathscr{L}^m))$.

PROOF. The proof is a simplified version of the proof of Thm. 1.8 in [Kn]. We may assume that S is affine. We want to use the theorem on formal functions ([H] p. 277 and remark 11.1.1. on p. 279):

$$(R^i\pi_*\mathscr{L})_s^\wedge \xrightarrow{\sim} \varprojlim H^i(X_m, \mathscr{L}_m)$$

where $s \in S$ and where

$$X_m = X \times_S \text{spec } (R_m) \xrightarrow{v} X \text{ with } R_m = \mathscr{O}_{S,s}/\underline{m}_{S,s}^{n+1} \text{ and } \mathscr{L}_m = v^*\mathscr{L}.$$

First we calculate $H^i(X_0, \mathscr{L}_0)$. Consider the exact sequence:

$$0 \to \mathscr{L}_0 \to \bigoplus_L \mathscr{L}_0|L \to \bigoplus_d \mathscr{L}_0|d \to 0$$

when L denotes the components of X_0 and where d denotes the double points of X_0. The cohomology of this sequence is:

$$\begin{aligned} &0 \to H^0(X_0, \mathscr{L}_0) \xrightarrow{\alpha} \bigoplus_L H^0(L, \mathscr{L}_0|L) \xrightarrow{\beta} \bigoplus_d \mathscr{L}_0(d) \to H^1(X_0, \mathscr{L}_0) \to \\ &\to \bigoplus H^1(L, \mathscr{L}_0|L)) \to \cdots \end{aligned}$$

It is easily seen that $H^i(L, \mathscr{L}_0|L) = 0$ for $i \geq 1$ and that β is surjective. Hence $H^0(X_0, \mathscr{L}_0)$ has dimension $n-1$ and $H^i(X_0, \mathscr{L}_0) = 0$ for $i > 0$.

Next we consider $\mathscr{L}_m$ on X_m. The cohomology of $\mathscr{L}_m$ can be calculated with a Čech-complex $0 \to \bigoplus \mathscr{L}_m(U_i) \to \bigoplus \mathscr{L}_m(U_i \cap U_j) \to \cdots$.

Let H^0, H^1 etc. denote the cohomology groups of this complex. Then $H^i = 0$ for $i \geq 2$ because dim $\mathscr{L}_m = 1$. Further one has exact sequences:

$$0 \to H^0(X_0, \mathscr{L}_0) \to H^0 \otimes_{R_m} R_0 \to \text{Tor}_1^{R_m}(H^1, R_0) \to 0$$

$$0 \to H^1(X_0, \mathscr{L}_0) \to H^1 \otimes_{R_m} R_0 \to \text{Tor}_1^{R_m}(H^2, R_0) \to 0.$$

One knows that H^1 is a finitely generated R_m-module. The second exact sequences implies now $H^1 = 0$. The first sequence implies $H^0(X_0, \mathscr{L}_0) \xrightarrow{\sim}$ $\xrightarrow{\sim} H^0 \otimes_{R_m} R_0$.

The augmented Čech-complex

$$0 \to H^0 \to \oplus \mathscr{L}_m(U_i) \to \oplus \mathscr{L}_m(U_i \cap U_j) \to \cdots$$

is now exact.

Since X/S is flat, each term $\mathscr{L}_m(U_i \cap U_j \cap \cdots)$ is a flat R_m-module. It follows that H^0 is a flat R_m-module. Since R_m is a local ring it follows that H^0 is a free R_m-module of rank $n-1$.

Taking projective limits, one finds $(R^i\pi_*\mathscr{L})_s^\wedge = 0$ for $i>0$ and so $R^i\pi_*\mathscr{L}=0$ for $i\geq 1$. Further $(\pi_*\mathscr{L})_s^\wedge$ is a free module of rank $n-1$ and so $\pi_*\mathscr{L}$ is a vector bundle on S of rank $n-1$.

Now we prove the last part of the lemma. We take S affine and small enough such that $\pi_*\mathscr{L}$ is free of rank $(n-1)$. Consider the graded $\mathscr{O}(S)$-algebra

$$\mathscr{A} = \bigoplus_{m\geq 0} H^0(X, \mathscr{L}^{\otimes m}) = \bigoplus_{m\geq 0} H^0(S, \pi_*(\mathscr{L}^{\otimes m})) = \bigoplus_{m\geq 0} \mathscr{A}_m.$$

We note the following properties of $\mathscr{A}$:

(a) $\mathscr{A}_1$ generates $\mathscr{A}$ over $\mathscr{O}(S)$. Indeed let $\mathscr{B}\subset\mathscr{A}$ be generated by $\mathscr{A}_1$ then $\mathscr{A}_m/\mathscr{B}_m$ is a finitely generated $\mathscr{O}(S)$-module for every m. For every $s\in S$, $\mathscr{L}\,|X_s$ is very ample and so $\mathscr{A}_m/\mathscr{B}_m \otimes k(s)=0$. This shows $\mathscr{A}=\mathscr{B}$.

(b) $\mathscr{A}_1$ is a free $\mathscr{O}(S)$ module with free basis $f_0, \ldots, f_{n-2}$. Then

$$X = \bigcup_{i=0}^{n-2} \{x\in X\,|\,f_i(x)\neq 0\}.$$

Indeed, the analogous statement for any fibre X_s is true.

From (a) and (b) it follows that Proj ($\mathscr{A}$) is a closed subspace $Y\subset \mathbb{P}_S^{n-2}$ and a well defined morphism

$$\begin{tikzcd} X \arrow[rr, "\sigma"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

We note further:

(c) σ is bijective. Indeed every $\sigma_s : X_s \to Y_s$ is an isomorphism.

(d) $\hat{\mathscr{O}}_{Y,\sigma(x)} \xrightarrow{\hat{\sigma}^*} \hat{\mathscr{O}}_{X,x}$ is an isomorphism for every $x\in X$.

Indeed let $s=\pi(x)$, then one has isomorphisms

$$\hat{\mathscr{O}}_{Y,\sigma(x)}/\underline{\hat{m}}_s\hat{\mathscr{O}}_{Y,\sigma(x)} \xrightarrow{\sim} \hat{\mathscr{O}}_{X,x}/\underline{\hat{m}}_s\hat{\mathscr{O}}_{X,x}.$$

It follows at once that $\hat{\sigma}^*$ is surjective. Let I denote the kernel of $\hat{\sigma}^*$. The flatness of $\hat{\mathscr{O}}_{X,x}$ over $\hat{\mathscr{O}}_{S,s}$ implies $I\underline{\hat{m}}_{S,s}=I$ and so $IM=I$ where M denotes the maximal ideal of $\hat{\mathscr{O}}_{Y,\sigma(x)}$. From $\bigcap_{m\geq 1} M^m=0$ it follows that $I=0$.

From (d) one concludes that $\mathscr{O}_{Y,\sigma(x)} \to \mathscr{O}_{X,x}$ is an isomorphism and this finishes the proof that $\sigma : X\to Y$ is an isomorphism.

COROLLARY. *Let $X\to S$ be a stable 3-pointed tree, then there exists a unique isomorphism*

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & \mathbb{P}\times S \\
& S \arrow[ur] &
\end{tikzcd}$$

such that $\sigma\circ\phi_i$ $(i=1,2,3)$ *are the sections* $S\to\mathbb{P}\times S$ *given by* 0, ∞ *and* 1.

PROOF. We only need to verify this for S affine and small enough such that $\pi_*\mathscr{L}$ is free of rank 2 over S. Then lemma 4 yields an isomorphism which can be normalized in a unique way such that $\sigma\circ\phi_i$ is the section 0, ∞, or 1 for $i=1$, 2 or 3.

LEMMA 5. (*Contraction*) *Let* $\pi: X\to S$ *be a n-pointed stable tree of projective lines. Let* $\mathscr{L}_1=\omega_{X/S}(\phi_1(s)+\cdots+\phi_{n-1}(s))$. *The morphism c,*

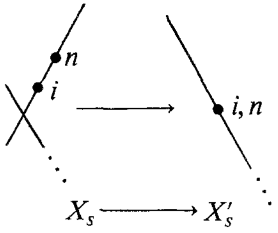
$$\begin{tikzcd}
X \arrow[rr, "c"] \arrow[dr, "\pi"'] & & X'=\operatorname{Proj}\left(\bigoplus\limits_{m\geq 0}\pi_*(\mathscr{L}_1^{\otimes m})\right) \arrow[dl, "\pi'"] \\
& S &
\end{tikzcd}$$

has the following properties:

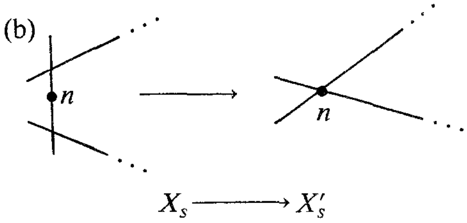
(1) $X'\to S$ *with* $c\circ\phi_i$ $(i=1,\dots,n-1)$ *is a* $(n-1)$*-pointed stable tree.*

(2) *c is a proper morphism and is called the contraction morphism.*

(3) $c_s: X_s\to X'_s$ *is an isomorphism except in the following two cases:*



and



PROOF. One may assume that S is affine and small enough. As in Lemma 4 one shows that $R^i\pi_*\mathscr{L}_1=0$ for $i>0$ and $\pi_*\mathscr{L}_1$ is a free $\mathscr{O}_S$-module of rank $n-2$. Similarly one shows that

$$\mathscr{A}'=\bigoplus_{m\geq 0}H^0(X,\mathscr{L}_1^{\otimes m})$$

is generated by the terms $\mathscr{A}'_1$. Further, each $\mathscr{A}'_m$ is a projective $\mathscr{O}(S)$-module. Hence $X'\xrightarrow{\pi'}S$ is proper and flat. One has to see that $c: X\to X'$ is well-defined, that amounts to showing that

$$X=\bigcup_{i=0}^{n-3}\{x\in X\,|\,f_i(x)\neq 0\}$$

where $f_0, \dots, f_{n-3}$ is a free basis of the $\mathcal{O}(S)$-module $H^0(X, \mathcal{L}_1)$. For the calculation of the fibres one has to calculate explicitely

$$\mathrm{Proj}\ (\bigoplus_{m \geq 0} H(X_s, \mathcal{L}_1^{\otimes m} | X_s)).$$

This is easily done and one finds that $X' \to S$ is an $(n-1)$-tree and moreover one finds property (3). Lemma 5 is proved.

Now we can give the proof of Proposition 3. Fix $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4) \in V_n$. There is a uniquely defined morphism $u_\upsilon : S \to \mathbb{P}$ defined as follows: contract in some order all the sections ϕ_i with $i \neq \upsilon_1, \upsilon_2, \upsilon_3$ of $X \to S$. This yields a diagram

$$X \xrightarrow{c} X_{\upsilon_1 \upsilon_2 \upsilon_3 \upsilon_4} \xrightarrow[\sigma]{\sim} \mathbb{P} \times S$$

where σ is the isomorphism of the corollary. Then $u_\upsilon : = p \circ \sigma \circ c \circ \phi_{\upsilon_4}$ where p is the projection $\mathbb{P} \times S \to \mathbb{P}$. The morphism u of proposition 3 is clearly $u = \prod_{\upsilon \in V_n} u_\upsilon$.

3. THE UNIVERSAL STABLE n-POINTED TREE

In this section we study properties of the projection

$$\pi : = \pi_n : B_{n+1} \to B_n,$$

where B_n is the closed subscheme of $\mathbb{P}^{V_n}$ introduced in (1.4). First we show that the fibres of π are stable n-pointed trees, thus π is a family of stable n-pointed trees. The main result of this section (Proposition 4) is that this family is in fact the universal family of stable n-pointed trees of projective lines. In the proof we use a covering of B_n by open affine subsets of $\mathbb{A}^{n-3}$ which also shows that B_n is nonsingular.

We also define the fibre product

$$Z_n : = B_n \times_{B_{n-1}} B_n$$

formed with respect to two different projections $B_n \to B_{n-1}$. We determine the singularities of Z_n and show that it is a contraction of B_{n+1}. The section ends with examples for small n.

(3.1) LEMMA 1. *Let k be a field and $q \in B_n(k)$. Then the fibre*

$$B : = B_{n+1} \times_{B_n} \mathrm{Spec}\ k$$

is isomorphic to the stable n-pointed tree $C : = C(q)$ over k associated with q in Prop. 1.

PROOF. For the k-valued points we find a bijective map $\alpha : B(k) \to C(k)$ easily as follows: Let $q' : \mathrm{Spec}\ k \to B$ be a point in $B(k)$ and let C' be the stable $(n+1)$-pointed tree over k associated with q'. Then omitting the point ϕ_{n+1} on C' induces a contraction map $p : C' \to C$. Now define $\alpha(q')$ as the image of ϕ_{n+1} under p.

This idea leads to a proof of the proposition in the following way: Let $\mathscr{L}$ be the set of components of C. For every $L \in \mathscr{L}$ we can find $d^L = (d_1^L, d_2^L, d_3^L) \in D_n$ such that L is the median component relative to d^L. By (1.1) the morphism $\lambda_{d^L} : C \to \mathbb{P}_L = \mathbb{P}_k^1$ is an isomorphism on L. Thus the product map

$$\beta := \prod_{L \in \mathscr{L}} \lambda_{d^L} : C \to \prod_{L \in \mathscr{L}} \mathbb{P}_L =: \mathbb{P}^{\mathscr{L}}$$

is an isomorphism of C onto its image $\tilde{C}$ in $\mathbb{P}^{\mathscr{L}}$.

If we consider λ_{d^L} as (inhomogeneous) coordinate on $\mathbb{P}_L$ then $\beta(L)$ is given by the equations $\lambda_{d^{L'}} = q_{d^{L'}i}$, $L' \in \mathscr{L}$, $L' \neq L$, where $i \in \{d_1^L, d_2^L, d_3^L\}$ is chosen in such a way that the $\underset{d'}{\sim}$-equivalence class of i contains at least two of the indices d_1^L, d_2^L, d_3^L: this condition ensures that $\pi_{L'}(\phi_i) = \pi_{L'}(L)$, so $q_{d^{L'}i} = \lambda_{d^{L'}}(\phi_i)$ is the constant that $\lambda_{d^{L'}}$ takes on the component L.

Now for every $L \in \mathscr{L}$ the map $u_{d_1^L d_2^L d_3^L n+1} : B_{n+1} \to \mathbb{P}_{\mathbb{Z}}^1$ defined in (2.3) induces a morphism $\gamma_L : B \to \mathbb{P}_L$. We claim that the product morphism

$$\gamma := \prod_{L \in \mathscr{L}} \gamma_L : B \to \mathbb{P}^{\mathscr{L}}$$

has its image in $\tilde{C}$. Indeed if k' is a field extension of k and $q' \in B(k')$ then by construction $\gamma(q') = \beta \circ \alpha(q')$, where $\alpha : B(k') \to C(k')$ is defined as at the beginning of the proof. To show that the morphism $\alpha := \beta^{-1} \circ \gamma : B \to C$ is an isomorphism we construct a map $\delta : \mathbb{P}^{\mathscr{L}} \to \mathbb{P}^{V_{n+1}}$ and show that $\delta(\tilde{C})$ is contained in B, considered as subspace of $\mathbb{P}^{V_{n+1}}$ by the canonical embedding of $B_{n+1} \times \operatorname{Spec} k$ into $\mathbb{P}^{V_{n+1}}$. Let $v \in V_{n+1}$. If $v \in V_n$, δ is defined by sending λ_v to q_v. Otherwise we may after permutation assume $v_4 = n+1$. Let $L := L_v$ denote the median component of C with respect to v_1, v_2, v_3, and let $d := d^L$. Then there is a unique automorphism τ_v of $\mathbb{P}_L$ which maps $\lambda_d(\pi_L(\phi(v_i)))$, $i = 1, 2, 3$ to 0, ∞, and 1, resp. Now let δ be given by sending λ_v to $\tau_v \circ \lambda_d$. By construction it is clear that δ maps $\tilde{C}$ onto B and that $\delta | \tilde{C}$ and γ are mutually inverse, so $\alpha := \beta^{-1} \circ \gamma : B \to C$ is an isomorphism.

(3.2) For $q \in B_{n+1}$ let

$$U_q := \{\lambda_v \neq 0 \text{ for all } v \in V_{n+1} \text{ such that } \lambda_v(q) \neq 0\}.$$

U_q is an affine open subset of B_{n+1} as for any $v \in V_{n+1}$ we have $U_q \subset \{\lambda_v \neq 0\}$ or $U_q \subset \{\lambda_v \neq \infty\}$. Therefore any v there is $\varepsilon(v) \in \{+1, -1\}$ such that

$$U_q = \bigcap_{v \in V_{n+1}} \{\lambda_v^{\varepsilon(v)} \neq \infty\}.$$

This clearly is the intersection of an open affine subset of $\mathbb{P}^{V_{n+1}}$ with B_{n+1}.

Let $\mathscr{O} := \mathscr{O}_{B_{n+1}}$ be the structure sheaf on B_{n+1}. Then $\mathscr{O}(U_q)$ is a $\mathbb{Z}$-algebra generated by the λ_v, $v \in V_{n+1}$, such that $\lambda_v(q) \neq \infty$. If moreover $\lambda_v(q) \neq 0$ then λ_v is a unit in $\mathscr{O}(U_q)$.

Let (C, ϕ) be the stable $(n+1)$-pointed tree of projective lines associated with q and let (T, ψ) be the combinatorial type of (C, ϕ). Then U_q consists of all

$q' \in B_{n+1}$ whose associated combinatorial type is a contraction of (T, ψ). Thus

$$U_q = B_{n+1} - \cup B(T', \psi')$$

where the union is taken over all (T', ψ') to which (T, ψ) cannot be contracted. In particular, U_q depends only on (T, ψ).

LEMMA 2. *Let $p := \pi(q)$. Then $\pi(U_q) = U_p$, and we have the following cases:*

(i) *ϕ_{n+1} lies on a component of valence ≥ 4. Then for $d \in D_n$ such that $\phi_{n+1} \in L_d$, $\mathscr{O}(U_q)$ is a localisation of $\mathscr{O}(U_p)[\lambda_{d,n+1}]$.*

(ii) *ϕ_{n+1} lies on an end component L of valence 3. Choose $d \in D_n$ such that L_d intersects L and $\phi_{d_3} \in L$. Again $\mathscr{O}(U_q)$ is a localisation of $\mathscr{O}(U_p)[\lambda_{d,n+1}]$.*

(iii) *ϕ_{n+1} lies on a component L of valence 3 that meets two other components L', L''.*

Choose $d \in V_n$ such that $L' = L_{d_1 d_3 n+1}$, $L'' = L_{d_2 d_4 n+1}$. Then for $x := = \lambda_{d_1 d_2 d_4 n+1}$ and $y := \lambda_{d_2 d_1 d_3 n+1}$, we have $x \cdot y = \lambda_{d_1 d_2 d_4 d_3} \in \mathscr{O}(U_p)$, and $\mathscr{O}(U_q)$ is a localisation of $\mathscr{O}(U_p)[x, y]$.

PROOF. Let $m := n+1$. We shall show that for any $e \in D_n$ such that $\lambda_{em}(q) \neq \infty$ we have $\lambda_{em} \in \mathscr{O}(U_p)[\lambda_{dm}][1/f]$ for a suitable f. We give the proof for the first case:

1. If e is a permutation of d, then $\lambda_{em}(q) \neq \infty$, and λ_{em} is one of the functions $\lambda_{dm}^{\pm 1}$, $(1 - \lambda_{dm})^{\pm 1}$ $(\lambda_{dm}(1 - \lambda_{dm}))^{\pm 1}$.

Thus λ_{em} is contained in

$$A_1 := \mathscr{O}(U_p)[\lambda_{dm}][(\lambda_{dm}(1 - \lambda_{dm}))^{-1}].$$

2. Let $e = d_1 d_2 e_3$ such that e_3 is not $\underset{d}{\sim}$-equivalent to d_1 or d_2 relative to (T, ψ). Then $\lambda_{de_3} \in \mathscr{O}(U_p)^*$, and from

$$\lambda_{d_1 d_2 e_3 m} \cdot \lambda_{de_3} = \lambda_{dm}$$

we get $\lambda_{em} \in A_1$.

3. Let e define the same equivalence relation as d, $\underset{d}{\sim} = \underset{e}{\sim}$. Then we can apply step 2 and permutation several times to transform e into d. The permutations make a further localisation necessary, one sees that

$$\lambda_{em} \in A := A_1 \left[\frac{1}{f} \right],$$

where $f := \prod (\lambda_{de_3} - \lambda_{dm})$, the product being taken over all e_3 with $\underset{d_1 d_2 e_3}{\sim} = \underset{d}{\sim}$.

4. Let $e = d_1 d_2 e_3$; if $e_3 \underset{d}{\sim} d_1$ then $\lambda_{em}(q) = \infty$, so let $e_3 \underset{d}{\sim} d_2$. From

$$\lambda_{d_1 d_2 e_3 d_3}(p) = \lambda_{d_1 d_2 d_3 e_3}^{-1}(p) = 0$$

we see that $\lambda_{d_1 d_2 e_3 d_3} \in \mathscr{O}(U_p)$. Thus $\lambda_{em} = \lambda_{d_1 d_2 e_3 d_3} \in A_1$.

5. For general e we have to apply step 4, permutation, and step 3 to show that $\lambda_{em} \in A$.

The same proof holds for cases (ii) and (iii) with the following difference in the third case: if the median component of e is in the same connected component of $C-L$ as L' (resp. L'') one shows that λ_{em} is contained in a localisation of $\mathcal{O}(U_p)[x]$ (resp. of $\mathcal{O}(U_p)[y]$).

By induction on n one proves the following consequences of Lemma 2:

COROLLARY. (i) *B_n can be covered by open affine subsets of $\mathbb{A}^{n-3}$.*
(ii) *B_n is nonsingular.*
(iii) *$\pi : B_{n+1} \rightarrow B_n$ is flat.*

Note that to prove (i) we have to use all the projections $B_{n+1} \rightarrow B_n$ with respect to the different indices. Equivalently we could use the obvious action of the symmetric group S_{n+1} on B_{n+1} to obtain the desired covering.

(3.3) There are natural sections $\sigma_1, \dots, \sigma_n$ to our projection $\pi : B_{n+1} \rightarrow B_n : \sigma_i$ is the morphism defined by sending $\lambda_v^{(n+1)}$ to $\lambda_v^{(n)}$ if $v \in V_n$ and $\lambda_{v_1 v_2 v_3 n+1}$ to $\lambda_{v_1 v_2 v_3 i}$ if $i \notin \{v_1, v_2, v_3\}$ and to 0, ∞ and 1 if $i = v_1$, v_2 and v_3, respectively. By (3.1) and (3.2) these sections make $\pi : B_{n+1} \rightarrow B_n$ into a stable n-pointed tree of projective lines.

PROPOSITION 4. *$\pi : B_{n+1} \rightarrow B_n$ is the universal stable n-pointed tree of projective lines.*

This means that the functor which associates with every noetherian scheme S the set of stable n-pointed trees of projective lines over S, is represented by B_n.

In other words B_n is a fine moduli space for stable n-pointed trees of projective lines.

PROOF. We have to show that for any stable n-pointed tree of projective lines $f : X \rightarrow S$ there is a unique morphism $u : S \rightarrow B_n$ such that X becomes isomorphic to $B_{n+1} \times_{B_n} S$.

By Prop. 3 we have a morphism $u : S \rightarrow \mathbb{P}^{V_n}$. It clearly factors through B_n, so we consider u as a morphism $u : S \rightarrow B_n$.

For any triple $d = (d_1, d_2, d_3) \in D$ we have a corresponding contraction $c(d) : X \rightarrow X_d$ of X and a commutative diagram:

$$\begin{tikzcd}
g(d) : X \arrow[r, "c(d)"] \arrow[d] & X_d \xrightarrow{\sim} \mathbb{P}_d \times S \arrow[r, "(id, u)"] & \mathbb{P}_d \times B_n \arrow[d] \\
S \arrow[rr, "u"] & & B_n
\end{tikzcd}$$

The product $\prod_{d \in D} g(d) : X \rightarrow \prod_{d \in D} \mathbb{P}_d \times B_n$ factors over B_{n+1}. Here B_{n+1} is seen as a closed subscheme of $\prod \mathbb{P}_d \times B_n$. The corresponding $g : X \rightarrow B_{n+1}$

satisfies $\pi \circ g = u \circ f$ and $g \circ \phi_i = \sigma_i \circ u$ for $i = 1, \ldots, n$. Moreover g induces on any component of X_s, $s \in S$, either a constant map or an isomorphism. As the fibre X_s is a stable n-pointed tree over $k(s)$, and as g preserves the marked points, it cannot map a component onto a point. This shows that the induced morphism $X \to B_{n+1} \times_{B_n} S$ is an isomorphism. We still have to prove the uniqueness of u. Suppose that $u_1 : S \to B_n$ and a compatible isomorphism $X \to B_{n+1} \times_{B_n} S$ are given. The induced $g_1 : X \to B_{n+1}$ satisfies again:

(i) $\pi \circ g_1 = u_1 \circ f$ and $g_1 \circ \phi_i = \sigma_i \circ u_1$ for $i = 1, \ldots, n$.

(ii) g_1 induces for any $s \in S$ on any component of X_s either a constant map or an isomorphism.

We fix a triple $d = (v_1, v_2, v_3)$ and we contract both X and B_{n+1} with respect to all $i \notin \{v_1, v_2, v_3\}$. Since contraction commutes with base-change we find a morphism $g_{1,d}$ between the contractions X_d and $(B_{n+1})_d$. Now X is identified with $\mathbb{P}_1 \times S$ such that ϕ_{v_1}, ϕ_{v_2}, ϕ_{v_3} become the sections 0, ∞ and 1, and we have a similar identification $(B_{n+1})_d = \mathbb{P}_1 \times B_n$. Then $g_{1,d} = (id_{\mathbb{P}_1}, u_1)$. For any $v_1 \notin \{v_1, v_2, v_3\}$ one has that $pr_1 \circ g_{1,d} \circ \phi_{v_4} : S \to \mathbb{P}_1$ coincides with $u_{v_1 v_2 v_3 v_4}$. Hence $u = u_1$ and $g = g_1$ since $g_{1,d} = g_d$ for all d.

(3.4) We fix two indices i, j such that $1 \leq i \leq j \leq n$, and form the fibre product

$$Z := Z_n^{ij} := B_n \times_{B_{n-1}} B_n$$

with respect to the projections π_n^i and π_n^j induced by omitting the index i and j, respectively.

The projection

$$pr_1 : Z \to B_n$$

onto the first factor is a stable $(n-1)$-pointed tree.

We also have an extra section, namely the diagonal map $\Delta : B_n \to Z$. In this situation Knudsen defines in [Kn], § 2 the stabilization of $pr_1 : Z \to B_n$ with respect to Δ. We claim that this stabilization is isomorphic to $\pi : B_{n+1} \to B_n$. By [Kn], Cor. 2.6 we can equivalently show that Z is isomorphic to the contraction of $\pi : B_{n+1} \to B_n$ with respect to the i-th section σ_i. Now the projections π_{n+1}^i and π_{n+1}^{j+1} from B_{n+1} to B_n satisfy

$$\pi_n^j \circ \pi_{n+1}^i = \pi_n^i \circ \pi_{n+1}^{j+1}.$$

Hence we get a proper morphism

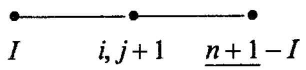
$$f : B_{n+1} \to Z.$$

Now for any $q \in B_n$ the fibre $\pi^{-1}(q) := B_{n+1} \times_{B_n} \operatorname{Spec} k(q)$ is isomorphic to the n-pointed tree $C(q)$ by lemma 1.

On the other hand, $pr_1^{-1}(q)$ is isomorphic to $(\pi_n^j)^{-1}(\pi_n^i(q))$, and this is in fact the contraction of $C(q)$ with respect to the i-th point.

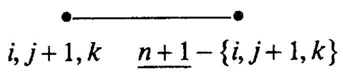
The above remark shows in particular that $f : B_{n+1} \to Z$ is birational. More precisely, f is an isomorphism on the open set $U := B_{n+1} - \bigcup B(T, \psi)$, where the union is taken over all stable $(n+1)$-marked trees of the following two types.

(a)



$I \subset \underline{n+1}-\{i, j+1\}$,

(b)



$k \in \underline{n+1}-\{i, j+1\}$.

The fibre over a point in $Z-f(U)$ is a projective line. Note that a subspace $B(T, \psi)$ of type (b) has codimension 1 in B_{n+1} whereas a subspace of type (a) has codimension 2, so their images in Z have codimension 2 and 3, resp.

There is no $(n+1)$-marked tree which can be contracted to different (T, ψ) of type (a) or (b), so the union in the definition of U is disjoint.

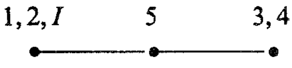
Although B_n and B_{n-1} are smooth varieties by (3.2) the morphism π_n^i is not smooth because of the singularities in the fibres. So Z may have singularities. In fact we have

PROPOSITION 5. *The singular set of Z is $S: = \cup f(B(T, \psi))$ where the union is taken over all (T, ψ) of type (a).*

PROOF. We first show that the singularities of Z are contained in S. Since f is biregular on U, it suffices to show that $f(B(T, \psi))$ is nonsingular for (T, ψ) of type (b). But any point z in such a set is mapped into B_{n-1}^* by $\pi_n^i \circ pr_1$, so it lies in a smooth fibre. Hence we can find an open $V \subset \bar{B}_n$ containing $pr_1(z)$ such that $\pi_n^i | V$ is smooth. Then $pr_2 | U \times_{B_{n-1}} B_n$ is also smooth, and thus z is a regular point of Z.

To prove that any $z \in S$ is in fact singular we calculate the local ring $\mathcal{O}_{Z,z}$: for simplicity we assume $i=4, j=5$, and the combinatorial type of $z_1: =pr_1(z)$ and $z_2: =pr_2(z)$ is

and



resp.

Here $I: =\underline{n}-\underline{5}$; one easily reduces to the case $n=5$, so we may take I to be empty.

The proof of lemma 2 shows that we may take $\alpha = : \lambda_{1342}$ and $\beta: =\lambda_{1354}$ as local coordinates in z_1 and $\gamma: =\mu_{1352}$ and $\delta: =\mu_{1345}$ as local coordinates in z_2. (We write λ and μ in order to distinguish the two copies of B_5).

Now π_5^4 is given by sending the coordinate $\nu=\nu_{1234}$ of B_4 onto λ_{1235}, whereas π_5^5 is given by $\nu \rightarrow \mu_{1234}$. In local coordinates we have $\lambda_{1235}=1-\alpha\beta$ and $\mu_{1234}=1-\gamma\delta$. Thus

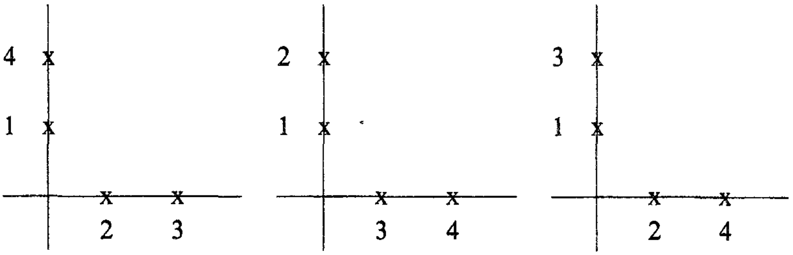
$$\mathcal{O}_{Z,z}=\mathbb{Z}[\alpha, \beta, \gamma, \delta]/(\alpha\beta-\gamma\delta),$$

and Z has a "conic" singularity in z.

COROLLARY. *The singular locus S of Z has codimension 3. Note that $f: B_{n+1} \to Z$ is a desingularization of Z which is not obtained by blowing up the singular locus.*

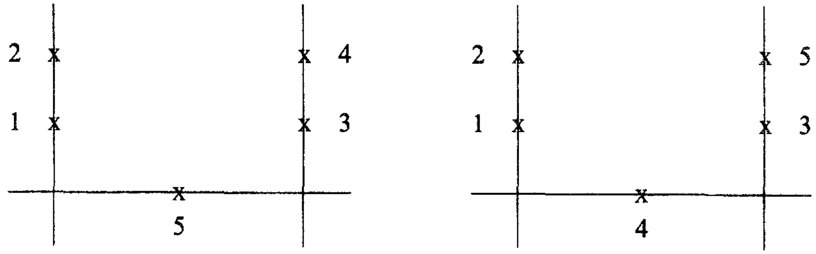
EXAMPLES. 1) From the definition we see that $B_4 \cong \mathbb{P}^1$.

2) B_5 is the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1 = B_4 \times B_4$ at the three points $(0,0)$, $(1,1)$ and (∞, ∞) that correspond respectively to the following 4-pointed trees (the same on both components):

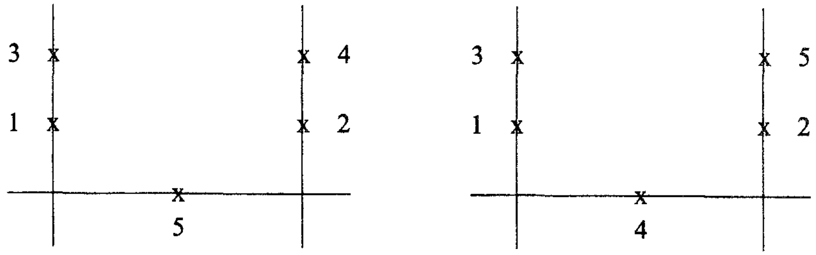


3) $B_5 \times_{B_4} B_5$ has three singular points: namely let the fibre product be defined by π_5^5 and π_5^4; then the projection of the singular points s_1, s_2, s_3 on the two factors B_5 are

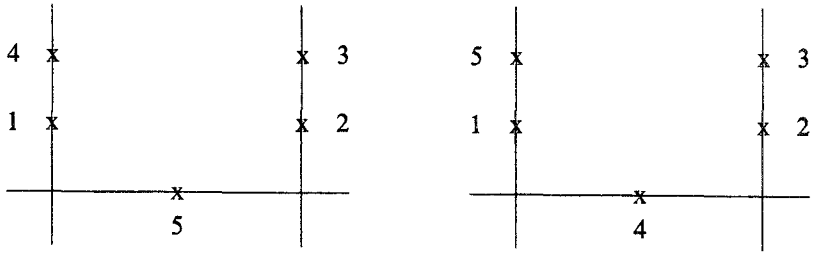
s_1:



s_2:

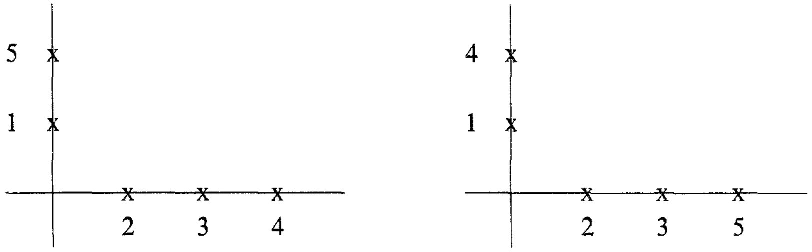


s_3:

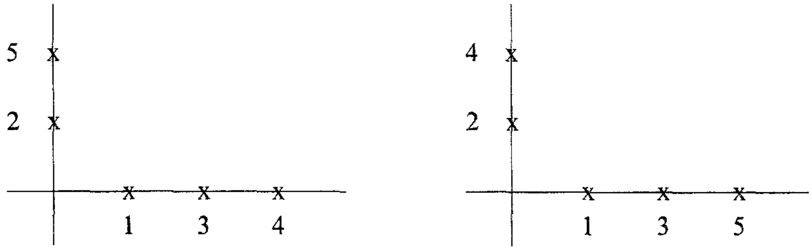


The map $f: B_6 \to Z = Z_5^{45}$ has nontrivial fibres over s_1, s_2, s_3, and over the four disjoint projective lines

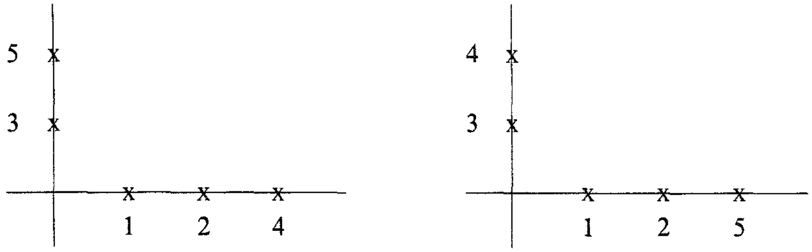
L_1:



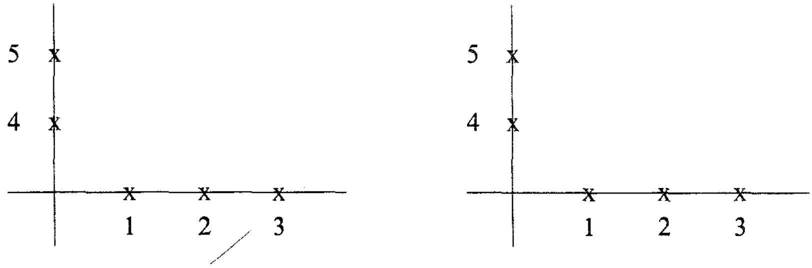
L_2:



L_3:



L_4:



4. PICARD GROUP AND BETTI NUMBERS OF B_n

(4.1) PROPOSITION 6. *Pic* (B_n) *is a free group of rank*

$$2^{n-1}-(n+1)-\frac{n(n-3)}{2}.$$

PROOF. 1) For any $S \subseteq \underline{n}$ with $2 \leq \# S \leq n-2$ we denote by $D(S)$ the divisor $B(T,\phi)$ where T has two vertices t_1, t_2 and $\phi^{-1}(t_1)=S$, $\phi^{-1}(t_2)=S^*: =\underline{n}-S$. From Proposition 2 in (1.5) and the Corollary in (3.2) it follows that $D(S)$ is an irreducible divisor.

It is obvious from the definition that $D(S)=D(T)$ if and only if $S=T$ or

$S=T^*=\underline{n}-T$, and that the irreducible components of $B_n-B_n^*$ are exactly the subschemes of the form $D(S)$ because any n-marked tree (T', ψ') with more than one vertex contracts to a n-marked tree with two vertices (T, ψ) and $B(T', \psi')\subseteq B(T, \psi)$, see (1.5).

$D(S)\cap D(T)\neq\emptyset$ if and only if one of the following cases occurs: $S\subset T$ or $T\subset S$ or $S\subset T^*$ or $T^*\subset S$ because if $D(S)\cap D(T)\neq\emptyset$ then there is a n-marked tree (T, ψ) with ≥ 3 vertices which contracts to the n-marked trees belonging to the subsets S and T of $\underline{n}$.

Now we prove that the divisor of the rational function $\lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}$ is equal to

$$\sum_{\substack{\upsilon_1, \upsilon_4\in S\\ \upsilon_2, \upsilon_3\in S^*}} D(S)-\sum_{\substack{\upsilon_1, \upsilon_3\in S\\ \upsilon_2, \upsilon_4\in S^*}} D(S)$$

The function $\lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}$ has no zeros nor poles in B_n^*. Further $\lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}$ is zero on $D(S)$ if and only if $\upsilon_1, \upsilon_4\in S$ and $\upsilon_2, \upsilon_3\notin S$ (or $\upsilon_1, \upsilon_4\notin S$ and $\upsilon_2, \upsilon_3\in S$) and $\lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}$ has a pole on $D(S)$ if and only if $\upsilon_1, \upsilon_3\in S$ and $\upsilon_2, \upsilon_4\notin S$ (or $\upsilon_1, \upsilon_3\notin S$ and $\upsilon_2, \upsilon_4\in S$). We only have to show that all the multiplicities are 1 or (-1). Fix some $S\subset\underline{N}$ with $2\leq \#S\leq n-2$. Then $D(S)^*$ is given by

$$\lambda_{abcd}=0 \text{ if } a, d\in S \text{ and } b, c\notin S$$

$$\lambda_{abcd}\neq 0,\ 1,\ \infty \text{ if } \#\{a, b, c, d\}\cap S\geq 3$$

$$\text{or if } \#\{a, b, c, d\}\cap S\leq 1.$$

Let U denote the open subset of B_n given by $\lambda_{abcd}\neq 0,\ 1,\ \infty$ if

$$\#\{a, b, c, d\}\cap S\geq 3 \text{ or } \#\{a, b, c, d\}\cap S\leq 1.$$

Clearly $D(S)^*=U\cap D(S)$.

Fix now a, b, c, d with $a, d\in S$ and $b, c\notin S$.

Let $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$ also satisfy $\upsilon_1, \upsilon_4\in S$ and $\upsilon_2, \upsilon_3\notin S$. Consider the following equations:

$$\lambda_{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}=\lambda_{\upsilon_1\upsilon_2\upsilon_3 c}\cdot\lambda_{\upsilon_1\upsilon_2 c\upsilon_4}$$

$$\lambda_{\upsilon_1\upsilon_2 c\upsilon_4}=\lambda_{c\upsilon_4\upsilon_1\upsilon_2}=\lambda_{c\upsilon_4\upsilon_1 b}\cdot\lambda_{c\upsilon_4 b\upsilon_2}=\lambda_{\upsilon_1 bc\upsilon_4}\cdot\lambda_{c\upsilon_4 b\upsilon_2}$$

$$\lambda_{\upsilon_1 bc\upsilon_4}=\lambda_{\upsilon_1 bcd}\cdot\lambda_{\upsilon_1 bd\upsilon_4}$$

$$\lambda_{\upsilon_1 bcd}=\lambda_{cd\upsilon_1 b}=\lambda_{cd\upsilon_1 a}\cdot\lambda_{cdab}=\lambda_{cd\upsilon_1 a}\cdot\lambda_{abcd}.$$

This shows that $\lambda_{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4}=u.\ \lambda_{abcd}$ where u is a unit on U. Hence $D(S)\cap U$ is defined by the principal ideal (λ_{abcd}) on U. Since a, b, c, d were arbitrary, except for $a, d\in S$ and $b, c\notin S$, we have shown that all multiplicities are 1 and -1.

2) Pic (B_n) is generated by the $D(S)$ because B_n^* is factorial. The number of generators is $2^{n-1}-(n+1)$. The relations in Pic (B_n) are given by the divisors of $t_i:=\lambda_{1,2,3,i}$, t_i-1 and $(t_i-t_j)t_j^{-1}$ with $i, j\geq 4$

$$\left(\text{in number } \frac{n(n-3)}{2}\right).$$

The proposition will be proved if we can find some

$$\frac{n(n-3)}{2} \times \frac{n(n-3)}{2}$$

submatrix of the relations with determinant ± 1. In order to find this submatrix we only look at the $D(S)$ with $\# S=2$.

An easy calculation yields:

$$t_i=\lambda_{123i} \text{ and div } (t_i)=\underline{D(\{1,i\})}+D(\{2,3\})-D(\{2,i\})-D(\{1,3\})$$

$$-t_i+1=\lambda_{23i1} \text{ and div } (1-t_i)=D(\{1,2\})+\underline{D(\{3,i\})}-D(\{2,i\})-D(\{1,3\})$$

$$t_j^{-1}(t_i-\lambda_j)=\lambda_{2ji1} \text{ and div } (\lambda_{2ji1})=D(\{1,2\})+\underline{D(\{i,j\})}-D(\{2,i\})-D(\{1,j\}).$$

The underlined $D(S)$ occur just once; they give a submatrix of size

$$\frac{n(n-3)}{2} \times \frac{n(n-3)}{2}$$

of determinant 1.

(4.2) Let $F_n(x)$ be the polynomial

$$\sum_{(T,\psi)} (x-2)^{r_4(T,\psi)} \cdot \ldots \cdot (x-n+2)^{r_n(T,\psi)}$$

where the summation is over all isomorphism classes of n-marked stable trees and where $r_i(T,\psi)$ is the number of vertices of T of valence $\geq i$. The term

$$F_{(T,\psi)}(x)=(x-2)^{r_4(T,\psi)} \cdot \ldots \cdot (x-n+2)^{r_n(T,\psi)}$$

has degree $(n-3)$-number of edges of T. If (T,ψ) is obtained from (T',ψ') by contracting one edge, then

$$\deg F_{(T',\psi')}=\sum_{i=4}^{n} r_i(T',\psi')=\deg F_{(T,\psi)}-1$$

because

$$\sum_{i=4}^{n} r_i(T,\psi)=\sum_{t\in T_0} (\text{val } t-3) \text{ and } \sum_{t\in T_0} \text{val } t=\sum_{t'\in T_0'} \text{val } t'-2$$

while $\# T_0'=1+\# T_0$.

In the sum there is one term of degree $n-3$ which is $F_{(T^0,\psi^0)}$ where T^0 has just one vertex and

$$F_{(T^0,\psi^0)}(x)=(x-2)(x-3)\cdot \ldots \cdot(x-n+2).$$

Thus $\deg F_n(x)=n-3$. Let

$$F_n(x)=\sum_{i=0}^{n-3} F_{ni}x^i \in \mathbb{Z}[x].$$

PROPOSITION 7. *Let h_i be the rank of the cohomology group $H^i(B_n(\mathbb{C}),\mathbb{Z})$ which is the i-th Betti number of the complex manifold of $\mathbb{C}$-valued points of B_n. Then*

$$h_i=\begin{cases} F_{n,n-3-r} : \text{if } i=2r \text{ is even} \\ 0 \quad : \text{if } i \text{ is odd.} \end{cases}$$

Moreover

$$h_2=2^{n-1}-n-\binom{n-1}{2},$$

$F_{n,n-3-r}=F_{n,r}$, $\sum_{i=0}^{n-3} F_{n,i}\cdot 2^i$ *is the number of* 3*-regular n-marked stable trees.*

PROOF. 1) The number of $\mathbb{F}_{p^r}$-valued points of B_n is $N_r=\sum_{i=0}^{n-3} F_{ni}(p^r)^i$ by the corollary to proposition 2 in (1.5), where p is any prime number. The zeta-function of $B_n\times\mathbb{F}_p$ is thus

$$\begin{aligned} Z(t)&=\exp\left(\sum_{\nu=1}^{\infty}\sum_{i=0}^{n-3} F_{ni}(p^i)^\nu \frac{t^r}{\nu}\right) \\ &=\exp\sum_{i=0}^{n-3} F_{ni}\left(\sum_{\nu=1}^{\infty}\frac{(p^it)^\nu}{\nu}\right) \\ &=\prod_{i=0}^{n-3}\frac{1}{(1-p^it)F_{ni}} \end{aligned}$$

B_n is smooth and projective and therefore by the Riemann hypothesis for $Z(t)$ one gets that all F_{ni} are ≥ 0, see [D].

Moreover also through the Weil conjectures one knows that

$$h_{2i}=F_{ni}$$

$$h_{2i+1}=0.$$

The functional equation for $Z(t)$ tells that $F_{n,n-3-r}=F_{n,r}$.

2) We determine the n-marked trees (T,ψ) with deg $(T,\psi)=n-4$. Any such (T,ψ) is uniquely given by a pair of subsets A, $\underline{n}-A$ of $\underline{n}$ with $2\leq \#A\leq n-2$. The number of these subset pairs is $2^{n-1}-n-1$.

Now

$$F_{(T^0,\psi^0)}(x)=x^{n-3}-(2+3+\cdots+n-2)x^{n-4}$$

+lower terms and $F_{(T,\psi)}(x)=x^{n-4}+$lower terms if deg $(T,\psi)=n-4$. Thus

$$F_n(x)=x^{n-3}+(2^{n-1}-n-1-(2+3+\cdots+n-2))x^{n-4}$$

+lower terms. This shows that

$$F_{n,n-4}=2^{n-1}-n-\binom{n-1}{2}.$$

If $F_{(T,\psi)}$ is not a constant then $F_{(T,\psi)}(x)=(x-2)^{r_4}\cdots(x-n-2)^{r_n}$ with $r_4\geq 1$ and thus $F_{(T,\psi)}(2)=0$. Thus $\sum_{i=0}^{n-3} F_{ni}2^i$ is the number of stable n-marked trees (T,ψ) for which $F_{(T,\psi)}$ is constant. This is the case iff (T,ψ) is 3-regular and for 3-regular n-marked trees (T,ψ) one has $F_{(T,\psi)}=1$.

5. B_n AS BLOW UP OF $\mathbb{P}_1^n/PGL_2$

(5.1) The natural action of PGL_2 on $(\mathbb{P}_1)^n$, given by $\sigma(x_1,\dots,x_n)=$ $=(\sigma(x_1),\dots,(x_n))$, has been studied in detail in [GIT], [MS] and [G]. We denote by $(\mathbb{P}_1^n)_{ss}$ the set of points in $\mathbb{P}_1^n$ that are semistable for this action, and by Q_n the quotient $(\mathbb{P}_1^n)_{ss}/PGL_2$. In this section we show that B_n is a blow up of Q_n and describe this blow up explicitly.

We begin by recalling the basic results on Q_n from [GIT], [MS], or [G]: $x=(x_1,\dots,x_n)\in\mathbb{P}_1^n$ is semistable if and only if $a_y(x):=\#\{i:x_i=y\}\leq n/2$ for any $y\in\mathbb{P}_1$; x is stable if and only if $a_y(x)<n/2$ for any $y\in\mathbb{P}_1$.

For n odd, the sets $(\mathbb{P}_1^n)_s$ of stable points and $(\mathbb{P}_1^n)_{ss}$ coincide, and Q_n is a geometric quotient. Moreover Q_n is smooth and projective over $\mathbb{Z}$ of relative dimension $n-3$.

For $n=2m$, $\tilde{Q}_n:=(\mathbb{P}_1^n)_s/PGL_2$ is again a geometric quotient, but not complete, whereas Q_n is projective but not a geometric quotient. In fact $Q_n-\tilde{Q}_n$ consists of $\frac{1}{2}\binom{2m}{m}$ points corresponding to the orbits of points where exactly m entries coincide. (Note that the point $(0,\dots,0,\infty,\dots,\infty)$ is contained in the boundary of the orbit of any point of the form $(x,\dots,x,x_{m+1},\dots,x_n)$ or $(x_1,\dots,x_m,x,\dots,x)$.) $\tilde{Q}_n$ is smooth over $\mathbb{Z}$, but we have

PROPOSITION 8. *For n even, $Q_n-\tilde{Q}_n$ is the singular locus of Q_n.*

PROOF. According to [G], $Q_{2m}=\operatorname{Proj} A$, where $A=\bigoplus_{k=0}^{\infty} A_k$ is a graded $\mathbb{Z}$-algebra with $A_0=\mathbb{Z}$.

A_1 generates A, and A_1 as a $\mathbb{Z}$-module is generated by the expressions $p_{a_1,b_1}p_{a_2,b_2}\cdots p_{a_m,b_m}$ where

(1) $\{1,2,\dots,2m\}=\{a_1,b_1,\dots,a_m,b_m\}$

(2) $p_{a,b}=x_a(0)x_b(1)-x_a(1)x_b(0)$ and $(x_i(0),x_i(1))$ denote the homogeneous coordinates of the i^{th} factor in $(\mathbb{P}_1^n)$.

Let $U\subset Q_{2m}$ be given by $p_{1,b_1}\cdots p_{m,b_m}\neq 0$ for all $b_1,\dots,b_m$ with $\{b_1,\dots,b_m\}=$ $=\{m+1,\dots,2m\}$. Clearly U is affine. Let $\tau:(\mathbb{P}_1^n)_{ss}\to Q_{2m}$ denote the canonical map. Then $\tau^{-1}(U)=\{(p_1,\dots,p_{2m})\in\mathbb{P}_1^n: p_i\neq p_j \text{ for } i\leq m<j\}$. Since $\tau:(\mathbb{P}_1^n)_{ss}\to$ $\to Q_{2m}$ is a good quotient we find that $\mathcal{O}(U)=\mathcal{O}(\tau^{-1}U)^{PGL_2}$. Consider $W\subset\tau^{-1}U$ given as $W=\{(p_1,\dots,p_{2m})\in\tau^{-1}(U):p_1=0,p_{m+1}=\infty\}$. Clearly $\mathcal{O}(\tau^{-1}U)^{PGL_2}=\mathcal{O}(W)^{\mathbb{G}_m}$.

Put $q_i=p_{i+m}^{-1}$ for $i\geq 2$ and identify W with the open part

$$\{(p_2,\dots,p_m,q_2,\dots,q_m)\in\mathbb{A}_{\mathbb{Z}}^{m-1}\times\mathbb{A}_{\mathbb{Z}}^{m-1}:p_iq_j\neq 1 \text{ for all } i,j\}$$

of the affine space $\mathbb{A}_{\mathbb{Z}}^{2m-2}$. The action of $\mathbb{G}_m$ is given by

$$a(p_2,\dots,p_m,q_2,\dots,q_m)=(ap_2,\dots,ap_m,a^{-1}q_2,\dots,a^{-1}q_m).$$

One easily sees that

$$\mathcal{O}(W)^{\mathbb{G}_m} = \mathbb{Z}[p_i q_j \mid i, j = 2, \ldots, m]_{\text{loc}} =$$
$$= \mathbb{Z}[A_{ij} \mid i, j = 2, \ldots, m]/(A_{ij}A_{kl} - A_{il}A_{kj}) \text{ loc},$$

where A_{ij} stands for $p_i q_j$ and where loc means localisation at $\prod_{i,j} (A_{ij} - 1)$.

Let $\mathbb{A}^{m-1} \otimes \mathbb{A}^{m-1}$ denote the $(m-1)^2$-dimensional affine space over $\mathbb{Z}$ and let $\mathbb{A}^{m-1} \circ \mathbb{A}^{m-1}$ be the closed subscheme of $\mathbb{A}^{m-1} \otimes \mathbb{A}^{m-1}$ consisting of the simple tensors, i.e. the elements of the form $v_1 \otimes v_2$. Then

$$\mathbb{A}^{m-1} \circ \mathbb{A}^{m-1} \cong \text{spec } (\mathbb{Z}[A_{ij} : i, j = 2, \ldots, m]/(A_{ij}A_{kl} - A_{kj}A_{il})).$$

In particular, the singular locus of U corresponds to the 0-section of $\mathbb{A}^{m-1} \circ \mathbb{A}^{m-1}$, and corresponds to the prime-ideal $(A_{ij} : \text{all } i, j)$ of the ring above.

The isomorphism of U with the open subset of $\mathbb{A}^{m-1} \circ \mathbb{A}^{m-1}$ given by $A_{ij} \neq 1$ for all i, j can be easily described by the morphism $\tau^{-1}U \rightarrow \mathbb{A}^{m-1} \circ \mathbb{A}^{m-1}$ given by the formula

$$(p_1, \ldots, p_{2m}) \rightarrow (\sigma(p_2), \ldots, \sigma(p_m)) \otimes (\sigma(p_{m+1})^{-1}, \ldots, \sigma(p_{2m})^{-1}),$$

in which

$$\sigma(z) = \frac{z - p_1}{z - p_{m+1}}.$$

Note that for $m = 3$ the singularities of Q_6 are isomorphic to those of $B_5 \times_{B_4} B_5$, see (3.5).

(5.2) Let $(\mathbb{P}_1^n)^* = \{(x_1, \ldots, x_n) \in \mathbb{P}_1^n : x_i \neq x_j \text{ for } i \neq j\} \subset (\mathbb{P}_1^n)_s$ and let $Q_n^* : = (\mathbb{P}_1^n)^*/PGL_2$. Then there is a natural isomorphism $p_n^* : B_n^* \rightarrow Q_n^*$. We can extend p_n^* to a morphism $p_n : B_n \rightarrow Q_n$ in the following way:

For $q \in B_n$ there exists a component L of the associated stable n-pointed tree $C(q)$ such that $(\pi_L(x_1), \ldots, \pi_L(x_n)) \in (\mathbb{P}_1^n)_{ss}(k(q))$ where $\pi_L : C(q) \rightarrow L$ is the projection onto L and $x_1 : = \phi_q(i) \in C(q)(k(q))$ is the i-th marked point on $C(q)$ and where we identify L with $\mathbb{P}_1(k(q))$ in some way. The existence of such an L can easily be proved by induction on n. Note however that for n odd, L is unique, whereas for n even there may be two intersecting components L, L' with the required property. We call L (or L and L') the center of $C(q)$. Now we define $p_n(q)$ as the PGL_2-orbit of $(\pi_L(x_1), \ldots, \pi_L(x_n))$.

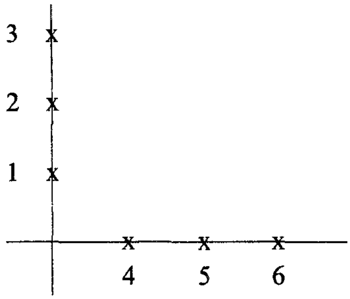
Let $d \in D_n$ be a triple of indices such lthat L is the median component $L(d)$ of d. In an open affine neighbourhood U_q of q, $L(d)$ is still the center of $C(q')$, $q' \in U_q$. Now it is not hard to see that $q \rightarrow p_n(q)$ is a morphism on U_q and that these morphisms on a covering of B_n by U_q's fit together to a projective morphism $p_n : B_n \rightarrow Q_n$. Since p_n is an isomorphism on B_n^*, thus birational, we have proved (see [H], Ch. II, Thm. 7.17):

PROPOSITION 9. *B_n is the blowing-up of Q_n with respect to some coherent sheaf of ideals.*

REMARK. $p_4 : B_4 \to Q_4$ is obviously an isomorphism.

$p_5 : B_5 \to Q_5$ is also an isomorphism since for $q \in B_5$ any component of $C(q)$ which is different from the center, has order 3.

The first nontrivial blowing up occurs when $n=6$. The nontrivial fibres of p_6 are the 10 disjoint subschemes in B_6, each isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$, for which the associated 6-pointed tree contracts to a permutation of



These subvarieties are mapped onto the 10 singular points of Q_6. Surprisingly, although Q_6 and $B_5 \times_{B_4} B_5$ are locally isomorphic around the singular points, the maps $B_6 \to Q_6$ and $B_6 \to B_5 \times_{B_4} B_5$ are completely different.

(5.3) In order to find on explicit description of the blow up $B_n \to Q_n$ we introduce the notion of a stable (d,n)-tree:

DEFINITION. a) Let k be a field and d, n integers satisfying $1 \le d \le (n-1)/2$. A tree of projective lines X over k (see (1.1)) together with a map $\phi : \{1, \dots, n\} \to X(k)$ is called a stable (d,n)-tree over k if

(i) $\phi(i)$ is nonsingular for all i

(ii) $\# \phi^{-1}(a) \le d$ for all $a \in X(k)$

(iii) $\operatorname{ord}_d(L) > 2$ for any irreducible component of X, where $\operatorname{ord}_d(L) : =$ $= \# \{L' : L' \text{ component of } X, L' \cap L \neq \emptyset\} + 1/d \# \phi^{-1}(L)$.

b) A stable (d,n)-tree X over a scheme S is a flat morphism $\pi : X \to S$ together with sections $\phi_1, \dots, \phi_n : S \to X$ such that for any $s \in S$, the fibre $X_s = X \times_S \operatorname{spec} k(s)$ together with the map $\phi_s : i \to \phi_i(s)$ is a stable (d,n)-tree over $k(s)$.

Note that a stable $(1,n)$-tree is just a stable n-pointed tree. On the other hand a stable (d,n)-tree is in general not stable as $(d+1,n)$-tree.

We want to show the existence of fine moduli spaces $B_{n,d}$ for stable (d,n)-trees and begin with the case $(d, 2d+1)$; it will turn out that $B_{2d+1,d}$ is just Q_{2d+1}. In order to describe the universal family we have to introduce some other blow ups of Q_n: For disjoint subsets $I_1, \dots, I_r$ of $\{1, \dots, n\}$ let $Q_n(I_1, \dots, I_r)$ be the image in Q_n of $\{(x_1, \dots, x_n) \in (\mathbb{P}_1^n)_{ss} : x_i = x_j$ if there exists ν such that $\{i, j\} \subset I_\nu\}$. Clearly $Q_n(I_1, \dots, I_r)$ is a closed subscheme of Q_n.

Recall from the proof of Prop. 1 that Q_{2m} is locally at its singular points isomorphic to $\mathbb{A}^{m-1} \circ \mathbb{A}^{m-1}$. Consider the scheme

$$(\mathbb{A}^{m-1} \circ \mathbb{A}^{m-1})' : = \{(v_1 \otimes v_2, \bar{w}) \in (\mathbb{A}^{m-1} \circ \mathbb{A}^{m-1}) \times \mathbb{P}_{m-2} : \bar{v}_1 = \bar{w}\}$$

(here we consider $\mathbb{P}_{m-2}$ as $\mathbb{P}(\mathbb{A}^{m-1})$ and denote by $\bar{v}_1$ the class of v_1). Clearly $(\mathbb{A}^{m-1}\circ\mathbb{A}^{m-1})'$ is a desingularization of $\mathbb{A}^{m-1}\circ\mathbb{A}^{m-1}$, and the fibre over the 0-section is $\mathbb{P}_{m-2}$. Let Q'_{2m} be the desingularization of Q_{2m} which locally over the singular points is isomorphic to $(\mathbb{A}^{m-1}\circ\mathbb{A}^{m-1})'$. Q'_{2m} is not isomorphic to the blow up Q''_{2m} of Q_{2m} at all singular points since the fibre in Q''_{2m} over a singular point is isomorphic to $\mathbb{P}_{m-2}\times\mathbb{P}_{m-2}$. Moreover Q'_{2m} is not symmetric in the indices $1,\dots,2m$.

We can embed Q'_{2m} as a closed subscheme of Q_{4m-1} as follows: Let $Y:=Q_{4m-1}(\{1,2\},\{3,4\},\dots,\{4m-3,4m-2\})$; thus Y consists of the orbits of the points $(p_1,p_1,p_2,p_2,\dots,p_{2m-1},p_{2m-1},p_{2m})\in\mathbb{P}_1^{4m-1}$. There is an obvious morphism $Y\to Q_{2m}$ induced by $(p_1,p_1,\dots,p_{2m-1},p_{2m-1},p_{2m})\to(p_1,p_2,\dots,p_{2m})$; it is an isomorphism outside the singular set in Q_{2m}. The fibres over the singular points are $\mathbb{P}_{m-2}$: the preimage of the point

$$(p_1,\dots,p_m,p_{m+1},p_{m+1},\dots,p_{m+1})$$

consists of all points $(p_1,p_1,\dots,p_m,p_m,p_{m+1},\dots,p_{m+1})\in\mathbb{P}_1^{4m-1}$ such that $p_i\neq p_{m+1}$ for $i=1,\dots,m$ and $p_1,\dots,p_m$ are not all equal. Taking $p_1=0$, $p_{m+1}=\infty$ leaves us with a $\mathbb{G}_m$*-action on* $\{(p_2,\dots,p_m)\neq(0,\dots,0)\}=\mathbb{A}^{m-1}-\{(0,\dots,0)\}$. This shows that Y is isomorphic to Q'_{2m}.
On the other hand, the morphism $Y\to Q_{2m-1}$ induced by

$$(p_1,p_1,\dots,p_{2m-1},p_{2m-1},p_{2m})\to(p_1,\dots,p_{2m-1})$$

has fibres isomorphic to $\mathbb{P}_1$, in fact Y is a locally trivial $\mathbb{P}_1$-bundle. There are sections $\sigma_i: Q_{2m-1}\to Y$, $i=1,\dots,2m-1$ given by

$$\sigma_i(p_1,\dots,p_{2m-1})=(p_1,p_1,\dots,p_{2m-1},p_{2m-1},p_i).$$

We conclude that if we take $d=m-1$ and identify Y with Q'_{2m}, then $Q'_{2d+2}\to Q_{2d+1}$ is a stable $(d,2d+1)$-tree.

PROPOSITION 10. $Q'_{2d+2}\to Q_{2d+1}$ *is the universal stable* $(d,2d+1)$*-tree.*

PROOF. Let $\pi: X\to S$ be any stable $(d,2d+1)$-tree. For any triple

$$v=(v_1,v_2,v_3)\in D_{2d+1}$$

let $S(v)$ be the open subscheme of S on which ϕ_{v_1},ϕ_{v_2} and ϕ_{v_3} do not meet. Since the $S(v)$ cover S it is sufficient to prove the proposition for $S=S(1,2,3)$.

With respect to ϕ_1,ϕ_2,ϕ_3, $\pi: X\to S$ is a stable 3-tree. Thus by (2,3) we get morphisms $\phi'_i: S\to\mathbb{P}_1$, $i=4,\dots,2d+1$, and by (2,3) we have isomorphisms $\psi_i: X\to\mathbb{P}_1\times S$ such that $\psi_i\circ\phi_i=(\phi'_i,id)$. Letting ϕ'_1,ϕ'_2,ϕ'_3 be the constant morphisms 0, ∞, 1, the ϕ'_i define a morphism $\phi: S\to(\mathbb{P}_1^{2d+1})_{ss}\to Q_{2d+1}$. Clearly ϕ induces an isomorphism $X\xrightarrow{\sim}Q'_{2d+2}\times_{Q_{2d+1}}S$.

(5.4) PROPOSITION 11. $Q'_{2d+3}\to Q''_{2d+2}$ *is the universal stable* $(d,2d+2)$*-tree.*
Here Q''_{2d+2} *denotes the blow up of* Q_{2d+2} *at all singular points, and* Q'_{2d+3} *is the blow up of* Q_{2d+3} *at all* $Q_{2d+3}(I)$ *such that* $\# I=d+1$ *and* $2d+3\notin I$.

PROOF. On the open parts

$$Q_{2d+3} - \bigcup_{\substack{\#I=d+1\\ 2d+3\notin I}} Q_{2d+3}(I) \text{ and } \tilde{Q}_{2d+2},$$

the morphism π is given by $(x_1, \ldots, x_{2d+3}) \to (x_1, \ldots, x_{2d+2})$. With the help of the obvious sections $\sigma_i : (x_1, \ldots, x_{2d+2}) \to (x_1, \ldots, x_{2d+2}, x_i)$, $i = 1, \ldots, 2d+2$, π clearly becomes a stable $(d, 2d+2)$-tree. We have to extend π to a morphism $Q'_{2d+3} \to Q''_{2d+2}$.

Let $V = \{(x_1, \ldots, x_{2d+2}) \in \mathbb{P}_1^{2d+2} : x_i \neq x_j \text{ for } 1 \leq i \leq d+1 < j \leq 2d+2\}/PGL_2$. V is a neighbourhood of the singular point $(0, \ldots, 0, \infty, \ldots, \infty)$ in Q_{2d+2}. Let V'' be the inverse image of V in Q''_{2d+2}.

On the other hand, let U be the inverse image of V in Q_{2d+3} (for the projectives π). Clearly U is the union of the open subspaces

$$U_{ij} = \{(x_1, \ldots, x_{2d+3}) \in \mathbb{P}_1^{2d+3} : x_\nu \neq x_\mu \text{ for } 1 \leq \nu \leq d+1 < \mu \leq 2d+2,$$

$$x_\nu \neq x_{2d+3} \neq x_\mu\}/PGL_2 \text{ for } i = 1, \ldots, d+1, j = d+2, \ldots, 2d+2.$$

Any U_{ij} contains $Q_{2d+3}(\{1, \ldots, d+1\})$ and $Q_{2d+3}(\{d+2, \ldots, 2d+2\})$, and has empty intersection with all other $Q_{2d+3}(I)$, $\#I = d+1$, $2d+3 \notin I$.

Let U' and U'_{ij} be the inverse images of U and U_{ij} in Q'_{2d+3}, respectively. Then π can be extended to $U'_{1,d+2}$ as follows:

First note that $U_{1,d+2}$ has an open immersion into $\mathbb{A}^d \times \mathbb{A}^d$ by putting $x_1 = 0$, $x_{d+2} = \infty$ and $x_{2d+3} = 1$. The immersion is explicitely given by

$$(x_1, \ldots, x_{2d+3}) \to ((\sigma(x_2), \ldots, \sigma(x_{d+1})), (\sigma(x_{d+2})^{-1}, \ldots, \sigma(x_{2d+2})^{-1})$$

where

$$\sigma(z) = \frac{z - x_1}{z - x_{d+2}} \cdot \frac{x_{2d+3} - x_{d+2}}{x_{2d+3} - x_1}.$$

This identifies $U_{1,d+2}$ with $\{((x_1, \ldots, x_d), (y_1, \ldots, y_d)) \in \mathbb{A}^d \times \mathbb{A}^d | x_i y_j \neq 1$ for all i and $j\}$.

In particular

$$Q_{2d+3}(\{1, \ldots, d+1\}) \cap U_{1,d+2} = \{0\} \times \mathbb{A}^d$$

and

$$Q_{2d+3}(\{d+2, \ldots, 2d+2\}) \cap U_{1,d+2} = \mathbb{A}^d \times \{0\}.$$

So the blow up $U'_{1,d+2}$ is an open subset of:

$$\{(v_1, v_2, \bar{w}_1, \bar{w}_2) \in \mathbb{A}^d \times \mathbb{A}^d \times \mathbb{P}(\mathbb{A}^d) \times \mathbb{P}(\mathbb{A}^d) | v_i \text{ and } w_i$$

$$\text{are dependent for } i = 1, 2\}.$$

We know already the explicit form of V'', namely an open part of

$$\{(v_1 \otimes v_2, \bar{w}_1, \bar{w}_2) \in \mathbb{A}^d \circ \mathbb{A}^d \times \mathbb{P}(\mathbb{A}^d) \times \mathbb{P}(\mathbb{A}^d) | v_1 \otimes v_2 \text{ and } w_1 \otimes w_2$$

$$\text{are dependent.}\}$$

The extension of our morphism π is now given by:

$$(v_1, v_2, \bar{w}_1, \bar{w}_2) \in U'_{1,d+2} \to (v_1 \otimes v_2, \bar{w}_1, \bar{w}_2) \in V''.$$

On affine parts of both of the $\mathbb{P}(\mathbb{A}_d)$'s this morphism reads:

$$(\lambda w_1, \mu w_2, w_1, w_2) \to (\lambda\mu w_1 \otimes w_2, w_1, w_2).$$

This morphism is identity in the last two factors and has the form $\mathbb{A}_1 \times \mathbb{A}_1 \to \to \mathbb{A}_1$, $(\lambda, \mu) \to \lambda\mu$, in the first factors. Hence the map is flat. The fibre above a point $\neq 0$ is isomorphic to $\mathbb{A}_1 - \{0\}$. The fibre of 0 is $(\mathbb{A}_1 \times \{0\}) \cup \cup (\{0\} \times \mathbb{A}_1)$. Glueing the various $U'_{i,j}$ together one finds that $U' \to V''$ is flat with fibre $\mathbb{P}_1$ or $(\mathbb{P}_1 \times \{0\}) \cup (\{0\} \times \mathbb{P}_1)$.

The sections $\sigma_i : V'' \to U'$ can also be described: For i with $2 \leq i \leq d+1$, we consider the open part of V'' where the i-th coordinate of $\bar{w}_1$ is not zero. On this open part σ_i has the form $(v_1 \otimes v_2, \bar{w}_1, \bar{w}_2) \to (\lambda w_1, \mu w_2, \bar{w}_1, \bar{w}_2)$ where λ is determined by: the i^{th} coordinate of λw_1 equals 1, and where μ is determined by $\lambda\mu w_1 \otimes w_2 = v_1 \otimes v_2$.

Glueing over the various $U'_{i,j}$ yields all the σ_i on all of V''. So we have shown that $Q'_{2d+3} \to Q''_{2d+2}$ with the σ_i is indeed a $(d, 2d+2)$-tree.

We now prove that this $(d, 2d+2)$-tree is universal.

Let $X \xrightarrow{\pi} S$, with sections ϕ_i, denote any stable $(d, 2d+2)$-tree. Locally on S we have to show existence and uniqueness of a morphism $f : S \to Q''_{2d+2}$ such that X is isomorphic to $Q'_{2d+3} \times_{Q''_{2d+2}} S$. For a point $s \in S$ the fibre X_s has one or two components. Let us consider the case where X_s has two components L_1 and L_2. We may then suppose that $\phi_i(s) \in L_1$ for $i = 1, \ldots, d+1$; that $\phi_i(s) \in L_2$ for $i = d+2, \ldots, 2d+2$; that $\phi_1(s) \neq \phi_2(s)$ and that $\phi_{d+2}(s) \neq \phi_{d+3}(s)$. After shrinking S we may suppose that for all $t \in S$ and (i, j) of the form $(1,2)$, $(d+2, d+3)$ or $1 \leq i \leq d+1 < j \leq 2d+2$ one has $\phi_i(t) \neq \phi_j(t)$.

$X \to S$ with the 4 sections $\phi_1, \phi_2, \phi_{d+2}, \phi_{d+3}$ is a stable 4-tree and so there exists a morphism $u : S \to B_4 = \mathbb{P}_1$ such that $X \xrightarrow{\sim} B_5 \times_{B_4} S$. We may suppose $u(s) = o \in \mathbb{P}_1$ and $u(S) \subset \mathbb{A}_1 - \{1\}$. According to (3.4), B_5 is the blow up of $\mathbb{P}_1 \times \mathbb{P}_1$ in the 3 sections $(0,0)$, $(1,1)$, (∞, ∞), and $B_5 \to B_4$ is derived from the projection on the second factor. Since $u(S)$ does not meet 1 and ∞ we may replace $B_4 = \mathbb{P}_1$ by $\mathbb{A}_1 - \{1\}$ and B_5 by Z, the blow up of $\mathbb{P}_1 \times (\mathbb{A}_1 - \{1\})$ in $(0,0)$. This Z is the closed subspace of $\mathbb{P}_1 \times \mathbb{P}_1 \times (\mathbb{A}_1 - \{1\})$ given by the equation $x_0 y_0 z - x_1 y_1 = 0$, where we have used $(x_0, x_1), (y_0, y_1), z$ as coordinates for the three factors. Hence X is isomorphic to the closed subscheme of $\mathbb{P}_1 \times \mathbb{P} \times S$ given by the equation $x_0 y_0 u - x_1 y_1 = 0$ where $u \in \mathcal{O}_S(S)$ satisfies $(u-1) \in \mathcal{O}_S(S)^*$.

We note in passing that this implies that $\{t \in S \mid X_t \text{ has two components}\} = = \{t \in S \mid u(t) = 0\}$ is closed.

We identify X with this closed subset of $\mathbb{P}_1 \times \mathbb{P}_1 \times S$ and we write $\phi_i(t) = = (\alpha_i(t), \beta_i(t), t)$ for $i = 1, \ldots, 2d+2$. The morphism $f : S \to V''$ is now given by $t \to (v_1 \otimes v_2, \bar{w}_1, \bar{w}_2)$ where

$$w_1 = (\sigma_1(\alpha_2(t)), \ldots, \sigma_1(\alpha_{d+1}(t)))$$

$$w_2=(\sigma_2(\beta_{d+3}(t)),\ldots,\sigma_2(\beta_{2d+2}(t)))$$

$$v_1\otimes v_2=u(t)w_1\otimes w_2$$

and

$$\sigma_1(z)=\frac{z-\alpha_1(t)}{z-\alpha_{d+2}(t)}\,\frac{\alpha_2(t)-\alpha_{d+2}(t)}{\alpha_2(t)-\alpha_1(t)}$$

$$\sigma_2(z)=\frac{z-\beta_{d+2}(t)}{z-\beta_1(t)}\,\frac{\beta_{d+3}(t)-\beta_1(t)}{\beta_{d+3}(t)-\beta_{d+2}(t)}$$

In order to verify that $X\to S$ is isomorphic to $U'\times_{V''}S\to S$ we consider an open part X' of X defined by $x_0\neq 0$ and $y_0\neq 0$. On this open part one can define a morphism $g:X'\to U'_{1,d+2}$ by $t\to(v_1,v_2,\bar{w}_1,\bar{w}_2)$ where w_1,w_2 are defined as before and where

$$v_1=\frac{x_1}{x_0}\,w_1 \text{ and } v_2=\frac{y_1}{y_0}\,w_2.$$

The diagram

$$\begin{tikzcd}
X' \arrow[r, dash, "g"] \arrow[d] & U'_{1,d+2} \arrow[d] \\
S \arrow[r, dash, "f"] & V''
\end{tikzcd}$$

is commutative and can easily be shown to be cartesian. (Indeed the morphism $X'\to U'_{1,d+2}\times_{V''}S$ is an isomorphism in every fibre and is therefore an isomorphism). Glueing yields an isomorphism $X\xrightarrow{\sim}U'\times_{V''}S$. One can also prove uniqueness of f. A similar but easier verification can be done in the neighbourhood of a point $s\in S$ such that the fibre X_s has only one component.

(5.5) The proof of the previous proposition easily generalizes to an inductive construction of the fine moduli spaces $B_{n,d}$ for stable (d,n)-trees, where we still assume $1\leq d\leq(n-1)/2$: Let $B'_{n,d}\to B_{n,d}$ be the universal stable (d,n)-tree. By induction, $B_{n,d}$ is a blow up of Q_n and $B'_{n,d}$ is a blow up of Q_{n+1}. Now $B_{n,d-1}$ is obtained from $B_{n,d}$ by blowing up the preimages of all subspaces $Q_n(I)$ where $|I|=d$. To get $B'_{n,d-1}$ from $B'_{n,d}$ we have to blow up the preimages of all subspaces $Q_{n+1}(I)$ where $|I|=d$ and $n+1\notin I$.

For fixed n we thus obtain a sequence of blowing ups which finally leads to $B_n=B_{n,1}$ and in which every intermediate blow up is either a fine moduli space $B_{n,d}$ or a universal family $B'_{n-1,d}$ for some d. Only the singular spaces Q_{2m} have no interpretation in terms of moduli spaces.

The situation is illustrated in the following diagram where the horizontal arrows are the various blow ups described and where the vertical map labelled "(d,n)" is the universal stable (d,n)-tree:

$$\begin{tikzcd}
 & & & & & & & B_4 \arrow[r, "\sim"] & Q_4 \\
 & & & & & & B_5 \arrow[ur, "{(1,4)}"] \arrow[r, "\sim"] & Q_5 & \\
 & & & & & B_6 \arrow[ur, "{(1,5)}"] \arrow[r, "\sim"] & Q_6'' \arrow[r] & Q_6' \arrow[r] \arrow[u, "{(2,5)}"'] & Q_6 \\
 & & & & B_7 \arrow[ur, "{(1,6)}"] \arrow[r, "\sim"] & Q_7'' \arrow[r] & Q_7' \arrow[r] \arrow[u, "{(2,6)}"'] & Q_7 & \\
 & & & B_8 \arrow[ur, "{(1,7)}"] \arrow[r, "\sim"] & Q_8'''' \arrow[r] & Q_8''' \arrow[r] \arrow[u, "{(2,7)}"'] & Q_8'' \arrow[r] & Q_8' \arrow[r] \arrow[u, "{(3,7)}"'] & Q_8 \\
 & & B_9 \arrow[ur, "{(1,8)}"] \arrow[r, "\sim"] & Q_9'''' \arrow[r] & Q_9''' \arrow[r] \arrow[u, "{(2,8)}"'] & Q_9'' \arrow[r] & Q_9' \arrow[r] \arrow[u, "{(3,8)}"'] & Q_9 & \\
 & B_{10} \arrow[ur, "{(1,9)}"] \arrow[r, "\sim"] & Q_{10}^{(6)} \arrow[r] & Q_{10}^{(5)} \arrow[r] \arrow[u, "{(2,9)}"'] & Q_{10}'''' \arrow[r] & Q_{10}''' \arrow[r] \arrow[u, "{(3,9)}"'] & Q_{10}'' \arrow[r] & Q_{10}' \arrow[r] \arrow[u, "{(4,9)}"'] & Q_{10} \\
B_{11} \arrow[ur, "{(1,10)}"] \arrow[r, "\sim"] & Q_{11}^{(6)} \arrow[r] & Q_{11}^{(5)} \arrow[r] \arrow[u, "{(2,10)}"'] & Q_{11}'''' \arrow[r] & Q_{11}''' \arrow[r] \arrow[u, "{(3,10)}"'] & Q_{11}'' \arrow[r] & Q_{11}' \arrow[r] \arrow[u, "{(4,10)}"'] & Q_{11} &
\end{tikzcd}$$

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