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Abstract

This paper offers a new input-normal output-diagonal realization and model reduction procedure for nonlinear systems based on the differential eigenstructure of Hankel operators. Firstly, we refer to the preliminary results on input-normal realizations with original singular value functions and the differential eigenstructure of Hankel operators with axis singular value functions. Secondly, the relationship between the two different characterizations of singular value functions is clarified and, consequently, the new input-normal realization is characterized. Thirdly, we perform the model reduction based on the obtained realization. Furthermore numerical examples demonstrate the effectiveness of the proposed method.

1 Introduction

In the theory of continuous time linear systems, the system Hankel operator plays a central role in minimality theory, in model reduction problems, in realization theory, and related to these, in linear identification methods. Specifically, the Hankel operator supplies a set of similarity invariants, the so called Hankel singular values, which can be used to quantify the importance of each state in the corresponding input-output system [7]. The Hankel operator can also be factored into the composition of the observability and controllability operators, from which Gramian matrices can be defined and the notion of a balanced realization follows, firstly introduced in [9].The linear Hankel theory is rather complete and the relations and interpretations between state space and input-output settings are fully understood.

The nonlinear extension of the state space concept of balanced realizations has been introduced in [13], mainly based on studying the past input energy and the future output energy. Since then, many results on state space balancing, modifications, computational issues for model reduction and related minimality considerations for nonlinear systems have appeared in the literature. e.g. [5, 8, 11, 10, 14]. Further, the relation of the state space notion of balancing for nonlinear systems with the nonlinear Hankel operator has been considered, see e.g. [5, 15, 14]. In particular, *singular value functions* [13] which are nonlinear state space extension of the Hankel singular values in the linear case play an important role in the nonlinear Hankel theory. It has been shown that singular value functions are closely related to Hankel operators [15]. However, there are some major differences from the linear theory, i.e., studying similarity invariance of singular value functions in relation to the nonlinear Hankel operator can be done via several interpretations of the concept of similarity invariance and may result in different conclusions. Recently, *axis singular value functions* were introduced [2] as alternative nonlinear extension of the Hankel singular values, which are derived based on the differential eigenstructure of the self adjoint of nonlinear Hankel operators. This new characterization is defined only by input-output properties of the system and, consequently, does not depended on the choice of the coordinate in contrast with the original singular value functions. In addition, the relationship between the axis singular value functions and the original ones were not clear so far.

The main objective of this paper is to provide an input-normal/output-diagonal realization based on the axis singular value functions. To this end, we will prove that there exists an input-normal/output diagonal realization whose original singular value functions coincide with the axis singular value function on certain subspaces of the state space. Hence this realization connects the axis singular value functions and the original ones. Furthermore, we will perform the model reduction [12, 13] based on the newly obtained realization. Moreover, it will be proved that the axis singular value functions are preserved in the the model reduction procedure, and that the Hankel norm of the original system is also preserved as a result. Some numerical examples demonstrate the effectiveness of the proposed method.

2 Preliminaries

This section refers to preliminary results on input-normal/output-diagonal realizations (as a nonlinear extension

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of balanced realizations) [13, 4, 2].

2.1 Singular value functions

Here we refer to the observability and controllability functions, and relate them to singular value analysis of nonlinear dynamical operators. We only consider time invariant, input-affine, sufficiently smooth nonlinear systems without direct feed-through in the form of

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} . \quad (1)$$

Here $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$, and it is supposed to be asymptotically stable and L_2-stable in the sense that $u \in L_2^m(-\infty, 0]$ implies that $y = \Sigma(u)$ restricted to $[0, \infty)$ is in $L_2^r[0, \infty)$.

The observability and controllability functions of the system (1) are defined as follows.

Definition 1 The observability function $L_o(x)$ and the controllability function $L_c(x)$ of Σ in (1) are defined by

$$L_o(x^0) := \frac{1}{2}\int_0^\infty \|y(t)\|^2\,dt,\ x(0) = x^0,\ u(t) \equiv 0$$

$$L_c(x^1) := \inf_{\substack{u \in L_2^m(-\infty,0] \\ x(-\infty)=0,\, x(0)=x^1}} \frac{1}{2}\int_{-\infty}^0 \|u(t)\|^2\,dt.$$

In [13], these functions have been used for the basis of input-normal realizations and singular value functions of nonlinear systems. Also they fulfill corresponding Hamilton-Jacobi equations, in a similar way to the observability Gramian and the inverse of the controllability Gramian are solutions of a Lyapunov/Riccati equation.

Theorem 1 *[13] Consider the system Σ in (1). Suppose there exists a neighborhood W of 0 on which Σ is asymptotically stable and there exists a smooth observability function $L_o(x)$. Then $L_o(x)$ is the unique smooth solution of the Hamilton-Jacobi equation*

$$\frac{\partial L_o(x)}{\partial x} f(x) + \frac{1}{2}\, h(x)^{\mathrm{T}} h(x) = 0.$$

Furthermore, suppose that there exists a smooth controllability function $L_c(x)$ on W. Then $L_c(x)$ is the unique solution of the Hamilton-Jacobi equation

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2}\, \frac{\partial L_c(x)}{\partial x} g(x) g(x)^{\mathrm{T}} \frac{\partial L_c(x)}{\partial x}^{\mathrm{T}} = 0$$

such that $\dot{x} = -(f + gg^{\mathrm{T}}(\partial L_c/\partial x)^{\mathrm{T}})$ is asymptotically stable about 0 on W.

Next we review what we mean by input-normal/output-diagonal form. In the rest of this paper, the word *input-normal* is also used as shorthand for input-normal/output-diagonal.

Theorem 2 *[13] Consider an operator Σ with an asymptotically stable state-space realization (1). Suppose that the Jacobian linearization of Σ is controllable and observable and that there exists a neighborhood V of the origin where smooth L_c and L_o exist. Then there exists a smooth coordinate transformation $x = \Phi(z)$, $\Phi(0) = 0$, on V, which converts Σ into an input-normal/output-diagonal form, where*

$$L_c(\Phi(z)) = \frac{1}{2}\, z^{\mathrm{T}} z \quad (2)$$

$$L_o(\Phi(z)) = \frac{1}{2}\, z^{\mathrm{T}} \mathrm{diag}(\tau_1(z), \cdots, \tau_n(z)) z \quad (3)$$

with $\tau_1(z) \geq \ldots \geq \tau_n(z)$ being smooth singular value functions on V.

Example 1 We consider the system (1) which fulfills all technical assumptions with $x = (x_1, x_2) \in \mathbb{R}^2$, $u = (u_1, u_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and f, g and h as follows:

$$f(x) = \begin{pmatrix} -9x_1 + 6x_1^2 x_2 + 6x_2^3 - x_1^5 - 2x_1^3 x_2^2 - x_1 x_2^4 \\ -9x_2 - 6x_1^3 - 6x_1 x_2^2 - x_1^4 x_2 - 2x_1^2 x_2^3 - x_2^5 \end{pmatrix}$$

$$g(x) = \begin{pmatrix} \frac{3\sqrt{2}(9-6x_1x_2+x_1^4-x_2^4)}{9+x_1^4+2x_1^2x_2^2+x_2^4} & \frac{\sqrt{2}(-9x_1^2-27x_2^2+6x_1^3x_2+6x_1x_2^3-(x_1^2+x_2^2)^3)}{9+x_1^4+2x_1^2x_2^2+x_2^4} \\ \frac{\sqrt{2}(27x_1^2+9x_2^2+6x_1^3x_2+6x_1x_2^3+(x_1^2+x_2^2)^3)}{9+x_1^4+2x_1^2x_2^2+x_2^4} & \frac{3\sqrt{2}(9+6x_1x_2-x_1^4+x_2^4)}{9+x_1^4+2x_1^2x_2^2+x_2^4} \end{pmatrix}$$

$$h(x) = \begin{pmatrix} \frac{2\sqrt{2}(3x_1+x_1x_2^2+x_2^3)(3-x1^4-2x_1^2x_2^2-x_2^4)}{1+x1^4+2x_1^2x_2^2+x_2^4} \\ \frac{\sqrt{2}(3x_2-x_1^3-x_1x_2^2)(3-x1^4-2x_1^2x_2^2-x_2^4)}{1+x1^4+2x_1^2x_2^2+x_2^4} \end{pmatrix}.$$

This system is zero-state observable, and $\dot{x} = f(x)$ is asymptotically stable. Computing L_o and L_c by solving the HJB equations in Theorem 1, we obtain:

$$L_c(x) = \frac{1}{2} x^{\mathrm{T}} x$$

$$L_o(x) = \frac{1}{2} \frac{36x_1^2+9x_2^2+18x_1^3x_2+18x_1x_2^3+x_1^6+6x_1^4x_2^2+9x_1^2x_2^4+4x_2^6}{1+x_1^4+2x_1^2x_2^2+x_2^4}$$

We see that the controllability function is already in input-normal form and that the observability function is in output-diagonal form. A pair of singular value functions are:

$$\tau_1(x) = \frac{36 + 18x_1x_2 + x_1^4 + 6x_1^2x_2^2}{1 + x_1^4 + 2x_1^2x_2^2 + x_2^4}$$

$$\tau_2(x) = \frac{9 + 18x_1x_2 + 9x_1^2x_2^2 + 4x_2^4}{1 + x_1^4 + 2x_1^2x_2^2 + x_2^4}.$$

The neighborhood V of 0, where the number of distinct singular value functions is constant, i.e. $\tau_1(x) > \tau_2(x)$, is

$$V = \{\, x \mid (x_1^2 + x_2^2)(x_1^2 - 4x_2^2) + 27 > 0\}.$$

2.2 The differential eigenstructure of Hankel operators

The observability and controllability operators of Σ in (1) are mappings of $\mathbb{R}^n \to L_2^r[0, \infty)$ and $L_2^m[0, \infty) \to \mathbb{R}^n$, and their state space realizations are given by

$$y = \mathcal{O}_\Sigma(x^0): \begin{cases} \dot{x} = f(x), & x(0) = x^0 \\ y = h(x) \end{cases} \tag{4}$$

$$x^1 = \mathcal{C}_\Sigma(u): \begin{cases} \dot{x} = f(x) + g(x)\mathcal{F}_-(u), & x(-\infty) = 0 \\ x^1 = x(0) \end{cases} \tag{5}$$

where $\mathcal{F}_-$ is the time flipping operator defined by

$$\mathcal{F}_-(u) := \begin{cases} u(-t) & (t < 0) \\ 0 & (t \geq 0) \end{cases}. \tag{6}$$

The observability and controllability functions and operators are related to each other as

$$L_o(x^0) = \frac{1}{2}\|\mathcal{O}_\Sigma(x^0)\|_{L_2^r}^2, \quad L_c(x^1) = \frac{1}{2}\|\mathcal{C}_\Sigma^\dagger(x^0)\|_{L_2^m}^2$$

where $\mathcal{C}_\Sigma^\dagger$ is the pseudo inverse of $\mathcal{C}_\Sigma$ defined by

$$\mathcal{C}_\Sigma^\dagger(x^1) := \arg \min_{\mathcal{C}_\Sigma(u)=x^1} \|u\|_{L_2^m}. \tag{7}$$

Furthermore the Hankel operator $\mathcal{H}_\Sigma : L_2^m[0,\infty) \to L_2^r[0,\infty)$ of Σ is given by

$$\mathcal{H}_\Sigma = \mathcal{O}_\Sigma \circ \mathcal{C}_\Sigma. \tag{8}$$

Recall that if the system Σ is linear, then the square of the singular value σ_i is the eigenvalue of the self-adjoint of the Hankel operator, i.e.

$$\mathcal{H}_\Sigma^* \circ \mathcal{H}_\Sigma(v_i) = \sigma_i^2 v_i$$

for each eigenvector v_i. For nonlinear systems so far the extension of this linear result was not obtained. In the sequel we study another type of extension with the help of the differential of the Hankel operator: we consider an eigenstructure of the operator $u \mapsto (d\mathcal{H}_\Sigma(u))^* \circ \mathcal{H}_\Sigma(u)$ characterized by

$$(d\mathcal{H}_\Sigma(v))^* \circ \mathcal{H}_\Sigma(v) = \lambda\, v \tag{9}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue and $v \in L_2^m[0,\infty)$ the corresponding eigenvector. See [2] for the motivation of investigating this eigenstructure. Here the operator $d(\cdot)$ denotes the Gâteaux differential, which corresponds to the Fréchet derivative in our setting. The operator $(\cdot)^*$ is defined by

Definition 2 [15] Consider an operator $\Sigma : U \to Y$ with Hilbert spaces U and Y. An operator $\Sigma^* : Y \times U \to U$ satisfying

$$\langle \Sigma(u), y \rangle_Y = \langle u, \Sigma^*(y,u) \rangle_U, \ \forall u \in U, \ \forall y \in Y \tag{10}$$

which is linear in y is said to be a *nonlinear Hilbert adjoint* of Σ.

The state-space realizations of those operators are given in [4] and can be readily computed. The eigenstructure (9) has a close relationship with the *Hankel norm* of Σ defined by

$$\|\Sigma\|_H := \sup_{\substack{u \in L_2^m[0,\infty) \\ u \neq 0}} \frac{\|\mathcal{H}_\Sigma(u)\|_2}{\|u\|_2}. \tag{11}$$

We have the following result which characterizes the solution of (9) and yields an expression for the Hankel norm.

Theorem 3 *[2] Consider an operator Σ with the state-space realization (1). Suppose that the Hankel operator $\mathcal{H}_\Sigma$ is Gâteaux differentiable, that there exist sufficiently smooth energy functions $L_o(x)$ and $L_c(x)$ and that the Jacobian linearization of the system Σ has nonzero distinct Hankel singular values. Then there exist a neighborhood $U \subset \mathbb{R}$ of 0, n smooth functions $\rho_i : U \to [0,\infty) \subset \mathbb{R}$'s, $i \in \{1,2,\ldots,n\}$ such that*

$$\min\{\rho_i(s), \rho_i(-s)\} \geq \max\{\rho_{i+1}(s), \rho_{i+1}(-s)\}, \ \forall s \in U \tag{12}$$

holds for $\forall i \in \{1,2,\ldots,n-1\}$ and there exist n distinct smooth curves $\xi_i : U \to \mathbb{R}^n$ satisfying $\xi_i(0) = 0$ and

$$L_c(\xi_i(s)) = \frac{s^2}{2}, \quad L_o(\xi_i(s)) = \frac{\rho_i{}^2(s)\, s^2}{2} \tag{13}$$

$$\frac{\partial L_o}{\partial x}(\xi_i(s)) = \lambda_i(s) \frac{\partial L_c}{\partial x}(\xi_i(s)) \tag{14}$$

with

$$\lambda_i(s) := \rho_i{}^2(s) + \frac{s}{2}\frac{d\rho_i{}^2(s)}{ds}. \tag{15}$$

Here every pair

$$\lambda = \lambda_i(s), \quad v = \mathcal{C}_\Sigma^\dagger(\xi_i(s))$$

satisfies the differential eigenstructure (9) where $\mathcal{C}_\Sigma^\dagger$ is the pseudo inverse of $\mathcal{C}_\Sigma$ defined in (7). Furthermore, if $U = \mathbb{R}$, then

$$\|\Sigma\|_H = \sup_{s \in \mathbb{R}} \rho_1(s).$$

Here we call the functions ρ_i's *axis singular value functions*. The axis singular value functions are uniquely determined whereas the singular value functions τ_i's in Theorem 2 are not unique [5]. The relationship between those two functions will be clarified in the following section. Furthermore, it is noted that Theorem 3 gives an input-output characterization of the Hankel operator without using the state variable.

Example 2 (continued) Consider the state space system in the form of (1) which is in input-normal form, as given in Example 1. In order to obtain the curves $\xi_i(s)$'s, we have to compute the solution of

$$\begin{aligned} 0 &= \det \begin{pmatrix} \frac{\partial L_c}{\partial x} \\ \frac{\partial L_o}{\partial x} \end{pmatrix} \\ s^2 &= 2L_c(x) = x_1^2 + x_2^2. \end{aligned}$$

Here the first equation follows from the fact that $\partial L_c/\partial x$ is parallel to $\partial L_o/\partial x$, i.e. (14). These equations have two solutions parameterized by s:

$$\xi_1(s) = \begin{pmatrix} \frac{3s}{\sqrt{9+s^4}} \\ \frac{s^3}{\sqrt{9+s^4}} \end{pmatrix}, \quad \xi_2(s) = \begin{pmatrix} \frac{s^3}{\sqrt{9+s^4}} \\ -\frac{3s}{\sqrt{9+s^4}} \end{pmatrix}. \tag{16}$$

The $\rho_i(s)$'s can be obtained by a direct computation:

$$\begin{aligned}\rho_1(s) &= \sqrt{\tfrac{L_o(\xi_1(s))}{L_c(\xi_1(s))}} = 2\sqrt{\tfrac{9+s^4}{1+s^4}} \\ \rho_2(s) &= \sqrt{\tfrac{L_o(\xi_2(s))}{L_c(\xi_2(s))}} = \sqrt{\tfrac{9+s^4}{1+s^4}}.\end{aligned}$$

Note that both functions $\xi_i(s)$'s and $\rho_i(s)$'s are defined for all $s \in \mathbb{R}$. We can easily check that the $\lambda_i(s)$'s given by (15) satisfy the condition (14). Furthermore it can be observed that

$$\begin{aligned}\min\{\rho_1(s), \rho_1(-s)\} &= 2\sqrt{\tfrac{9+s^4}{1+s^4}} > \sqrt{\tfrac{9+s^4}{1+s^4}} \\ &= \max\{\rho_2(s), \rho_2(-s)\}\end{aligned}$$

holds for all $s \in \mathbb{R}$. This implies the equation (13) holds on $U = \mathbb{R}$. Therefore it follows that

$$\|\Sigma\|_H = \sup_{s\in\mathbb{R}} \rho_1(s) = 6.$$

3 Input-normal/output-diagonal realizations

The previous section gives a new characterization of the nonlinear extensions of the Hankel singular values, namely the *axis singular value functions* $\rho_i(s)$'s in Theorem 3. The corresponding coordinates ξ_i's are expected to play the role of the input-normal coordinates. The next theorem derives a procedure to obtain the input-normal realization whose (original) singular value functions coincide with the axis ones on certain subspaces of the state-space, by applying Theorem 3 repetitively. This result relates the axis singular value functions ρ_i's (given in Section 2.2) with the conventional singular value functions τ_i's (given in Section 2.1).

Theorem 4 *Consider an operator Σ with the state-space realization (1). Suppose that the Hankel operator $\mathcal{H}_\Sigma$ is Gâteaux differentiable, that there exist sufficiently smooth energy functions $L_o(x)$ and $L_c(x)$, and that the Jacobian linearization of the system Σ has nonzero distinct Hankel singular values. Then there exists a neighborhood $V \subset \mathbb{R}^n$ of 0 and a coordinate transformation $x = \Phi(z)$, $\Phi(0) = 0$, converting the system into an input-normal form, i.e. there exist n smooth functions $\tau_i : V \to \mathbb{R}$ satisfying (2) and (3), such that*

$$z_i = 0 \;\Leftrightarrow\; \frac{\partial L_c(\Phi(z))}{\partial z_i} = 0 \;\Leftrightarrow\; \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0 \quad (17)$$

holds for all $i \in \{1, 2, \ldots, n\}$ on V. Furthermore

$$\tau_i(0,\ldots,0,\overset{i}{\check{z}_i},0,\ldots,0) = \rho_i^2(z_i) \quad (18)$$

$$\frac{\partial \tau_i}{\partial z}(0,\ldots,0,\overset{i}{\check{z}_i},0,\ldots,0) = (0,\ldots,0,\overset{i}{\check{\frac{d\rho_i^2(z_i)}{dz_i}}},0,\ldots,0) \quad (19)$$

holds for all $i \in \{1, 2, \ldots, n\}$. Here the notation $\overset{i}{\check{(\cdot)}}$ denotes the i-th element of a given vector. In particular, if $V = \mathbb{R}^n$, then

$$\|\Sigma\|_H^2 = \sup_{z_1\in\mathbb{R}} \tau_1(z_1, 0, \ldots, 0).$$

The proof of this theorem is very long and tedious, and breaks down into several steps. Equation (17) is proved by induction, where the cases $n = 1, n = 2$ are studied. Then the case $n = k$ is proved using Theorem 3, which in itself breaks down into three steps. Finally the output-diagonal form in the z coordinates is proved. The complete proof is omitted for the sake of space, and it is available at [3, 1].

In Theorem 4 the existence of an input-normal form is proved so that the property in (17) and those in Theorem 3 along each axis (subspace) are achieved simultaneously. That is why the functions ρ_i's are called axis singular value functions. This result is a coordinate free characterization of the input-normal realization. We illustrate this theorem in the following example.

Example 3 (continued from Example 1) Consider the state space system of the form (1) as given in Examples 1 and 2 again. Equation (16) implies that the coordinate transformation $x = \Phi(z)$ is given by

$$x = \Phi(z) = \begin{pmatrix} \frac{3}{\sqrt{9+(z_1^2+z_2^2)^2}} & \frac{(z_1^2+z_2^2)}{\sqrt{9+(z_1^2+z_2^2)^2}} \\ \frac{-(z_1^2+z_2^2)}{\sqrt{9+(z_1^2+z_2^2)^2}} & \frac{3}{\sqrt{9+(z_1^2+z_2^2)^2}} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad (20)$$

by which the z_i-axis is mapped into the curve ξ_i, i.e.,

$$\begin{aligned}\xi_1(s) &= \Phi(s, 0) \\ \xi_2(s) &= \Phi(0, s).\end{aligned}$$

The coordinate transformation (20) converts the system vector fields and the output mapping into

$$\begin{cases} \dot{z} &= \tilde{f}(z) + \tilde{g}(z)u \\ y &= \tilde{h}(z) \end{cases} \quad (21)$$

with

$$\tilde{f} = \begin{pmatrix} -9z_1 - z_1^5 - 2z_1^3z_2^2 - z_1z_2^4 \\ -9z_2 - z_1^4z_2 - 2z_1^2z_2^3 - z_2^5 \end{pmatrix}$$

$$\tilde{g} = \begin{pmatrix} \sqrt{18 + 2z_1^4 + 4z_1^2z_2^2 + 2z_2^4} & 0 \\ 0 & \sqrt{18 + 2z_1^4 + 4z_1^2z_2^2 + 2z_2^4} \end{pmatrix}$$

$$\tilde{h} = \begin{pmatrix} \frac{(6z_1 - 2z_1^5 - 4z_1^3z_2^2 - 2z_1z_2^4)\sqrt{18+2z_1^4+4z_1^2z_2^2+2z_2^4}}{1+z_1^4+2z_1^2z_2^2+z_2^4} \\ \frac{(3z_2 - z_1^4z_2 + 2z_1^2z_2^3 - 2z_2^5)\sqrt{18+2z_1^4+4z_1^2z_2^2+2z_2^4}}{1+z_1^4+2z_1^2z_2^2+z_2^4} \end{pmatrix}.$$

The observability and controllability functions in the new coordinates are given as follows:

$$\begin{aligned}L_c(\Phi(z)) &= \frac{1}{2}z^{\mathrm{T}}z \\ L_o(\Phi(z)) &= \frac{1}{2}z^{\mathrm{T}}\mathrm{diag}(\tilde{\tau}_1(z), \tilde{\tau}_2(z))z \\ &= \frac{1}{2}z^{\mathrm{T}}\mathrm{diag}\left(\frac{4(9+z_1^4+2z_1^2z_2^2+z_2^4)}{1+z_1^4+2z_1^2z_2^2+z_2^4}, \frac{9+z_1^4+2z_1^2z_2^2+z_2^4}{1+z_1^4+2z_1^2z_2^2+z_2^4}\right)z\end{aligned}$$

satisfying (17), (18) and (19) on $V = \mathbb{R}^n$. Furthermore, we have

$$\|\Sigma\|_H^2 = \sup_{z_1 \in \mathbb{R}} \tilde{\tau}_1(z_1, 0) = 36.$$

which indeed equals the outcome of Example 2.

It can be observed from the above example that the singular value functions $\tilde{\tau}_i$'s in the new procedure have a closer relationship to the Hankel norm of the system than the conventional ones τ_i's in Example 1.

4 Model reduction

This subsection develops the procedure of model reduction method based on the balanced truncation [12, 13]. However, since our realization from Section 3 is an input-normal realization it is not in "strict" balanced form yet. The input-normal, output-diagonal realization essentially measures the importance of states such that the control energy is equally important for all states, and the output energy (or in other words, the observability properties of the output) are different for the different state components. Hence reduction of the input-normal form of Section 3, is only based on the output energy of the different state components. This corresponds in a certain sense with the linear cross Gramian thinking (see e.g. [6]), i.e., by noting that the Hankel norm equals

$$\|\Sigma\|_H = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \sqrt{\frac{L_o(x)}{L_c(x)}},$$

it can be seen that for the input-normal case, $L_o(x)$ contains the information that is given by the cross Gramians. The property (17) can be considered of importance for model reduction of nonlinear systems. Suppose the assumptions in Theorem 4 hold and the coordinate transformation $x = \Phi(z)$ gives the input-normal representation in the sense of Theorem 4, i.e. (17) holds. Suppose moreover that

$$\min\{\rho_k(z_k), \rho_k(-z_k)\} > \max\{\rho_{k+1}(z_{k+1}), \rho_{k+1}(-z_{k+1})\} \tag{22}$$

holds for all $z \in V$. Then the state variables $z_1, \ldots, z_k$ are more important in terms of energy than those $z_{k+1}, \ldots, z_n$ due to the ordering of the axis singular value functions ρ_i's.

Divide the coordinates into two parts corresponding to the division (22) as

$$\begin{aligned}
z &= (z^a, z^b) \in \mathbb{R}^n \\
z^a &:= (z_1, \ldots, z_k) \in \mathbb{R}^k \\
z^b &:= (z_{k+1}, \ldots, z_n) \in \mathbb{R}^{n-k} \\
\begin{pmatrix} f^a(z) \\ f^b(z) \end{pmatrix} &:= \tilde{f}(z) = \left.\frac{\partial \Phi^{-1}(x)}{\partial x}\right|_{x=\Phi(z)} f(\Phi(z)) \\
\begin{pmatrix} g^a(z) \\ g^b(z) \end{pmatrix} &:= \tilde{g}(z) = \left.\frac{\partial \Phi^{-1}(x)}{\partial x}\right|_{x=\Phi(z)} g(\Phi(z)).
\end{aligned}$$

Moreover, divide the system Σ into two subsystems accordingly as follows:

$$\Sigma^a \; : \; \left\{ \begin{array}{rcl} \dot{z}^a &=& f^a(z^a, 0) + g^a(z^a, 0) u^a \\ y^a &=& h(\Phi(z^a, 0)) \end{array} \right. \tag{23}$$

$$\Sigma^b \; : \; \left\{ \begin{array}{rcl} \dot{z}^b &=& f^b(0, z^b) + g^b(0, z^b) u^b \\ y^b &=& h(\Phi(0, z^b)) \end{array} \right. . \tag{24}$$

Then we obtain the following properties.

Lemma 1 *Consider the system Σ in (1) and the divided systems (23) and (24). Suppose the assumptions in Theorem 4 hold. Then the controllability and observability functions $L_c^a(z^a)$, $L_o^a(z^a)$, $L_c^b(z^b)$ and $L_o^b(z^b)$ of the systems Σ^a and Σ^b satisfy*

$$\begin{aligned}
L_c^a(z^a) &= L_c(\Phi(z^a, 0)) && (25) \\
L_o^a(z^a) &= L_o(\Phi(z^a, 0)) && (26) \\
L_c^b(z^b) &= L_c(\Phi(0, z^b)) && (27) \\
L_o^b(z^b) &= L_o(\Phi(0, z^b)). && (28)
\end{aligned}$$

Proof: It follows from Theorems 1 and 4 that the controllability and observability functions satisfy

$$\begin{aligned}
0 &= \frac{\partial L_c(\Phi(z))}{\partial z} \tilde{f}(z) + \frac{1}{2} \frac{\partial L_c(\Phi(z))}{\partial z} \tilde{g}(z) \tilde{g}(z)^{\mathrm{T}} \frac{\partial L_c(\Phi(z))}{\partial z}^{\mathrm{T}} \\
0 &= \frac{\partial L_o(\Phi(z))}{\partial z} \tilde{f}(z) + \frac{1}{2} \tilde{h}(z)^{\mathrm{T}} \tilde{h}(z).
\end{aligned} \tag{29}$$

Then, by (17), we obtain

$$\frac{\partial L_c(\Phi(z))}{\partial z^b}(z^a, 0) = \frac{\partial L_o(\Phi(z))}{\partial z^b}(z^a, 0) = 0.$$

Hence, substituting $z = (z^a, 0)$ for (29) yields

$$\begin{aligned}
0 &= \frac{\partial L_c^a(z^a)}{\partial z^a} f^a(z^a, 0) \\
&\quad + \frac{1}{2} \frac{\partial L_c^a(z^a)}{\partial z^a} g^a(z^a, 0)^{\mathrm{T}} g^a(z^a, 0) \frac{\partial L_c^a(z^a)}{\partial z^a}^{\mathrm{T}}, \\
0 &= \frac{\partial L_o^a(z^a)}{\partial z^a} f^a(z^a, 0) + \frac{1}{2} h(\Phi(z^a, 0))^{\mathrm{T}} h(\Phi(z^a, 0)).
\end{aligned}$$

These relations prove (25) and (26) again by Theorem 1. Equations (27) and (28) can be proved in the same way and this completes the proof. ∎

Lemma 1 implies the following preservation property in the model reduction procedure which is a natural generalization of the linear case results in [12].

Theorem 5 *Consider the system Σ in (1) and the divided systems (23) and (24). Suppose the assumptions in Theorem 4 hold. Then the reduced systems Σ^a and Σ^b are in the input-normal form with the properties (17), and*

$$\begin{aligned}
\rho_i^a(z_i^a) &= \rho_i(z_i^a) \quad i \in \{1, 2, \ldots, k\} && (30) \\
\rho_i^b(z_i^b) &= \rho_{i+k}(z_i^b) \quad i \in \{1, 2, \ldots, n-k\} && (31)
\end{aligned}$$

hold with ρ_i^a's and ρ_i^b's the singular value functions of the systems Σ^a and Σ^b, respectively. In particular, if $V = \mathbb{R}^n$, then

$$\|\Sigma^a\|_H = \|\Sigma\|_H. \qquad (32)$$

Proof: The fact that Σ^a and Σ^b are again in the input-normal form is obvious because of

$$\begin{aligned} z_i^a = 0 \quad &\Leftrightarrow \quad \frac{\partial L_c^a(z^a)}{\partial z_i^a} = \frac{\partial L_c(\Phi(z^a,0))}{\partial z_i^a} = 0 \\ &\Leftrightarrow \quad \frac{\partial L_o^a(z^a)}{\partial z_i^a} = \frac{\partial L_o(\Phi(z^a,0))}{\partial z_i^a} = 0 \end{aligned}$$

which is obtained from Lemma 1. Also the equations (30)–(32) follow straightforwardly from Lemma 1. This completes the proof. ∎

The model reduction of Σ into Σ^a (and Σ^b) is uniquely determined (coordinate free), although the input-normal coordinates $z = \Phi^{-1}(x)$ are not unique. On the other hand, the model reduction method based on the conventional model reduction procedure [13] is coordinate dependent.

Example 4 (continued from Example 3) Consider again the state-space system (21), which already has input-normal form, obtained in Example 3. According to the above model reduction procedure, one obtains

$$\Sigma^a : \begin{cases} \dot{z}^a &= f^a(z^a) + g^a(z^a)u \\ y &= h^a(z^a) \end{cases}$$

with

$$\begin{aligned} f^a(z^a) &= -9z^a - (z^a)^5 \\ g^a(z^a) &= \left(\sqrt{18 + 2(z^a)^4}, \ \ 0 \right) \\ h^a(z^a) &= \left(\frac{(6z^a - 2(z^a)^5)\sqrt{18+2(z^a)^4}}{1+(z^a)^4}, \ 0 \right)^{\mathrm{T}}. \end{aligned}$$

The observability and controllability functions of the reduced system Σ^a are given as follows:

$$\begin{aligned} L_c^a(z^a) &= \frac{1}{2}(z^a)^2 \\ L_o^a(z^a) &= \frac{1}{2}\frac{4(z^a)^2(9 + (z^a)^4)}{1 + (z^a)^4} =: \frac{1}{2}(z^a)^2\, \tau^a(z^a). \end{aligned}$$

Furthermore, the square of the Hankel norm of the reduced system Σ^a can be computed as

$$\begin{aligned} \|\Sigma^a\|_H^2 &= \sup_{z^a \in \mathbb{R}} \frac{L_o^a(z^a)}{L_c^a(z^a)} = \sup_{z^a \in \mathbb{R}} \tau^a(z^a) = \sup_{z^a \in \mathbb{R}} \frac{4(9 + (z^a)^4)}{1 + (z^a)^4} \\ &= 36 = \|\Sigma\|_H^2. \end{aligned}$$

which indeed equals that of the original Σ. Thus the Hankel norm is preserved.

This example exhibits the effectiveness of the new characterization of input-normal realization and model reduction method based on it.

5 Conclusion

This paper has provided a new input-normal/output-diagonal realization and model reduction procedure for nonlinear systems based on the differential eigenstructure of Hankel operators. The relationship between the two different characterizations of singular value functions has been clarified and, consequently, the new input-normal realization has been identified. More precisely, the existence of an input-normal realization has been proved whose singular value functions coincide with the axis singular value functions in certain subspaces of the state space. We have also performed the model reduction based on this realization. Furthermore numerical examples have demonstrated the effectiveness of the proposed method.

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