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# Sampled-data and discrete-time $H_2$ optimal control

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## Abstract

This paper deals with the sampled-data $H_2$ optimal control problem. Given a linear time-invariant continuous-time system, the problem of minimizing the $H_2$ performance over all sampled-data controllers with a fixed sampling period can be reduced to a pure discrete-time $H_2$ optimal control problem. This discrete-time $H_2$ problem is always singular. Motivated by this, in this paper we give a treatment of the discrete-time $H_2$ optimal control problem in its full generality. The results we obtain are then applied to the singular discrete-time $H_2$ problem arising from the sampled-data $H_2$ problem. In particular, we give conditions for the existence of optimal sampled data controllers. We also show that the $H_2$ performance of a continuous-time controller can always be recovered asymptotically by choosing the sampling period sufficiently small. Finally, we show that the optimal sampled-data $H_2$ performance converges to the continuous-time optimal $H_2$ performance as the sampling period converges to zero.

## 1 Introduction

Recently, much attention has been paid to $H_2$ and $H_\infty$ optimal control of linear systems using sampled-data control (see [2,4,6,7,11] and [1,3,5,9,10,14,17]). For a given continuous-time plant, a sampled-data controller consists of the cascade connection of an A/D converter, a discrete-time controller, and a D/A converter. The A/D device converts the continuous-time measured plant output into a discrete-time signal, which is used as an input for the discrete-time controller. The discrete-time controller generates a discrete-time output signal, which, in turn, is converted into a continuous-time signal that is used as a control input for the continuous-time plant.

Apart from a control input and a measurement output, the plant under consideration has an exogenous input and an output to be controlled. The quality of a controller is given by the performance of the corresponding closed-loop system. This performance measures the influence of the exogenous input on the output to be controlled. In the present paper, we will take as performance measure the $H_2$ performance of the closed loop system.

In contrast to the $H_\infty$ performance of a sampled-data control system, which in analogy with the pure continuous-time context can simply be defined as the norm of the input/output operator between the exogenous inputs and the outputs to

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be controlled, it is not clear from the outset how one should define the $H_2$ performance of a sampled-data control system. An, in our opinion, natural definition was given independently in [2,11]. In these references, the crucial observation is that the closed-loop system resulting from a sampled data controller, albeit time-varying, is in fact a periodic system, with period equal to the sampling period. It is then argued that, instead of aplying impulsive inputs at time $t = 0$, one should in fact apply these inputs at all time instances between 0 and the sampling period, and take the mean of the integral squares of the resulting outputs. This leads to an $H_2$ performance measure that captures the essential features of a sampled-data closed-loop system more satisfactorily. For a given continuous-time plant, the sampled-data $H_2$ optimal control problem is then to minimize the $H_2$ performance of the closed-loop system over all internally stabilizing sampled data controllers with a fixed sampling period. It is the latter problem that will be studied in the present paper.

It was shown in [2,4,11] that the sampled-data $H_2$ optimal control problem can be reduced to a pure discrete-time $H_2$ optimal control problem in the following way. First one defines an auxiliary time-invariant discrete-time system. Next, one expresses the sampled-data $H_2$ performance in terms of the 'normal' $H_2$ performance of the closed loop system obtained by interconnecting the auxiliary discrete-time system and the discrete-time controller defining the sampled data controller. Thus, the sampled data $H_2$ optimal control problem under consideration is completely resolved once the auxiliary discrete-time $H_2$ problem is. This procedure makes use of the so-called *lifting technique* (see [1,3,19]).

Now, it turns out that the auxiliary discrete-time $H_2$ problem obtained in this way *is always a singular problem*: the direct feedthrough matrix from the exogenous input to the measurement output is always equal to 0. Apart from this, in the auxiliary discrete-time system the direct feedthrough matrix from the control input to the output to be controlled is in general not injective. In [11], this difficulty is partly removed by introducing an additional noise on the sampled measured output signal and by assuming the corresponding feedthrough matrix to be surjective.

In the present paper we want to consider the completely general formulation of the sampled-data $H_2$ problem. We will take as a starting point the auxiliary discrete-time $H_2$ problem derived in [2,11]. As noted, this problem is inherently singular. To our best knowledge, no resolution of the discrete-time singular $H_2$ optimal is known in the literature. Therefore, a substantial part of this paper is devoted to a study of the completely general discrete-time $H_2$ problem (no assumptions on the direct feedthrough matrices, no assumptions on the absence of zeros on the unit circle).

We will describe a complete resolution to this problem, including a characterization of the optimal performance, and necessary *and* sufficient conditions for the existence of optimal controllers. The expression for the optimal performance is different from the one that might be expected in analogy with the continuous-time case (see [15]). Due to the fact that the role of the imaginary axis is taken over by the unit circle, for the discrete-time $H_2$ performance to be finite it is no longer required that the closed loop transfer matrix is *strictly proper*. Intuitively, this enlarges the class of admissible controllers and yields a smaller optimal performance.

We will apply our results on the discrete-time $H_2$ optimal control problem to the sampled-data $H_2$ problem by simply applying them to the auxiliary discrete-time system derived in [2, 11]. Our expression for the optimal sampled-data $H_2$ performance will be an immediate consequence of these results. We will however also be interested in conditions guaranteeing the existence of optimal sampled-data controllers. Our results on the general discrete-time $H_2$ problem give such conditions in terms of the auxiliary discrete-time system, but we will reformulate these conditions *in terms of the original continuous-time plant*. Preliminary results in that direction were also found in [11].

Obviously, the sampled-data $H_2$ optimal performance is a function of the sampling period. An important question is, what happens if the sampling period tends to zero. In particular, we will answer the following two questions. Firstly, if we control the original continuous-time plant by a 'normal' continuous-time compensator, is it then possible to recover this performance asymptotically by using a sampled data controller with sufficiently small sampling period? This question was also studied for the $H_\infty$ performance and for the $H_2$ performance à la Chen and Francis in [6]. A second, related, question that we will answer is: does the optimal sampled data $H_2$ performance converge to the optimal continuous-time $H_2$ performance as the sampling period decreases to zero?

The outline of this paper is as follows. In section 2 we will define the sampled data $H_2$ opimal control problem and recall the main results of [2, 11]. We will also introduce some notation. In section 3 we deal with the discrete-time $H_2$ optimal control problem. Then, in section 4, we return to the sampled-data context, and apply the results of sections 3 to the sampled-data $H_2$ optimal control problem. In particular, we will derive conditions in terms of the original continuous-time plant that guarantee the existence of optimal controllers for the sampled-data $H_2$ problem. Finally, in section 5 we study the afore-mentioned questions regarding the behaviour of the (optimal) performance as the sampling period tends to zero.

## 2 Problem formulation

Consider a continuous-time, linear, time-invariant, finite-dimensional plant $\Sigma$. Let $\Sigma$ have inputs $d$ and $u$, and outputs $z$ and $y$, where $d$ is an exogenous input, $u$ is a control input, $z$ is an output to be controlled, and $y$ is a measured output. We want to control $\Sigma$ by means of sampled data feedback control. We take a fixed $\Delta > 0$, called the *sampling period*. From the measured output $y$ we obtain a discrete-time signal $\bar{y} = \{y_k\}$ defined by $y_k := (S_\Delta y)_k$, where $S_\Delta$ denotes the sampling operator defined by $(S_\Delta y)_k := y(k\Delta)$. This discrete-time signal

is taken as input for a discrete-time, linear, time-invariant, finite-dimensional compensator $\Gamma_{\text{dis}}$. The latter compensator generates a discrete-time signal $\bar{u} = \{u_k\}$ which, in turn, yields a (piecewise constant) continuous-time input signal $u$ for the plant by defining $u(t) := (H_\Delta \bar{u})(t)$, where $H_\Delta$ is the hold operator defined by $(H_\Delta \bar{u})(t) := u_k$ ($t \in [k\Delta, (k+1)\Delta)$). This type of feedback control is depicted in the following diagram.
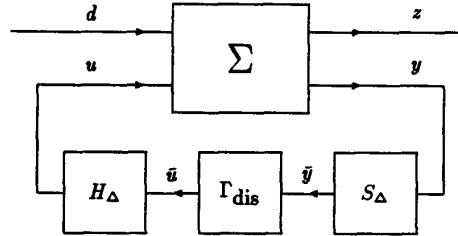


If we control the system $\Sigma$ by means of a sampled data controller with sampling period $\Delta$, then the resulting closed loop system will no longer be time-invariant. In [2, 11] the following definition of $H_2$ performance in the context of sampled data control is proposed. First, it is observed that the closed loop system resulting from a sampled data controller with sampling period $\Delta$ is always a time-varying, $\Delta$-periodic system. Then, for $\Delta$-periodic systems the notion of $H_2$ performance is defined as follows. Suppose we have a finite-dimensional, time-varying, $\Delta$-periodic system $\Sigma_{\text{per}}$ described by

$$z(t) = \int_0^t G(t, s)d(s)ds. \tag{2.1}$$

It is argued in [11] and [2] that a natural way to define the $H_2$ performance of (2.1) is

$$\|\Sigma_{\text{per}}\|_2^2 := \frac{1}{\Delta} \int_0^\Delta \text{tr} \int_s^\infty G^\top(t, s)G(t, s)dtds. \tag{2.2}$$

Next, if $\Gamma$ is a sampled data controller with sampling period $\Delta$, the performance is defined as $J_{\Sigma, \Delta}(\Gamma) := \|\Sigma \times \Gamma\|_2^2$, i.e. the $H_2$ performance of the ($\Delta$-periodic) closed loop system $\Sigma \times \Gamma$. The sampled data $H_2$ problem is then to minimize, for a fixed sampling period $\Delta$, the performance criterion $J_{\Sigma, \Delta}(\Gamma)$ over all internally stabilizing sampled data controllers $\Gamma$ with sampling period $\Delta$. It as shown in [2, 11] that this problem can be reduced to a discrete-time 'normal' $H_2$ optimal control problem. To be specific, let the plant $\Sigma$ be given by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ed(t), \\ y(t) &= C_1 x(t), \\ z(t) &= C_2 x(t) + D_2 u(t), \end{aligned} \tag{2.3}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $d(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^p$ and $z(t) \in \mathbb{R}^q$. It will be a standing assumption in this paper that $(A, B)$ is stabilizable and that $(C_1, A)$ is detectable, both with respect to $C^- := \{s \in C \mid \Re e\ s < 0\}$. Introduce a finite-dimensional linear time-invariant discrete-time system $\Sigma_\Delta$:

$$x_{k+1} = A_\Delta x_k + B_\Delta u_k + E_\Delta d_k ,$$
$$y_k = C_1 x_k , \qquad (2.4)$$
$$z_k = C_{2,\Delta} x_k + D_{2,\Delta} u_k ,$$

where we define

$$A_\Delta := e^{\Delta A}, \quad B_\Delta := \int_0^\Delta e^{tA} dt B,$$

where $E_\Delta$ is any matrix satisfying

$$E_\Delta E_\Delta^T = \int_0^\Delta e^{tA} EE^T e^{tA^T} dt, \qquad (2.5)$$

and where $C_{2,\Delta}$ and $D_{2,\Delta}$ are matrices satisfying

$$\begin{pmatrix} C_{2,\Delta}^T \\ D_{2,\Delta}^T \end{pmatrix} \begin{pmatrix} C_{2,\Delta} & D_{2,\Delta} \end{pmatrix} =$$

$$\int_0^\Delta e^{t\underline{A}^T} \begin{pmatrix} C_2^T \\ D_2^T \end{pmatrix} \begin{pmatrix} C_2 & D_2 \end{pmatrix} e^{t\underline{A}} dt. \qquad (2.6)$$

Here we have denoted

$$\underline{A} := \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}. \qquad (2.7)$$

Let $\Delta$ denote the set of sampling periods for which either $(A_\Delta, B_\Delta)$ is not stabilizable or $(C_1, A_\Delta)$ is not detectable, both with respect to the open unit disc $\{z \in \mathcal{C} \mid |z| < 1\}$. It is well known ( [8,12]) that if $(A, B)$ is stabilizable and $(C_1, A)$ is detectable, then every bounded subset of $\mathbb{R}^+$ contains only finitely many elements of $\Delta$. We will restrict ourselves to sampling periods that are not in $\Delta$. The plant $\Sigma$ is controlled using sampled data controllers $\Gamma := H_\Delta \Gamma_{dis} S_\Delta$, with $\Gamma_{dis}$ given by the equations

$$w_{k+1} = K w_k + L y_k ,$$
$$u_k = M w_k + N y_k . \qquad (2.8)$$

Let us denote by $J_{\Sigma_\Delta}(\Gamma_{dis})$ the discrete-time $H_2$ performance of the closed loop system $\Sigma_\Delta \times \Gamma_{dis}$, i.e. the value $\sum_k \text{tr} (G_k G_k^T)$, where $\{G_k\}$ denotes the pulse response of the closed loop system. The main result of [2,11] is the following:

**Theorem 2.1** : *Assume that $\Delta \notin \Delta$. Then there exists a sampled data controller $\Gamma$ with sampling period $\Delta$ such that the closed loop system $\Sigma \times \Gamma$ is internally stable. The sampled data controller $\Gamma = H_\Delta \Gamma_{dis} S_\Delta$ internally stabilizes $\Sigma$ if and only if the discrete time controller $\Gamma_{dis}$ internally stabilizes $\Sigma_\Delta$. Furthermore, for every such controller the closed loop cost $J_{\Sigma,\Delta}(\Gamma)$ is equal to*

$$\frac{1}{\Delta} \int_0^\Delta \int_0^{\Delta-s} \text{tr} \left( C_1 e^{tA} EE^T e^{tA^T} C_1^T \right) dt ds + \frac{1}{\Delta} J_{\Sigma_\Delta}(\Gamma_{dis}).$$

We shall use this theorem as a starting point and study in this paper the discrete-time $H_2$ optimal control problem for the discrete-time system $\Sigma_\Delta$ given by (2.4). This $H_2$ problem is inherently singular, due to the fact that the direct feedthrough matrix from the disturbance input to the measured output is always equal to zero.

# 3 The discrete-time $H_2$ problem

In this section we shall consider the discrete-time $H_2$ problem. Consider the system $\Sigma_{dis}$ given by the equations

$$x_{k+1} = A x_k + B u_k + E d_k ,$$
$$y_k = C_1 x_k + D_1 d_k , \qquad (3.1)$$
$$z_k = C_2 x_k + D_2 u_k .$$

It will be a standing assumption that $(A, B)$ is stabilizable and that $(C_1, A)$ is detectable, both with respect to the open unit disc. Besides these assumptions which are needed to guarantee the existence of a stabilizing controller, no other assumptions are made on the system.

We will consider discrete-time controllers $\Gamma_{dis}$ given by (2.8). For any internally stabilizing controller $\Gamma_{dis}$, let $J_{\Sigma_{dis}}(\Gamma_{dis})$ be its $H_2$ performance. Denote by $J^*$ the optimal performance, i.e. the infimum over all internallly stabilizing controllers $\Gamma_{dis}$.

For a given matrix $M$, we will denote by $M^+$ its Moore-Penrose inverse. The solution of the discrete-time $H_2$ optimal control problem centers around the following two algebraic Riccati equations:

$$P = A^T P A + C_2^T C_2 - (C_2^T D_2 + A^T P B) \times$$

$$(D_2^T D_2 + B^T P B)^+ (D_2^T C_2 + B^T P A), \qquad (3.2)$$

$$Q = A Q A^T + E E^T - (A Q C_1^T + E D_1^T) \times$$

$$(D_1 D_1^T + C_1 Q C_1^T)^+ (D_1 E^T + C_1 Q A^T). \qquad (3.3)$$

For any real symmetric matrix $P$, we shall denote:

$$D_P := (D_2^T D_2 + B^T P B)^{\frac{1}{2}}, \qquad (3.4)$$

$$C_P := D_P^+ (D_2^T C_2 + B^T P A). \qquad (3.5)$$

Note that, since for any matrix $M \geq 0$ we have $(M^{\frac{1}{2}})^+ = (M^+)^{\frac{1}{2}}$, we have $D_P^+ C_P = (D_2^T D_2 + B^T P B)^+ (D_2^T C_2 + B^T P A)$. If, in addition, $P$ is a real symmetric solution of (3.2), then $C_P^T C_P = A^T P A - P + C_2^T C_2$. Note also that $D_P$ is symmetric by definition. Finally, since $\text{im} (D_2^T C_2 + B^T P A) \subset \text{im } D_P$, we have $D_P C_P = D_2^T C_2 + B^T P A$. (Note: it is a property of the Moore-Penrose inverse that $M M^+$ is the orthogonal projection onto $\text{im } M$.)

The following is a corrected and slightly extended version of a theorem from [13]. A proof can be given along the lines of the proof of [13, theorem 18].

**Theorem 3.1** : *Consider the system $(A, B, C_2, D_2)$ together with the algebraic Riccati equation (3.2). The algebraic Riccati equation (3.2) has a real symmetric solution $P$ with the following property: there exists a matrix $F_1$ such that*

$$|\sigma(A - B D_P^+ C_P + B(I - D_P^+ D_P) F_1)| \leq 1. \qquad (3.6)$$

*Furthermore, if $P$ satisfies this condition, it is the unique real symmetric solution of (3.2) for which this condition holds. In addition, $P$ is positive semi-definite and is in fact the largest real symmetric solution of (3.2).*

Next we consider the dual algebraic Riccati equation (3.3). For any real symmetric matrix $Q$, denote

$$D_Q := (D_1 D_1^T + C_1 Q C_1^T)^{\frac{1}{2}}, \tag{3.7}$$

$$E_Q := (AQC_1^T + ED_1^T)D_Q^+. \tag{3.8}$$

By dualizing the previous theorem, the corresponding result on the Riccati equation (3.3) can be found:

**Theorem 3.2** : *Consider the system $(A, E, C_1, D_1)$ together with the algebraic Riccati equation (3.3). The algebraic Riccati equation (3.3) has a real symmetric solution $Q$ with the following property: there exists a matrix $K_1$ such that*

$$|\sigma(A - E_Q D_Q^+ C_1 + K_1(I - D_Q D_Q^+)C_1)| \le 1. \tag{3.9}$$

*Furthermore, if $Q$ satisfies this condition, it is the unique real symmetric solution of (3.3) for which this condition holds. In addition, $Q$ is positive semi-definite and is in fact the largest real symmetric solution of (3.3).*

In the remainder of this section we will always denote by $P$ and $Q$ the largest real symmetric solution of (3.2) and (3.3), respectively. We will now state the main result of this section:

**Theorem 3.3** : *Consider the system (3.1). Assume that $(A, B)$ is stabilizable and $(C_1, A)$ is detectable. Then we have:*

$$J^* = \text{tr } (E^T P E) + \text{tr } (C_P Q C_P^T)$$
$$- \text{tr } ((D_P N^* D_Q)(D_P N^* D_Q)^T), \tag{3.10}$$

*where $N^*$ is defined by*

$$N^* := -(D_P^+)^2(D_P C_P Q C_1^T + B^T P E D_1^T)(D_Q^+)^2. \tag{3.11}$$

Under two standard assuptions we can actually guarantee the existence of optimal controllers:

**Theorem 3.4** : *Assume that the systems $(A, B, C_2, D_2)$ and $(A, E, C_1, D_1)$ have no zeros on the unit circle. There exists an optimal controller, i.e. an internally stabilizing controller $\Gamma_{dis}^*$ such that $J_{\Sigma_{dis}}(\Gamma_{dis}^*) = J^*$. One such optimal controller is given by the following 'construction':*

(i) *Choose a state feedback $F$ such that $|\sigma(A + BF)| < 1$ and $C_P + D_P F = 0$.*

(ii) *Choose an output injection $G$ such that $|\sigma(A+GC_1)| < 1$ and $E_Q + GD_Q = 0$.*

(iii) *Define $\Gamma_{dis}^* = (K^*, L^*, M^*, N^*)$ by choosing $N^*$ according to (3.11), and by choosing $K^* := A + BF + GC_1 - BN^*C_1$, $L^* := BN^* - G$, and $M^* := F - N^*C_1$.*

Although the above theorem guarantees the existence of a suitable controller under some standard assumptions, we will derive necessary and sufficient conditions for the existence of an optimal controller which are weaker than the two assumptions made in the above theorem.

In [18], it is shown that the existence of an optimal controller is equivalent to the existence of a strictly proper controller

which achieves disturbance decoupling with internal stability for a related system. This system can be constructed explicitly. The disturbance-decoupling problem has been studied extensively in [16]. One of the main results of [16] gives necessary and sufficient conditions for the existence of an internally stabilizing, strictly proper compensator $\Gamma_{dis}^0$ which achieves disturbance decoupling for a system $\Sigma_{dis}$ of the form 3.1.

We have used the above method to derive necessary and sufficient conditions for the existence of an optimal controller. We define the subspace $\mathcal{V}_g$ by:

$$\mathcal{V}_g := \mathcal{X}_g(A - BD_P^+ C_P) + < A - BD_P^+ C_P \mid B \ker D_P >, \tag{3.12}$$

where for a given matrix $M$, $\mathcal{X}_g(M)$ is the sum of the generalized eigenspace of $M$ associated with its eigenvalues in $|z| < 1$, and where $< M \mid \mathcal{L} >$ is the smallest $M$-invariant subspace contained in $\mathcal{L}$. Moreover, we define the subspace $\mathcal{S}_g$ by

$$\mathcal{S}_g := \mathcal{X}_b(A - E_Q D_Q^+ C_1) \cap < C_1^{-1} \text{im } D_Q \mid A - E_Q D_Q^+ C_1 >, \tag{3.13}$$

where $\mathcal{X}_b(M)$ is the sum of the generalized eigenspaces of $M$ associated with its eigenvalues in $|z| \ge 1$ and where $< \mathcal{L} \mid M >$ is the largest $M$-invariant subspace containing $\mathcal{L}$. We obtain the following necessary and sufficient conditions for the existence of an optimal controller for the discrete-time $H_2$ optimal control problem associated with the system $\Sigma_{dis}$:

**Theorem 3.5** : *Consider the system (3.1). Assume that $(A, B)$ is stabilizable and $(C_1, A)$ is detectable. Let $P$ and $Q$ be the largest real symmetric solution of (3.2) and (3.3), respectively. Let $\mathcal{V}_g$ and $\mathcal{S}_g$ be given by (3.12) and (3.13). Then we have: there exists an optimal controller, i.e. an internally stabilizing controller $\Gamma_{dis}^* = (K^*, L^*, M^*, N^*)$ such that $J_{\Sigma_{dis}}(\Gamma_{dis}^*) = J^*$, if and only if the four subspace inclusions are satisfied:*

$$\mathcal{V}_g \supset \text{im } (E_Q - BD_P^+ R^*),$$
$$\mathcal{S}_g \subset \ker(C_P - R^* D_Q^+ C_1).$$
$$\mathcal{V}_g \supset (A - BD_P^+ R^* D_Q^+ C_1)\mathcal{S}_g.$$
$$\mathcal{S}_g \subset \mathcal{V}_g$$

# 4 The sampled data $H_2$ problem

We now return to the sampled data $H_2$ problem. Consider the continuous-time system $\Sigma$ given by (2.3), and let $\Delta \notin \bar{\Delta}$ be a given sampling period. Let the discrete-time system $\Sigma_\Delta$ be given by (2.4). According to theorem 2.1, the optimal sampled data $H_2$ performance $J_{\Sigma, \Delta}^*$ is equal to

$$\frac{1}{\Delta} \int_0^\Delta \int_0^{\Delta - s} \text{tr } \left( C_1 e^{tA} EE^T e^{tA^T} C_1^T \right) dt ds + \frac{1}{\Delta} J_{\Sigma_\Delta}^*,$$

where $J_{\Sigma_\Delta}^*$ is the optimal discrete-time $H_2$ performance associated with $\Sigma_\Delta$. According to theorem 3.3, the optimal performance $J_{\Sigma_\Delta}^*$ can be found in terms of two algebraic

Riccati equations associated with $\Sigma_\Delta$. According to theorem 3.5, an optimal compensator $\Gamma_{dis,\Delta}$ exists if and only if four subspace inclusions involving subspaces associated with the system $\Sigma_\Delta$ are satisfied. According to theorem 3.4, if the systems $(A_\Delta, B_\Delta, C_{2,\Delta}, D_{2,\Delta})$ and $(A_\Delta, E_\Delta, C_1, 0)$ have no zeros on the unit circle, then an optimal compensator $\Gamma_{dis,\Delta}$ exists and can be calculated using the 'construction' in the statement of theorem 3.4. The sampled data controller $\Gamma := H_\Delta \Gamma_{dis,\Delta} S_\Delta$ is then optimal for the sampled data $H_2$ problem under consideration.

In this section we study the following question: what are conditions *in terms of the original system* $\Sigma$ that guarantee that there exists an optimal compensator for the sampled data $H_2$ problem? Instead of being completely general, we will study the following question: what are necessary and sufficient conditions in terms of the original system $\Sigma$ such that $(A_\Delta, B_\Delta, C_{2,\Delta}, D_{2,\Delta})$ and $(A_\Delta, E_\Delta, C_1, 0)$ have no zeros on the unit circle?

We define *the controllability subspace* $\mathcal{R}$ of $(A, B, C, D)$ as follows:

$$\mathcal{R} := < A + BF \mid \mathcal{V} \cap B \ker D >,$$

for any $F$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$ and $(C + DF)\mathcal{V} = 0$ (any such $F$ yields the same $\mathcal{R}$). It was shown in [13] that the system $(A, B, C, D)$ is left-invertible if and only if $\ker B \cap \ker D = 0$ and $\mathcal{V} \cap B \ker D = 0$. Note that $\mathcal{V} \cap B \ker D = 0$ if and only if $\mathcal{R} = 0$.

The main results of this section are the following:

**Theorem 4.1 :** *Consider the system $\Sigma$. Let $\Delta > 0$.*

(i) *Let $\lambda$ be a zero of $(A_\Delta, B_\Delta, C_{2,\Delta}, D_{2,\Delta})$, $\lambda \neq 0$. Then there exists an unobservable eigenvalue $\mu$ of $(C_2, A)$ such that $\lambda = e^{\mu\Delta}$.*

(ii) *If $(A, B, C_2, D_2)$ is left-invertible then also the converse of (i) holds: if $\mu$ is an unobservable eigenvalue of $(C_2, A)$, then $e^{\mu\Delta}$ is a zero of $(A_\Delta, B_\Delta, C_{2,\Delta}, D_{2,\Delta})$.*

(iii) *1 is a zero of $(A_\Delta, B_\Delta, C_{2,\Delta}, D_{2,\Delta})$ if and only if at least one of the following two conditions hold:*

    (a) *0 is a zero of $(A, B, C_2, D_2)$,*

    (b)   $\mathcal{R} \not\subset < \ker C_2 \mid A >$.     (4.1)

(iv) *If $(A, B, C_2, D_2)$ is left-invertible then 0 is a zero of $(A, B, C_2, D_2)$ if and only if 1 is a zero of $(A_\Delta, B_\Delta, C_{2,\Delta}, D_{2,\Delta})$*

**Corollary 4.2 :** *Consider the system $\Sigma$. Let $\Delta > 0$.*

(i) *If $(C_2, A)$ has no unobservable eigenvalues on the imaginary axis, 0 is not a zero of $(A, B, C_2, D_2)$, and $\mathcal{R} \subset < \ker C_2 \mid A >$, then $(A_\Delta, B_\Delta, C_{2,\Delta}, D_{2,\Delta})$ has no zeros on the unit circle.*

(ii) *If $(A, B, C_2, D_2)$ is left-invertible then $(C_2, A)$ has no unobservable eigenvalues on the imaginary axis and 0 is not a zero of $(A, B, C_2, D_2)$ if and only if $(A_\Delta, B_\Delta, C_{2,\Delta}, D_{2,\Delta})$ has no zeros on the unit circle.*

Clearly the above theorem and corollary have a dual. However, due to the different structure the results are quite different.

**Theorem 4.3 :** *Consider the system $\Sigma$. Let $\Delta > 0$.*

(i) *Let $\lambda$ be a zero of $(A_\Delta, E_\Delta, C_1, 0)$. Then there exists an uncontrollable eigenvalue $\mu$ of $(A, E)$ such that $\lambda = e^{\mu\Delta}$.*

(ii) *If $(A, E, C_1, 0)$ is right-invertible then also the converse of (i) holds. i.e., if $\mu$ is an uncontrollable eigenvalue of $(A, E)$ then $e^{\mu\Delta}$ is a zero of $(A_\Delta, E_\Delta, C_1, 0)$.*

**Corollary 4.4 :** *Consider the system $\Sigma$. Let $\Delta > 0$. If $(A, E)$ has no uncontrollable eigenvalues on the imaginary axis, then $(A_\Delta, E_\Delta, C_1, 0)$ has no zeros on the unit circle. If, in addition, $(A, E, C_1, 0)$ is right-invertible then also the converse holds: $(A_\Delta, E_\Delta, C_1, 0)$ has no zeros on the unit circle if and only if $(A, E)$ has no uncontrollable eigenvalues on the imaginary axis.*

## 5 Performance recovery and convergence of optimal performance

In this section we study the connection between the 'ordinary' continuous-time $H_2$ problem and the sampled data $H_2$ problem. In particular, we are interested in the following questions:

- Suppose that we control the system $\Sigma$ by means of an internally stabilizing continuous-time compensator $\Gamma_{con}$, yielding continuous-time $H_2$ performance $J_\Sigma(\Gamma_{con})$. Is it true that for all $\epsilon > 0$ there exists $\Delta > 0$ and an internally stabilizing sampled data controller $\Gamma$ with sampling period $\Delta$ such that $|J_\Sigma(\Gamma_{con}) - J_{\Sigma,\Delta}(\Gamma)| < \epsilon$?

- Suppose that $J^*_{\Sigma,con}$ is the optimal continuous-time $H_2$ performance associated with the system $\Sigma$ and, as before, denote the optimal sampled data $H_2$ performance by $J^*_{\Sigma,\Delta}$. Is it true that $\lim_{\Delta\downarrow 0} J^*_{\Sigma,\Delta} = J^*_{\Sigma,con}$?

The first question above was studied before in [6, theorem 4] using a different definition of $H_2$ performance, and for the $H_\infty$ performance criterion ( [6, theorem 5]). In this section we will show that both questions have an affirmative answer. Let $\Sigma$ be given by (2.2). If the system $\Sigma$ is controlled by a continuous-time compensator $\Gamma_{con}$ given by the equations

$$\begin{aligned} \dot{w}(t) &= \bar{K}w(t) + \bar{L}y(t), \\ u(t) &= \bar{M}w(t) + \bar{N}y(t), \end{aligned} \qquad (5.1)$$

with $w(t) \in \mathbb{R}^\ell$, then the associated closed-loop system $\Sigma \times \Gamma_{con}$ is given by

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + E_e y(t), \\ z(t) &= C_e x_e(t), \end{aligned}$$

with

$$A_e := \begin{pmatrix} A + B\bar{N}C_1 & B\bar{M} \\ \bar{L}C_1 & \bar{K} \end{pmatrix}, \quad E_e := \begin{pmatrix} E \\ 0 \end{pmatrix},$$

$$C_e := \begin{pmatrix} C_2 + D_2\bar{N}C_1 & D_2\bar{M} \end{pmatrix}.$$

If $\Gamma_{con}$ is internally stabilizing, i.e., $\sigma(A_e) \subset \mathcal{C}^-$, then the $H_2$ performance of the closed loop system $\Sigma \times \Gamma_{con}$ is equal to

$$J_\Sigma(\Gamma_{con}) = \text{tr}\,(E_e P_e E_e^T),$$

where $P_e$ is the unique solution of the Lyapunov equation

$$A_e^T P_e + P_e A_e + C_e^T C_e = 0. \qquad (5.2)$$

On the other hand, if the system $\Sigma$ is controlled by the sampled data controller $\Gamma = H_\Delta \Gamma_{dis} S_\Delta$, with $\Gamma_{dis}$ given by (2.8), then the discrete-time closed loop system $\Sigma_\Delta \times \Gamma_{dis}$ is given by the equations

$$\begin{aligned} x_{e,k+1} &= A_{e,\Delta} x_{e,k} \ E_{e,\Delta} y_k \ , \\ z_k &= C_{e,\Delta} x_{e,k} \ , \end{aligned}$$

with

$$A_{e,\Delta} := \begin{pmatrix} A_\Delta + B_\Delta NC_1 & B_\Delta M \\ LC_1 & K \end{pmatrix}, \ E_{e,\Delta} := \begin{pmatrix} E_\Delta \\ 0 \end{pmatrix},$$

$$C_{e,\Delta} := \begin{pmatrix} C_{2,\Delta} + D_{2,\Delta} NC_1 & D_{2,\Delta} M \end{pmatrix}.$$

If $\Gamma$ is internally stabilizing, equivalently $|\sigma(A_{e,\Delta})| < 1$, then the $H_2$ performance of the closed-loop system $\Sigma \times \Gamma$ is given by

$$J_{\Sigma,\Delta}(\Gamma) = \frac{1}{\Delta} \int_0^\Delta \int_0^{\Delta-s} \mathrm{tr} \left( C_1 e^{tA} EE^T e^{tA^T} C_1^T \right) dt ds$$

$$+ \frac{1}{\Delta} \mathrm{tr} \left( E_{e,\Delta} P_{e,\Delta} E_{e,\Delta}^T \right), \qquad (5.3)$$

where $P_{e,\Delta}$ is the unique solution of the Lyapunov equation

$$A_{e,\Delta}^T P_{e,\Delta} A_{e,\Delta} - P_{e,\Delta} + C_{e,\Delta}^T C_{e,\Delta} = 0. \qquad (5.4)$$

The following theorem shows that our first question above indeed has an affirmative answer:

**Theorem 5.1 :** *Let $\Gamma_{con}$ be a continuous-time compensator which stabilizes $\Sigma$. For any $\Delta > 0$ define a discrete-time controller $\Gamma_{dis}$ by $\Gamma_{dis} := S_\Delta \Gamma_{con} H_\Delta$, and let $\Gamma_\Delta := H_\Delta \Gamma_{dis} S_\Delta$ be the corresponding sampled data controller with sampling period $\Delta$. Then we have: there exists $\Delta_1 > 0$ such that for all $\Delta \in \Delta$ with $0 < \Delta < \Delta_1$, $\Gamma_\Delta$ is internally stabilizing. Furthermore,*

$$J_{\Sigma,\Delta}(\Gamma_\Delta) \to J_\Sigma(\Gamma_{con}) \ (\Delta \downarrow 0).$$

We now turn to the second question posed above. Let $J^*_{\Sigma,con}$ be the optimal continuous-time $H_2$ performance, i.e., the infimum of $J_\Sigma(\Gamma_{con})$ over all internally stabilizing continuous-time compensators (5.1). It was shown in [15] that if $(A,B)$ is stabilizable and $(C_1,A)$ is detectable, then

$$J^*_{\Sigma,con} = \mathrm{tr} \left( EE^T P \right) + \mathrm{tr} \left( (A^T P + PA + C_2^T C_2) Q \right), \quad (5.5)$$

where $P$ and $Q$ are the largest real symmetric solutions of the following two linear matrix inequalities

$$\begin{pmatrix} AQ + QA^T + EE^T & C_1^T Q \\ QC_1 & 0 \end{pmatrix} \geq 0. \qquad (5.6)$$

$$\begin{pmatrix} A^T P + PA^T + C_2^T C_2 & PB + C_2^T D_2 \\ B^T P + D_2^T C_2 & D_2^T D_2 \end{pmatrix} \geq 0. \qquad (5.7)$$

Let $J^*_{\Sigma,\Delta}$ be the optimal sampled data $H_2$ performance. Our next theorem gives an affirmative answer to the second question posed in the introduction to this section.

**Theorem 5.2 :** *Let $(A,B)$ be stabilizable and $(C_1,A)$ be detectable. Then there exists $\Delta_1$ such that for all $0 < \Delta < \Delta_1$, $J^*_{\Sigma,\Delta} < \infty$. We have $\lim_{\Delta \downarrow 0} J^*_{\Sigma,\Delta} = J^*_{\Sigma,con}$.*

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