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A reversible bifurcation analysis of the inverted pendulum

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Abstract

The inverted pendulum with a periodic parametric forcing is considered as a bifurcation problem in the reversible setting. Parameters are given by the size of the forcing and the frequency ratio. Normal form theory provides an integrable approximation of the Poincaré map generated by a planar vector field. Genericity of the model is studied by a perturbation analysis, where the spatial symmetry is optional. Here equivariant singularity theory is used.

Keywords: Parametrically forced oscillator; Spatio-temporal symmetry; Hamiltonian system; Normal form theory; Equivariant singularity theory

1. Introduction

The unstable upper equilibrium of a pendulum can be stabilized by a vertical periodic motion of the suspension point in a specific frequency domain. The corresponding stability analysis and its further dynamical aspects are elements of classical perturbation theory, e.g. see [16,25,29]. For a textbook analysis see [1]. The problem is considered in the $1\frac{1}{2}$-degree-of-freedom Hamiltonian setting where the motion of the suspension point is regarded as a parametric forcing.

The subject of the present paper is the nonlinear dynamics, studied by its period – or Poincaré map. In particular it is investigated how changes of stability correspond to bifurcations in this dynamics, following the approach of Broer and Vegter [12]. Leading question will be how persistent (generic) the results are. In order to answer this we perform small perturbations, duely respecting the symmetries of the system. Here we resort to equivariant singularity theory.

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1.1. Setting of the problem

The equation of motion of the *inverted pendulum* is given by

$$\ddot{x} = (\alpha + \beta\rho(t))V'(x),$$

where the angle x is the deviation from the upper equilibrium. The potential energy is $V(x) = 1 - \cos x$. The function ρ gives the forcing, periodic in the time t. Time is scaled such that ρ has period 2π. For simplicity we take the time-average of ρ equal to zero. Then, $\sqrt{\alpha}$ denotes the ratio of the 'eigenfrequency' of the pendulum and the forcing frequency, while β controls the amplitude of the forcing.

By periodicity we take both x and $t \in \mathbb{R}/(2\pi\mathbb{Z}) =: \mathbb{S}^1$, the standard circle. Putting $y := \dot{x}$ the equation of motion so rewrites to the following vector field in the extended phase space $\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1 = \{x, y, t\}$:

$$X_{\alpha,\beta}(x, y, t) = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x}$$
$$+ (\alpha + \beta\rho(t))V'(x)\frac{\partial}{\partial y} \qquad (1)$$

depending on parameters $(\alpha, \beta) \in \mathbb{R}^2$. The t-dependent Hamiltonian reads $H_{\alpha,\beta}(x, y, t) = \frac{1}{2}y^2 - (\alpha + \beta\rho(t))V(x)$. Since $V'(0) = 0$, it follows that $x = y = 0$ is always a periodic orbit of $X_{\alpha,\beta}$, referred to as the *upper equilibrium*.

Throughout we assume that the function ρ is *even* in t, meaning that the system is time-reversible. In the extended phase space this gives the involution $\mathcal{R}$: $(x, y, t) \mapsto (x, -y, -t)$ and $X = X_{\alpha,\beta}(x, y, t)$ is $\mathcal{R}$-reversible in the sense that $\mathcal{R}_* X = -X$. Also $H = H_{\alpha,\beta}$ is $\mathcal{R}$-equivariant in the sense that $H \circ \mathcal{R} = H$.

Note that this problem by evenness of V also has a spatial symmetry which can be incorporated as follows. Consider the involution $\mathcal{S}$: $(x, y, t) \mapsto (-x, y, -t)$: by evenness of V the vector field X is $\mathcal{S}$-reversible and H is likewise $\mathcal{S}$-equivariant.

In our perturbation analysis we shall restrict to $\mathcal{R}$-reversible vector fields, while $\mathcal{S}$-reversibility is optional. Also we shall maintain the 'upper equilibrium' periodic solution $x = y = 0$.

Remark. In our case $V'(x) = -V'(x + \pi)$ holds, giving rise to another symmetry. Indeed, if we denote $\mathcal{T}$: $(x, y, t; \alpha, \beta) \mapsto (x + \pi, y, t; -\alpha, -\beta)$ then X and H are $\mathcal{T}$-equivariant.

Stability of the upper equilibrium is determined by the linearized system

$$\ddot{x} - (\alpha + \beta\rho(t))x = 0.$$

Classically the Mathieu case with $\rho(t) = \cos t$ and the 'square' case with $\rho(t) = \text{sgn}\cos t$ are best known, see [21,25,29] also compare the above references. Fig. 1 depicts the stability diagram for the Mathieu case. For recent topological and geometric results on these linear systems see [19] or [9]. A nonlinear treatment of the inverted pendulum can be found for example in Hale [16], who focusses on stable periodic solutions. Similar approaches seem to exist a lot in the engineering literature. Our aim is to give a more general description of the nonlinear dynamics, by systematically exploring an integrable approximation of the Poincaré map. Doing so, we incorporate the possible symmetries and so end up in equivariant singularity theory.
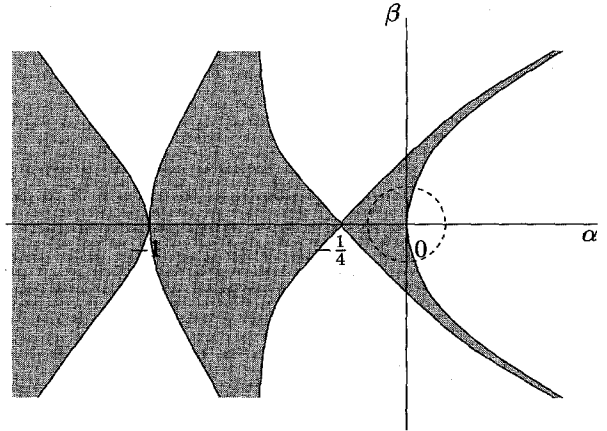

Fig. 1. Stability diagram for Mathieu's equation, shading indicates stability of the upper equilibrium. Our present interest is inside the dashed circle.

Broer and Vegter [12] contains a bifurcation analysis of the parametrically forced pendulum, with both V and ρ of class C^∞, and perturbations thereof, near the lower equilibrium in resonance. The present paper attempts a similar analysis of (1), i.e. near the upper equilibrium, duely keeping track of the $\mathcal{R}$- and $\mathcal{S}$- or just the $\mathcal{R}$-symmetry. We study the bifurcations concerning harmonic periodic solutions, i.e. of period 2π, the behaviour of their stable and unstable manifolds and the dynamics supported by this framework.

The velocity y is taken small, while the angle x varies globally. It turns out that the dynamics near $(x, y) = (0, 0)$ is the most interesting. The parameters (α, β) are taken near $(0, 0)$, see Fig. 1: β serves as a perturbation parameter. Since the forcing ρ has period 2π, the eigenfrequency α is small, meaning that we restrict to the case of a long pendulum. In future work we report on a formal and numerical continuation of our results to larger values of α and β. Here asymptotic information for $\alpha \to \infty$ plays a role, compare [21]. This work fits in a general program where computer algebra is combined with numerical methods.

1.2. Method

System (1) is studied by means of its Poincaré map

$$P_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}^2$$

given by

$$X^{2\pi}_{\alpha,\beta}(x, y, 0) = (P_{\alpha,\beta}(x, y), 2\pi).$$

Here X^τ denotes the flow of X over time τ. Since the 'upper equilibrium' always is a closed orbit of $X_{\alpha,\beta}$ we have $P_{\alpha,\beta}(0, 0) = (0, 0)$ throughout. Furthermore X is Hamiltonian, so P is area preserving (symplectic). The symmetries now are expressed as follows. Given the planar involutions $R : (x, y) \mapsto (x, -y)$ and $S : (x, y) \mapsto (-x, y)$, the map P is R- and S-reversible in the sense that

$$RPR = P^{-1} \quad \text{resp.} \quad SPS = P^{-1}.$$

Similarly for the involution $T : (x, y; \alpha, \beta) \mapsto (x + \pi, y; -\alpha, -\beta)$. Our problem translates to studying the bifurcations of fixed points of $P_{\alpha,\beta}$, their stable and unstable manifolds, and the dynamics this framework supports, like KAM-cylinders of invariant circles.

An arbitrary area preserving map with discrete symmetry is far too general an object to study. However, in this case near a fixed point, we may resort to the classical approximation of P by an *integrable* map. Indeed, up to a canonical transformation we approximate

$$P \approx \tilde{X}^{2\pi},$$

where $\tilde{X}$ is a *planar* Hamiltonian vector field, reversible with respect to the involutions R and S, hence having $(x, y) = (0, 0)$ as an equilibrium for all α and β. Compare, e.g. [2,11,12,26]. In the next section a formulation of this normal form theory is given, which amounts to averaging out the time-dependence from (1) to arbitrarily high order in $(y; \alpha, \beta)$.

The phase portrait of the map P then is approximated by that of the planar vector field $\tilde{X}$, which is just the collection of level curves of the corresponding planar Hamiltonian $\tilde{H}$. Notice that the equivariance relations $H \circ R = H$, $H \circ S = H$ hold and (hence) that $\tilde{H}$ has the origin $(x, y) = (0, 0)$ as a critical point for all values of α and β.

The last step of this analysis simplifies the planar family of functions $\tilde{H}_{\alpha,\beta}$ by a further transformation. From then on we restrict to more qualitative considerations where only the configuration of level curves

plays a role. This allows us to drop the canonical (symplectic) nature of the transformations, thereby forgetting the time parametrization of the integral curves, compare, e.g. [6,7]. Thus we are lead into singularity theory. The transformations respect the symmetries R and S (and hence the fact that $(x, y) = (0, 0)$ always is critical). So we obtain polynomial normal forms given by the first few terms of the Taylor series, from which the dynamics and its bifurcations can be easily read off. Equivariant singularity theory guarantees persistence under perturbation within the class of symmetric systems that can be obtained by normal form truncations.

1.3. Summary of the results

The upper part of Fig. 2 shows simulations of Poincaré map phase portraits in case of the inverted pendulum for two different values of (α, β), obtained by the program DsTool [3]. In the lower part we show integrable approximations of these by means of vector field phase portraits, i.e. the level curves of corresponding Hamiltonians. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry corresponding to the R- and S-reversibility is clearly illustrated. Also the local stability diagram is shown, in which the stability boundary turns out to be a line of Hamiltonian pitchfork bifurcations.

We also consider perturbations of the inverted pendulum, studying how the stability diagram and the corresponding dynamical scenario may change, compare Fig. 3. In the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric context, where both R- and S- reversibility holds, the dynamics of the normal form approximation is persistent, in particular the pitchfork bifurcation. For the corresponding planar functions this amounts to structural stability of the equivariant cusp catastrophe.

If the S-symmetry is broken, we end up in the $\mathbb{Z}_2$-symmetric context of only R-reversibility. Now the pitchfork bifurcation falls apart into a Hamiltonian saddle–node (or saddle–centre) and a transcritical bifurcation, at the level of functions corresponding to a fold and an 'exchange' catastrophe. Here it is important that the origin $(x, y) = (0, 0)$ is maintained as a critical point.
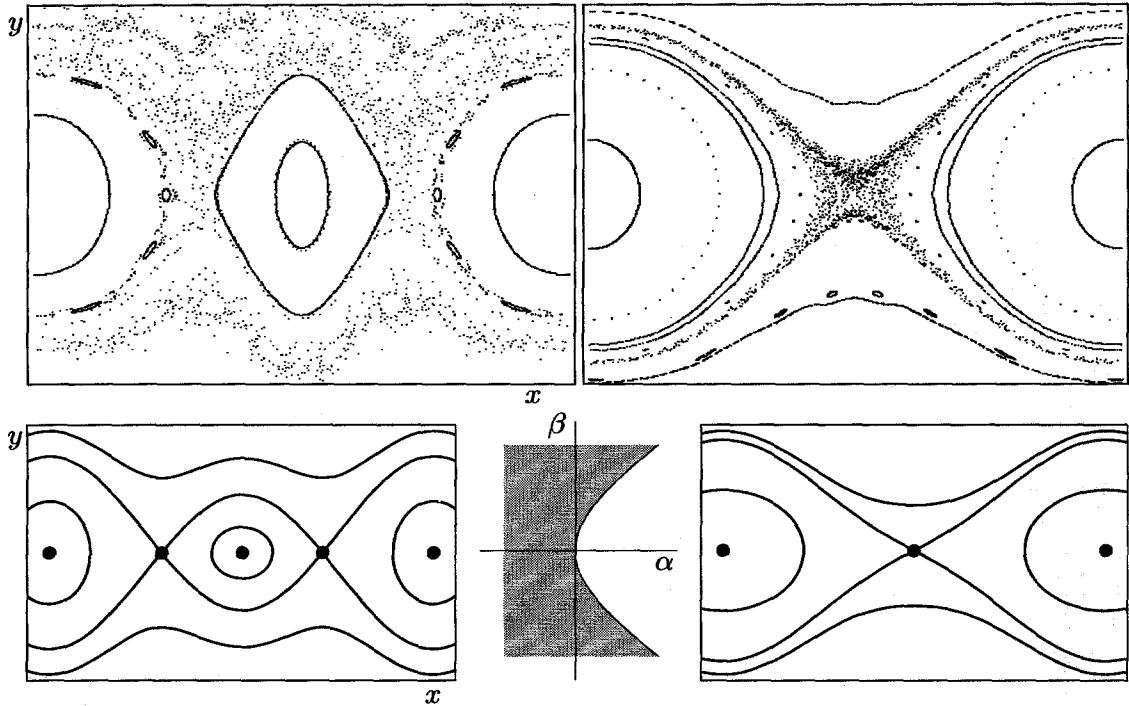
Fig. 2. Top: Poincaré map for two values of (α, β). Bottom centre: local stability diagram, shading indicates stability of upper equilibrium. Bottom left and right: corresponding normal form phase portraits. The stability boundary is a line of Hamiltonian pitchfork bifurcations.
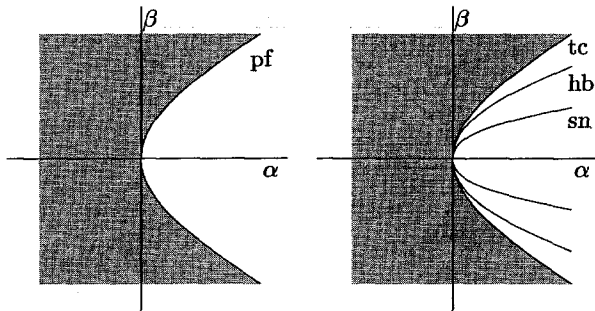

Fig. 3. Local stability diagram in the $\mathbb{Z}_2 \times \mathbb{Z}_2$- (left) and in the $\mathbb{Z}_2$-symmetric context (right). Stability of the origin (upper equilibrium) again is indicated by shading. The coding is explained in Section 3.

In the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case two heteroclinic connections are persistent, while in the $\mathbb{Z}_2$-symmetric case the normal form exhibits a codimension 1 heteroclinic bifurcation. Details of this are shown in Fig. 3, which is a stability diagram in which also the bifurcations

are indicated. The normal form dynamics in the $\mathbb{Z}_2$-symmetric context is sketched in Fig. 4. We note that all nonlinear dynamical information is restricted to wedge-shaped neighbourhoods of the stability boundary, vanishing quadratically at $\beta = 0$.

We briefly summarize how the planar normal form corresponds to the integrable Poincaré map. First, critical points of the planar function (i.e. equilibria in the planar vector field) correspond to fixed points of the integrable Poincaré map. Level sets of saddle critical points then become stable and unstable manifolds of the fixed points. Homo- or heteroclinic connections can be easily read off. Closed level curves of the function (i.e. closed orbits of the planar vector field) correspond to invariant circles of the approximating Poincaré map. Thus we are left with a perturbation problem between Poincaré maps. Since this is quite involved, we just give a few remarks, for details and further reference compare [6,7,12].
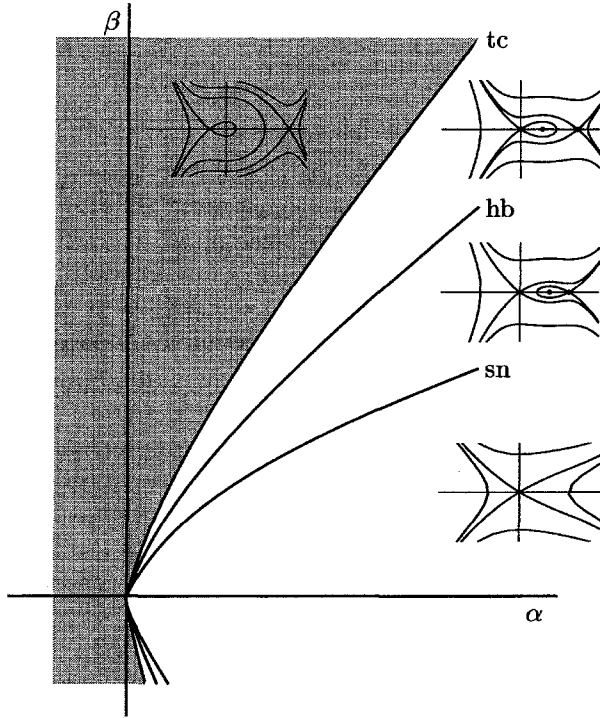
Fig. 4. Bifurcation diagram of the normal form in the $\mathbb{Z}_2$-symmetric context.

Concerning the fixed points, their type and the local behaviour of stable and unstable manifolds the perturbation theory is rather standard. Indeed, persistence of these objects can be handled by the implicit function theorem and related contraction arguments. The cylinders of invariant circles will not persist as a continuum, but by KAM-theory the circles with Diophantine rotation number do persist, thus forming a Cantor foliation of invariant circles of positive measure. Here we forego problems of small twist, non-monotonicity of the period function, etc. The resonant circles, in a better approximation, are expected to break up into isolated periodic points arranged as a string of 'pendulum beads'. For this, a further appropriate normal form analysis is needed, see, e.g. [12].

Finally there is chaos. Every homo- and heteroclinic connection of the integrable approximation is expected to split and give rise to a 'chaotic sea' as in Fig. 2. However, these effects are not easily detectable asymptotically, as they are infinitely flat in β at $\beta = 0$.

2. The integrable approximation

In this section we formulate a normal form or averaging theory appropriate for our purpose. Although the general approach is well-known, compare, e.g. [4,11–13,15,17,24,28] and references therein, the details are somewhat involved. The reason is that we consider $x \in \mathbb{S}^1$ globally and hence can be formal only in the variables $(y; \alpha, \beta)$ at $(0; 0, 0)$.

As said before, we assume the functions ρ and V to be of class C^∞. We then consider the Taylor series of the system (1) in (α, β), with coefficients that are periodic both in t and in x, and that are formal power series in y. This series is simplified to increasing order in (α, β) by successive changes of coordinates, where 'simple' means t-independent. All transformations respect the R-, or R- and S-reversing symmetries at hand. In principle the normalization can be carried out to infinite order, after which the Borel theorem gives C^∞ representations of the formal coordinate changes with the same preservation of structure, compare [4,12].

Theorem 1 (Normalization of the vector field). Let the C^∞ vector field X on $\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1 = \{x, y, t\}$ have the form (1) with $V'(0) = 0$. Then there exists a C^∞ canonical transformation $\Psi_1 : \mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1$, preserving the time t, such that

$$(\Psi_{1\,*}X)(x, y, t; \alpha, \beta)$$
$$= \frac{\partial}{\partial t} + X_1(x, y; \alpha, \beta) + p_1(x, y, t; \alpha, \beta),$$

where X_1 has the time-independent form

$$X_1(x, y; \alpha, \beta)$$
$$= (y + O(|\alpha, \beta|^2 y))\frac{\partial}{\partial x} - (U'(x; \alpha, \beta)$$
$$+ O(|\alpha, \beta|^3))\frac{\partial}{\partial y}$$

with

$$U(x; \alpha, \beta) = \tfrac{1}{2}\beta^2(V'(x))^2 - \alpha V(x)$$

and

$$p_1(x, y, t; \alpha, \beta) = O(|y; \alpha, \beta|^\infty).$$

The remainder $O(|\alpha, \beta|^2 y)$ is independent of x and $O(|\alpha, \beta|^3))$ independent of y. Moreover, if X is $\mathcal{R}$-reversible or $\mathcal{R}$- and $\mathcal{S}$-reversible, then so are $\Psi_{1*}X$ and X_1. The same holds with respect to $\mathcal{T}$.

For a proof and more background, see Appendix A. Next consider the Poincaré map P of X. The normalized system $\Psi_{1*}X$ again is 2π-periodic in t (and x) so again consider its Poincaré map. By Theorem 1 we can write a conjugate of P as a small perturbation of the 2π-flow of the planar vector field X_1:

Corollary 2 (*Normalization of the Poincaré map*). Let P be the Poincaré map of X, as above. Then there exists a C^∞ area preserving (symplectic) transformation Φ_1 of the plane such that

$$\Phi_1 \circ P \circ \Phi_1^{-1} = X_1^{2\pi} + p_2,$$

where $p_2(x, y; \alpha, \beta) = O(|y; \alpha, \beta|^\infty)$. Moreover, if X is $\mathcal{R}$-reversible or $\mathcal{R}$- and $\mathcal{S}$-reversible, the maps $\Phi_1 \circ P \circ \Phi_1^{-1}$ and $X_1^{2\pi}$ are R-reversible or R- and S-reversible. The same holds with respect to $\mathcal{T}$.

Proof. It is easy to see that the Poincaré map of $\Psi_{1*}X$ has the form $\Phi_1 \circ P \circ \Phi_1^{-1}$, where $\Phi_1 := \Psi_1|_{t=0}$. Since Ψ_1 is canonical and t-preserving the map Φ_1 is area preserving. The symmetry properties are direct, also compare [12,18]. $\square$

Recall that we are after the understanding of the Poincaré map P. The corollary says that up to the infinitely flat perturbation p_2, small for $(y; \alpha, \beta)$ near $(0; 0, 0)$, instead of P, we may as well consider the integrable map $X_1^{2\pi}$, which is again area preserving and has all desired symmetries. Hence, we consider the planar vector field X_1, Hamiltonian with respect to a planar function $H_1 = H_1(x, y; \alpha, \beta)$, which by Theorem 1 satisfies

$$H_1(x, y; \alpha, \beta) = \tfrac{1}{2}y^2 + U(x; \alpha, \beta)$$
$$+ O(|\alpha, \beta|^2 y^2) + O(|\alpha, \beta|^3).$$

We simplify further by removing the remainder term $O(|\alpha, \beta|^2 y^2)$. Again we apply successive coordinate transformations. Here we use non-symplectic transformations, which is allowed in one degree of freedom.

Indeed, if $\Psi_2 : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ is an arbitrary transformation, then $\Psi_{2*}X_1 = (\det D\Psi_2) X_{H_1 \circ \Psi_2}$, where $X_{H_1 \circ \Psi_2}$ is the vector field with Hamiltonian $H_1 \circ \Psi_2$. So *equivalence* of the vector fields exactly corresponds to *right-equivalence* of the Hamiltonians. This rather qualitative way of looking changes the perspective to the collection of level sets of Hamiltonian functions, compare, e.g. [6,7]. The next result is a kind of equivariant splitting lemma, compare [14].

Theorem 3 (*Normalization of the planar Hamiltonian*). Let $H_1 = H_1(x, y; \alpha, \beta)$ be as before. Then there exists a C^∞ coordinate transformation $\Psi_2 : \mathbb{S}^1 \times \mathbb{R} \longrightarrow \mathbb{S}^1 \times \mathbb{R}$ such that

$$(H_1 \circ \Psi_2)(x, y; \alpha, \beta) = H_2(x, y; \alpha, \beta),$$

where

$$H_2(x, y; \alpha, \beta) = \tfrac{1}{2}y^2 + U(x; \alpha, \beta) + O(|\alpha, \beta|^3).$$

Here the remainder $O(|\alpha, \beta|^3)$ is independent of y. Moreover, if X is $\mathcal{R}$-reversible or $\mathcal{R}$- and $\mathcal{S}$-reversible, H_2 is R-equivariant or R- and S-equivariant. The same holds with respect to $\mathcal{T}$.

3. Bifurcation analysis of the planar normal form

We now study the level sets of the family of planar functions $H_2(x, y; \alpha, \beta)$ as obtained in the previous section. The idea is to consider the function $N(x, y; \alpha, \beta) := \tfrac{1}{2}y^2 + U(x; \alpha, \beta)$, realizing that by Theorem 3

$$H_2(x, y; \alpha, \beta) = N(x, y; \alpha, \beta) + O(|\alpha, \beta|^3), \qquad (2)$$

where the O-term, for small $|(\alpha, \beta)|$, is a small perturbation of N. The truncation $N(x, y; \alpha, \beta)$ turns out to be 'sufficient' for the full planar Hamiltonian $H_2(x, y; \alpha, \beta)$. To establish this we use equivariant singularity theory, on the family of 'potential' functions $U(x; \alpha, \beta) = \tfrac{1}{2}\beta^2(V'(x))^2 - \alpha V(x)$. At this moment we only require that $V'(0) = 0$.

For similar methods to study bifurcations in Hamiltonian systems, see [6,7,12,14] or [10], this volume, and the references quoted there. For background on Hamiltonian bifurcations, compare [22].

First observe that $U(x; 0, 0) \equiv 0$, expressing a great degeneracy. To overcome this we perform a scaling, as suggested by the form of the stability boundary in Mathieu's equation, see Fig. 1. Indeed, we introduce new variables $\bar{x}$, $\bar{y}$ and new parameters $\bar{\alpha}$, $\bar{\beta}$ by the following relations:

$$\alpha = \bar{\beta}^2 \bar{\alpha}, \quad \beta = \bar{\beta}, \quad x = \bar{x}, \quad y = |\bar{\beta}|\bar{y}.$$

Note that $\beta = \bar{\beta}$ is considered small. Observe that this scaling is R-, S- and T-equivariant. We consider the scaled functions $\bar{H}_2$ and $\bar{N}$, defined by

$$\bar{N}(\bar{x}, \bar{y}; \bar{\alpha}) = \beta^{-2} N(x, y; \alpha, \beta)$$

and

$$\bar{H}_2 = \beta^{-2} H_2(x, y; \alpha, \beta).$$

Moreover consider the Fourier expansion $\rho(t) = \sum_{k \in \mathbb{Z}} a_k e^{ikt}$. Since ρ is real-valued with average zero, one has $\bar{a}_k = a_{-k}$ for all $k \in \mathbb{Z}$. By a preliminary scaling of β we can achieve that

$$\sum_{k \neq 0} \frac{|a_k|^2}{k^2} = 1. \tag{3}$$

A simple computation, using (2), shows that:

Lemma 4 (*Scaling*). Let V and ρ be C^∞ functions with $V'(0) = 0$, $a_0 = 0$ and such that (3) holds. Then $\bar{N}$ is independent of $\bar{\beta}$, while

$$\bar{N}(\bar{x}, \bar{y}; \bar{\alpha}) = \tfrac{1}{2}\bar{y}^2 + \bar{U}(\bar{x}; \bar{\alpha})$$

where

$$\bar{U}(\bar{x}; \bar{\alpha}) = \tfrac{1}{2}(V'(\bar{x}))^2 - \bar{\alpha} V(\bar{x}).$$

Moreover,

$$\bar{H}_2(\bar{x}, \bar{y}; \bar{\alpha}, \bar{\beta}) = \bar{N}(\bar{x}, \bar{y}; \bar{\alpha}) + O(\bar{\beta}),$$

where the term $O(\bar{\beta})$ is independent of $\bar{y}$.

To simplify notation, from now on we omit all bars.

3.1. The inverted pendulum

In the case of the inverted pendulum with $V(x) = 1 - \cos x$, we have $N(x, y; \alpha) = \tfrac{1}{2}y^2 + U(x; \alpha)$, with 'potential' function

$$U(x; \alpha) = \tfrac{1}{2}\sin^2 x - \alpha(1 - \cos x).$$

Recall that this case has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry, generated by R- and S-reversibility, implying that $V'(0) = 0 \equiv U'(0; \alpha)$. By the T-equivariance we only need to consider the case $\alpha \geq 0$. In Fig. 5 we plotted the 'potential energy' $U = U(x; \alpha)$, $x \in \mathbb{S}^1$, for several values of α. Fig. 6 shows the corresponding planar phase portraits. The only bifurcation of interest takes place at $(x, y; \alpha) = (0, 0; 1)$, which we now consider more closely.

3.2. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case

We discuss the bifurcation at $(x, y; \alpha) = (0, 0; 1)$ using equivariant singularity theory, beginning with a definition. A universal model of the *cusp catastrophe*
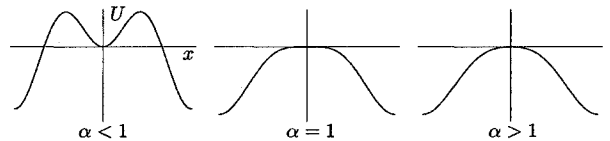
Fig. 5. 'Potential energy' U of the inverted pendulum for $x \in \mathbb{S}^1$. For $\alpha = 1$ a $\mathbb{Z}_2$-equivariant cusp catastrophe occurs at $x = 0$.
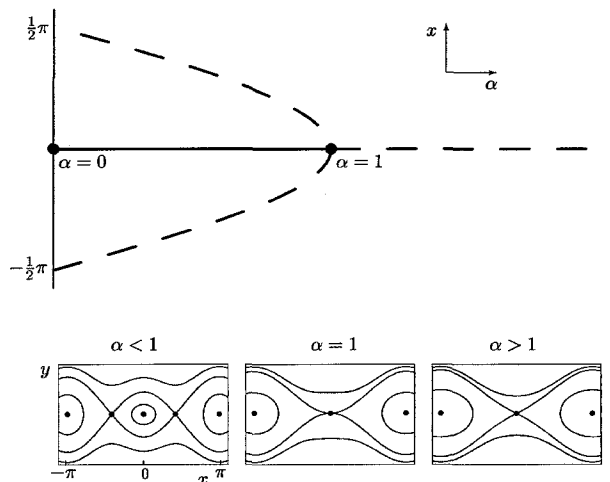

Fig. 6. The inverted pendulum for $x \in \mathbb{S}^1$. Top: Bifurcation diagram in the (α, x)-plane. Dashed lines indicate unstable equilibria. Bottom: Corresponding global phase portraits. For $\alpha = 1$ a subcritical Hamiltonian pitchfork bifurcation occurs at $(x, y) = (0, 0)$.
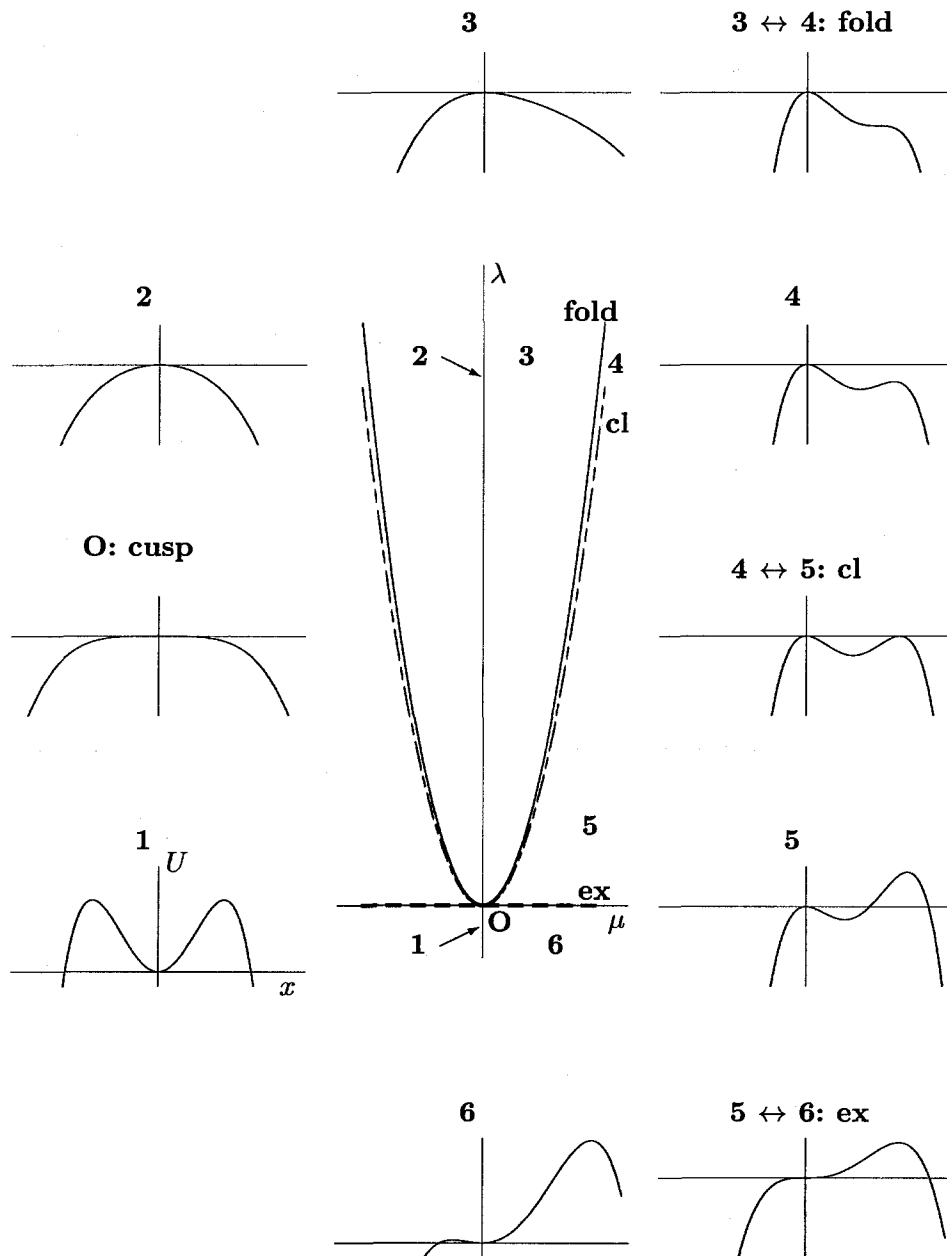
Fig. 7. 'Potential energy' of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric and $\mathbb{Z}_2$-symmetric universal models, organized around the central diagram in the (μ, λ)-plane. The left column of pictures shows the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case, corresponding to $\mu = 0$. The numbering corresponds to the numbers of the graphs, also compare the coding below.

within this $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric context is given by

$$\tfrac{1}{2}y^2 - \lambda x^2 - x^4,\tag{4}$$

see Figs. 7 and 8.

Theorem 5 (Universal model in the $\mathbb{Z}_2\mathbb{Z}_2$-symmetric case). The one parameter family N at $(x, y; \alpha) = (0, 0; 1)$ is a universal $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant unfolding of the cusp singularity, i.e. up to a local equivariant
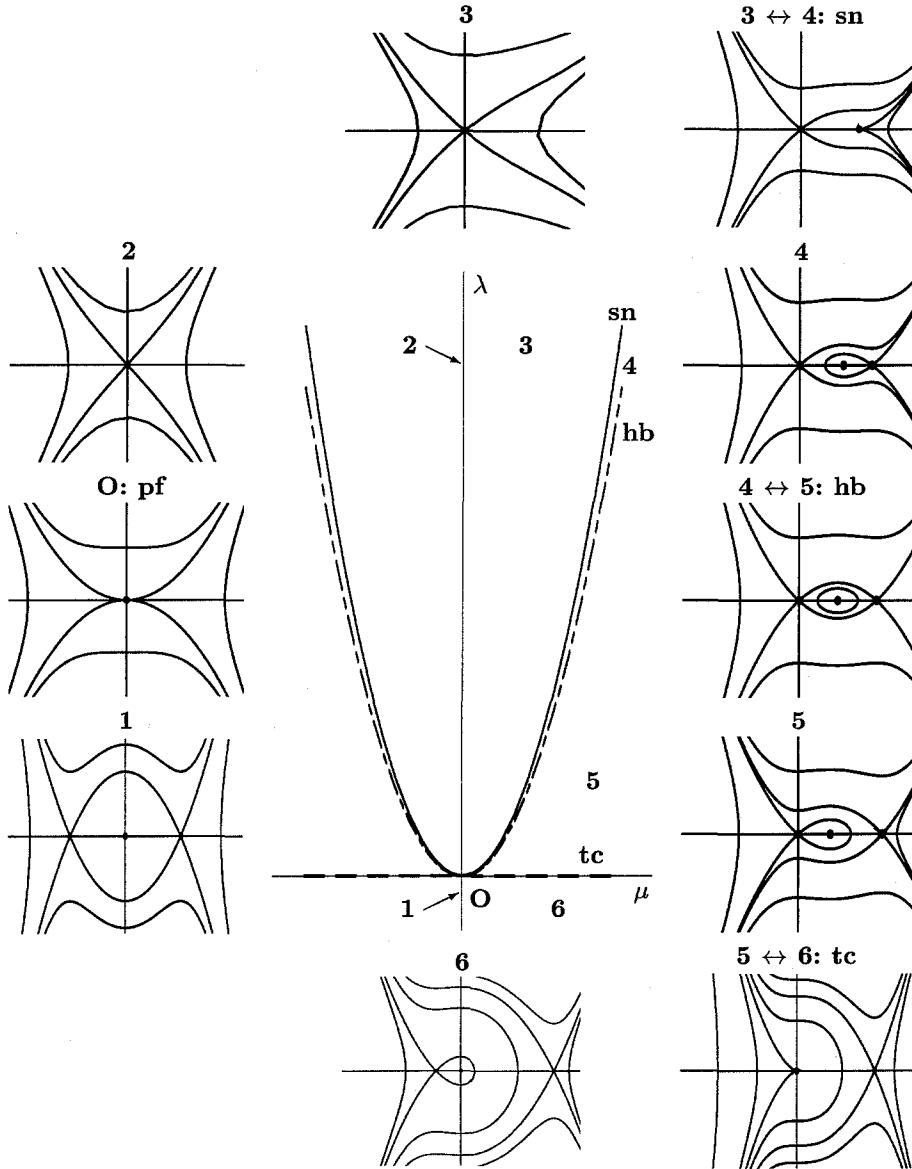
Fig. 8. The universal models in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric and $\mathbb{Z}_2$-symmetric cases. Central: parameter diagram as before, organizing the phase portraits around. Along the line $\mu = 0$ the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case is recovered, as shown in the left column of pictures.

right equivalence and a local reparametrization N is equal to (4).

Proof. Consider the Hamiltonian $N(x; \alpha) = \frac{1}{2}y^2 + U(x; \alpha)$. For $\alpha < 1$ the function N has four crit-

ical points and for $\alpha > 1$ exactly two, where the point $(x, y) = (\pm\pi, 0)$ is counted only once. All these points are non-degenerate. The origin $(x, y) = (0, 0)$ is a minimum for $\alpha < 1$ and a saddle point for $\alpha > 1$. In this sense $\alpha = 1$ gives an approximation of the stability boundary.

For $\alpha = 1$ the function N has a degenerate critical point at $(x, y) = (0, 0)$, the central singularity. The Taylor expansion of U at this point reads

$$U(x; \alpha) = \tfrac{1}{2}(1 - \alpha)x^2 - \tfrac{1}{8}x^4$$
$$+ O(|x; \alpha - 1|^5). \quad (5)$$

From expansion (5) it follows that U at $(x; \alpha) = (0; 1)$ has an $\mathbb{Z}_2$-*equivariant cusp* catastrophe, which implies our result, compare [14,20] or [12]. $\square$

Remarks.
1. Also H_2 (for small $|\beta|$) is a versal unfolding in this context, i.e. H_2 is equivalent to (4). Among other things this means that the reparametrization

$$(\alpha, \beta) \mapsto \lambda(\alpha, \beta)$$

has maximal rank.
2. The same result holds if V and ρ are slightly perturbed, while keeping them both even, implying that the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry is not broken.

3.3. Breaking the S-symmetry: The $\mathbb{Z}_2$-symmetric case

We simply perturb the potential energy V to

$$V + \varepsilon W,$$

where $V(x) = 1 - \cos x$, as before, and where $W = W(x)$ is an arbitrary 2π-periodic C^∞-function with $W'(0) = 0$. In general the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry will be broken, but $(x, y) = (0, 0)$ persists as 'upper equilibrium' solution and as an equilibrium for the normal form. We focus on the bifurcation at $(x, y; \alpha) = (0, 0; 1)$.

It turns out that the pitchfork bifurcation generically breaks up into a *transcritical* and a *Hamiltonian saddle–node* bifurcation. Application of Lemma 4 to this situation yields (in scaled coordinates and parameters):

$$H_2(x, y; \alpha, \beta, \varepsilon) = N(x, y; \alpha, \varepsilon) + O(\beta),$$

where

$$N(x, y; \alpha, \varepsilon) = \tfrac{1}{2}y^2 + U(x; \alpha, \varepsilon)$$

and

$$U(x; \alpha, \varepsilon) = \tfrac{1}{2}(\sin x + \varepsilon W'(x))^2$$
$$+ \alpha(\cos x - 1 - \varepsilon W(x)).$$

Now the normalized system just has a $\mathbb{Z}_2$-symmetry, generated by R-reversibility, while $V'(0) = W'(0) = 0 \equiv U'(0; \alpha, \varepsilon)$, by assumption. As before our discussion uses singularity theory. Again we start by giving a universal model for N, which is just the *cusp catastrophe* in the present context

$$\tfrac{1}{2}y^2 - \lambda x^2 + \mu x^3 - x^4, \quad (6)$$

compare [8], also see Figs. 7 and 8.

Theorem 6 (Universal model in the $\mathbb{Z}_2$-symmetric case). Suppose that $W'''(0) \neq 0$. Then, the two parameter family N at $(x, y; \alpha, \varepsilon) = (0, 0; 1, 0)$ is a universal $\mathbb{Z}_2$-equivariant unfolding of the cusp singularity, within the context where $(x, y) = (0, 0)$ is kept singular. This means that, up to a local right equivalence, respecting this structure, and a local reparametrization N is equal to (6).

Proof. Consider the Hamiltonian $N(x, y; \alpha, \varepsilon) = \tfrac{1}{2}y^2 + U(x; \alpha, \varepsilon)$. We expand the 'potential energy' U around the central singularity $(x; \alpha, \varepsilon) = (0; 1, 0)$:

$$U(x; \alpha, \varepsilon)$$
$$= -\tfrac{1}{2}(1 + \varepsilon W''(0))(\alpha - 1 - \varepsilon W''(0))x^2$$
$$+ \tfrac{1}{3}\varepsilon W'''(0)x^3 - \tfrac{1}{8}x^4 + O(|x; \alpha - 1, \varepsilon|^5). \quad (7)$$

By a suitable local reparametrization $(\mathbb{R}^2, (1, 0)) \to (\mathbb{R}^2, (0, 0))$,

$$(\alpha, \varepsilon) \mapsto (\lambda(\alpha, \varepsilon), \mu(\alpha, \varepsilon))$$

and a right equivalence, expression (7) now simplifies to

$$-\lambda x^2 + \mu x^3 - x^4.$$

This implies our result, again compare [8,12,14,20]. $\square$

Remarks.
1. Also H_2 (for small $|\beta|$) is a versal unfolding in this context, i.e. H_2 is equivalent to (4). Among other things this means that the reparametrization

$(\alpha, \varepsilon, \beta) \mapsto (\lambda(\alpha, \varepsilon, \beta), \mu(\alpha, \varepsilon, \beta))$

has maximal rank.

2. The same result holds if $V + \varepsilon W$ and ρ are slightly perturbed, while keeping $V'(0) = W'(0) = 0$ and ρ even, such that the $\mathbb{Z}_2$-symmetry is not broken. Indeed, under condition (3) our analysis is independent of the choice ρ.

For the understanding of Figs. 7 and 8 we introduce the following coding:

ex: exchange bifurcation
cl: coinciding levels
tc: transcritical bifurcation
sn: (Hamiltonian) saddle–node bifurcation
pf: (Hamiltonian) pitchfork bifurcation
hb: heteroclinic bifurcation

4. Conclusions

As said earlier, the (local) stability diagrams are related to the bifurcation diagrams. All bifurcations are obtained from universal unfoldings of the central singularity

$$\frac{1}{2}y^2 - x^4$$

within the context at hand. Indeed, the given families N and H_2 are locally equivalent to these, up to a reparametrization of full rank, meaning that for $\beta \neq 0$ they form versal unfoldings. Returning to the original parameters near $|\beta| = 0$ we take the scaling of the beginning of this section into account. Indeed, scaling (or blowing) down the appropriate parts of the bifurcation diagrams we get the following.

In Fig. 3 (left) the pitchfork bifurcation line scales down to the stability boundary for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case. Also compare with Figs. 6 and 2. In Fig. 3 (right) the bifurcation lines are scaled down for the $\mathbb{Z}_2$-symmetric case. The line of transcritical bifurcations then forms the stability boundary.

Remark. If we abandon the condition that $V'(0) = 0$ and $W'(0) = 0$, we recover the 'ordinary' catastrophe with universal model

$$\frac{1}{2}y^2 - \mu x - \lambda x^2 - x^4$$

(which is $\mathbb{Z}_2$-symmetric by its dependence on y). Also the evenness of ρ can be broken in which case we again expect an ordinary cusp catastrophe as universal model.

4.1. The inverted pendulum

The inverted pendulum (1) belongs to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric category. Hence Figs. 5 and 6 apply here as well as Theorem 5. In the integrable approximation, this system undergoes a persistent subcritical pitchfork bifurcation. Note that by the symmetry the connection between the two saddle points persists.

In Fig. 6 let us follow the two saddle points that are born for $\bar{\alpha} \leq 1$, once more using a scaled parameter. Observe that these move to $(x, y) = (\pm\frac{1}{2}\pi, 0)$ as $\bar{\alpha}$ varies from 1 to 0. For a perturbation discussion regarding the flat term, see the end of Section 1.

4.2. Comparing symmetries

In the universal models (4) and (6) we see that for $\mu = 0$ the $\mathbb{Z}_2$-symmetric case reduces to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case. Indeed, in Figs. 7 and 8 the graphs/diagrams **1**, **O** and **2** correspond to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case, where the cusp **O** is located at $(\mu, \lambda) = (0, 0)$. Compare with Figs. 5 and 6.

Generally the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case has infinite codimension in the $\mathbb{Z}_2$-symmetric case, but in the universal models this codimension only seems to be one. This is a common phenomenon, e.g. compare [12].

Appendix A. A normal form theorem

We develop a suitable adaptation of the standard normal form theory for the case of the inverted pendulum, cf. the C^∞ vector field (1). As stated before, the theory serves to average out the time dependence to sufficiently high order in the local variables $(y; \alpha, \beta)$, while the position variable x is kept global. It is important that the normalizing transformations preserve the structure present in the original system (1), such

as the canonical character and the (reversing) symmetries. For background and details we refer to [27] and to [4,5,11,12], also see [13] and the references given in the main text.

The normalization deals with formal vector fields

$$
\begin{aligned}
Z&(x, y, t; \alpha, \beta) \\
&= K(x, y, t; \alpha, \beta)\frac{\partial}{\partial t} \\
&\quad + M(x, y, t; \alpha, \beta)\frac{\partial}{\partial x} \\
&\quad + N(x, y, t; \alpha, \beta)\frac{\partial}{\partial y}
\end{aligned}
\tag{A.1}
$$

on $\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1 = \{x, y, t\}$ with parameters $(\alpha, \beta) \in \mathbb{R}^2$. We require that $Z(0, 0, t; 0, 0) = 0$. Here K, M and N are formal power series in (α, β) with coefficients that are 2π-periodic C^∞ functions of x and t and formal power series in y.

The set $\mathcal{H}$ of all these vector fields is a Lie algebra by the usual bracket. Moreover $\mathcal{H} = \prod_{k \geq 0}\mathcal{H}_k$ is a graded Lie algebra with

$$Z \in \mathcal{H}_k$$

$$\Leftrightarrow K, M, N \text{ are homogeneous in } (\alpha, \beta)$$

$$\text{with degree}(K) = \text{degree}(M)$$

$$= \text{degree}(N) = k.$$

This means that $[\mathcal{H}_k, \mathcal{H}_l] \subseteq \mathcal{H}_{k+l}$ for all integers k and $l \geq 0$.

Given $L \in \mathcal{H}_0$ we define the adjoint operator $\operatorname{ad} L : \mathcal{H} \rightarrow \mathcal{H}$ by $\operatorname{ad} L(Y) = [L, Y]$, which nicely splits into $\operatorname{ad}_k L = \operatorname{ad} L|_{\mathcal{H}_k} : \mathcal{H}_k \rightarrow \mathcal{H}_k$. The theory further requires linear subspaces $\mathcal{G}_k \subseteq \mathcal{H}_k$ such that $\operatorname{im}\operatorname{ad}_k L + \mathcal{G}_k = \mathcal{H}_k$, i.e., complementary to the image of $\operatorname{ad}_k L$. Otherwise the $\mathcal{G}_k$ are arbitrary.

The normal form theory roughly says that any formal vector field $X \in \mathcal{H}$ with $X = L + \sum_{j \geq 1} X_j$, with $X_j \in \mathcal{H}_j$, by a formal change of coordinates can be normalized to

$$\Psi_* X = L + \sum_{j \geq 1} G_j,$$

with $G_j \in \mathcal{G}_j$, for all $j \geq 1$. To be more precise:

Theorem A.1 (*Normal form procedure*). Let $n \geq 0$ be given. Suppose $X \in \mathcal{H}$ with $X = L + \sum_{j=1}^n G_j + \sum_{j=n+1}^\infty X_j$, with $G_j \in \mathcal{G}_j$ ($1 \leq j \leq n$) and $X_j \in \mathcal{H}_j$ ($j > n$). Then there exists a transformation Ψ_{n+1}, infinitesimally generated by a vector field $Y_{n+1} \in \mathcal{H}_{n+1}$, such that

$$(\Psi_{n+1})_* X = L + \sum_{j=1}^{n+1} G_j + R, \tag{A.2}$$

where $G_{n+1} \in \mathcal{G}_{n+1}$ and $R \in \prod_{k > n+1}\mathcal{H}_k$.

This formalism produces an algorithm that automatically preserves the structures present in our case (1), where we take

$$L = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x}.$$

To be precise:

1. X is a time-dependent Hamiltonian vector field and $K(x, y, t; \alpha, \beta) \equiv 1$. Then all infinitesimal generators Y_n can be chosen with these same properties. This means that Ψ_n is time preserving and canonical. In turn this implies that the transformed system (A.2) again is time-dependent Hamiltonian. The 2π-periodicity in time is preserved as well.

2. In the case where X respects the (reversing) symmetries $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$, the generators Y_n can be chosen equivariant with respect to these symmetries, such that Ψ_n commutes with these. This implies that the transformed system (A.2) respects the same (reversing) symmetries.

The choice of the 'good' spaces $\mathcal{G}_k$ is essential. Let $\operatorname{ad}_k L = \operatorname{ad}_k L_S + \operatorname{ad}_k L_N$ be the Jordan–Chevalley decomposition of $\operatorname{ad} L$, where $\operatorname{ad}_k L_S$ is semisimple and $\operatorname{ad}_k L_N$ is nilpotent. It is easy to see that

$$\operatorname{ad}_k L_S = \operatorname{ad}_k \frac{\partial}{\partial t} \quad \text{and} \quad \operatorname{ad}_k L_N = \operatorname{ad}_k y\frac{\partial}{\partial x}.$$

Moreover, it is known that

$$\operatorname{im}\operatorname{ad}_k L + \ker\operatorname{ad}_k L_S = \mathcal{H}_k,$$

which means that we may choose $\mathcal{G}_k = \ker\operatorname{ad}_k L_S = \ker\operatorname{ad}_k(\partial/\partial t)$. In that case the elements $G_k \in \mathcal{G}_k$ are time-independent, indeed, they are just the *time averages* of the X_k, so this already would give us

a time-independent normal form. The nilpotent part, however, can be used to *further restrict* the choice of $\mathcal{G}_k$ to

$$\mathcal{G}_k = \ker \mathrm{ad}_k \, L_S \setminus \mathrm{im} \, \mathrm{ad}_k \, L_N.$$

Compare [5,28]. In this context '$\setminus$' should be interpreted as follows: for linear spaces A, B, C we write $C = A \setminus B$ if $(B \cap A) \oplus C = A$.

In our case where the vector field X has the form (1), a computation leads to the following expressions:

$$\mathcal{G}_k = \mathrm{span} \left\{ \alpha^m \beta^{k-m} h(y) \frac{\partial}{\partial x} \mid m = 0, 1, \ldots, k \right.$$

$$\left. \text{and } h(y) = O(y) \text{ a formal power series} \right\}$$

$$\oplus \mathrm{span} \left\{ \alpha^m \beta^{k-m} \mathrm{e}^{\mathrm{i}nx} \frac{\partial}{\partial y} \mid m = 0, 1, \ldots, k \right.$$

$$\left. \text{and } n \in \mathbb{Z} \setminus \{0\} \right\}.$$

As a consequence we obtain

$$(\Psi_* X)(x, y, t; \alpha, \beta)$$

$$= L + yf(y; \alpha, \beta) \frac{\partial}{\partial x} + g(x; \alpha, \beta) \frac{\partial}{\partial y},$$

where f is a formal power series in (y, α, β), and g a formal power series in (α, β), 2π-periodic in x. Both series are of first order in (α, β).

In this way the map Ψ is obtained as a formal power series in $(y; \alpha, \beta)$, with coefficients that are periodic in x and t. The Borel theorem then gives a C^∞-map, defined on a neighbourhood of $y = \alpha = \beta = 0$, with this ay, mptotic data. As a result we obtain Theorem 1, for similar arguments compare [4,12]. For background on the Borel theorem see [23].

Acknowledgements

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