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Multifractal Analysis of Local Entropies for Expansive Homeomorphisms with Specification

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Abstract: In the present paper we study the multifractal spectrum of local entropies. We obtain results, similar to those of the multifractal analysis of pointwise dimensions, but under much weaker assumptions on the dynamical systems. We assume our dynamical system to be defined by an expansive homeomorphism with the specification property. We establish the variational relation between the multifractal spectrum and other thermodynamical characteristics of the dynamical system, including the spectrum of correlation entropies.

1. Introduction

Recently in the series of papers [10, 11, 2] L. Barreira, Ya. B. Pesin, J. Schmeling, and H. Weiss performed a complete multifractal analysis of local dimensions, entropies and Lyapunov exponents for conformal expanding maps and surface Axiom A diffeomorphisms with Gibbs measures. The main goal of these papers was primarily the analysis of the local (pointwise) dimensions. This is an extremely difficult problem and, for example, similar results for hyperbolic systems in dimensions 3 and higher have not been obtained.

In the present work we concentrate our attention on the multifractal analysis of the local (pointwise) entropies. We are able to obtain results, which are similar to those mentioned above, for Gibbs measures of the expansive homeomorphisms with specification property.

Note that such dynamical systems may not have Markov partitions, which is a crucial condition in the previous works. However, due to the fact that less is known about thermodynamical properties of these dynamical systems we were able to obtain only the continuous differentiabilty of the multifractal spectrum of local entropies (compare: the same spectra for the dynamical systems with Markov partitions are analytic). We believe that the smoothness of the multifractal spectrum in our case can be improved.

We have related the mutifractal spectrum of the local entropies to the spectrum of correlation entropies. These correlation entropies serve as entopy-like analogues of the Hentschel–Procaccia and Renyi spectra of generalized dimensions. This allows us to complete the duality between the mutifractal analyses of local dimensions and entropies.

2. Expansiveness and Specification

The following definitions and fundametal results are taken from [6,8,17], for a compact presentation see [9, Chap.20].

Throughout this paper we assume (X, d) to be a compact metric space.

Definition 2.1. *A homeomorphism $f : X \to X$ is called expansive if there exists a constant $\gamma > 0$ such that if*

$$d(f^n(x), f^n(y)) < \gamma \quad \text{for all } n \in \mathbb{Z} \quad \text{then} \quad x = y. \tag{2.1}$$

The maximal γ with such a property is called the expansivity constant.

Another important property is the following.

Definition 2.2 (Bowen [6]**).** *We say that $f : X \to X$ is a homeomorphism with the specification property (abbreviated to "a homeomorphism with specification") if for each $\delta > 0$ there exists an integer $p = p(\delta)$ such that the following holds: if*

a) $I_1, \ldots, I_n$ *are intervals of integers,* $I_j \subseteq [a, b]$ *for some* $a, b \in \mathbb{Z}$ *and all j,*
b) *dist*$(I_i, I_j) \geq p(\delta)$ *for* $i \neq j$*, then for arbitrary* $x_1, \ldots, x_n \in X$ *there exists a point* $x \in X$ *such that*
   1) $f^{b-a+p(\delta)}(x) = x$,
   2) $d(f^k(x), f^k(x_i)) < \delta$ *for* $k \in I_i$.

The specification property guarantees good mixing properties of f and a sufficient number of periodic orbits. Homeomorphisms that are expansive and with specification, form a general class of "strongly chaotic" dynamical systems. For example, the following holds:

Theorem 2.3 ([9, Theorem 18.3.9]**).** *Let Λ be a topologically mixing compact locally maximal hyperbolic set for a diffeomorphism f. Then $f|_\Lambda$ has the specification property.*

Remark. A generalization of the notion of a space with a hyperbolic diffeomorpism is the so-called Smale space [16]. Also for the Smale spaces mixing implies specification as well.

3. Equilibrium States

For the multifractal analysis one needs an invariant probability measure. On an attractor there is usually one physically relevant measure (density of a generic orbit) called the SRB (Sinai-Ruelle-Bowen) measure, which often belongs to the class of equilibrum states or Gibbs measures. We introduce the last notion now. Again, let (X, d) be a compact space, $f : X \to X$ a continuous map and $\varphi : X \to \mathbb{R}$ a continuous function. We shall use the following notation.

Definition 3.1. *For every $n \in \mathbb{N}$ and any $x, y \in X$ define a new metric*

$$d_n(x, y) = \max_{i=0,\dots,n-1} d(f^i(x), f^i(y)),$$

and let $\mathcal{B}_n(x, \varepsilon) = \{y \in X : \ d_n(x, y) < \varepsilon\}$ for $\varepsilon > 0$.

The set $E \subset X$ is said to be (n, ε)-separated if for every $x, y \in E$ such that $x \neq y$ we have $d_n(x, y) > \varepsilon$.

We say that a set $F \subset X$ is (n, ε)-spanning if for every $y \in X$ there exist $x \in F$ such that $d_n(x, y) < \varepsilon$.

For any function $\varphi : X \to \mathbb{R}$ and $x \in X$ put

$$(S_n\varphi)(x) = \sum_{k=0}^{n-1} \varphi(f^k(x)).$$

Now we introduce the topological pressure which will be defined on the space $C(X)$ of all continuous functions on (X, d).

Definition 3.2. *For $n \in \mathbb{N}$ and $\varepsilon > 0$ define*

$$Z_n(\varphi, \varepsilon) = \sup_E \left\{ \sum_{x \in E} \exp\Big((S_n\varphi)(x) \Big) \right\}, \tag{3.1}$$

where the supremum is taken over all (n, ε)-separated sets E. The pressure is then defined as

$$P(\varphi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, \varepsilon). \tag{3.2}$$

The topological entropy of f, denoted by $h_{top}(f)$, is by definition the topological pressure of $\varphi \equiv 0$. The topological pressure admits other equivalent definitions, for this, see [21]. In particular, the following statement is known as the Variational Principle.

Theorem 3.3. *Denote by $\mathcal{M}_f(X)$ the set of all f-invariant Borel probability measures on X. Let $\varphi \in C(X)$. Then*

$$P(\varphi) = \sup_{\mu \in \mathcal{M}_f(X)} \left(h_\mu(f) + \int \varphi d\mu \right).$$

This result inspires the following definition.

Definition 3.4. *An element μ of $\mathcal{M}_f(X)$ is called an equilibrium state for the potential φ if*

$$P(\varphi) = h_\mu(f) + \int \varphi \ d\mu.$$

The equilibrium state for $\varphi \equiv 0$ (if it exists) is called a measure of maximal entopy.

We recall some other basic properties of the topological pressure:

1. $P : C(X) \to \mathbb{R}$ is continuous and monotonously increasing, i.e.,

$$\varphi \le \psi \Rightarrow P(\varphi) \le P(\psi).$$

2. One of the following holds:

$$P(\varphi) = +\infty \quad \forall \varphi \in C(X),$$
$$P(\varphi) < +\infty \quad \forall \varphi \in C(X).$$

Expansive homeomorphisms, which we will consider in the next sections, always have finite topological entropy and hence the pressure of every continuous function is finite.

3. $P : C(X) \to \mathbb{R}$ is convex, i.e., $\forall \lambda \in [0, 1]$,

$$P(\lambda\varphi + (1-\lambda)\psi) \leq \lambda P(\varphi) + (1-\lambda)P(\psi).$$

4. For any $\varphi \in C(X)$ and $c \in \mathbb{R}$ one has $P(\varphi + c) = P(\varphi) + c$.

We impose additional conditions on the class of potentials under consideration.

We say that $\varphi \in \mathcal{V}_f(X)$ if it is continuous and there exist $\varepsilon > 0$ and $K > 0$ such that for all $n \in \mathbb{N}$,

$$d(f^k(x), f^k(y)) < \varepsilon \text{ for } k = 0, \dots, n-1 \Rightarrow \left|(S_n\varphi)(x) - (S_n\varphi)(y)\right| < K.$$

For example, for a hyperbolic diffeomorphism f, any Hölder continuous function φ is in $\mathcal{V}_f(X)$ [9, Prop.20.2.6].

Theorem 3.5 ([6,16,9]**).** *If f is an expansive homeomorphism with specification and $\varphi \in \mathcal{V}_f(X)$ then there exists a unique measure μ_φ such that*

$$P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi d\mu_\varphi.$$

Moreover, μ_φ is ergodic, positive on open sets and mixing.

The equilibrium state μ_φ can be constructed from the measures concentrated on periodic points in the following way. For every $n \geq 1$ define a probability measure $\mu_{\varphi,n}$ supported on the set of periodic points $Fix(f^n) = \{x \in X : f^n(x) = x\}$ as follows:

$$\mu_{\varphi,n} = \frac{1}{P(f,\varphi,n)} \sum_{x \in Fix(f^n)} e^{(S_n\varphi)(x)} \delta_x, \tag{3.3}$$

where δ_x is a unit measure at x and $P(f,\varphi,n) = \sum_{x \in Fix(f^n)} e^{(S_n\varphi)(x)}$ is a normalizing constant.

Theorem 3.6 ([6,9]**).** *An equilibrium state μ_φ is a weak* limit of the sequence $\{\mu_{\varphi,n}\}$, i.e., for every $h \in C(X)$,*

$$\int h(x)d\mu_{\varphi,n} \to \int h(x)d\mu_\varphi \quad \text{as} \quad n \to \infty.$$

For our purposes of analysis of local entropies the following result will play a key role.

Theorem 3.7 ([8, Proposition 2.1], [9, Theorem 20.3.4]**).** *Let f be an expansive homeomorphism with the specification property. Let $\varphi \in \mathcal{V}_f(X)$ and denote its equilibrium state by μ_φ. Then for a sufficiently small $\varepsilon > 0$ there exist constants $A_\varepsilon, B_\varepsilon > 0$ such that for all $x \in X$ and $n \geq 0$,*

$$A_\varepsilon \leq \frac{\mu_\varphi\big(\{y \in X : \ d(f^k(x), f^k(y)) < \varepsilon \text{ for } k = 0, \ldots, n-1\}\big)}{\exp\left(-nP(\varphi) + (S_n\varphi)(x)\right)} \leq B_\varepsilon. \tag{3.4}$$

Remark. Actually, the result above states that for expansive homeomorphisms with specification the equilibrium states are the so-called Gibbs measures (states) as well. See [8] for detailed discussion.

We have seen that for every $\varphi \in \mathcal{V}_f(X)$ there exists a unique equilibrium state. Using (3.3) and (3.4) we are able to give necessary and sufficient conditions for potentials $\varphi, \psi \in \mathcal{V}_f(X)$ to have the same equilibrium states $\mu_\varphi = \mu_\psi$.

Theorem 3.8. *Let f be an expansive homeomorphism with specification. The equilibrium states μ_φ and μ_ψ corresponding to the potentials $\varphi, \psi \in \mathcal{V}_f(X)$ coincide if and only if there exists a constant $c \in \mathbb{R}$ such that*

$$(S_n\varphi)(x) = (S_n\psi)(x) + nc \tag{3.5}$$

for all $x \in Fix(f^n)$ and all n.

Proof. If (3.5) holds for all $x \in Fix(f^n)$ and n, then by (3.3) one has $\mu_{\varphi,n} = \mu_{\psi,n}$ for all n. Thus $\mu_\varphi = \mu_\psi$.

Suppose that $\mu_\varphi = \mu_\psi =: \mu$. Consider "adjusted" potentials $\widetilde{\varphi} = \varphi - P(\varphi)$ and $\widetilde{\psi} = \psi - P(\psi)$. Let $x \in Fix(f^n)$ for some $n \in \mathbb{N}$, applying (3.4) for sufficiently small $\varepsilon > 0$, we conclude that

$$A_\varepsilon^\varphi \exp\big((S_n\widetilde{\varphi})(x)\big) \leq \mu(\mathcal{B}_n(x, \varepsilon)) \leq B_\varepsilon^\psi \exp\big((S_n\widetilde{\psi})(x)\big).$$

This implies that $(S_n\widetilde{\varphi})(x) \leq (S_n\widetilde{\psi})(x) + C'$ for some constant C' independent of x and n. Since $x \in Fix(f^{kn})$ for all $k \in \mathbb{N}$ we have that

$$(S_n\widetilde{\varphi})(x) = \lim_{k\to\infty} \frac{(S_{kn}\widetilde{\varphi})(x)}{k} \leq \lim_{k\to\infty} \frac{(S_{kn}\widetilde{\psi})(x)}{k} = (S_n\widetilde{\psi})(x).$$

By symmetry we obtain the opposite inequality. Hence

$$(S_n\widetilde{\varphi})(x) = (S_n\widetilde{\psi})(x)$$

for all $x \in Fix(f^n)$ and $n \in \mathbb{N}$. This implies (3.5) with $c = P(\varphi) - P(\psi)$. □

4. Thermodynamical Formalism for Expansive Homeomorphisms with Specification

In this section we establish some technical results on the properties of the pressure for expansive homeomorphisms which will be exploited later in the proof of the main result.

Lemma 4.1. *Suppose $f : X \to X$ is an expansive homeomorphism with specification. Let $\varphi \in \mathcal{V}_f(X)$. Then the function $P(q\varphi)$, $q \in \mathbb{R}$, is continuously differentiable with respect to q and its derivative is given by*

$$\frac{dP(q\varphi)}{dq} = \int \varphi d\mu_q,$$

where μ_q is the equlibrium state corresponding to the potential $q\varphi$. Moreover, $P(q\varphi)$ is a strictly convex function of q provided the equilibrium state μ_φ for φ is not a measure of maximal entropy.

If μ_φ is the measure of maximal entropy then $P(q\varphi) - qP(\varphi) = (1-q)h_{top}(f)$ for all $q \in \mathbb{R}$.

Proof. We shall use several results from [21] to show that $P(q\varphi)$ is a differentiable function of q.

For a moment we are going to use the fact that $f : X \to X$ is a continuous map on a compact metric space (X, d) with finite topological entropy. Since the topological pressure is a continuous and convex function on $C(X)$, for every $\varphi, \psi \in C(X)$, the function

$$t \to \frac{P(\varphi + t\psi) - P(\varphi)}{t}$$

is non-increasing as $t \downarrow 0$. Hence there exist right and left derivatives of $P(\varphi)$ in the direction of ψ, i.e.,

$$d^+P(\varphi)(\psi) = \lim_{t\to 0+} \frac{P(\varphi + t\psi) - P(\varphi)}{t},$$
$$d^-P(\varphi)(\psi) = \lim_{t\to 0-} \frac{P(\varphi + t\psi) - P(\varphi)}{t}.$$

We say that the pressure P is Gâteaux differentiable at φ if for every ψ the following holds

$$d^+P(\varphi)(\psi) = d^-P(\varphi)(\psi).$$

This turns out to be equivalent to the condition that the map $\psi \to d^+P(\varphi)(\psi)$ is linear.

A linear functional α on $C(X)$ is called a tangent functional (subdifferential) to $P(\cdot)$ at φ if

$$P(\varphi + \psi) - P(\varphi) \geq \alpha(\psi)$$

for all $\psi \in C(X)$. Applying the Riesz representation theorem we conclude that there exist a finite signed measure $\nu = \nu(\alpha)$ on X such that

$$\alpha(\psi) = \int \psi d\nu$$

for all $\psi \in C(X)$. From now on we identify the tangent functional α with the corresponding measure ν from the Riesz representation.

Denote by $t_\varphi(P)$ the set of all tangent functionals to P at φ and by $M_\varphi(X)$ the set of all equilibrium states corresponding to the potential φ. Applying the Variational Principle one concludes

$$M_\varphi(X) \subset t_\varphi(P).$$

One can easily check that the pressure P is Gâteaux differentiable at φ if and only if there is a unique tangent functional ν to P at φ [21, Corollary 2] and that

$$dP(\varphi)(\psi) = \int \psi d\nu.$$

Combining the results of Theorems 8.2 and 9.15 from [21] one has that for expansive homeomorphism $f : X \to X$,

$$M_\varphi(X) = t_\varphi(X)$$

for every $\varphi \in C(X)$.

Since for every $\varphi \in \mathcal{V}_f(X)$ the set $M_\varphi(X)$ consists of a single element (uniqueness of equilibrium states), we have that the pressure P is Gâteaux differentiable at any $\varphi \in \mathcal{V}_f(X)$ and

$$\frac{d}{dt}P(\varphi + t\psi)\Big|_{t=0} = \int \psi d\mu_\varphi \tag{4.1}$$

for all $\psi \in C(X)$. This proves the differentiability of the pressure function $P(q\varphi)$ at $q = 1$. The result for all other q follows in the same manner since $q\varphi \in \mathcal{V}_f(X)$ for every $q \in \mathbb{R}$ if $\varphi \in \mathcal{V}_f(X)$.

If a convex function is differentiable, then its derivative is continuous. Since we have already established the differentiability of $P(q\varphi)$ (and it is convex) we obtain the desired result.

Now we are going to establish the strict convexity of $P(q\varphi)$. Suppose, μ_φ is not a measure of maximal entropy. Then applying the result of Theorem 3.8 we conclude that the equilibrium states μ_{q_1} and μ_{q_2}, corresponding to potentials $q_1\varphi$ and $q_2\varphi$ respectively, are not equal if $q_1 \neq q_2$. Indeed, assume $\mu_{q_1} = \mu_{q_2}$ for some $q_1 \neq q_2$. Then by Theorem 3.8 we conclude that for some constant c,

$$(S_n q_1\varphi)(x) = (S_n q_2\varphi)(x) + nc$$

for all n and $x \in Fix(f^n)$. This implies that $(S_n\varphi)(x) = n\tilde{c}$ with $\tilde{c} = c/(q_1 - q_2)$. Appying again Theorem 3.8 one has that the equilibrium state μ_φ and the equilibium state μ_0, corresponding to potential $\psi \equiv 0$, are equal. It means that μ_φ is the measure of maximal entropy. Hence we have arrived at a contradiction with the assumption. Therefore $\mu_{q_1} \neq \mu_{q_2}$ if $q_1 \neq q_2$.

The function $h : \mathbb{R} \to \mathbb{R}$ is called strictly convex if for every $q_0 \in \mathbb{R}$ there exists $\lambda(q_0) \in \mathbb{R}$ such that

$$h(q) > h(q_0) + \lambda(q_0)(q - q_0) \quad \text{for all } q \neq q_0.$$

Put $\lambda(q_0) = \int \varphi d\mu_{q_0}$ for any $q_0 \in \mathbb{R}$. Since $\mu_q \neq \mu_{q_0}$ for $q \neq q_0$ and μ_q is the unique equilibrium state for $q\varphi$, one has

$$\begin{aligned}
P(q\varphi) &= h_{\mu_q}(f) + \int q\varphi d\mu_q \\
&= \sup_{\mu \in \mathcal{M}_f(X)} \Big(h_\mu(f) + \int q\varphi d\mu \Big) \\
&> h_{\mu_{q_0}}(f) + \int q\varphi d\mu_{q_0} \\
&= h_{\mu_{q_0}}(f) + \int q_0\varphi d\mu_{q_0} + (q - q_0)\int \varphi d\mu_{q_0} \\
&= P(q_0\varphi) + \lambda(q_0)(q - q_0).
\end{aligned}$$

This means that $P(q\varphi)$ is a strictly convex function.

If the equilibrium state μ_φ is indeed a measure of maximal entropy, then $\mu_\varphi = \mu_{q\varphi} =: \mu$ for all $q \in \mathbb{R}$. This is a consequence of Theorems 3.5 and 3.8. Then applying the Variational Principle to μ_φ and $\mu_{q\varphi}$ we conclude that

$$P(q\varphi) = h_\mu(f) + q \int \varphi d\mu,$$
$$P(\varphi) = h_\mu(f) + \int \varphi d\mu,$$

where $h_\mu(f) = h_{top}(f)$ since μ is the measure of maximal entropy. The result follows immediately. □

Remark. Much stronger result on smoothness of the pressure are known. For example, the analyticity of pressure has been established for Smale spaces [16], i.e., generalizations of Axiom A diffeomorphisms. The key property which these systems inherit from hyperbolic dynamical systems is the so-called local product structure, which in turn guarantees the existence of Markov partitions. The known methods of establishing the analyticity of pressure strongly rely on this Markov structure. Expansive homeomorphism with specification do not necessarily have Markov partitions. For expansive homeomorpshism with specification we were able to prove only the continuous differentiability of the pressure. However we believe that this result can be improved.

Definition 4.2. *We say that E is a maximal (n, ε)-separated set if it can not be enlarged by adding new points preserving the separation condition.*

It is easy to see that every maximal (n, ε)-separated set E is an (n, ε)-spanning set as well.

The following estimates from [8] will be used later.

Lemma 4.3. *Let f be an expansive homeomorphism and $\gamma > 0$ be its expansivity constant. Let $\varphi \in \mathcal{V}_f(X)$. For every finite set E put*

$$Z_n(\varphi, E) = \sum_{x\in E} \exp\Big((S_n\varphi)(x)\Big).$$

1. *If $\varepsilon, \varepsilon' < \gamma/2$ and E, E' are the maximal (n, ε)- and (n, ε')-separated sets respectively then one has*
$$Z_n(\varphi, E) \le C Z_n(\varphi, E'),$$
where the constant $C = C(\varepsilon, \varepsilon')$ is independent of n. In particular,
$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi, E_n), \tag{4.2}$$
where E_n are the arbirary maximal (n, ε)-separated sets.
2. *If furthermore f satisfies the specification property and $\varepsilon < \gamma/2$, then there exists a constant $D = D(\varphi, \varepsilon) > 0$ such that*
$$|\log Z_n(\varphi, E_n) - nP(\varphi)| < D \tag{4.3}$$
for all n and all maximal (n, ε)-separated sets.

5. Topological Entropy for Non-Compact Sets

The generalization of the topological entropy to non-compact or non-invariant sets goes back to Bowen [5]. Later Pesin and Pitskel [13] generalized the notion of pressure to the case of non-compact sets. Note that by definition topological entropy is the topological pressure for $\varphi \equiv 0$. Now we give the formal definition of the topological entropy of a non-compact or non-invariant set.

Suppose $f : X \to X$ is a continuous map on a compact metric space (X, d). Let $\mathfrak{U} = \{U_1, \dots, U_M\}$ be a finite open cover of X. By defintion, a string $\mathbf{U}$ is a sequence $U_{i_1} \dots U_{i_n}$ with $i_k \in \{1, \dots, M\}$, its length n is denoted by $n(\mathbf{U})$. The collection of all strings of length n is denoted by $W_n(\mathfrak{U})$. For each $\mathbf{U} \in W_n(\mathfrak{U})$ define the open set

$$\begin{aligned} X(\mathbf{U}) &= U_1 \cap f^{-1}U_2 \cap \dots \cap f^{-n+1}U_n \\ &= \{x \in X : \ f^{k-1}x \in U_k, \ k = 1, \dots, n\}. \end{aligned}$$

We say that a collection of strings Γ covers a set $Z \subset X$ if

$$\bigcup_{\mathbf{U} \in \Gamma} X(\mathbf{U}) \supset Z.$$

For every real number s introduce

$$M(Z, s, \mathfrak{U}) = \lim_{N \to \infty} \inf_{\Gamma} \sum_{\mathbf{U} \in \Gamma} \exp(-n(\mathbf{U})s),$$

where the infinum is taken over all collections $\Gamma \subseteq \bigcup_{n \geq N} W_n(\mathfrak{U})$ covering Z. There exists a unique value s such that $M(Z, \cdot, \mathfrak{U})$ jumps from $+\infty$ to 0,

$$h(Z, \mathfrak{U}) := s = \sup\{s : M(Z, s, \mathfrak{U}) = +\infty\} = \inf\{s : M(Z, s, \mathfrak{U}) = 0\}$$

Finally, one can show that the following limit exists:

$$h_{top}(f|_Z) := \lim_{\operatorname{diam}(\mathfrak{U}) \to 0} h(Z, \mathfrak{U}).$$

Definition 5.1. *The number $h_{top}(f|_Z)$ is called the topological entropy of f restricted to the set Z, or, simply, the topological entropy of Z.*

This definition of the topological entropy is similar to the definition of the Hausdorff dimension (the diameters of the covering open sets are substituted by $\exp(-n(\mathbf{U}))$, which can be treated as a "dynamical diameter" of $X(\mathbf{U})$). Indeed, these definitions are particular cases of the so-called Carathéodory dimension characteristics [14].

Theorem 5.2 ([12] **).** *The topological entropy as defined above has the following properties:*

1. $h_{top}(f|_{Z_1}) \leq h_{top}(f|_{Z_2})$ *for any* $Z_1 \subset Z_2 \subset X$*;*
2. $h_{top}(f|_Z) = \sup_i h_{top}(f|_{Z_i})$*, where* $Z = \cup_{i=1}^{\infty} Z_i \subset X$*;*
3. *if* μ *is an invariant measure such that* $\mu(Z) = 1$*, then* $h_{top}(f|_Z) \geq h_\mu(f)$*.*

6. Local Entropy

In this section we give the definition of local entropy. The fundamental result on its existence and properties is the Brin–Katok formula below.

Using the notation from Sect. 3 we introduce the lower and upper local entropies at $x \in X$ as follows

$$\underline{h}_{\mu}(f, x) := \lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu(\mathcal{B}_n(x, \varepsilon)), \tag{6.1}$$

$$\overline{h}_{\mu}(f, x) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(\mathcal{B}_n(x, \varepsilon)). \tag{6.2}$$

Note that the limits in ε exist due to the monotonicity.

We say that the local entropy exists at x if

$$\underline{h}_{\mu}(f, x) = \overline{h}_{\mu}(f, x). \tag{6.3}$$

In this case the common value will be denoted by $h_{\mu}(f, x)$.

Theorem 6.1 **(**Brin–Katok formula, [7]**).** *Let $f : X \to X$ be a continuous map on a compact metric space (X, d) preserving a non-atomic Borel measure μ, then*

1. *for μ-a.e. $x \in X$ the local entropy exists, i.e.,*

$$h_{\mu}(f, x) = \underline{h}_{\mu}(f, x) = \overline{h}_{\mu}(f, x);$$

2. *$h_{\mu}(f, x)$ is a f–invariant function of x, and*

$$\int h_{\mu}(f, x) \, d\mu = h_{\mu}(f),$$

where $h_{\mu}(f)$ is the measure–theoretic entropy of f.

Remark. If μ is ergodic then $h_{\mu}(f, x) = h_{\mu}(f)$ for μ-a.e. $x \in X$.

Lemma 6.2. *Let f be an expansive homeomorphism with specification. Consider an equilibrium state μ_{φ} for the potential $\varphi \in \mathcal{V}_f(X)$. For every $x \in X$ put*

$$\underline{\varphi}^*(x) = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)),$$

$$\overline{\varphi}^*(x) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)).$$

Then

$$\underline{h}_{\mu}(f, x) = P(\varphi) - \overline{\varphi}^*(x),$$

$$\overline{h}_{\mu}(f, x) = P(\varphi) - \underline{\varphi}^*(x),$$

for all $x \in X$. Therefore

$$\underline{h}_{\mu}(f, x) = \overline{h}_{\mu}(f, x) \quad \text{if and only if} \quad \underline{\varphi}^*(x) = \overline{\varphi}^*(x).$$

Proof. Using the estimate from Theorem 3.7 we conclude that for every sufficiently small $\varepsilon > 0$ and some constants C_1, C_2 one has

$$\frac{C_1}{n} + P(\varphi) - \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) \le -\frac{1}{n}\log\mu(\mathcal{B}_n(x,\varepsilon)) \le \frac{C_2}{n} + P(\varphi) - \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))$$

for all $n \ge 1$ and every $x \in X$. The statement follows easily.

7. Multifractal Spectrum for Local Entropies

Following [2] we introduce a multifractal spectrum for (local) entropies. For every α consider a level set of local entropy

$$K_\alpha = \{x \in X : \ h_\mu(f,x) = \alpha\}, \tag{7.1}$$

and the corresponding multifractal decomposition on level sets

$$X = \bigcup_\alpha K_\alpha \bigcup \{x \in X : \ h_\mu(f,x) \text{ does not exist}\}. \tag{7.2}$$

We use the topological entropy, defined in Sect. 5, to measure the "size" of sets $\{K_\alpha\}$. Namely, define a multifractal spectrum for local entropies as follows:

$$\mathcal{E}_E(\alpha) = h_{top}(f|_{K_\alpha}). \tag{7.3}$$

This notation needs a brief explanation: two E's stand for the topological $\mathcal{E}$ntropy of level set of local Entropy. For other multifractal spectra $\mathcal{D}_E, \mathcal{E}_D, \mathcal{D}_D$, see [2].

From a general multifractal formalism one expects $\mathcal{E}_E(\alpha)$ to be smooth and concave on a certain interval of α's. We are able to establish this in the case of equilibrium states for expansive homeomorphisms with specification. The crucial observation which we exploit in the proof is the following. Let $\mu = \mu_\varphi$ be an equilibrium state for a potential φ. Then applying the result of the previous section one gets that

$$x \in K_\alpha \iff h_\mu(f,x) = \alpha \iff \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^i(x)) = P(\varphi) - \alpha. \tag{7.4}$$

Therefore, the level sets of local entropies are exactly the level sets of limits of ergodic averages of φ. From the Ergodic Theorem one concludes that only one of these level sets has full measure, while others are of measure 0. We adopt a technique of estimation of the topological entropy of these level sets from [2]. The main idea is the following: we introduce a 1-parameter family of measures such that for each α with $K_\alpha \neq \varnothing$ there is exactly one measure in the family for which K_α has full measure. These measures μ_q are the equilibrium states for potentials $\varphi_q = q\varphi - P(q\varphi)$. However, for the correspondence between levels $\{K_\alpha\}$ and measures $\{\mu_q\}$ we need a parameterization $\alpha(q)$ such that

$$\mu_q(K_{\tilde\alpha}) = \begin{cases} 1, & \text{if } \tilde\alpha = \alpha(q), \\ 0, & \text{if } \tilde\alpha \neq \alpha(q). \end{cases}$$

The parameterization can be given as follows: first define $T(q) = P(q\varphi) - qP(\varphi)$, and $\alpha(q) = -T'(q)$ (note that T is C^1 by Lemma 4.1). Below we will establish that

$$h_{top}(f|_{K_{\alpha(q)}}) = h_{\mu_q}(f),$$

i.e., μ_q is the measure with maximal metric entropy among all invariant measures $\{\nu\}$ such that $\nu(K_{\alpha(q)}) = 1$. In order to complete the analysis we have to show that $K_\alpha = \varnothing$ for every $\alpha \notin [\inf_q \alpha(q), \sup_q \alpha(q)]$.

8. Main Result

In this section we state our main result. It is exactly in the form of the corresponding results from [2,10] for the multifractal analysis of local (pointwise) dimensions. We are following the same notation and order of statements. The last statement of our theorem is analogous to Remark 5 in [10]. It relates the multifractal spectra of the local entropies to the spectra $h_\mu(f,q)$ of the correlation entropies (analogue of the Hentschel–Procaccia spectra for dimensions $HP(q)$) and $R_\mu(f,q)$ (analogue of the Renyi spectra of dimensions $R(q)$). Although it would be natural to call $R_\mu(f,q)$ the Renyi spectra of entropies, it might cause some confusion, since there exists a different notion called the Renyi entropy of order q [4,20].

Theorem 8.1. *Let f be an expansive homeomorphism with the specification property of a compact metric space (X,d). Let $\varphi \in \mathcal{V}_f(X)$ and $\mu = \mu_\varphi$ be the corresponding equilibrium state. Then*

1. *For μ-a.e. $x \in X$ the local entropy at x exists and*

$$h_\mu(f,x) = h_\mu(f) = P(\varphi) - \int \varphi\, d\mu.$$

2. *For any $q \in \mathbb{R}$ define the function*

$$T(q) = P(q\varphi) - qP(\varphi).$$

   Then $T(q)$ is a convex C^1 function of q. Moreover, $T(0) = h_{top}(f)$, $T(1) = 0$; for every $q \in \mathbb{R}$ one has $T'(q) = \int \varphi d\mu_q - P(\varphi) \le 0$, where μ_q is the equilibrium state for $\varphi_q = q\varphi - P(q\varphi)$.

3. *Put $\alpha(q) = -T'(q)$. Then*

$$\mathcal{E}_E(\alpha(q)) := h_{top}(f|_{K_{\alpha(q)}}) = T(q) + q\alpha(q).$$

   Define

$$\underline{\alpha} = \inf_q \alpha(q) = \lim_{q\to+\infty} \alpha(q),$$
$$\overline{\alpha} = \sup_q \alpha(q) = \lim_{q\to-\infty} \alpha(q).$$

   Then $K_\alpha = \varnothing$ if $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$. It means that the domain of the multifractal spectrum for local entropies $\alpha \to \mathcal{E}_E(\alpha)$ is the range of the function $q \to -T'(q)$.

4. *If the equilibrium state μ for the potential φ is not a measure of maximal entropy, then the relation between $\mathcal{E}_E$ and $T(q)$ can be written in the following variational form:*

$$\mathcal{E}_E(\alpha) = \inf_{q\in\mathbb{R}} (T(q) + q\alpha) \quad \textit{for } \alpha \in (\underline{\alpha}, \overline{\alpha}),$$
$$T(q) = \sup_{\alpha\in(\underline{\alpha},\overline{\alpha})} (\mathcal{E}_E(\alpha) - q\alpha) \quad \textit{for } q \in \mathbb{R}.$$

This implies that $\mathcal{E}_E$ is strictly concave and continuously differentiable on $(\underline{\alpha}, \overline{\alpha})$ with the derivative given by $\mathcal{E}'_E(\alpha) = q$, where $q \in \mathbb{R}$ is such that $\alpha = -T'(q)$.

5. *For every $q \in \mathbb{R}$, $q \neq 1$, the following limits exist:*

$$h_\mu(f, q) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} -\frac{1}{n(q-1)} \log \int \mu(\mathcal{B}_n(x, \varepsilon))^{q-1} d\mu,$$
$$R_\mu(f, q) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} -\frac{1}{n(q-1)} \log\Big(\sup_E \sum_{x\in E} \mu(\mathcal{B}_n(x, \varepsilon))^q\Big),$$

where the supremum is taken over all (n, ε)-separated sets E.
For $q \neq 1$ one has

$$h_\mu(f, q) = R_\mu(f, q) = -\frac{T(q)}{q-1}.$$

The family of correlation entropies $h_\mu(f, q)$ depends continuously on q and

$$h_\mu(f, 0) = h_{top}(f),$$
$$h_\mu(f, 1) := \lim_{q\to 1} h_\mu(f, q) = h_\mu(f).$$

Proof. (1) The first statement is a consequence of the Brin-Katok formula for ergodic dynamical systems (Theorem 6.1).

(2) The smoothness and convexity properties of T follow directly from Lemma 4.1. We calculate the derivative of T with respect to q. Using the formula from Lemma 4.1 one gets

$$T'(q) = \int \varphi d\mu_q - P(\varphi), \tag{8.1}$$

where μ_q is the equilibrium state for the potential $\varphi_q = q\varphi - P(q\varphi)$. The inequality $T'(q) \leq 0$ follows from the Variational Principle applied to φ.

(3) This statement is taken from [2] where it has not been proved. For the sake of completeness we give the proof here.

Let us first calculate the measure–theoretic entropy of the equilibrium state μ_q. From the Variational Principle for μ_q we have

$$\begin{aligned} h_{\mu_q}(f) &= P(\varphi_q) - \int \varphi_q d\mu_q \\ &= 0 + T(q) + qP(\varphi) - q\int \varphi d\mu_q \\ &= T(q) + q\big(P(\varphi) - \int \varphi d\mu_q\big) \\ &= T(q) + q\alpha(q), \end{aligned} \tag{8.2}$$

where $\alpha(q) = -T'(q)$ and we use formula (8.1) for the derivative of $T(q)$.

As we have seen in Lemma 6.2 for any α one has

$$h_\mu(f, x) = \alpha \quad \text{if and only if } \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = P(\varphi) - \alpha.$$

Let us apply now Lemma 6.2 to the equilibrium state μ_q corresponding to the potential $q\varphi$. Similarly one gets that for every β,

$$h_{\mu_q}(f, x) = \beta \quad \text{if and only if} \quad q \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = P(q\varphi) - \beta.$$

Hence one concludes that

$$h_\mu(f, x) = \alpha \quad \text{if and only if} \quad h_{\mu_q}(f, x) = P(q\varphi) - qP(\varphi) + q\alpha.$$

For $\alpha = \alpha(q)$ we get

$$x \in K_{\alpha(q)} \quad \text{if and only if} \quad h_{\mu_q}(f, x) = T(q) + q\alpha(q). \tag{8.3}$$

Combining the results of (8.2) and (8.3) one gets

$$\begin{aligned} h_{\mu_q}(f) &= T(q) + q\alpha(q), \\ h_{\mu_q}(f, x) &= T(q) + q\alpha(q) \quad \text{if and only if } x \in K_{\alpha(q)}. \end{aligned}$$

This means that $h_{\mu_q}(f, x) = h_{\mu_q}(f)$ if and only if $x \in K_{\alpha(q)}$. Since μ_q is ergodic, we know from the Brin–Katok formula that $h_{\mu_q}(f, x) = h_{\mu_q}(f)$ for μ_q-a.e. $x \in X$. Hence we conclude that

$$\mu_q(K_{\alpha_q}) = \mu_q(\{x : h_{\mu_q}(f, x) = h_{\mu_q}(f)\}) = 1.$$

Therefore we obtained the desired parametrization of the level sets.

We have to compute the topological entropy of f restricted to $K_{\alpha(q)}$,

$$\mathcal{E}_E(\alpha(q)) := h_{top}\left(f|_{K_{\alpha(q)}}\right).$$

Using the properties of the topological entropy from Theorem 5.2 we conclude that

$$\mathcal{E}_E(\alpha(q)) = h_{top}\left(f|_{K_{\alpha(q)}}\right) \geq h_{\mu_q}(f) = T(q) + q\alpha(q),$$

since $\mu_q(K_{\alpha(q)}) = 1$. We have to prove the oposite inequality. For this it would be sufficient to show that $h_{top}(f|_{K_{\alpha(q)}}) \leq \lambda$ for any $\lambda > T(q) + q\alpha(q)$. Choose such λ and let $\delta = \lambda - T(q) - q\alpha(q) > 0$.

Rewriting the definition of $K_{\alpha(q)}$ in terms of μ_q and φ_q one has

$$\begin{aligned} K_{\alpha(q)} &= \left\{x \in X : \ h_{\mu_q}(f, x) = h_{\mu_q}(f) = T(q) + q\alpha(q)\right\} \\ &= \left\{x \in X : \ \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi_q(f^i x) = -T(q) - q\alpha(q)\right\}. \end{aligned}$$

For every $x \in K_{\alpha(q)}$ there exists an integer $n(x)$ such that

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \varphi_q(f^i x) + T(q) + q\alpha(q) \right| \leq \frac{\delta}{2} \tag{8.4}$$

for all $n \geq n(x)$. For every integer N consider the set

$$K_{\alpha(q),N} = \{x \in K_{\alpha(q)} : \ n(x) \leq N\}.$$

Obviously we have

$$K_{\alpha(q)} = \bigcup_{N \geq 1} K_{\alpha(q),N}, \quad K_{\alpha(q),N} \subset K_{\alpha(q),N+1}.$$

Using the properties of the topological entropy from Theorem 5.2 we conclude that

$$h_{top}(f|_{K_{\alpha(q)}}) = \lim_{N \to \infty} h_{top}(f|_{K_{\alpha(q),N}}).$$

We are going to show that for any $N \in \mathbb{N}$ one has $h_{top}(f|_{K_{\alpha(q),N}}) \leq \lambda$; this in turn will imply $h_{top}(f|_{K_{\alpha(q)}}) \leq \lambda$.

Consider an arbitrary finite cover $\mathcal{U} = \left\{\mathcal{B}(x_i, \varepsilon/2)\right\}_{i=1}^{M}$ of X by open balls of radius $\varepsilon/2$, with $\varepsilon < \gamma/2$, where γ is the expansivity constant for f. Together with $\mathcal{U}$ we consider $\tilde{\mathcal{U}}$ an open cover by balls with centers at x_i and radii ε. Let $E = \{y_j\}$ be a maximal $(n, \varepsilon/2)$-separated set in X. Define a subset E' of E by choosing those y_j which have a point from $K_{\alpha(q),N}$ close to them, namely

$$E' = \{y_j \in E : \ K_{\alpha(q),N} \cap \mathcal{B}_n(y_j, \varepsilon/2) \neq \varnothing\}.$$

This implies that

$$K_{\alpha(q),N} \subset \bigcup_{y_j \in E'} \mathcal{B}_n(y_j, \varepsilon/2).$$

For every $y_j \in E'$ there exists at least one string $U_{i_0,\dots,i_{n-1}}$ from $\mathcal{W}_n(\mathcal{U})$ such that $y_j \in X(U_{i_0,\dots,i_{n-1}})$. It is easy to see that if

$$y_j \in X(U_{i_0,\dots,i_{n-1}}) = U_{i_0} \cap f^{-1}U_{i_1} \cap \dots f^{-n+1}U_{i_{n-1}},$$

then

$$\mathcal{B}_n(y_j, \varepsilon/2) \subset S(\tilde{U}_{i_0,\dots,i_{n-1}}) = \tilde{U}_{i_0} \cap f^{-1}\tilde{U}_{i_1} \cap \dots f^{-n+1}\tilde{U}_{i_{n-1}}.$$

In other words the collection of strings $\tilde{\Gamma} = \{\tilde{U}_{i_0,\dots,i_{n-1}}\}$ covers $K_{\alpha(q),N}$. Therefore

$$\begin{aligned} m(K_{\alpha(q),N}, \lambda, \tilde{\mathcal{U}}, n) &= \inf_{\substack{\Gamma \subset \cup_{k \geq n} \mathcal{W}_k(\tilde{\mathcal{U}}) \\ \Gamma \text{ covers } K_{\alpha(q),N}}} \sum_{\mathbf{U} \in \Gamma} \exp(-m(\mathbf{U})\lambda) \\ &\leq \sum_{\tilde{\mathbf{U}} \in \tilde{\Gamma}} \exp(-m(\tilde{\mathbf{U}})\lambda) \\ &= e^{-n\delta} \sum_{\tilde{\mathbf{U}} \in \tilde{\Gamma}} \exp\big(-n(T(q) + q\alpha(q))\big) \\ &= e^{-n\delta} \sum_{y_j \in E'} \exp\Big(-n\big(T(q) + q\alpha(q)\big)\Big). \end{aligned} \tag{8.5}$$

Since the potential $\varphi \in \mathcal{V}_f(X)$, so is φ_q, and

$$\left|(S_n\varphi_q)(x) - (S_n\varphi_q)(y)\right| = \left|\sum_{k=0}^{n-1}\varphi_q(f^k(x)) - \sum_{k=0}^{n-1}\varphi_q(f^k(y))\right| \le |q|K$$

for all $x, y \in X$ with $d_n(x, y) < \varepsilon/2$.

For any $y_j \in E'$ let x_j be an arbitrary point from $K_{\alpha(q),N} \cap \mathcal{B}_n(y_j, \varepsilon/2)$. Since $x_j \in K_{\alpha(q),N}$ and $n \ge N$ from (8.4) we have

$$\begin{aligned}-n(T(q) + q\alpha(q)) &\le -n(T(q) + q\alpha(q)) - (S_n\varphi_q)(x_j) + (S_n\varphi_q)(y_j) + |q|K \\ &\le \frac{n\delta}{2} + (S_n\varphi_q)(y_j) + |q|K.\end{aligned}$$

Thus we can continue the estimate (8.5) as follows:

$$\begin{aligned}m(K_{\alpha(q),N}, \lambda, \widetilde{\mathcal{U}}, n) &\le e^{-n\delta/2+|q|K} \sum_{y_j\in E'} \exp((S_n\varphi_q)(y_j)) \\ &\le C' e^{-n\delta/2} Z_n(\varphi_q, E).\end{aligned}$$

Using the estimates from Lemma (4.3) and the fact that $P(\varphi_q) = 0$ we conclude that

$$m(K_{\alpha(q),N}, \lambda, \tilde{\mathcal{U}}, n) \le C'' e^{-n\delta/2}.$$

Hence

$$m(K_{\alpha(q),N}, \lambda, \tilde{\mathcal{U}}) = \lim_{n\to\infty} m(K_{\alpha(q),N}, \lambda, \tilde{\mathcal{U}}, n) = 0,$$

and since $\mathcal{U}$ was an open cover by balls of radius $\varepsilon/2$ we get

$$m(K_{\alpha(q),N}, \lambda) = \lim_{\mathrm{diam}(\mathcal{U})\to 0} m(K_{\alpha(q),N}, \lambda, \tilde{\mathcal{U}}) = 0.$$

Then by definition of the topological entropy we have

$$h_{top}(f|_{K_{\alpha(q),N}}) \le \lambda$$

for all N. Hence $h_{top}(f|_{K_{\alpha(q)}}) \le \lambda$ for all $\lambda > T(q) + q\alpha(q)$. This completes the proof that $h_{top}(f|_{K_{\alpha(q),}}) \le T(q) + q\alpha(q)$.

The rest of the statement is taken from [18]. It states that we have a complete description of the spectra for local entropies.

(4) If the equilibrium state for the potential φ is not a measure maximal entropy then it was shown in Lemma 4.1 that $T(q)$ is strictly convex, i.e., the following holds for every $q, q_0 \in \mathbb{R}$, $q \ne q_0$:

$$T(q) > T(q_0) + T'(q_0)(q - q_0).$$

Therefore, if $\alpha \in (\underline{\alpha}, \overline{\alpha})$ then there exists $q_0 \in \mathbb{R}$ such that $\alpha = -T'(q_0)$. We have seen that in this case $\mathcal{E}_E(\alpha) = T(q_0) + \alpha q_0$. Using the strict convexity of $T(q)$ we obtain that for $q \in \mathbb{R}$, $q \ne q_0$ the following holds

$$\mathcal{E}_E(\alpha) = T(q_0) + \alpha q_0 < T(q) + \alpha q.$$

Hence, $\mathcal{E}_E(\alpha) = \inf_{q\in\mathbb{R}} (T(q) + \alpha q)$ for $\alpha \in (\underline{\alpha}, \overline{\alpha})$.

In a similar manner one obtains the second relation $T(q) = \sup_{\alpha\in(\underline{\alpha},\overline{\alpha})}(\mathcal{E}_E(\alpha) - q\alpha)$.

Using the notion of the Legendre transform [15] we can say that actually functions $T(q)$ and $F(\alpha) := -\mathcal{E}_E(-\alpha)$ form a Legendre pair, i.e., one is the Legendre transform of another. Therefore the convexity and differentiabilty of $\mathcal{E}_E$ follow from the properties of the Legendre transform. In particular, for $\alpha \in (\underline{\alpha}, \overline{\alpha})$ one has $\mathcal{E}'_E(\alpha) = q$, where $q \in \mathbb{R}$ is such that $\alpha = -T'(q)$.

In the case when μ is the measure of maximal entropy one has

$$h_\mu(f, x) = h_\mu(f) = h_{top}(f)$$

for all $x \in X$. It means that $\mathcal{E}_E$ is a delta-like function

$$\mathcal{E}_E(\alpha) = \begin{cases} h_{top}(f), & \text{if } \alpha = h_{top}(f), \\ 0, & \text{otherwise.} \end{cases}$$

This "degenerate" behaviour of the multifractal spectrum for the measure of maximal entropy can be successfuly exploited. For this see [2], where it has been used for the calculations of the mutifractal spectra for Lyapunov exponents.

(5) This is an essentially new result. We prove it by means of standard thermodynamical technique.

Let $q > 1$ and E be an arbitrary (n, ε)-separated set. One has

$$\begin{aligned} \int \mu(\mathcal{B}_n(x, \varepsilon))^{q-1}\, d\mu &\geq \sum_{x_i\in E} \int_{\mathcal{B}_n(x_i,\varepsilon/2)} \mu(\mathcal{B}_n(x, \varepsilon)^{q-1}\, d\mu \\ &\geq \sum_{x_i\in E} \mu(\mathcal{B}_n(x_i, \varepsilon/2))^q, \end{aligned}$$

since $x \in \mathcal{B}_n(x_i, \varepsilon/2)$ implies $\mathcal{B}_n(x_i, \varepsilon/2) \subset \mathcal{B}_n(x, \varepsilon)$.

Applying inequality (3.4), and using the fact that E is an (n, ε)-separated set, we get

$$\int \mu(\mathcal{B}_n(x, \varepsilon))^{q-1}\, d\mu \geq \sup_E \left(\sum_{x_i\in E} A^q_{\varepsilon/2} \exp\Big(-qPn + \sum_{j=0}^{n-1} q\varphi(f^j x_i)\Big)\right),$$

where the supremum is taken over all (n, ε)-separated sets. Taking logarithms and applying estimates from Lemma 4.3 we conclude that in the limit

$$h_\mu(f, q) \leq R_\mu(f, q) \leq \frac{P(q\varphi) - qP(\varphi)}{1-q}.$$

To finish the proof we have to show the opposite inequality. We do it in a similar manner.

Let now E be a maximal $(n, \varepsilon/2)$-separated set, then

$$\begin{aligned} \int \mu(\mathcal{B}_n(x, \varepsilon/2))^{q-1} d\mu &\leq \sum_{x_i\in E} \int_{\mathcal{B}_n(x_i,\varepsilon/2)} \mu(\mathcal{B}_n(x, \varepsilon/2))^{q-1} d\mu \\ &\leq \sum_{x_i\in E} \mu(\mathcal{B}_n(x_i, \varepsilon))^q, \end{aligned}$$

since $x \in \mathcal{B}_n(x_i, \varepsilon/2)$ implies that $\mathcal{B}_n(x, \varepsilon/2) \subset \mathcal{B}_n(x_i, \varepsilon)$.

Again since E is an arbitrary $(n, \varepsilon/2)$-separated set and applying the inequality (3.4) we obtain

$$\int \mu(\mathcal{B}_n(x, \varepsilon/2))^{q-1} d\mu \leq \sup_E \left(\sum_{x_i \in F} B_\varepsilon^q \exp\Big(-qPn + \sum_{j=0}^{n-1} q\varphi(f^j x_i) \Big) \right).$$

Taking logarithms and using estimates from Lemma 4.3 in the limit $n \to \infty$ we get

$$h_\mu(f, q) \geq R_\mu(f, q) \geq \frac{P(q\varphi) - qP(\varphi)}{1 - q}.$$

Combining all together we get the statement in the case $q > 1$. The case $q < 1$ is completely analogous. The continuity and other properties of $h_\mu(f, q)$ follow from the corresponding properties of $T(q)$. □

9. Final Remarks

A. Consider an irregular set

$$\begin{aligned} B &= \{x \in X : \ h_\mu(f, x) \text{ does not exist } \} \\ &= \{x \in X : \ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) \text{ does not exist}\}. \end{aligned}$$

We have seen that for the measure of maximal entropy m_E this is an empty set. It was shown in [3] that in a number of cases, the set B is either empty or has full topological entropy and Hausdorff dimension.

B. There exists another way of defining local (pointwise) entropies. Namely, consider an arbitrary finite measurable partition ξ of X. We can define a local entropy at x with respect to ξ as follows (if the limit exists):

$$h_\mu(f, x, \xi) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(\xi^{(n)}(x)),$$

where $\xi^{(n)} = \xi \vee f^{-1}\xi \vee \ldots \vee f^{-n+1}\xi$ and $\xi^{(n)}(x)$ is the element of $\xi^{(n)}$ containing x. We can define a spectrum of local entropies with respect to ξ as follows:

$$\mathcal{E}_E(\alpha) = h_{top}(f|_{\{x : h_\mu(f,x,\xi)=\alpha)\}}).$$

The situation when ξ is a finite Markov partition for an expanding dynamical system has been studied in [2,1]. One can eaily check that in this case the two spectra coincide.

C. The results of this paper can be extended to the case of expansive endomorphisms (i.e., non-invertible maps) with the specification property. They are defined in exactly the same way as the expansive homeomorphisms with specification except that the set $\mathbb{Z}$ in (2.1) is substituted by $\mathbb{N}$ (positive expansiveness). The characteristic property of the equilibrium states (Theorem 3.7) remains valid [17]. Therefore our analysis works without any modifications.

In the case of expansive homeomorphisms we can give another definition of local entropies. Namely, for any $n \geq 1$ define

$$\mathcal{B}_n^{\pm}(x, \varepsilon) = \{y \in X : \ d(f^i(x), f^i(y)) < \varepsilon \quad \text{for all} \quad i = -n+1, \ldots, n-1\},$$

and

$$\underline{h}_\mu^{\pm}(f,x) = \lim_{\varepsilon\to 0}\liminf_{n\to\infty} -\frac{1}{2n-1}\log\mu(\mathcal{B}_n^{\pm}(x,\varepsilon)),$$
$$\overline{h}_\mu^{\pm}(f,x) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{2n-1}\log\mu(\mathcal{B}_n^{\pm}(x,\varepsilon)).$$

Then the level sets of these local entropies will be in one-to-one correspondence with the level sets of two-sided ergodic averages of φ,

$$\lim_{n\to\infty}\frac{1}{2n-1}\sum_{k=-n+1}^{n-1}\varphi(f^k(x)).$$

The level sets of two-sided and one-sided ergodic averages of φ can be different. However, they have the same topological entropy with respect to f. Therefore the multifractal spectrum based on $h_\mu^{\pm}(f,x)$ will be the same.

D. A requirement of the existence of a Markov partition is stronger than a specification property, provided the dynamical system is mixing.

Consider the family of one-dimensional interval maps T_β, defined by $T_\beta(x) = \beta x \pmod 1$. For $\beta > 1$ these maps are expanding and therefore expansive. The ergodic properties of T_β depend on the number-theoretic properties of β. For these systems it turns out [19] that:

i) the set of β's for which T_β has a finite Markov partition is at most countable;
ii) the set of β's for which T_β has the specification property is uncountable and has Hausdorff dimension 1, but still has Lebesgue measure 0.

Therefore, we can see that in the family $\{T_\beta\}_{\beta>1}$, specification is a much more general property than the property of having a finite Markov partition.

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