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Published in:
Journal of Difference Equations and Applications

Publication date:
2001

Journal of Difference Equations and Applications,
2001, Vol. 7, pp. 63–104
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Formal Theory for Differential–Difference Operators

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(Received 28 January 2000; In final form 4 February 2000)

Differential–difference operators are linear operators involving both d/dz and the shift $z \mapsto z+1$ (or $z(d/dz)$ and $z \mapsto qz$). The aim is to give a formal classification and to provide solutions for these equations. Differential–difference operators can be considered as formal differential operators of infinite order. For the latter one studies Newton polygons, factorizations, solutions and developes a theory of symbolic solutions. This theory applied to differential–difference operators seems in many case adequate. In other cases, one cannot produce enough symbolic solutions. Independent from differential operators of infinite order, certain systems of differential–difference are treated. Here the theory seems complete. Many examples illustrate the theory.

Keywords: Differential operator; Difference operator; Formal solutions

1 INTRODUCTION

We first introduce some notations. The field of the convergent Laurent series over $\mathbf{C}$ in the variable z^{-1} is denoted by $K = \mathbf{C}(\{z^{-1}\})$. The field of the formal Laurent series is written as $\hat{K} = \mathbf{C}((z^{-1}))$. The fields K and $\hat{K}$ are equipped with the usual derivation d/dz and with the automorphism ϕ, given by $\phi(z) = z+1$ (or, equivalently, $\phi(z^{-1}) = z^{-1}/(1+z^{-1})$). The fields $\mathcal{P}$ and $\hat{\mathcal{P}}$ of the convergent and the formal Puiseux series, are the algebraic closures of K and $\hat{K}$, respectively. More precisely, $\mathcal{P} := \bigcup_{p \in \mathbf{N}} \mathcal{P}_p$ with $\mathcal{P}_p = \mathbf{C}(\{z^{-1/p}\})$, and $\hat{\mathcal{P}} := \bigcup_{p \in \mathbf{N}} \hat{\mathcal{P}}_p$, with $\hat{\mathcal{P}}_p = \mathbf{C}((z^{-1/p}))$. (Here, and in what follows,

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$\mathbf{N}=\{1,2,\dots\}$, and $\mathbf{N}_0=\mathbf{N}\cup\{0\}$.) On the field $\hat{\mathcal{P}}$ (and its subfields) one defines the usual additive valuation $v:\hat{\mathcal{P}}\to\mathbf{Q}\cup\{\infty\}$ by

$$v(0)=\infty \text{ and } v\left(\sum_{\lambda} a_\lambda z^{-\lambda}\right)=\mu, \quad \text{if } a_\mu\neq 0,\ a_\lambda=0,\ \forall\lambda<\mu\,.$$

The fields $\hat{\mathcal{P}}_p$ are complete with respect to the topology derived from v. On the field $\hat{\mathcal{P}}$ and its subfields we will also use the finer topology for which a sequence $f_n=\sum f_{n,\lambda}z^{-\lambda}$ has limit $f=\sum f_\lambda z^{-\lambda}$ if $\lim f_{n,\lambda}=f_\lambda$ holds in $\mathbf{C}$ for every λ.

The derivation $\mathrm{d}/\mathrm{d}z$ of K and $\hat{K}$ extends in a unique way to $\mathcal{P}$ and $\hat{\mathcal{P}}$. The automorphism ϕ extends in many ways to these fields. We will choose the extension of ϕ given by $\phi(z^{-\lambda})=z^{-\lambda}(1+z^{-1})^{-\lambda}$ for all $\lambda\in\mathbf{Q}$.

We consider skew rings of operators $A=\mathcal{P}[\tau,\tau^{-1},\partial]$ and $\hat{A}=\hat{\mathcal{P}}[\tau,\tau^{-1},\partial]$ given by the rules for multiplication $\tau\partial=\partial\tau$, $\partial f=f\partial+\mathrm{d}f/\mathrm{d}z$ and $\tau f=\phi(f)\tau$. Further δ will denote the operator $z\partial$. The elements of $\hat{A}$ are called *differential–difference operators*. The main theme of this paper is to produce formal or symbolic solutions for a differential–difference operator L or for a left ideal in A or $\hat{A}$. We note that $\hat{A}$ acts on $\hat{P}$ by the formulas $\tau(f)=\phi(f)$ and $\partial(f)=\mathrm{d}f/\mathrm{d}z$.

Certain differential operators of infinite order operate on $\hat{\mathcal{P}}$. In Section 2 the "largest" class $\hat{\mathcal{E}}$ of such operators is studied. The algebra $\hat{\mathcal{E}}$ has a sequence of subalgebras. For those subalgebras "division with remainder" and "factorization" are proved. This and the Newton polygons are the tools for finding formal or symbolic solutions of an operator in $\hat{\mathcal{E}}$. It is shown that $\hat{A}$ embeds into $\hat{\mathcal{E}}$ by sending ∂,τ,τ^{-1} to the operators ∂, $e^{\partial}=\sum_{n\geq 0}(\partial^n/n!)$, $e^{-\partial}=\sum_{n\geq 0}((-\partial)^n/n!)$. In this way, one can consider a differential–difference operator L as a differential operator of infinite order. In some cases this produces symbolic solutions for L. It is shown that the space $\hat{A}/\hat{A}L$ has (if L is not invertible) a countable dimension over $\hat{P}$. This leads to the idea that the vector space of symbolic solutions of L should also have a countable dimension over $\mathbf{C}$. Some examples confirm this idea. In other examples one finds few (or no) symbolic solutions. A study of rings of symbols tries to remedy this situation.

For left ideals $I\subset\hat{A}$ of finite codimension over $\hat{\mathcal{P}}$ the situation is quite different. In this case one finds (under an additional hypothesis) a space of symbolic solutions over **C** which has the same dimension as $\hat{A}/I$ over $\hat{\mathcal{P}}$. However, it turns out that the left ideal I is essentially equivalent to a single differential equation over $\hat{\mathcal{P}}$.

Finally we also study the parallel case of equations involving differentiation and the q-difference operator. Similar results are obtained.

2 DIFFERENTIAL OPERATORS OF INFINITE ORDER

By Ent we denote the ring of entire functions on **C** (i.e., the power series with infinite radius of convergence). We recall that δ stands for the operator $z\partial$. For any $A \in \mathrm{Ent}$ we consider $A(\delta)$ as operator on $\hat{K}$. Explicitly this reads for $A = \sum a_n t^n$ and $f = \sum_{m \gg -\infty} f_m z^{-m} \in \hat{K}$ as $A(\delta)(f) = \sum_{m \gg -\infty} A(-m) f_m z^{-m}$. We note that here the topology of termwise convergence on $\hat{K}$ is used. One defines $\hat{\mathcal{E}}_1$ as the set of expressions $\sum_{n \gg -\infty} z^{-n} A_n(\delta)$ with all $A_n \in \mathrm{Ent}$. The addition on $\hat{\mathcal{E}}_1$ is the obvious one. The multiplication is defined by

$$\sum_{n \gg -\infty} z^{-n} A_n(\delta) \cdot \sum_{m \gg -\infty} z^{-m} B_m(\delta) = \sum_{k \gg -\infty} z^{-k} \sum_{n+m=k} A_n(\delta - m) B_m(\delta).$$

The elements of $\hat{\mathcal{E}}_1$ act upon $\hat{K}$ by the formula

$$\sum_{n \gg -\infty} z^{-n} A_n(\delta) \left(\sum_{m \gg -\infty} b_m z^{-m} \right) = \sum_k \left(\sum_{n+m=k} A_n(-m) b_m \right) z^{-k}.$$

For any integer $p > 1$ one defines $\hat{\mathcal{E}}_p$ to be the set of expressions $\sum_{n \gg -\infty} z^{-n/p} A_n(\delta)$ with all $A_n \in \mathrm{Ent}$ and further $\hat{\mathcal{E}} := \cup_{p \geq 1} \hat{\mathcal{E}}_p$.

LEMMA 1 *$\hat{\mathcal{E}}_p$ and $\hat{\mathcal{E}}$ are algebras and operate on the fields $\hat{\mathcal{P}}_p$ and $\hat{\mathcal{P}}$.*

Proof The algebra structure of $\hat{\mathcal{E}}_1$ and the action of this algebra on $\hat{\mathcal{P}}_1 = \hat{K}$ has already been made explicit. Similar formulas hold for $\hat{\mathcal{E}}_p$ and $\hat{\mathcal{E}}$. □

2.1 Algebras of Differential Operators

We will introduce some families of subalgebras of $\hat{\mathcal{E}}$. Define $\hat{\mathcal{D}}_1 \subset \hat{\mathcal{E}}_1$ as the set of elements $\sum_{n \gg -\infty} z^{-n} A_n(\delta)$ such that each A_n is a polynomial. This is clearly a subalgebra. Similarly, one defines the subalgebra $\hat{\mathcal{D}}_p \subset \hat{\mathcal{E}}_p$ as the set of elements $\sum_{n\gg-\infty} z^{-n/p}A_n(\delta)$ where all A_n are polynomials. Thus $\hat{\mathcal{D}} := \cup_{p\geq 1}\hat{\mathcal{D}}_p$ is a subalgebra of $\hat{\mathcal{E}}$.

For the next subalgebras that we define, the role of δ is taken over by the operator $\Delta := z^{-k}\delta$, which is also a derivation of the fields $\hat{\mathcal{P}}_p$. To be more precise, for any positive integer p and any $k \in (1/p)\mathbf{N}$ one defines $\hat{\mathcal{D}}_{p,k}$ as the set of expressions $\sum_{n\gg-\infty} z^{-n/p}A_n(z^{-k}\delta)$ where the A_n are polynomials. It is clear that $\hat{\mathcal{D}}_{p,k}$ is a subset of $\hat{\mathcal{D}}_p$. One can rewrite an element of $\hat{\mathcal{D}}_{p,k}$ as a power series in $\Delta = (z^{-k}\delta)$ and with coefficients in $\hat{\mathcal{P}}_p$. In doing so one finds that $\hat{\mathcal{D}}_{p,k}$ consists of expressions of the form $\sum_{m\geq 0} a_m\Delta^m$ with all $a_m \in \hat{\mathcal{P}}_p$ and $\lim v(a_m) = +\infty$. The formula

$$\Delta^n f = f\Delta^n + \binom{n}{1}\Delta(f)\Delta^{n-1} + \cdots + \binom{n}{n-1}\Delta^{n-1}(f)\Delta + \binom{n}{n}\Delta^n(f)$$

easily implies that $\hat{\mathcal{D}}_{p,k}$ is a subalgebra of $\hat{\mathcal{D}}_p$. For notational convenience we will write $\hat{\mathcal{D}}_{p,0} = \hat{\mathcal{D}}_p$.

For any positive integer p and any $k \in (1/p)\mathbf{N}$ one defines $\hat{\mathcal{E}}_{p,k}$ as the set of expressions $\sum_{n\geq 0} a_n\Delta^n$, where $\Delta = z^{-k}\delta$ and where the $a_n \in \hat{\mathcal{P}}_p$ satisfy $\inf v(a_n) > -\infty$. One easily sees that $\hat{\mathcal{E}}_{p,k} \subset \hat{\mathcal{E}}_p$. An element of $\hat{\mathcal{E}}_{p,k}$ can be rewritten as an expression $\sum_{n\gg-\infty} z^{-n/p}A_n(\Delta)$, where the A_n are *formal power series* with coefficients in $\mathbf{C}$. For notational convenience we write $\hat{\mathcal{E}}_{p,0} := \hat{\mathcal{E}}_p$.

We have the following inclusions:

$$\hat{\mathcal{D}}_{p,k} \subset \hat{\mathcal{D}}_{mp,k}, \quad \hat{\mathcal{E}}_{p,k} \subset \hat{\mathcal{E}}_{mp,k}, \qquad \text{for } m \in \mathbf{N}, \text{ and } k \in \frac{1}{p}\mathbf{N},$$

$$\hat{\mathcal{D}}_{p,k} \subset \hat{\mathcal{E}}_{p,k} \subset \hat{\mathcal{D}}_{p,k-1/p} \subset \hat{\mathcal{D}}_p \subset \hat{\mathcal{E}}_p \quad \text{for } k \in \frac{1}{p}\mathbf{N}.$$

The second inclusion easily follows from the equality

$$(z^{-k}\delta)^n = \sum_{m=1}^{n} c_n z^{-(n-m)k-m/p}(z^{-k-1/p}\delta)^m,$$

for some real constants c_n.

PROPOSITION 2 *The $\hat{\mathcal{D}}_{p,k}$ and $\hat{\mathcal{E}}_{p,k}$ are subalgebras of $\hat{\mathcal{E}}_p$.*

Proof For $\hat{\mathcal{D}}_{p,k}$ we have already indicated the proof. Consider now two elements $A := \sum_{n\geq 0} a_n\Delta^n$ and $B := \sum_{m\geq 0} b_m\Delta^m$ in $\hat{\mathcal{E}}_{p,k}$ (with $k>0$ and with the notation $\Delta = z^{-k}\delta$). The product $a_n\Delta^n b_m\Delta^m$ can be written as $a_n \sum_{i=0}^{n}\binom{n}{i}\Delta^i(b_m)\Delta^{n+m-i}$. Thus, formally speaking, AB should be

$$\sum_{s\geq 0} c_s\Delta^s \quad \text{with } c_s := \sum_{i\leq n, n+m-i=s} a_n\binom{n}{i}\Delta^i(b_m).$$

The sums c_s converge in $\hat{\mathcal{P}}_p$ for the topology induced by the additive valuation v, since they are finite. We note that for any element $b \in \hat{\mathcal{P}}_p$ one has $v(\Delta(b)) \geq 1/p + v(b)$. This implies that the $v(c_s)$ are bounded from below. □

By $\hat{\mathcal{D}}^+_{p,k}$ and $\hat{\mathcal{E}}^+_{p,k}$ we denote the subalgebras of $\hat{\mathcal{D}}_{p,k}$ and $\hat{\mathcal{E}}_{p,k}$, respectively, of the operators $\sum_{n\geq 0} a_n(z^{-k}\delta)^n$ with all the coefficients a_n in $\hat{\mathcal{P}}^+_p := \mathbf{C}[[z^{-1/p}]]$. We have $\hat{\mathcal{D}}_{p,k} = \hat{\mathcal{P}}_p \otimes_{\hat{\mathcal{P}}^+_p} \hat{\mathcal{D}}^+_{p,k}$ and $\hat{\mathcal{E}}_{p,k} = \hat{\mathcal{P}}_p \otimes_{\hat{\mathcal{P}}^+_p} \hat{\mathcal{E}}^+_{p,k}$. We claim that the left ideal of $\hat{\mathcal{D}}_{p,k}$ generated by $z^{-1/p}$ is in fact a two-sided ideal. This follows easily from the formula $z^{-1/p}\Delta^n = (\Delta + z^{-k}/p)^n z^{-1/p}$ with $\Delta = z^{-k}\delta$. In the same way one sees that the left ideal in $\hat{\mathcal{E}}^+_{p,k}$ generated by $z^{-1/p}$ is a two-sided ideal. Moreover one has

$$\hat{\mathcal{D}}^+_{p,k}/z^{-1/p}\hat{\mathcal{D}}^+_{p,k} \cong \mathbf{C}[z^{-k}\delta] \quad \text{for } k=0 \text{ or } k \in \frac{1}{p}\mathbf{N},$$

$$\hat{\mathcal{E}}^+_{p,k}/z^{-1/p}\hat{\mathcal{E}}^+_{p,k} \cong \mathbf{C}[[z^{-k}\delta]] \quad \text{for } k \in \frac{1}{p}\mathbf{N},$$

$$\hat{\mathcal{E}}^+_{p,0}/z^{-1/p}\hat{\mathcal{E}}^+_{p,0} \cong \text{Ent}, \quad \text{the ring of entire functions.}$$

An element M of $\hat{\mathcal{D}}_{p,k}$ is called *k-normalized* if $M \in \hat{\mathcal{D}}^+_{p,k}$ and its image $\bar{M}$ in $\mathbf{C}[z^{-k}\delta]$ is not zero. The *degree of M*, $\deg_k(M)$, is defined as the degree of $\bar{M}$ as polynomial with respect to the variable $z^{-k}\delta$. Any non-zero element L of $\hat{\mathcal{D}}_{p,k}$ can be k-normalized by left multiplication with a suitable power of z, i.e., $M = z^{\lambda}L$ is the k-normalization of L. The degree of L, denoted as $\deg_k(L)$, is defined to be the degree of M.

THEOREM 3 *Let* $p \in \mathbf{N}$ *and* $k \in (1/p)\mathbf{N}$ *or* $k=0$. *Suppose that* $L \in \hat{\mathcal{D}}_{p,k}$ *is non-zero. Then* $\hat{\mathcal{D}}_{p,k}/\hat{\mathcal{D}}_{p,k}L$ *is a vector space over* $\hat{\mathcal{P}}_p$ *of dimension equal to* $\deg_k(L)$. *In particular*, $\deg_k(L) = 0$ *if and only if* L *is invertible in* $\hat{\mathcal{D}}_{p,k}$.

Proof We may and will suppose that L is k-normalized. For elements R of $\hat{\mathcal{P}}_p[z^{-k}\delta]$, we define $\deg(R)$ as the degree with respect to the variable $z^{-k}\delta$. Put $d := \deg_k(L)$. The theorem will follow from the following statement on division with remainder:

For any $F \in \hat{\mathcal{D}}_{p,k}$ there exist unique $Q \in \hat{\mathcal{D}}_{p,k}$ and $R \in \hat{\mathcal{P}}_p[z^{-k}\delta]$, of degree $< d$, such that $F = QL + R$.

Let us first prove the *uniqueness* of Q and R. Suppose that $Q_1L + R_1 = Q_2L + R_2$ with $R_1 \neq R_2$ polynomials in the variable $z^{-k}\delta$ of degrees $< d$. Then $(Q_1 - Q_2)L = R_2 - R_1$. After k-normalization we may suppose that $Q_1 - Q_2$ lies in $\hat{\mathcal{D}}^+_{p,k}$ and has a non-zero image in $\mathbf{C}[z^{-k}\delta]$. Then $R_2 - R_1$ is also in $\hat{\mathcal{D}}^+_{p,k}$ and one has $0 \neq \overline{Q_1 - Q_2}\bar{L} = \overline{R_2 - R_1}$. This is a contradiction since the degree of $\overline{R_2 - R_1}$ is $< d$.

For the *existence* of a division with remainder we may restrict ourselves to elements $F_0 := F \in \hat{\mathcal{D}}^+_{p,k}$. Division with remainder by $\bar{L}$ in the polynomial ring $\mathbf{C}[z^{-k}\delta]$ implies that there exist Q_0, $R_0 \in \mathbf{C}[z^{-k}\delta]$, R_0 of degree $< d$, such that

$$F_0 = Q_0L + R_0 + z^{-1/p}F_1 \quad \text{with } F_1 \in \hat{\mathcal{D}}^+_{p,k}.$$

Similarly, there exist Q_1, $R_1 \in \mathbf{C}[z^{-k}\delta]$, R_1 of degree $< d$, such that

$$F_1 = Q_1L + R_1 + z^{-1/p}F_2,$$

for some $F_2 \in \hat{\mathcal{D}}^+_{p,k}$. By induction one finds polynomials Q_n, R_n in $\mathbf{C}[z^{-k}\delta]$ and elements $F_n \in \hat{\mathcal{D}}^+_{p,k}$ with relations $F_n = Q_nL + R_n + z^{-1/p}F_{n+1}$ and all R_n of degree $< d$. Taking infinite sums one finds

$$F_0 + \sum_{n\geq 1} z^{-n/p}F_n = \left(\sum_{n\geq 0} z^{-n/p}Q_n\right)L + \left(\sum_{n\geq 0} z^{-n/p}R_n\right) + \sum_{n\geq 1} z^{-n/p}F_n\,.$$

It is easily seen that the sums converge in v-adic topology of $\hat{\mathcal{D}}_{p,k}$ and that the term $\sum_{n\geq 0} z^{-n/p}R_n$ is a polynomial w.r.t. $z^{-k}\delta$ of degree $< d$. Then

$$F_0 = \left(\sum_{n\geq 0} z^{-n/p}Q_n\right)L + \left(\sum_{n\geq 0} z^{-n/p}R_n\right)$$

is the required division with remainder.

If $\deg_k(L) = 0$ then $\hat{\mathcal{D}}_{p,k} = \hat{\mathcal{D}}_{p,k}L$ and L obviously is invertible. ◻

Remark 4 Suppose that F belongs to $\hat{\mathcal{D}}^+_{p,k}$ then $F=QL+R$ holds with R a polynomial in $z^{-k}\delta$ and coefficients in $\hat{\mathcal{P}}^+_p$.

COROLLARY 5 *Let* $L\in\hat{\mathcal{D}}_{p,k}$, $L\neq 0$ *with* $p\in\mathbf{N}$, $k\in(1/p)\mathbf{N}$ *or* $k=0$. *Then* L *has a unique factorization* $L=E\cdot D$ *with* $D\in\hat{\mathcal{P}}^+_p[z^{-k}\delta]$ *a monic polynomial and* E *an invertible element of* $\hat{\mathcal{D}}_{p,k}$. *The degree of* D *is equal to* $\deg_k(L)$.

Proof Put $d=\deg_k(L)$. For $d=0$ the corollary is trivially true. Suppose that $d>0$. According to the previous theorem and the Remark 4, we have that $(z^{-k}\delta)^d=QL+R$, for some unique $Q\in\hat{\mathcal{D}}_{p,k}$ and $R\in\hat{\mathcal{P}}^+_p[z^{-k}\delta]$ of degree $<d$. Thus, $QL=(z^{-k}\delta)^d-R$. After k-normalizing L one finds that $\deg_k(Q)=0$. Thus, Q is invertible and we find the required expression $L=Q^{-1}((z^{-k}\delta)^d-R)$. In proving the uniqueness we have to consider the situation $F_1=EF_2$ with $F_1,F_2\in\mathcal{P}^+_p[z^{-k}\delta]$ monic and $E\in\hat{\mathcal{D}}_{p,k}$ invertible. We have to prove that $F_1=F_2$. In the ring $\mathcal{P}^+_p[z^{-k}\delta]$ one can also divide F_1 by F_2 with remainder since F_2 is monic. Let this division with remainder be $F_1=QF_2+R$ with $Q,R\in\mathcal{P}^+_p[z^{-k}\delta]$ and the degree of R strictly less than the degree of F_2. This division with remainder can also be seen as one in the ring $\hat{\mathcal{D}}_{p,k}$ and by uniqueness one has $R=0$ and thus, $E=Q$ lies in $\mathcal{P}^+_p[z^{-k}\delta]$. By interchanging the roles of F_1 and F_2 one finds that E is actually an invertible element of $\mathcal{P}^+_p[z^{-k}\delta]$ and thus, belongs to $\mathcal{P}^+_p$. Since F_1 and F_2 are monic one finds $E=1$ and $F_1=F_2$. □

An element $M\in\hat{\mathcal{E}}_{p,k}$ is called *k-normalized* if $M\in\hat{\mathcal{E}}^+_{p,k}$ and its image $\bar{M}$ in $\mathbf{C}[[z^{-k}\delta]]$, for $k>0$, or in Ent for $k=0$, is not zero. In the first case $\bar{M}$ is a formal power series in the variable $z^{-k}\delta$ over $\mathbf{C}$. One defines the order of M, denoted by $\mathrm{ord}_k(M)$, as the order of $\bar{M}$, i.e., the lowest power occurring in the power series for $\bar{M}$. We note that the order of $\bar{M}$ is also equal to the dimension of the complex vector space $\mathbf{C}[[z^{-k}\delta]]/\bar{M}\mathbf{C}[[z^{-k}\delta]]$.

In the second case, i.e., $k=0$, we define the order of M, denoted by $\mathrm{ord}_0(M)$ as the dimension of the complex vector space $\mathrm{Ent}/\bar{M}\,\mathrm{Ent}$. If $\bar{M}$ happens to be an entire function with only finitely many zeros, then one can write $\bar{M}=pq$, where p is a polynomial and q is an invertible entire function. Thus, $\mathrm{ord}_0(M)$ is the number of zeros of $\bar{M}$ counted with multiplicity. If $\bar{M}$ has infinitely many zeros then $\mathrm{ord}_0(M)$ is infinite. Any non-zero $L\in\hat{\mathcal{E}}_{p,k}$ can be k-normalized to M by multiplying L

with a suitable power of z. Then $\operatorname{ord}_k(L)$ is defined as $\operatorname{ord}_k(M)$. For $\hat{\mathcal{E}}_{p,k}$ there are similar results on division with remainder and factorization of operators. We have to treat the cases $k > 0$ and $k = 0$ separately.

THEOREM 6 *For $L \in \hat{\mathcal{E}}_{p,k}$, $L \neq 0$ with $k > 0$ the vector space $\hat{\mathcal{E}}_{p,k}/\hat{\mathcal{E}}_{p,k}L$ has dimension* $\operatorname{ord}_k(L)$ *over $\hat{\mathcal{P}}_p$. In particular*, $\operatorname{ord}_k(L) = 0$ *if and only if L is invertible in $\hat{\mathcal{E}}_{p,k}$.*

COROLLARY 7 *For any $L \in \hat{\mathcal{E}}_{p,k}$, $L \neq 0$ with $k > 0$ there is a unique factorization $L = E \cdot D$ with D a polynomial in $z^{-k}\delta$ of the form $D = (z^{-k}\delta)^d + a_{d-1}(z^{-k}\delta)^{d-1} + \cdots + a_0$; $a_0, \ldots, a_{d-1} \in z^{-1/p}\hat{\mathcal{P}}_p^+$, and with E an invertible element of $\hat{\mathcal{E}}_{p,k}$. Moreover $d = \operatorname{ord}_k(L)$.*

The proofs of Theorem 6 and Corollary 7 are similar to that for Theorem 3 and Corollary 5, respectively. For $\hat{\mathcal{E}}_{p,0}$ the situation is different.

THEOREM 8 *Let $L \in \hat{\mathcal{E}}_{p,0}$, $L \neq 0$. If* $\operatorname{ord}_0(L) < \infty$, *then $\hat{\mathcal{E}}_{p,0}/\hat{\mathcal{E}}_{p,0}L$ has dimension* $\operatorname{ord}_0(L)$ *over $\hat{\mathcal{P}}_p$.*

If $\operatorname{ord}_0(L) = \infty$, *then $\hat{\mathcal{E}}_{p,0}/\hat{\mathcal{E}}_{p,0}L$ has uncountable dimension over $\hat{\mathcal{P}}_p$.*
In particular L is invertible in $\hat{\mathcal{E}}_{p,0}$ if and only if $\operatorname{ord}_0(L) = 0$.

Proof We suppose that L is already 0-normalized and that the image $\bar{L}$ is a monic polynomial of degree $d = \operatorname{ord}_0(L) < \infty$. For a polynomial $R \in \hat{\mathcal{P}}_p[\delta]$ we will write $\deg(R)$ for its degree w.r.t. δ. Division with remainder by L can be stated as

For every $F \in \hat{\mathcal{E}}_{p,0}$ there is a unique expression $F = QL + R$ with $Q \in \hat{\mathcal{E}}_{p,0}$ and $R \in \hat{\mathcal{P}}_p[\delta]$ of degree strictly less than d.

We first consider the *uniqueness*. Suppose $Q_1L + R_1 = Q_2L + R_2$ with $Q_1 \neq Q_2$. After normalizing $Q_1 - Q_2$ one finds $\overline{Q_1 - Q_2}\bar{L} = \overline{R_2 - R_1}$. The left-hand side is non-zero and has at least d zeros (counted with multiplicity). The right-hand side is a polynomial of degree $< d$. Thus we find a contradiction.

For the *existence* we only need to consider a $F_0 := F \in \hat{\mathcal{E}}^+_{p,0}$. In the ring Ent one has a division with remainder of the form: any element $A \in \text{Ent}$ can uniquely be written as $A = Q\bar{L} + R$, where $Q \in \text{Ent}$ and R a polynomial of degree $< d$. This implies a first step division with remainder: $F_0 = Q_0 L + R_0 + z^{-1/p} F_1$ with $Q_0 \in \mathrm{Ent}$ and R_0 a polynomial in δ of degree $< d$. By induction one finds expressions $F_n = Q_n L + R_n + z^{-1/p} F_{n+1}$ with all $Q_n \in \mathrm{Ent}$ and all R_n polynomials of degree $< d$. Formally one finds $F = F_0 = \left(\sum_{n \geq 0} z^{-n/p} Q_n\right) L + \left(\sum_{n \geq 0} z^{-n/p} R_n\right)$. The term $\sum_{n \geq 0} z^{-n/p} Q_n$ belongs to $\hat{\mathcal{E}}^+_{p,0}$ and $\sum_{n \geq 0} z^{-n/p} R_n$ lies in $\hat{\mathcal{P}}^+_p[\delta]$ and has degree $< d$. This proves the first part of the theorem.

For the second part, we consider $M := \hat{\mathcal{E}}^+_{p,0} / \hat{\mathcal{E}}^+_{p,0} L$. This module over $\hat{\mathcal{P}}^+_p$ has the properties $\hat{\mathcal{P}}_p \otimes M \cong \hat{\mathcal{E}}_{p,0} / \hat{\mathcal{E}}_{p,0} L$ and $M / M z^{-1/p} \cong \mathrm{Ent} / \bar{L}\,\mathrm{Ent}$. It suffices now to prove that $\mathrm{Ent}/\bar{L}\,\mathrm{Ent}$ has an uncountable dimension over $\mathbf{C}$. Let $\alpha_1, \alpha_2, \ldots$ and $m_1, m_2, \ldots$ denote the zeros of $\bar{L}$ with their multiplicities. There is a homomorphism of $\mathbf{C}$-algebras $ev : \mathrm{Ent} \to \prod_i (\mathbf{C}[z - \alpha_i]/(z - \alpha_i)^{m_i})$, which assigns to any $f \in \mathrm{Ent}$ the first part of its Taylor series at each point α_i. The Weierstrass existence theorem implies that ev is surjective and moreover the kernel of ev is seen to be the ideal $\mathrm{Ent}\,\bar{L}$. It follows that we can identify $\mathrm{Ent}/\mathrm{Ent}\,\bar{L}$ with the infinite product $\prod_i (\mathbf{C}[z - \alpha_i]/(z - \alpha_i)^{m_i})$. The dimension of this complex vector space is uncountable. □

The proof of the next result can be copied from the proof of Corollary 5.

COROLLARY 9 *Let $L \in \hat{\mathcal{E}}_{p,0}$ satisfy $\mathrm{ord}_0(L) < \infty$. Then L has a unique factorization $L = E \cdot D$, with $D \in \hat{\mathcal{P}}^+_p[\delta]$ monic and E an invertible element of $\hat{\mathcal{E}}_{p,0}$. Moreover the degree of D in δ is equal to $\mathrm{ord}_0(L)$.*

2.2 Newton Polygons and Decomposition

For a non-zero element $L \in \hat{\mathcal{E}}$ we will define a Newton polygon. One writes L as an infinite sum $\sum_{\lambda,n} a_{\lambda,n} z^{-\lambda} \delta^n$ and calls the set of pairs (n,λ) with $a_{\lambda,n} \neq 0$ the *support of* L. Let $Q_2 = \{(x,y) \in \mathbf{R}^2 \mid x \leq 0,\ y \geq 0\}$ denote the second quadrant. Then $\mathcal{M}(L)$ is defined as the closure of the convex hull of the set $\cup_{a_{\lambda,n} \neq 0} (n, \lambda) + Q_2$. The *Newton polygon* $\mathcal{N}(L)$ of L is the part of the boundary of $\mathcal{M}(L)$ which lies in the right-half plane. For an ordinary differential operator L, this definition coincides with the usual one. A "finite" side of the Newton polygon has the form $\{(x, kx + a) \mid b \leq x \leq c\}$. Then the number $k > 0$ is rational and will be called the *slope* of this side and the $b < c$ are non-negative integers. The side itself is denoted by $\mathcal{N}_k(L)$ and we define the numbers $v_k(L), b_k(L), e_k(L)$ by $\mathcal{N}_k(L) = \{(x, kx + v_k(L)) | b_k(L) \leq x \leq e_k(L)\}$.

Further $N_k(L)$ denotes the set of integers n such that $(n, kn + v_k(L))$ lies in the support of L.

The Newton polygon has at most one infinite side. If the operator L has finite order then this is a vertical line (and one could consider $+\infty$ as a slope of L). If the operator L has infinite order and there is an infinite side, then this side has the form $\mathcal{N}_k(L) := \{(x, kx + v_k(L)) \mid b_k(L) \leq x \leq \infty\}$. Its *slope* k need not be rational, but $b_k(L)$ is still a non-negative integer. One defines $N_k(L)$ as the set of integers n such that $(n, kn + v_k(L))$ is in the support of L. Further $e_k(L)$ denotes the supremum of this set. Thus $b_k(L) \leq e_k(L) \leq +\infty$.

For any real number $k \geq 0$ one can define $v_k(L)$ as the infimum of the function $(x,y) \mapsto y - kx$ on the support of L. This infimum can be $-\infty$. By the definition of $\hat{\mathcal{E}}$ one has $v_0(L) > -\infty$. For $k > 0$ and rational, one easily sees that $v_k(L) > -\infty$ if and only if L belongs to $\hat{\mathcal{E}}_{p,k}$ for a suitable p. For a slope k of L the new definition of $v_k(L)$ coincides with the earlier definition of $v_k(L)$.

Examples

(1) $\mathcal{N}(L)$ *with infinitely many slopes.*
 (i) Put $s(n) = \frac{1}{2}n(n+1)$. We define $L = 1 + \sum_{n\geq 1} z^{-s(n-1)}\delta^{s(n)}$. Then $L \in \hat{\mathcal{E}}$, and $\mathcal{N}(L)$ has the slopes $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$
 (ii) The operator $L := 1 + \sum_{n\geq 1} z^{-s(n-1)}\delta^n$ is an element of $\hat{\mathcal{E}}$ and $\mathcal{N}(L)$ has slopes $1, 2, 3, \ldots$

(2) $\mathcal{N}(L)$ *has finitely many slopes, and the last slope is irrational.*
 Let $\alpha > 0$ be irrational. Define $a_n(z) := z^{-[\alpha n]-1}$, $n \geq 1$, and $L := 1 + \sum_{n\geq 1} a_n\delta^n$. Then $\mathcal{N}(L)$ is the half line $y = \alpha x$, $x \geq 0$.

(3) $\mathcal{N}(L)$ *has finitely many slopes, the last slope, say* α, *is rational and* $e_\alpha(L)$ *is finite.*
 (i) Put $L = 1 + \delta + \delta^2 + \sum_{n\geq 3} z^{-[\log n]\delta^n}$. Then $\mathcal{N}(L)$ is the non-negative x-axis and intersects the support of L in $\{(n,0) \mid n = 0,1,2\}$ and $e_0(L) = 2$.
 (ii) The operator $L = 1 + z^{-1}e^{\partial}$, with $e^{\partial} := \sum_{n\geq 0} 1/n!\partial^n$, only has slope 1. The intersection of the infinite side with slope 1 and the support of L is just the point (0,0).

(4) $\mathcal{N}(L)$ *has finitely many slopes, the last slope,* α, *is rational and* $N_\alpha(L)$ *is infinite.*
 The operator $L = e^{\partial}$ has only one slope, namely 1. The intersection of the infinite side with the support of L is $\{(n,n) \mid n \geq 0\}$.

Remark 10 We collect here some properties relating the subalgebras of $\hat{\mathcal{E}}$ and Newton polygons:

(1) Let $L = \sum z^{-\lambda} a_\lambda(\delta) \in \hat{\mathcal{E}}$. Then $\mathcal{N}(L)$ consists of one horizontal line only unless all a_λ are polynomials and their degrees are bounded by a constant multiple of λ. In particular, if $L \in \hat{\mathcal{E}} \backslash \hat{\mathcal{D}}$ then $\mathcal{N}(L)$ consist of one horizontal line.

(2) Let $L \in \hat{\mathcal{E}}$ have slope $k > 0$, then $L \in \hat{\mathcal{D}}_{p,\tilde{k}}$ for all suitable combinations of p and $\tilde{k}$ with $\tilde{k} \in (1/p)\mathbf{N}$, $\tilde{k} < k$.

   (a) If, moreover, k is rational, then $L \in \hat{\mathcal{E}}_{p,k}$ for some p, and $\mathrm{ord}_k(L) = b_k(L)$.

   (b) If k is rational and the side with slope k has finite length, then $L \in \hat{\mathcal{D}}_{p,k}$ for some p, and $\deg_k(L) = e_k(L)$.

PROPOSITION 11

(1) *Let L_1, L_2 be non-zero elements of $\hat{\mathcal{E}}$ and $L = L_1 L_2$. Then $\mathcal{M}(L) \subset \mathcal{M}(L_1) + \mathcal{M}(L_2)$. This inclusion can only be strict in the following exceptional cases:*

   (a) *L_1 and L_2 both have as Newton polygon a horizontal line. Then the slope* 0 *part of $\mathcal{N}(L_1 L_2)$ can be finite or empty.*

   (b) *L_1 and L_2 have a common slope* $k > 0$, *which is an infinite edge for L_1 or L_2. In this case the slope k part of $\mathcal{N}(L_1 L_2)$ can be finite or empty.*

(2) *Let $L \in \hat{\mathcal{D}}_{p,k}$ and let $L = E \cdot D$ be the decomposition of Corollary* 5. *Then $\mathcal{M}(L) = \mathcal{M}(E) + \mathcal{M}(D)$.*

(3) *Let $L \in \hat{\mathcal{E}}_{p,k}$ with* $\mathrm{ord}_k(L) < \infty$ *and let $L = E \cdot D$ be the decomposition of Corollary* 7 *or* 9. Then $\mathcal{M}(L) = \mathcal{M}(E) + \mathcal{M}(D)$.

Proof (1) For notational convenience we suppose that $L_1, L_2 \in \hat{\mathcal{E}}_1^+$ and that the constant terms of L_1 and L_2 are 1. Write $L_1 = \sum_{n \geq 0, m \geq 0} a_{m,n} z^{-m} \delta^n$ and $L_2 = \sum_{n \geq 0, m \geq 0} b_{m,n} z^{-m} \delta^n$. Then

$$L_1 L_2 = \sum_{n \geq 0, m \geq 0} \sum_{m_1 + m_2 = m, n_1 + n_2 = n} a_{m_1,n_1} b_{m_2,n_2} z^{-m} (\delta - m_2)^{n_1} \delta^{n_2}$$

and the inclusion $\mathcal{M}(L) \subset \mathcal{M}(L_1) + \mathcal{M}(L_2)$ follows at once. Now we have to consider special cases:

(i) *Suppose that $\mathcal{M}(L_2)$ is the first quadrant and $\mathcal{M}(L_1)$ is not. Then $\mathcal{M}(L)$ also is the first quadrant and thus equal to $\mathcal{M}(L_1) + \mathcal{M}(L_2)$.*

Suppose the contrary, then $L_1L_2 \in \hat{\mathcal{D}}_{p',k'}$ for some $p' \in \mathbf{N}$ and $k' \in (1/p')\mathbf{N}$. By assumption $L_1 \in \hat{\mathcal{D}}_{p,k}$ for suitable $p,k > 0$. We may and will suppose that $p = p'$. Let $L_1 = EL_3$ be the decomposition of Corollary 5. For $k'' = \min(k,k')$ we have that E is invertible in $\hat{\mathcal{D}}_{p,k''}$ and $L_3L_2 = E^{-1}L_1L_2 \in \hat{\mathcal{D}}_{p,k''}$. Now L_3 is polynomial in δ and using the first statement of Remark 10 one sees that L_3L_2 has as Newton polygon a horizontal line. This contradicts the assumption on L_3L_2.

A similar argument proves that for an L_1 with a horizontal line as Newton polygon and an L_2 with a different Newton polygon one has $\mathcal{M}(L_1L_2) = \mathcal{M}(L_1) + \mathcal{M}(L_2)$. Thus, a strict inclusion effecting the slope 0 can only happen if both L_1 and L_2 have a horizontal line as Newton polygon. We give two examples of strict inclusion connected with the 0-slopes:

* $L_1 = e^{\delta}$ and $L_2 = e^{-\delta}$. Thus, $L_1L_2 = 1$.
* $L_1 = 1 - R$, where $R = \sum_{n\geq 1} z^{-n}a_n(\delta)$ with all $a_n \in \mathrm{Ent}$. We suppose that either some a_n is not a polynomial or that all the a_n are polynomials and the supremum of $\deg a_n/n$ is infinite. Then the Newton polygon of L_1 is the positive x-axis. Then $L_2 := 1 + R + R^2 + R^3 + \cdots$ is easily seen to be an element of $\hat{\mathcal{E}}_1^+$ with the positive x-axis as Newton polygon. Further $L_1L_2 = 1$.

(ii) The case that L_1 or L_2 has an infinite edge with slope 0 (or equivalently a horizontal line as Newton polygon) has to be dealt with in (i). We now suppose that both L_1 and L_2 have a finite or empty edge with slope 0. By truncating L_1 and L_2 and using that for operators of finite order the equality $\mathcal{M}(\tilde{L}_1\tilde{L}_2) = \mathcal{M}(\tilde{L}_1) + \mathcal{M}(\tilde{L}_2)$ holds, one sees that a strict inclusion $\mathcal{M}(L_1L_2) \subset \mathcal{M}(L_1) + \mathcal{M}(L_2)$ can only occur if there is a slope $k > 0$ common to L_1 and L_2 which is an infinite edge for at least one of them. We give two examples of this situation:

* $L_1 = 1 + (z^{-k}\delta) + (z^{-k}\delta)^2 + \cdots$ and $L_2 = 1 - (z^{-k}\delta)^n$. Then $L_1L_2 = 1 + (z^{-k}\delta) + \cdots + (z^{-k}\delta)^{n-1}$.
* $L_1 = H(z^{-k}\delta)$, where H is any power series with constant term 1. Let $L_2 := K(z^{-k}\delta)$, where K is any power series with constant term 1, such that the product is a polynomial. Again L_1L_2 is a polynomial in $z^{-k}\delta$. If it is, for instance, given that H is not the power series of a

rational function, then both $\mathcal{N}(L_1)$ and $\mathcal{N}(L_2)$ have an infinite edge with slope k.

This proves (1). For (2) and (3) one can easily verify that the two factors E, D of the decomposition $L = ED$ are not in one of the exceptional cases. ∎

The following result generalizes the well-known decomposition of a differential operator of finite order, to the case of a differential operator of infinite order. The proof follows easily from Corollaries 5, 7 and 9 and Proposition 11.

COROLLARY 12 *Decomposition according to the Newton polygon. Let $L \in \hat{\mathcal{E}}$ be an operator of infinite order.*

(1) *Suppose that $\mathcal{N}(L)$ has only one slope k. If k is irrational then we cannot decompose L. If k is rational, then $L \in \hat{\mathcal{E}}_{p,k}$ for a suitable p and:*
   (a) *If $k = 0$ and $\operatorname{ord}_0(L) = \infty$ or $\operatorname{ord}_0(L) = 0$, then we cannot decompose L.*
   (b) *If $k = 0$ and $0 < \operatorname{ord}_0(L) < \infty$, then $L = ED$ with D a monic polynomial in δ of degree $\operatorname{ord}_0(L)$ and coefficients in $\hat{\mathcal{P}}_p^+$ and E invertible in $\hat{\mathcal{E}}_{p,0}$.*
   (c) *If $k > 0$ is rational and $\operatorname{ord}_k(L) = 0$ then we cannot decompose L. If $\operatorname{ord}_k(L) > 0$, then $L = ED$ with D a monic polynomial in $z^{-k}\delta$ of degree $\operatorname{ord}_k(L)$ and all other coefficients in $z^{-1/p}\hat{\mathcal{P}}_p^+$ and E invertible in $\hat{\mathcal{E}}_{p,k}$.*

(2) *Suppose that $\mathcal{N}(L)$ has finitely many slopes $0 \le k_1 \le k_2 \le \cdots \le k_r$ with $r > 1$. Then $k_1, \ldots, k_{r-1}$ are rational and $L \in \hat{\mathcal{E}}_{p,k}$ for all rational numbers k with $0 \le k \le k_r$ and suitable p (dependent on the coefficients of L and on k). Further, L decomposes uniquely as $L = E_r D_{r-1} \cdots D_1$, where each D_i is a monic polynomial in $z^{-k_i}\delta$ with coefficients in $\hat{\mathcal{P}}_p^+$. The element $E_r \in \hat{\mathcal{E}}_{p,k}$ for all rational k with $0 \le k \le k_r$ can be further composed if k_r is rational and $\operatorname{ord}_{k_r}(L) > 0$ (see (1) part (c)).*

(3) *Suppose that L has infinitely many slopes $0 \le k_1 < k_2 < \cdots$. Then all the k_i are rational and $L \in \hat{\mathcal{D}}_{p_i,k_i}$ for some suitable p_i, for all i. Further L has for every $r > 1$ a unique decomposition $L = E_r D_{r-1} \cdots D_1$ with each D_i a polynomial in $z^{-k_i}\delta$ and coefficients in $\hat{\mathcal{P}}_p^+$, and E_r an invertible element of $\hat{\mathcal{D}}_{p,k_r}$.*

2.3 Rings of Symbols

Consider the first order scalar differential equation

$$y' = \left(\sum_{m\geq M} a_m z^{-m/p}\right) y, \quad a_m \in \mathbf{C},\ a_M \neq 0,\ ' = \frac{\mathrm{d}}{\mathrm{d}z}. \qquad (2.1)$$

If $M > p$, then this equation has a non-zero solution in $\hat{\mathcal{P}}$, but if $M \leq p$, this is in general not the case. We will define formal objects (or symbols), with an action of $\mathrm{d}/\mathrm{d}z$ on them, enabling us to solve any first order equation as above in terms of these formal objects and elements of $\hat{\mathcal{P}}$.

Let $\mathcal{Q} := \cup_{p\geq 1} z^{1/p}\mathbf{C}[z^{1/p}]$. We define objects $e(q)$, $q \in \mathcal{Q}$, and Z^a, $a \in \mathbf{C}$, with the following rules of computation ($' = \mathrm{d}/\mathrm{d}z$):

- $e(q_1)e(q_2) = e(q_1 + q_2)$, $e(q)' = q'e(q)$;
- $Z^a Z^b = Z^{a+b}$, $Z^a = z^a$ if $a \in \mathbf{Q}$, $(Z^a)' = a\, Z^{a-1}$.

In the (commutative) ring $\hat{\mathcal{P}}[\{e(q)\}_{q\in\mathcal{Q}}, \{Z^a\}_{a\in\mathbf{C}}]$, Eq. (2.1) now has the non-zero solution

$$y = e(q)Z^a h,\ q(z) = \int_0^z (a_M \zeta^{-M/p} + \cdots + a_{p-1}\zeta^{-1+1/p})\mathrm{d}\zeta,\ a = a_p$$

($M \leq p-1$) and $h \in \hat{\mathcal{P}}$ satisfies $h' = \left(\sum_{m>p} a_m z^{-m/p}\right) h$.

In order that we can also solve higher order equations, we only need a formal equivalent of the logarithm: we define an object l, satisfying $l' = z^{-1}$. Indeed, in [HvdP95] it is shown that

$$R_\partial := \hat{\mathcal{P}}[\{e(q)\}_{q\in\mathcal{Q}}, \{Z^a\}_{a\in\mathbf{C}}, l]$$

is the smallest ring such that $\dim_{\mathbf{C}} \ker(D, R_\partial) = \deg(D)$, for any $D \in \hat{\mathcal{P}}[\partial]$ (if we make R_∂ a left $\hat{\mathcal{P}}[\partial]$-module by defining $\partial(f) = \mathrm{d}/\mathrm{d}z(f)$, $f \in R_\partial$). We call R_∂ the *universal differential ring*. More properties of this ring can be found in [HvdP95]. For $k \in \mathbf{Q}^+$, let $\mathcal{Q}_{\leq k}$ (resp. $\mathcal{Q}_{<k}$) $:= \{q \in \mathcal{Q} \mid \deg_z(q) \leq k$ (resp. $\deg_z(q) < k$)$\}$. We also introduce the subrings

$$R_{\partial,\leq k} := \hat{\mathcal{P}}[\{e(q)\}_{q\in\mathcal{Q}_{\leq k}}, \{Z^a\}_{a\in\mathbf{C}}, l], \quad k \in \mathbf{Q}^+;$$
$$R_{\partial,\leq 0} := \hat{\mathcal{P}}[\{Z^a\}_{a\in\mathbf{C}}, l];$$
$$R_{\partial,<k} := \hat{\mathcal{P}}[\{e(q)\}_{q\in\mathcal{Q}_{<k}}, \{Z^a\}_{a\in\mathbf{C}}, l], \quad k \in \mathbf{Q}^+.$$

LEMMA 13 *Let* $p \in \mathbf{N}$, $k \in (1/p)\mathbf{N}_0$, *and* $L = \sum_{h \gg -\infty} z^{-h/p} A_h(z^{-k}\delta) \in \hat{\mathcal{E}}_{p,k}$. (*Hence, the* $A_h(x)$ *are formal power series in* x *if* $k > 0$ *and entire functions if* $k = 0$.) *Then*:

(i) L *acts on* $R_{\partial,\leq k}$ *if and only if* $A_h(x)$ *is an entire function for all* h.
(ii) *If* $k > 0$, *then* L *acts on* $R_{\partial,<k}$.

Proof We note that $z(\mathrm{d}/\mathrm{d}z)(Z^b) = bZ^b$. Thus,

$$\sum_{h \gg -\infty} z^{-h/p} A_h(\delta)(Z^b) = Z^b \sum_{h \gg -\infty} A_h(b) z^{-h/p}.$$

This proves the statement for $k = 0$.

Let $k > 0$. From the formula $z^{1-k}(\mathrm{d}/\mathrm{d}z)(e(cz^k)) = cke(cz^k)$ we readily derive that

$$L(e(cz^k)) = e(cz^k) \sum_{h \gg -\infty} A_h(ck) z^{-h/p}.$$

The expression on the right-hand side makes sense, for all $c \in \mathbf{C}$, if and only if all $A_h(x)$ are entire functions. This proves the statement for positive k.

To prove the second statement, note the following: if $L \in \hat{\mathcal{E}}_{p,k}$, then L belongs to $\hat{\mathcal{D}}_{r,\tilde{k}} \subset \hat{\mathcal{E}}_{t,\tilde{k}}$. So, by (i), L acts on $R_{\partial,\leq \tilde{k}}$, for any $r \in p\mathbf{N}$ and any $\tilde{k} \in (1/r)\mathbf{N}$, $\tilde{k} < k$. Since the union of the spaces is $R_{\partial,<k}$, we conclude that L acts on $R_{\partial,<k}$. □

Remark The Newton polygon of the operator $L = 1 + \sum_{n \geq 1} z^{-s(n-1)} \delta^{s(n)}$, $s(n) = \frac{1}{2}n(n+1)$, has infinitely many slopes, and the supremum of the sequence of slopes is 1. Therefore, L acts on $R_{\partial,<1}$. However, one can easily check that $v_1(L) = -\infty$, so that $L \notin \hat{\mathcal{E}}_{1,1}$. This shows that the converse of the second statement of the lemma is not true.

With Lemma 13 we find

PROPOSITION 14 *Let* $p \in \mathbf{N}$, *and* $L \in \hat{\mathcal{E}}_p$. *Suppose* $\mathcal{N}(L)$ *has slopes* $0 \leq k_1 < \cdots < k_r$, $r > 1$ *and no other slopes* $\leq k_r$. *Let* $L = E_r D_{r-1} \cdots D_1$ *be the unique decomposition of* L (*cf. Corollary* 12).

Then L *acts on* $R_{\partial,\leq k_{r-1}}$ *and* $\ker(L, R_{\partial,\leq k_{r-1}}) = \ker(D_{r-1} \cdots D_1, R_{\partial,\leq k_{r-1}})$, *which is a vector space over* $\mathbf{C}$ *of dimension* $\mathrm{ord}_{k_r}(L) = \deg_{k_1}(D_1) + \cdots + \deg_{k_{r-1}}(D_{r-1}) = \deg(D_{r-1} \cdots D_1)$.

Proof E_r is invertible in $\hat{\mathcal{E}}_{p,k}$ for any rational k with $k_{r-1} \leq k \leq k_r$. Thus E_r is also invertible in $\hat{\mathcal{D}}_{p,k_{r-1}}$ and acts bijectively on $R_{\partial,k_{r-1}}$.

As a corollary we have

COROLLARY 15 *Suppose $L \in \hat{\mathcal{E}}$.*

(1) *If $\mathcal{N}(L)$ has infinitely many slopes and the supremum of the slopes is ∞, then L acts on R_∂ and* $\ker(L, R_\partial)$ *has countably infinite dimension over* $\mathbf{C}$.
(2) *If $\mathcal{N}(L)$ has infinitely many slopes, and the supremum of the slopes is $\alpha < \infty$, then L acts on $R := \cup_{k<\alpha, k \in \mathbf{Q}^+} R_{\partial,\leq k}$ and* $\ker(L, R)$ *has countably infinite dimension over* $\mathbf{C}$ *(we note that $R = R_{\partial,<\alpha}$ if α is rational).*
(3) *If $\mathcal{N}(L)$ has finitely many slopes and $L \in \hat{\mathcal{E}}_{p,k}$ for some $p \in \mathbf{N}$ and $k \in (1/p)\mathbf{N}$, then L acts on $R_{\partial,<k}$ and* $\dim_{\mathbf{C}} \ker(L, R_{\partial,<k}) = \mathrm{ord}_k(L) < \infty$.

Proof (1) and (2) If $\mathcal{N}(L)$ has infinitely many slopes, then all slopes are rational and the statement is an immediate consequence of the previous proposition.

(3) L acts on $R_{\partial,<k}$ because of Lemma 13. Let $L = ED$ be the factorization of Corollary 7. In particular, D is a finite order operator of degree $\mathrm{ord}_k(L)$ whose Newton polygon has only slopes smaller than k, and E an invertible operator in $\hat{\mathcal{E}}_{p,k}$. Hence, $\ker(L, R_{\partial,<k}) = \ker(D, R_{\partial,<k})$. Since $\deg(D) = \mathrm{ord}_k(L) < \infty$ and $\mathcal{N}(D)$ only has (finite) slopes $< k$, it follows from well-known theory for finite order differential operators, the last set is a $\mathbf{C}$-linear vector space of dimension $\deg(D) = \mathrm{ord}_k(L) = \infty$. □

The previous results can be made more explicit. Let $L \in \hat{\mathcal{E}}$ and suppose that k is a rational slope of $\mathcal{N}(L)$. Let $\bar{L}$ be the image in Ent or $\mathbf{C}[[z^{-k}\delta]]$ or $\mathbf{C}[z^{-k}\delta]$ (according to the situation) of the k-normalization of L. Then $\bar{L} = \sum_n c_n (z^{-k}\delta)^n$. We define $F_{L,k}(\xi) := \sum_n c_n \xi^n$, and we call $F_{L,k}$ *the k-characteristic* of L.

PROPOSITION 16 *Let $L \in \hat{\mathcal{E}}$ and suppose that $\mathcal{N}(L)$ has a horizontal slope.*

(1) *Suppose that* $\mathrm{ord}_0(L)$ *is finite. Then L factors according to Corollary 9 as ED and the entire function $F_{L,0}(\xi)$ factors as $F_{E,0}(\xi) F_{D,0}(\xi)$, where the first term is an invertible entire function*

and the second term is the indicial polynomial of the regular singular differential operator D. The kernels of L and D on $R_{\partial,\leq 0}$ *coincide and have dimension* $\mathrm{ord}_0(L)$ *over* $\mathbf{C}$.

(2) *Suppose that* $\mathrm{ord}_0(L)=\infty$. *Let* α *be a zero of* $\bar{L}$ *such that* $\alpha-n$ *is not a zero of* $\bar{L}$ *for all* $n\in\mathbf{N}$. *Fix a positive integer N and consider the polynomial* $P=\prod_{i=0,\ldots,N}(x-\alpha-i)^{m_i}$, *where* m_i *is the order of* $\bar{L}$ *at the point* $\alpha+i$. *Then L factors as* L_1D *with* $L_1\in\hat{\mathcal{E}}$ *and D a regular singular differential operator, monic in* δ *of degree* $\sum m_i$, *and with indicial polynomial P. The kernel of L on* $R_{\partial,\leq 0}$ *contains the kernel of D on* $R_{\partial,\leq 0}$, *which is a vector space of dimension* $\sum m_i$.

Proof The first statement is rather obvious. The second statement extends Corollary 15 (and also Corollary 12-1(a) and Theorem 8). We will do more factoring in $\hat{\mathcal{E}}$. For notational convenience we suppose that $L=l_0+z^{-1}l_1+z^{-2}l_2+\cdots$ with all $l_i\in\mathrm{Ent}$ and $l_0\neq 0$ having infinitely many zeros. We would like to factor L as AB with $A=a_0+z^{-1}a_1+\cdots$ and $B=b_0+z^{-1}b_1+\cdots$. This produces a sequence of equations for the entire functions $a_0(x),b_0(x),a_1(x),b_1(x),\ldots$:

$$a_0(x)b_0(x)=l_0(x),\quad a_0(x-1)b_1(x)+a_1(x)b_0(x)=l_1(x),$$
$$a_0(x-2)b_2(x)+a_1(x-1)b_1(x)+a_2(x)b_0(x)=l_2(x),\ldots$$

We choose a_0,b_0 such that $a_0(x-n)$ and $b_0(x)$ have no common zero for all $n\in\mathbf{N}$. In other words, if $b_0(\alpha)=0$ then $a_0(\alpha-n)\neq 0$ for all $n\in\mathbf{N}$. One can show that two elements $f,g\in\mathrm{Ent}$ generate the unit ideal in Ent if and only if f and g have no common zeros. Using this one finds that a decomposition of L of the required form exists if a_0,b_0 satisfy the above condition. We apply this to $l_0=\bar{L}$, $b_0=P$ and $a_0=l_0/P$. This gives a factorization AB with $\bar{B}=P$. To B we apply Corollary 9, i.e., $B=ED$. Then the decomposition $L=(AE)D$ has the required properties. The rest of the second statement is now obvious. □

Example There are operators $L\in\hat{\mathcal{E}}$ with $\mathrm{ord}_0(L)=\infty$ and the kernel of L on $R_{\partial,\leq 0}$ is 0. An example is $L=l_0+z^{-1}$ with $l_0=\bar{L}=e^{2\pi i\delta}-1$. A calculation shows that if the kernel of L on $R_{\partial,\leq 0}$ is non-trivial, then there is also a non-trivial solution $L(f)=0$ with $f\in\mathbf{C}((z^{-1}))$. However, for such a f one has $e^{2\pi i\delta}f=f$ and thus $L(f)=z^{-1}f$.

PROPOSITION 17 *Let $L = \sum_{h \geq H} z^{-h/p} A_h(z^{-k}\delta) \in \hat{\mathcal{E}}_{p,k}$ have slope k and suppose that $A_H \in \mathrm{Ent}$ is non-zero. If $A_H(0) = 0$ then $\mathcal{N}(L)$ has a slope $< k$. Let $c \in \mathbf{C}$ be any zero of $F_{L,k}$. Define $\tilde{L} := e(-(c/k)z^k)Le((c/k)z^k)$. Then $\tilde{L} = \sum_{h \gg -\infty} z^{-h/p} A_h(c + z^{-k}\delta) \in \hat{\mathcal{E}}_{p,k}$ and $\mathcal{N}(\tilde{L})$ has a slope smaller than k. If the edge with slope k has finite length, then $e_k(\tilde{L}) = e_k(L) < \infty$.*

Proof The obvious formula $(z^{-k}\delta)^n \cdot e((c/k)z^k) = e((c/k)z^k)(c + z^{-k}\delta)^n$ implies

$$\tilde{L} = e\Big(-\frac{c}{k}z^k\Big) L e\Big(\frac{c}{k}z^k\Big) = \sum_{h \geq H} z^{-h/p} A_h(c + z^{-k}\delta) \in \hat{\mathcal{E}}_{p,k};$$

$F_{L,k}(\xi) = A_H(\xi)$ holds and thus $F_{\tilde{L},k}(\xi) = A_H(\xi + c)$. In particular $F_{\tilde{L},k}(0) = 0$ and one easily sees that the Newton polygon of $\tilde{L}$ has a slope $< k$. □

3 SCALAR DIFFERENTIAL–DIFFERENCE EQUATIONS

We recall the definition of the ring of differential–difference operators $\hat{A} = \hat{\mathcal{P}}[\tau^{-1}, \tau, \partial]$, with $\tau\partial = \partial\tau$, $\tau f = \phi(f)\tau$, $\partial f = f\partial + (\mathrm{d}/\mathrm{d}z)(f)$ for any f in $\hat{\mathcal{P}}$.

PROPOSITION 18 *The $\hat{\mathcal{P}}$-algebra homomorphism $\hat{A} \to \hat{\mathcal{E}}$, given by $\tau \mapsto e^{\partial}$ and $\partial \mapsto \partial$, is an embedding. This embedding is compatible with the actions on $\hat{\mathcal{P}}$.*

Proof The mapping is well defined, and its kernel I is a two-sided ideal $\neq \hat{A}$. We will show that $\hat{A}$ only has trivial two-sided ideals. This implies that $I = 0$.

First we prove that $\hat{\mathcal{P}}[\partial]$ only has trivial two-sided ideals. Suppose that $I \neq 0$ is a two-sided ideal in $\hat{\mathcal{P}}[\partial]$. Choose $L \in I$ of minimal degree d. If $d = 0$, then $I = \hat{\mathcal{P}}[\partial]$, and we are done. If $d > 0$, then $L = a_0 + a_1\partial + \cdots + a_d\partial^d$, $a_d \neq 0$. We have $Lz - zL \in I$, and

$$Lz - zL = a_1 + 2a_2\partial + \cdots + (d-1)a_{d-1}\partial^{d-2} + da_d\partial^{d-1},$$

because $\partial^k z - z\partial^k = k\partial^{k-1}$ for all k. As d was chosen minimal it follows, that $Lz - zL = 0$, hence, in particular $a_d = 0$, which yields a contradiction.

Next we suppose that $I \neq 0$ is a two-sided ideal in $\hat{A}$. Choose $d \geq 0$ minimal such that

$$I_d := I \cap \{a_0 + a_1\tau + \cdots + a_d\tau^d \mid a_i \in \hat{\mathcal{P}}[\partial], i = 0, \ldots, d\} \neq 0 .$$

If $d = 0$, then $I = \hat{A}$. So suppose that $d > 0$. Let $L \in I_d$, $L \neq 0$. The minimality of d implies that $L = a_0 + a_1\tau + \cdots + a_d\tau^d$ with $a_0, a_d \neq 0$. Note that L is completely determined by a_0. The coefficients a_0 of elements of I_d form a non-zero two-sided ideal in $\hat{\mathcal{P}}[\partial]$, so we may suppose that $a_0 = 1$. Now consider $(Lz - zL)\tau^{-1}$. It is an element of I, and

$$(Lz - zL)\tau^{-1} = (a_1(z+1) - za_1) + \cdots + (a_d(z+d) - za_d)\tau^{d-1} .$$

As d was chosen minimal, this expression must equal zero. In particular, it follows that $a_d = 0$, which gives a contradiction. □

Remark 19 For $\alpha \in \mathbf{C}$ we denote by τ^α the shift operator $z \mapsto z + \alpha$. This operator acts on $\hat{\mathcal{P}}$ by the formula $z^\lambda \mapsto (z+\alpha)^\lambda := z^\lambda(1 + \alpha z^{-1})^\lambda$. The ring $\hat{\mathcal{P}}[\{\tau^\alpha\}_{\alpha \in \mathbf{C}}, \partial]$, defined in the obvious way, can also be embedded in $\hat{\mathcal{E}}$. In this embedding the element τ^α is mapped to $e^{\alpha\partial} \in \hat{\mathcal{E}}$.

3.1 The Dimension of the Solution Space

We define subrings $\hat{A}_p := \hat{\mathcal{P}}_p[\tau^{-1}, \tau, \partial]$, $p \in \mathbf{N}$, of $\hat{A}$. We recall that $\hat{\mathcal{P}}_p^+ := \mathbf{C}[[z^{-1/p}]]$, and define $\hat{A}_p^+ := \hat{\mathcal{P}}_p^+[\tau^{-1}, \tau, \partial]$. Then $z^{-1/p}\hat{A}_p^+$ is a two-sided ideal in $\hat{A}_p^+$, and $\hat{A}_p^+ / z^{-1/p}\hat{A}_p^+ \cong \mathbf{C}[\tau^{-1}, \tau, \partial]$.

Let $L \in \hat{A}_p$, for some $p \in \mathbf{N}$ and $L \neq 0$. Let $m \in \mathbf{Q}$ be minimal such that $z^{-m}L \in \hat{A}_p^+$. We call $z^{-m}L$ the *normalization* of L in $\hat{A}_p^+$. By $\bar{L}$ we denote the image in $\mathbf{C}[\tau^{-1},\tau,\partial]$ of the normalization of L. We will prove the following:

Proposition 20 *If $\bar{L} \notin \mathbf{C}^*\tau^{\mathbf{Z}}$, then $\hat{A}/\hat{A}L$ has a countable (infinite) dimension as vector space over $\hat{\mathcal{P}}$.*

Proof The dimension of $\hat{A}$ is countable and we have to show that the condition on $\bar{L}$ implies that $\hat{A}/\hat{A}L$ is not finite dimensional. Since $\hat{\mathcal{P}} \otimes_{\hat{\mathcal{P}}_p} \hat{A}_p / \hat{A}_p L \cong \hat{A}/\hat{A}L$ it suffices to prove that $\hat{A}/\hat{A}L$ is not finite dimensional over $\hat{\mathcal{P}}_p$. It is not difficult to prove that the $\mathcal{P}_p^+$-module $M := \hat{A}_p^+/\hat{A}_p^+L$ satisfies that $\hat{\mathcal{P}}_p \otimes M \to \hat{A}_p/\hat{A}_pL$ is an isomorphism. If the dimension of $\hat{A}_p/\hat{A}_pL$ were finite, then M would be a finitely generated and free module over $\hat{\mathcal{P}}_p^+$. The last condition implies that $M/z^{-1/p}M$ is a finite dimensional vector space over $\mathbf{C}$. We consider now the ring $M/z^{-1/p}M \cong \mathbf{C}[\tau,\tau^{-1},\partial]/(\bar{L})$. The set of the invertible elements of the ring $\mathbf{C}[\tau,\tau^{-1},\partial]$ is $\mathbf{C}^*\tau^{\mathbf{Z}}$. We have assumed that $\bar{L}$ does not belong to this set and thus, $\mathbf{C}[\tau,\tau^{-1},\partial]/(\bar{L}) \neq 0$. We now apply some theory of commutative noetherian rings. The Krull dimension of the ring $\mathbf{C}[\tau,\tau^{-1},\partial]$ is two and the Krull dimension of $\mathbf{C}[\tau,\tau^{-1},\partial]/(\bar{L})$ is one, since the ideal is generated by one element which is not a zero divisor. If $\mathbf{C}[\tau,\tau^{-1},\partial]/(\bar{L})$ were of finite dimension over $\mathbf{C}$, then its Krull dimension is 0. Thus, we conclude that the dimension of $\mathbf{C}[\tau,\tau^{-1},\partial]/(\bar{L})$ is countable (infinite). □

Conclusion Proposition 20 leads to the expectation that the space of "symbolic" or formal solutions of $L(y)=0$ has (in general) countably infinite dimension over $\hat{\mathcal{P}}$. This is in contrast with the situation for an ordinary differential operator.

3.2 Construction of a Differential–Difference Ring R

As remarked before, any finite order differential equation $D(y)=0$, $D \in \hat{\mathcal{P}}[\partial]$ has a fundamental system of solutions in R_∂. More precisely, D acts on R_∂, and if M is the degree of D, then the subset of R_∂ of solutions of $D(y)=0$ is an M-dimensional vector space over $\mathbf{C}$.

There exists a similar result for difference operators $T \in \hat{\mathcal{P}}[\tau^{-1},\tau]$. see [vdPS97, Chapter 6.2].

As before, let $\mathcal{Q} = \cup_{p\geq 1} z^{1/p}\mathbf{C}[z^{1/p}]$, and define $\mathcal{Q}_{<1} := \{q \in \mathcal{Q} \mid \deg_z(q) < 1\}$. Note that $\mathcal{Q}_{<1} = \{\sum_{0<\mu<1} a_\mu z^\mu \mid a_\mu \in \mathbf{C},\ \mu \in \mathbf{Q},\ \sum \text{ finite}\}$. Let

$$\mathcal{G} = \{z^\lambda c(1+z^{-1})^{a_0}\exp(\phi(q)-q) \mid \lambda \in \mathbf{Q}, c \in \mathbf{C}^*, a_0 \in V, q \in \mathcal{Q}_{<1}\},$$

where V denotes a $\mathbf{Q}$-linear subspace of $\mathbf{C}$, such that $V \oplus \mathbf{Q} = \mathbf{C}$. The ring

$$R_\tau := \hat{\mathcal{P}}[\{E(g)\}_{g\in\mathcal{G}}, l],$$

with generators $\{E(g)\}_{g\in\mathcal{G}}$ and l, relations $E(g_1g_2)=E(g_1)E(g_2)$ and $E(1)=1$, and an action of ϕ on R_τ given by $\phi(E(g))=gE(g)$, $\phi(l)=l+\log(l+z^{-1})$, and $\phi(z^{1/p})=(z+1)^{1/p}$ has the following properties:

- the action of ϕ on R_τ extends that of ϕ on $\hat{\mathcal{P}}$, and is bijective on R_τ;
- R_τ has only trivial ϕ-invariant ideals;
- $\{y\in R_\tau \mid \phi(y)=y\}=\mathbf{C}$;
- every (linear homogeneous) difference equation with coefficients in $\hat{\mathcal{P}}$ has a fundamental set of solutions in R_τ;
- R_τ is the smallest ring with the above properties.

This has been proved in [vdPS97, Chapter 6.2]. By defining $\tau(f):=\phi(f)$, for $f\in R_\tau$, R_τ becomes a left $\hat{\mathcal{P}}[\tau^{-1},\tau]$-module. We call R_τ the universal difference ring (for $\hat{\mathcal{P}}$).

We want to use different notations, and therefore give another description of R_τ. Recall that $\mathcal{Q}_{\le 1} := \{q\in\mathcal{Q} \mid \deg_z(q)\le 1\}$. We define a difference ring $\tilde{R}_\tau$,

$$\tilde{R}_\tau := \hat{\mathcal{P}}[\{\Gamma^\lambda\}_{\lambda\in\mathbf{Q}}, \{e(q)\}_{q\in\mathcal{Q}_{\le 1}}, \{Z^a\}_{a\in V}, l] \text{ with the only relations}$$

$$\begin{aligned} \Gamma^{\lambda_1}\Gamma^{\lambda_2} &= \Gamma^{\lambda_1+\lambda_2}, \quad \forall\lambda_1,\lambda_2\in\mathbf{Q};\\ e(q_1)e(q_2) &= e(q_1+q_2), \quad \forall q_1,q_2\in\mathcal{Q}_{\le 1};\\ Z^aZ^b &= Z^{a+b}, \quad \forall a,b\in V, \end{aligned}$$

and with ϕ-action

$$\begin{aligned} \phi(\Gamma) &= z\Gamma, \quad \text{where } \Gamma=\Gamma^1;\\ \phi(e(q)) &= \exp(\phi(q)-q)e(q);\\ \phi(Z^a) &= (1+z^{-1})^aZ^a;\\ \phi(l) &= l+\log(1+z^{-1});\\ \phi(z^{1/p}) &= (z+1)^{1/p}. \end{aligned}$$

There exists a difference homomorphism from $\tilde{R}_\tau$ to R_τ (i.e. a ring homomorphism that commutes with ϕ):

$$\begin{aligned} \Gamma^\lambda &\mapsto E(z^\lambda);\\ e(q) &\mapsto E(e^a\exp(\phi(\tilde{q})-\tilde{q})), \quad \text{if } q=az+\tilde{q},\ \tilde{q}\in\mathcal{Q}_{<1};\\ Z^a &\mapsto E((1+z^{-1})^a). \end{aligned}$$

Obviously, this mapping is surjective with kernel $(e(2\pi iz)-1)$, i.e., $\tilde{R}_\tau/(e(2\pi iz)-1)\cong R_\tau$. Since R_τ only has trivial ϕ-invariant ideals, $(e(2\pi iz)-1)$ is the maximal ϕ-invariant ideal in $\tilde{R}_\tau$. We also remark that $\tilde{R}_\tau$ has ϕ-invariant elements, which do not belong to $\mathbf{C}$, e.g. $e(2\pi inz)$, $n\in\mathbf{Z}$.

For differential–difference operators in $\hat{A}=\hat{\mathcal{P}}[\tau^{-1},\tau,\partial]$ one might like to combine the rings R_∂ and $\tilde{R}_\tau$. More precisely, we look for the smallest ring R that has the following properties:

(1) R is a $\hat{\mathcal{P}}$-algebra;
(2) d/dz and ϕ act on R, and ϕ defines a bijection on R; the actions of d/dz and ϕ on R extend their respective actions on $\hat{\mathcal{P}}$; d/dz and ϕ commute;
(3) R has no (non-trivial) ideals that are invariant under both ϕ and d/dz;
(4) $\{y\in R\mid y'=0\}=\mathbf{C}$;
(5) $R_\partial,\tilde{R}_\tau\subset R$;
(6) R is minimal with respect to properties 1–5.

For the construction of R we will use the symbols Γ^λ $(\lambda\in\mathbf{Q})$, $e(q)$ $(q\in\mathcal{Q})$, Z^a $(a\in V)$, and l. We define

$$R:=\hat{\mathcal{P}}[\{\Gamma^\lambda\}_{\lambda\in\mathbf{Q}},\{e(q)\}_{q\in\mathcal{Q}},\{Z^a\}_{a\in V},l]$$

with relations

$$\begin{aligned}&\Gamma^{\lambda_1}\Gamma^{\lambda_2}=\Gamma^{\lambda_1+\lambda_2},\quad\forall\lambda_1,\lambda_2\in\mathbf{Q};\\&e(q_1)e(q_2)=e(q_1+q_2),\quad\forall q_1,q_2\in\mathcal{Q};\\&Z^aZ^b=Z^{a+b},\quad\forall a,b\in V.\end{aligned}\tag{3.1}$$

The action of ϕ on Γ^λ, Z^a and l will be as above. For $q\in\mathcal{Q}$ we define

$$\phi(e(q)):=\exp(\phi(q)^-)e(\phi(q)^+),$$

if $\phi(q)=\phi(q)^-+\phi(q)^+$, $\phi(q)^-\in\cup_{p\in\mathbf{N}}\mathbf{C}\{z^{-1/p}\}$, $\phi(q)^+\in\mathcal{Q}$. Since $\phi(q)^+=q$ if $q\in\mathcal{Q}_{\le 1}$, this definition does indeed extend the previously defined operation of ϕ on $e(q)$ with $q\in\mathcal{Q}_{\le 1}$. Hence, $\tilde{R}_\tau\subset R$.

The action of $'=$d/dz is given by $e(q)'=q'e(q)$, $(Z^a)'=az^{-1}Z^a$ and $l'=z^{-1}$ (compare Section 2.3). This already implies $R_\partial\subset R$.

We further define $(\Gamma^\lambda)' = \lambda\Gamma^\lambda A$, for some $A \in R_\partial$ that we still have to determine.

To determine A we use the condition on commutativity of $\mathrm{d}/\mathrm{d}z$ and ϕ. Posing $(\phi(\Gamma^\lambda))' = \phi((\Gamma^\lambda)')$ leads to the equation $\phi(A) = A + z^{-1}$. Writing $A = \hat{h} + l$ we see that $\hat{h}$ should satisfy the equation $\phi(y) - y = z^{-1} - \log(1 + z^{-1})$. We define $\hat{h}$ to be the unique solution in $z^{-1}\mathbf{C}[[z^{-1}]]$ of this equation. Thus, we obtain

$$\Gamma' = (\hat{h} + l)\Gamma. \tag{3.2}$$

Remark $\hat{h} + l$ is in fact the asymptotic expansion of the logarithmic derivative of the Gamma-function.

The actions of ϕ and $'$ on the elements $e(q)$, Z^a and l also commute, and it follows that R satisfies conditions 1, 2 and 5.

LEMMA 21 *Let I be a non-zero,* $\mathrm{d}/\mathrm{d}z$*-invariant ideal of R. Then $I = R$.*

Proof The proof is based on the fact that R_∂ has only trivial $\mathrm{d}/\mathrm{d}z$-invariant ideals (cf. [HvdP95]).

Any $f \in I, f \neq 0$, can be uniquely written as

$$f = \sum f_\lambda \Gamma^\lambda, \quad f_\lambda \in \hat{\mathcal{P}}[\{e(q)\}_{q\in Q}, \{Z^a\}_{a\in V}, l].$$

Let now $S \subset \mathbf{Q}$ be a minimal set satisfying

$$\text{(i) } 0 \in S \quad \text{and} \quad \text{(ii) } \exists f \in I,\ f \neq 0, \quad \text{such that } f = \sum_{\lambda\in S} f_\lambda \Gamma^\lambda.$$

Then $J := \{v \in \hat{\mathcal{P}}[\{e(q)\}_{q\in Q}, \{Z^a\}_{a\in V}, l] \mid \exists f = \sum_{\lambda\in S} f_\lambda\Gamma^\lambda \in I$ with $f_0 = v\}$ is a $\mathrm{d}/\mathrm{d}z$-ideal in R_∂. Because of the minimality condition we posed on $S, J \neq 0$, hence, by [HvdP95], $J = R_\partial$. In particular, $1 \in J$, and there exists $f = 1 + \sum_{\lambda\in S\setminus\{0\}} f_\lambda\Gamma^\lambda \in I$.

If $S = \{0\}$, then $1 \in I$ and we are done.

If $S \neq \{0\}$, the fact that I is a $\mathrm{d}/\mathrm{d}z$-invariant ideal implies that

$$f' = \sum_{\lambda\in S\setminus\{0\}} (f'_\lambda + \lambda f_\lambda(\hat{h} + l))\Gamma^\lambda \in I.$$

As S was chosen minimal, we must have $f' = 0$, hence, $f'_\lambda + \lambda f_\lambda(\hat{h} + l) = 0$, $\forall \lambda \in S \backslash \{0\}$. Clearly, the assumption $f_\lambda \neq 0$, for some $\lambda \in S \backslash \{0\}$, implies the existence of a $g \in R_\partial$ such that $g'/g = l$. However, such a g does not exist. Indeed, $\left(\sum_{n=0}^N g_n l^n\right)' = \sum_{n=0}^N g'_n l^n + n g_n z^{-1} l^{n-1} \neq \sum_{n=0}^N g_n l^{n+1}$, if $g_N \neq 0$. Thus, it follows that, necessarily, $f_\lambda = 0$, $\forall \lambda \in S \backslash \{0\}$, which is in contradiction with $S \neq \{0\}$. □

LEMMA 22 *$f \in R, f' = 0$ implies $f \in \mathbf{C}$.*

Proof Let $f = \sum f_\lambda \Gamma^\lambda$, $f_\lambda \in R_\partial$. Then $f' = \sum (f'_\lambda + \lambda f_\lambda(\hat{h} + l))\Gamma^\lambda$, and $f' = 0$ implies $f_\lambda = 0$, $\forall \lambda \neq 0$, as we know from the proof of the previous lemma. For $\lambda = 0$ we obtain the condition $f'_0 = 0$. From [HvdP95], it follows that $f_0 \in \mathbf{C}$. □

We conclude that the ring R satisfies the conditions 1–6. R becomes a left $\hat{A}$-module by $\partial(f) := \mathrm{d}/\mathrm{d}z(f)$ and $\tau(f) := \phi(f), f \in R$.

Remark 23 By the definition of the ring R, the symbols Γ^λ, $e(q)$, Z^a, l only have the natural relations (3.1). One might ask the questions whether the same holds for the actual functions $\Gamma(z)^\lambda$, $\exp(q(z))$, z^a, $\log(z)$ on a given sector S at ∞. We have to make this question more precise, since the elements of $\hat{\mathcal{P}}$, belonging to R, have no obvious lift to functions on S.

For convenience we will suppose that the sector S does not contain the negative real axis, and we fix a branch of the logarithm $\log(z)$. On the sector S we consider the exact sequence

$$0 \to \mathcal{A}^0(S) \to \mathcal{A}(S) \xrightarrow{J} \hat{\mathcal{P}} \to 0,$$

where $\mathcal{A}(S)$ denotes the set of holomorphic functions on a neighbourhood of ∞ in S, having an asymptotic expansion in $\hat{\mathcal{P}}$, and $\mathcal{A}^0(S)$ consists of the functions of $\mathcal{A}(S)$ which have asymptotic expansion 0.

We consider the following three rings:

- $\mathcal{A}(S)[\{\Gamma^\lambda\}_{\lambda \in \mathbf{Q}}, \{e(q)\}_{q \in Q}, \{Z^a\}_{a \in V}, l]$, defined by the same relations (3.1). We define an element $H \in \mathcal{A}(S)$ by

$$H(z) := \frac{\Gamma'(z)}{\Gamma(z)} - \log(z) \quad \left(' = \frac{\mathrm{d}}{\mathrm{d}z}\right).$$

($H(z)$ has $\hat{h}$ (cf. (3.2)) as asymptotic expansion at ∞ on S.) We define $d/dz(\Gamma^\lambda) = \lambda(H+l)\Gamma^\lambda$. The action of d/dz on the other elements of the ring, and the action of ϕ on the ring is obvious.

- $\mathcal{A}(S)[\{\Gamma(z)^\lambda\}_{\lambda\in\mathbf{Q}}, \{\exp(q(z))\}_{q(z)\in\mathcal{Q}}, \{Z^a\}_{a\in V}, l]$, a subring of the ring of holomorphic functions on a neighbourhood of ∞ in S. The ring is invariant under the action of d/dz and ϕ.
- $R = \hat{\mathcal{P}}[\{\Gamma^\lambda\}_{\lambda\in\mathbf{Q}}, \{e(q)\}_{q\in\mathcal{Q}}, \{Z^a\}_{a\in V}, l]$.

There is a natural diagram of differential–difference rings:

$$\begin{array}{ccc} \mathcal{A}(S)[\{\Gamma^\lambda\}_{\lambda\in\mathbf{Q}}, \{e(q)\}_{q\in\mathcal{Q}}, \{Z^a\}_{a\in V}, l] & & \\ \Big\downarrow \beta & \overset{\alpha}{\searrow} & \\ & & \mathcal{A}(S)[\{\Gamma(z)^\lambda\}_{\lambda\in\mathbf{Q}}, \{\exp(q(z))\}_{q(z)\in\mathcal{Q}}, \{z^a\}_{a\in V}, l] \\ & \overset{\gamma ?}{\swarrow} & \\ R = \hat{\mathcal{P}}[\{\Gamma^\lambda\}_{\lambda\in\mathbf{Q}}, \{e(q)\}_{q\in\mathcal{Q}}, \{Z^a\}_{a\in V}, l] & & \end{array}$$

α is defined by the identity on $\mathcal{A}(S)$ and by $\Gamma^\lambda \mapsto \Gamma(z)^\lambda$, $e(q) \mapsto \exp(q(z))$, $Z^a \mapsto z^a$ and $l \mapsto \log(z)$; β is the extension of $\mathcal{A}(S) \xrightarrow{J} \hat{\mathcal{P}}$ given by $\Gamma^\lambda \mapsto \Gamma^\lambda$, $e(q) \mapsto e(q)$, $Z^a \mapsto Z^a$, $l \mapsto l$. The mappings α and β are homomorphisms which commute with the respective actions of d/dz and ϕ on the three rings. Obviously, α and β are surjective, and the kernel of β is easily seen to be $\mathcal{A}^0(S)[\{\Gamma^\lambda\}_{\lambda\in\mathbf{Q}}, \{e(q)\}_{q\in\mathcal{Q}}, \{Z^a\}_{a\in V}, l]$. It is more difficult to see that α has a non-zero kernel. This kernel depends on the sector S. Examples of elements in $\ker(\alpha)$:

- $e(-z^{1/2}) - \exp(-z^{1/2})$, for any S, since $\exp(-z^{1/2}) \in \mathcal{A}^0(S)$ for any S.
- $\Gamma - \Gamma(z)$ if S is contained in the left-half plane, since the Gamma-function is flat (i.e., has asymptotic expansion 0) on such a sector S (recall that S does not contain the negative real axis).

We want to show that there is a differential homomorphism

$$\gamma : \mathcal{A}(S)[\{\Gamma(z)^\lambda\}_{\lambda\in\mathbf{Q}}, \{\exp(q(z))\}_{q(z)\in\mathcal{Q}}, \{z^a\}_{a\in V}, l] \to R,$$

such that $\gamma\alpha = \beta$. We note that γ is unique and exists if and only if $\beta(\ker(\alpha)) = 0$. Now $\ker(\alpha)$ is a differential ideal. Since β is surjective,

$\beta(\ker(\alpha))$ also is a differential ideal. Furthermore, $1 \notin \beta(\ker(\alpha))$. By Lemma 21, it follows that $\beta(\ker(\alpha)) = 0$. This proves the existence of γ.

An answer to our question at the beginning of this remark is: there are new relations between $\Gamma(z)^\lambda$, $\exp(q(z))$, z^a, $\log(z)$, and $\mathcal{A}(S)$ on S, depending on S itself. All those relations arise from the fact that some expressions, like $\Gamma(z)$, $\exp(-z^{1/2})$, can be flat on S.

3.3 Determining Solutions in R

Let $D \in \hat{\mathcal{P}}[\partial]$. We have seen in Section 2.3 that we can determine a fundamental system of solutions in R_∂ of $D(y) = 0$ with help of the Newton polygon $\mathcal{N}(D)$, and the characteristics of D.

Again, there is a similar result for difference equations $T(y) = 0$, $T \in \hat{\mathcal{P}}[\tau^{-1}, \tau]$. However, in this case we need two Newton polygons. These Newton polygons have been introduced by Ecalle (cf. [Eca85]).

The *small Newton polygon* $\mathcal{N}_p(T)$ is by definition the Newton polygon of T considered as infinite order differential operator (recall Proposition 18). This polygon is needed to find solutions of $T(y) = 0$ in $\hat{\mathcal{P}}[\{e(q)\}_{q \in \mathbb{Q}_{\le 1}}, \{Z^a\}_{a \in V}, l]$.

The *large Newton polygon* $\mathcal{N}_g(T)$ is defined as follows. Suppose $T = \sum_{j=0}^J a_j \tau^j$, $a_0, a_J \neq 0$. (We may suppose that T does not contain negative powers of τ, since τ acts bijectively on R.) We define $\mathcal{M}_g^+(T)$ to be the convex hull of $\cup_{j=0}^J Q_2(j, v(a_j))$, and $\mathcal{M}_g^-(T)$ to be the convex hull of $\cup_{j=0}^J Q_1(j, v(a_j))$, where $Q_i(a,b)$ denotes the ith quadrant translated over (a,b). By definition $\mathcal{N}_g(T)$ is the boundary of the intersection of $\mathcal{M}_g^+(T)$ and $\mathcal{M}_g^-(T)$. The equation $T(y) = 0$ has a solution $y = u\Gamma^\lambda$, for some $u \in \hat{\mathcal{P}}[\{e(q)\}_{q \in \mathbb{Q}_{\le 1}}, \{Z^a\}_{a \in V}, l]$ if and only if $\mathcal{N}_g(T)$ has a slope λ.

Let us now look for symbolic solutions in R of differential–difference equations $L(y) = 0$, $L \in \hat{A}$. (This can be done since R is a left $\hat{A}$-module.)

Let $p \in \mathbf{N}$, $L \in \hat{A}_p = \hat{\mathcal{P}}_p[\tau^{-1}, \tau, \partial]$. Since τ acts bijectively on R we may assume that $L = \sum_{j=0}^J a_j(\partial)\tau^j$, $a_j(\partial) \in \hat{\mathcal{P}}_p[\partial]$, $a_0(\partial) \cdot a_J(\partial) \neq 0$. By $\mathrm{im}(L)$ we denote the image of L in $\hat{\mathcal{D}}_p$ under the embedding of Proposition 18. By definition the Newton polygon of L is that of $\mathrm{im}(L)$.

In the case that $J = 0$ we deal with a finite order differential operator. Since R contains R_∂ as subring, R contains a full set of solutions of $L(y)=0$, that we can determine with help of the Newton polygon of L and its characteristics (see Propositions 16 and 17).

Now suppose that $J>0$. From $\text{im}(\partial^i\tau^j)=\sum_{n\geq 0}(j^n/n!)\partial^{i+n}$ we see that the embedding of Proposition 18 in fact maps $\hat{A}_p$ into $\hat{\mathcal{E}}_{p,1}$, and that, in particular, $\text{im}(L)\in\hat{\mathcal{E}}_{p,1}\backslash\hat{\mathcal{D}}_{p,1}$. Moreover, $\text{im}(L)=\sum_{h\gg H} z^{-h/p} A_h(\partial)$, for some $H\in\mathbf{Z}$, $A_H\not\equiv 0$ and with the $A_h(x)$ entire functions. So, even though L acts on the whole of R, $\text{im}(L)$ only acts on $R_{\partial,\leq 1}$ by Lemma 13. Hence, $\text{im}(L)$ only helps us to find the solutions of $L(y)=0$ that lie in $R_{\partial,\leq 1}$ (considered as subring of R).

$A_H(\partial)$ is the image in $\mathbf{C}[[\partial]]$ of the 1-normalization of $\text{im}(L)$, and therefore $F_{\text{im}(L),1}(\xi)=A_H(\xi)$. So, the series $F_{\text{im}(L),1}(\xi)$ defines an entire function, which we denote by $F_{L,1}(\xi)$. It is not difficult to prove that $F_{L,1}(\xi)=\sum_{i,j}c_{ij}\xi^i e^{j\xi}$, if $\bar{L}=\sum_{i,j}c_{ij}\partial^i\tau^j$ is the image of the normalization of L in $\mathbf{C}[\tau^{-1},\tau,\partial]$ (see Sub-section 3.1); by definition, $F_{L,1}(\xi)$ is the 1-characteristic of differential–difference operator L. We have an analogue of Proposition 17:

Proposition 24 *Let $L\in\hat{A}$, and let $F_{L,1}(\xi)$ be its* 1-*characteristic. Let $c\in\mathbf{C}^*$ be a zero of $F_{L,1}$. Define $\tilde{L}:=e(-cz)Le(cz)$. Then $\tilde{L}\in\hat{A}$ and $\mathcal{N}(\tilde{L})$ has a slope smaller than* 1.

The actions of $\text{im}(L)$ on $R_{\partial,\leq 1}$, and of L on the same ring considered as subring of R, are compatible. With the help of Propositions 24, 16 and 17 *we can determine all solutions of $L(y)=0$, $L\in\hat{A}$, which lie in $R_{\partial,\leq 1}$.*

We do not know how to generalize the large polygon for difference operators to general operators $L\in\hat{A}$, and we do not know how to systematically find solutions of $L(y)=0$ in R, that are not in $R_{\partial,\leq 1}$, if L is a true differential–difference operator.

However, we can get some information on the structure of a general solution of $L(y)=0$, $L\in\hat{A}$, by writing R as direct sum of $\hat{A}$-invariant components.

Put $R_0:=R_{\partial,\leq 0}=\hat{\mathcal{P}}[\{Z^a\}_{a\in V}, l]$. Then $R_0\Gamma^\lambda$, $\lambda\in\mathbf{Q}$, is $\hat{A}$-invariant, and so is $R_0 e(q)$, for $q\in\mathcal{Q}_{\leq 1}$, since $\phi(q)^+=q$ in that case. If $q\in\mathcal{Q}$, $\deg_z(q)>1$, then $\phi^m(q)^+\neq q$, $\forall m\in\mathbf{Z}\backslash\{0\}$. For $q\in\mathcal{Q}$, $\deg_z(q)>1$, we define $S_q:=\{(\phi^m(q))^+\mid m\in\mathbf{Z}\}$. Note that for all $h\in\mathbf{Z}$, $S_{(\phi^h(q))^+}=S_q$, since $q=(\phi^{-h}(\tilde{q}))^+$ if $\tilde{q}=(\phi^h(q))^+$. Thus we have an equivalence relation $\sim$ on $\{q\in\mathcal{Q}\mid\deg_z(q)>1\}$ given by $q_1\sim q_2$ if $S_{q_1}=S_{q_2}$. Let S be a set of representatives of $\{q\in\mathcal{Q}\mid\deg_z(q)>1\}/\sim$. Then

$\oplus_{m\in\mathbf{Z}} R_0 e(\phi^m(q)^+)$ is invariant under $\hat{A}$, for all $q \in S$. Thus, we have the following decomposition of R into $\hat{A}$-invariant components:

$$R = \left(\bigoplus_{\lambda\in\mathbf{Q}, q\in\mathcal{Q}_{\leq 1}} R_0 \mathrm{e}(q)\Gamma^\lambda \right) \bigoplus \left(\bigoplus_{\lambda\in\mathbf{Q}, q\in S} \bigoplus_{m\in\mathbf{Z}} R_0 \mathrm{e}(\phi^m(q)^+)\Gamma^\lambda \right).$$

From this decomposition the next proposition immediately follows.

PROPOSITION 25 *Let $L \in \hat{A}$. Any solution $y \in R$ of $L(y)=0$ can be written as a finite sum of solutions $y_{\lambda,q} e(q)\Gamma^\lambda$, with $\lambda \in \mathbf{Q}$, $q \in \mathcal{Q}_{\leq 1}$ and $y_{\lambda,q} \in R_0$, and solutions $\sum_{m\in\mathbf{Z}} y_{\lambda,q,m} e(\phi^m(q)^+)\Gamma^\lambda$, where the sum is finite and $\lambda \in \mathbf{Q}$, $q \in S$ and $y_{\lambda,q,m} \in R_0$.*

In fact we have the following, stronger result.

THEOREM 26 *Let $L \in \hat{A}$, and let $L = \sum_{j=j_0}^{J} D_j \tau^j = \sum_{i=0}^{I} T_i \partial^i$, where $D_j \in \hat{\mathcal{P}}[\partial]$, $j = j_0, \ldots, J$, and $T_i \in \hat{\mathcal{P}}[\tau^{-1}, \tau]$, $i = 0, \ldots, I$.*

(1) *Let $\lambda \in \mathbf{Q}^*$. In order for $L(y)=0$ to have a solution in $R_0[\{e(q)\}_{q\in\mathcal{Q}_{\leq 1}}]\Gamma^\lambda$, it is necessary (but not sufficient) that $\mathcal{N}_g(T_I)$ has a slope λ.*
(2) *Let $\lambda \in \mathbf{Q}^*$ and $q \in S$. In order for $L(y)=0$ to have a solution in $\bigoplus_{m\in\mathbf{Z}} R_0 e(\phi^m(q)^+)\Gamma^\lambda$, it is necessary (but not sufficient) that $\lambda = 0$ and that $\mathcal{N}(D_J)$ has a slope* $\deg_z(q)$.

Proof (1) Let $y \in R_0[\{e(q)\}_{q\in\mathcal{Q}_{\leq 1}}]\Gamma^\lambda$ for some $\lambda \in \mathbf{Q}*$. Note that we can write

$$y = \sum_{h=0}^{H} y_h l^h \Gamma^\lambda, \quad \text{with } y_h \in \hat{\mathcal{P}}[\{e(q)\}_{q\in\mathcal{Q}_{\leq 1}}, \{Z^a\}_{a\in V}].$$

Now recall that $\Gamma' = (\hat{h}(z) + l)\Gamma$, so

$$(\mathrm{d}/\mathrm{d}z)^i(\Gamma^\lambda l^h) = \Gamma^\lambda \left(\lambda^i l^{h+i} + \sum_{k=0}^{h+i-1} f_k l^k \right), \quad \text{for some } f_k \in z^{-1}\mathbf{C}[[z^{-1}]]. \tag{3.3}$$

With this equality and $\phi(l) = l + \log(1+z^{-1})$ we obtain

$$L(y) = \sum_{k=0}^{H+I} v_k l^k,$$

with $v_k \in \hat{\mathcal{P}}[\{e(q)\}_{q\in Q}, \{Z^a\}_{a\in V}]\Gamma^\lambda$, and in particular $v_{H+I} = T_I(\lambda^I\Gamma^\lambda y_H)$.

We have $L(y)=0$ if and only if $v_k=0$, for $k=0,\dots,H+I$. In particular, it is necessary that $v_{H+I} = T_I(\lambda^I\Gamma^\lambda y_H)=0$, and this is only possible if $\mathcal{N}_g(T_I)$ contains a slope λ.

(2) Suppose $y = \sum_{m\in\mathbf{Z}} y_m e(\phi^m(q)^+)\Gamma^\lambda$ for sime $\lambda \in \mathbf{Q}$ and some $q \in S$, and with $y_m \in R_0$. Write $q_m := \phi^m(q)^+$. Then

$$L(y) = \sum_m \sum_j D_j(e(\phi^j(q_m)^+)\exp(\phi^j(q_m)^-)\phi^j(y_m\Gamma^\lambda))$$
$$= \sum_{j,m} \tilde{y}_{j,m} e(\phi^j(q_m)^+)\Gamma^\lambda,$$

for some $\tilde{y}_{j,m} \in R_0$. The second equality is a consequence of $R_0 e(q)\Gamma^\lambda$ being ∂-invariant for all $q \in Q$ and all $\lambda \in \mathbf{Q}$. (As remarked before, it is not τ-invariant if $\deg_z(q) > 1$.)

Suppose $L(y)=0$. Since $\phi^j(q_m)^+ = \phi^{j+m}(q)^+$, we obtain that

$$\sum_{j+m=const.} D_j(e(\phi^{j+m}(q)^+)\exp(\phi^j(q_m)^-)\phi^j(y_m\Gamma^\lambda)) = 0.$$

Taking $j=J$ and $m=M$ maximal, we get in particular

$$D_J(e(\phi^{J+M}(q)^+)\exp(\phi^J(q_M)^-)\phi^J(y_M\Gamma^\lambda)) = 0. \tag{3.4}$$

We must have $\lambda=0$, because (3.3). Then we are left with the action of D_J on an element of $R_0 e(\phi^{J+M}(q)^+)$, and it follows that the equation can only be fulfilled if $\mathcal{N}(D_J)$ has a slope $\deg_z(\phi^{J+M}(q)^+) = \deg_z(q)$. □

COROLLARY 27 *Let* $L = a_{0,0}(z) + \sum_{i=1}^I a_{i,0}(z)\partial^i + \sum_{j=1}^J a_{0,j}(z)\tau^j$, $I, J \geq 1$, $a_{I,0}, a_{0,J} \not\equiv 0$. *Then we have*

$$y \in R, \quad L(y) = 0 \Rightarrow y \in R_0\left[\{e(q)\}_{q\in Q_{\leq 1}}\right].$$

3.4 Examples

Example 28 $L = z^{-1} + \partial + \tau$ satisfies the conditions of Corollary 27, so all solutions in R of $L(y)=0$ can be found in the subring

$R_0[\{e(q)\}_{q\in\mathcal{Q}_{\leq 1}}]$. As $\phi(q)^+ = q$ if $q \in \mathcal{Q}_{\leq 1}$, we see that all solutions in $R_0[\{e(q)\}_{q\in\mathcal{Q}_{\leq 1}}]$ are finite linear combinations (over **C**) of solutions of the form $e(q)y$, with $y \in \hat{\mathcal{P}}[\{Z^a\}_{a\in\mathbf{C}}, l]$ and $q \in \mathcal{Q}_{\leq 1}$. These solutions can be found with help of the Newton polygon and the characteristics. The Newton polygon of L only has one edge of slope 1, and the 1-characteristic is $F_{L,1}(\xi) = \xi e^{\xi}$. If μ is a non zero $F_{L,1}(\xi)$, then $\tilde{L} := e(-\mu z)Le(\mu z) = z^{-1} + (1+e^{\mu})\partial + e^{\mu}\sum_{n\geq 2}\partial^n/n!$. The Newton polygon of $\tilde{L}$ has a horizontal edge of length 1, and an infinite edge of slope 1. We have $F_{\tilde{L},0}(\xi) = 1 + (1+e^{\mu})\xi$. Thus, we find infinitely many solutions

$$y_{\mu}(z) = e^{\mu z} z^{\alpha_{\mu}} \hat{f}_{\mu}(z),$$

with μ a zero of $F_{L,1}(\xi)$, $\alpha_{\mu} = -1/(1+e^{\mu})$, and $\hat{f}_{\mu}(z) \in \mathbf{C}[[z^{-1}]]$.

In [Fab] it is shown, that the formal series $\hat{f}_{\mu}(z)$ is 1-summable in all, but countably many directions. This yields us analytic solutions of $L(y) = 0$ with a certain asymptotic behaviour as $z \to \infty$.

Example 29 The solutions in R of $L(y) = 0$ with $L = z^{-1}\partial + \tau$ all lie in the subring $R_0[\{e(q)\}_{q\in\mathcal{Q}_{\leq 1}}]$. Again $\mathcal{N}(L)$ only has a slope 1, but now $F_{L,1}(\xi) = e^{\xi}$ has no zeros, and thus the equation has no solutions in R.

Example 30 The equation $L(y) = 0$ with $L = \partial + z^{-1}\tau$ has exactly one solution in R, and this solution belongs to $z^{-1}\mathbf{C}[[z^{-1}]]$. In [Fab] it is shown that this solution is of Gevrey-order 1, and that it is "1-summable" in directions in the right-half plane.

Example 31 Consider $L = (\frac{1}{2}z^{-1}\partial - 1) + (z^{-2}\partial^2 - (z^{-1} + 3z^{-2})\partial - 2)\tau + (z^{-1}\partial - (2 + 4z^{-1}))\tau^2$. Then the equation $L(y) = 0$ has the solution $e(z^2)$ in R. Indeed, the Newton polygons of the differential operators which are the coefficients of $\tau^j, j = 0, 1, 2$ all have a slope 2.

4 SYSTEMS OF DIFFERENTIAL–DIFFERENCE EQUATIONS

A *system of differential–difference equations* is a set of equations $L_1 y = 0, \ldots, L_r y = 0$ with $L_1, \ldots, L_r \in A$ or $\hat{A}$. One can consider the left ideal I in A or $\hat{A}$ generated by the L_i. The system of differential–difference equations is then equivalent to $Ly = 0$ for all $L \in I$.

We are interested in left ideals I of A or $\hat{A}$ such that A/I or $\hat{A}/I$ is finite dimensional. *In that case we expect that the system has a finite dimensional space over* $\mathbf{C}$, *of the same dimension as* A/I or $\hat{A}/I$, *of "symbolic solutions" of the system.*

We will consider more generally left modules M over A or $\hat{A}$ which are finite dimensional over $\mathcal{P}$ or $\hat{\mathcal{P}}$. In the first case M will be called a convergent (τ, ∂)-module and in the second case M will be called a formal (τ, ∂)-module. Thus, a convergent (resp. formal) (τ, ∂)-module M can also be described as

- M is a finite dimensional vector space over $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$).
- $\tau = \tau_M$ is a bijective, additive operator on M such that $\tau(fm) = \phi(f)\,\tau(m)$ for all f in $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) and $m \in M$.
- $\partial = \partial_M$ is an additive operator on M such that $\partial(fm) = f'm + f\partial(m)$ for f in $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) and $m \in M$.
- $\tau\partial = \partial\tau$.

If we choose a basis of M over $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) the above situation corresponds with matrix equations:

- $(\mathrm{d}/\mathrm{d}z)y = Ay$ and $\phi y = By$, where A and B are matrices with coefficient in $\mathcal{P}$ (or $\hat{\mathcal{P}}$), B invertible, and where
- $\phi(A)B = B' + BA$ (where $' = \mathrm{d}/\mathrm{d}z$), as a result of the condition that τ and ∂ commute.

Suppose M has dimension n. The definition of a (τ, ∂)-module assures that M is in particular a ∂-module, and thus, contains cyclic vectors. Let v be such a cyclic vector for M as ∂-module, i.e. $\{v, \partial v, \ldots, \partial^{n-1} v\}$ is a basis of M over $\mathcal{P}$ (or $\hat{\mathcal{P}}$). Then there exist $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$ in $\mathcal{P}$ (or $\hat{\mathcal{P}}$), such that

$$\partial^n v = -a_{n-1}\partial^{n-1} v - \cdots - a_0 v, \; \tau v = -b_{n-1}\partial^{n-1} v - \cdots - b_1 \partial v - b_0 v.$$

Define $L_1 := \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0$ and $L_2 := \tau + b_{n-1}\partial^{n-1} + \cdots + b_0$. Let $L \in A$ (resp. $\hat{A}$) be such that $Lv = 0$. We can write $L = \tau^m \sum_{j=0}^{J} c_j(\partial)\tau^j$, with $c_j(\partial) \in \mathcal{P}[\partial]$ (resp. $\hat{\mathcal{P}}[\partial]$). By substituting $\tau = L_2 - b_{n-1}\partial^{n-1} - \cdots - b_0$, we obtain $L = FL_2 + \tau^m \sum_{i=0}^{I} d_i \partial^i$, $F \in A$ (or $\hat{A}$) and $d_i \in \mathcal{P}$ (or $\hat{\mathcal{P}}$). By utilizing the expression for L_1 this can be further reduced to $L = FL_2 + GL_1 + \tau^m(g_{n-1}\partial^{n-1} + \cdots + g_0)$, $G \in A$ (resp. $\hat{A}$) and $g_i \in \mathcal{P}$ (resp. $\hat{\mathcal{P}}$). Since τ^m is an automorphism, the assumption that $Lv=0$ implies that $g_{n-1}\equiv\cdots\equiv g_0\equiv 0$. Hence, $L\in AL_1+AL_2$ (resp. $\hat{A}L_1+\hat{A}L_2$), and we have

$$M\cong A/(AL_1+AL_2)\quad(\text{resp. } \hat{A}/(\hat{A}L_1+\hat{A}L_2)).$$

Similarly, the definition of a (τ,∂)-module implies that M is a difference module. Hence, M has a basis $\{w,\tau w,\ldots,\tau^{n-1}w\}$, and by proceeding as above we obtain

$$M\cong A/(A\tilde{L}_1+A\tilde{L}_2)\quad(\text{resp. } \hat{A}/(\hat{A}\tilde{L}_1+\hat{A}\tilde{L}_2)),$$

$$\tilde{L}_1=\tau^n+\alpha_{n-1}\tau^{n-1}+\cdots+\alpha_0,\quad \tilde{L}_2=\partial+\beta_{n-1}\tau^{n-1}+\cdots+\beta_0.$$

Remark 32 Let $p\in\mathbf{N}$. We have already seen that the rings A_p and $\hat{A}_p$ have only trivial two-sided ideals. For the second ring one can easily proof that every left ideal is finitely generated. The idea of the proof is the following.

Recall that $\hat{A}_p$ has a subring $\hat{A}_p^+ := \cup_{p\geq 1}\mathbf{C}[[z^{-1/p}]][\tau^{-1},\tau,\partial]$, and that $z^{-1/p}\hat{A}_p^+$ is a two-sided ideal in $\hat{A}_p^+$ such that the factor ring $\hat{A}_p^+/z^{-1/p}\hat{A}_p^+$ is isomorphic to the two-dimensional commutative noetherean ring $\mathbf{C}[\tau^{-1},\tau,\partial]$. Let I be a left ideal of $\hat{A}_p$ and write J for the intersection $I\cap\hat{A}_p^+$. Let $L_1,\ldots,L_r\in J$ be chosen such that their images $\mathbf{C}[\tau^{-1},\tau,\partial]$ generate the image of J in this ring (which is an ideal). Then we claim that $L_1,\ldots,L_r$ generate the ideal J in $\hat{A}_p^+$ and thus, also the ideal I in $\hat{A}_p^+$.

Indeed, any $F\in J$ can be written as $F=\sum_{i-1}^r c_{i,0}L_i+z^{-1/p}F_1$ with $F_1\in J$. Again $F_1=\sum_{i-1}^r c_{i,1}L_i+z^{-1/p}F_2$, etc. All the coefficients $c_{i,m}$ are supposed to lie in $\hat{A}_p^+$. Taking infinite sums one finds

$$F=\sum_{i=1}^r\left(\sum_{m\geq 0}c_{i,m}z^{-m/p}\right)L_i.$$

This proves the claim. We further remark that this method does not seem to work for A_p and that we do not know whether every left ideal of A_p is finitely generated.

In the remaining part of this section we will first investigate the structure of the (τ,∂)-modules. A subclass of those modules will be called *realizable*. The category of the realizable (convergent or formal)

(τ, ∂)-modules is shown to be equivalent with the category of *mild* differential modules (convergent or formal). We will end this section with remarks on the asymptotics of realizable modules.

4.1 Classification of (τ, ∂)-Modules

We adopt here the terminology of [vdPS97]. A differential module M over $\hat{\mathcal{P}}$ (or $\mathcal{P}$) will be called *mild* if there is a basis $e_1, \ldots, e_m$ such that the $\mathbf{C}[[z^{-1}]]$-module $M_0 = \mathbf{C}[[z^{-1}]]e_1 + \cdots \mathbf{C}[[z^{-1}]]e_m$ is invariant under the action of ∂. This condition can be translated in various ways:

- The set of 'eigenvalues' for ∂ on M is contained in $\{\sum_\lambda a_\lambda z^\lambda | \lambda \in \mathbf{Q} \text{ and } -1 \leq \lambda \leq 0\}$.
- The differential module corresponds to a differential equation in matrix form $y' = Ay$ such that the coefficients of A belong to $\mathbf{C}[[z^{-1}]]$.
- Via cyclic vectors the module corresponds with scalar differential equations, which have a Newton polygon with only (finite) slopes smaller than or equal to 1.

Let a mild differential module M over $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) be given. Then one defines the operator τ_0 by the formula $\tau_0 = \sum_{m \geq 0} (\partial^m / m!)$. It has been verified in [vdPS97, Section 10], that this expression converges.

It is further easily verified that M, equipped with the operators τ_0, ∂, is a (τ, ∂)-module. In the sequel we will investigate whether every (τ, ∂)-module has this form.

Let M be a (τ, ∂)-module over $\mathcal{P}$ or $\hat{\mathcal{P}}$.

We consider M as ∂-module over $\hat{\mathcal{P}}$ and can then apply the classification of such modules. The result can be stated as

$$M = \oplus_q M_q \text{ (a finite direct sum)},$$
$$q \in \mathcal{Q} = \cup_{p \geq 1} z^{1/p} \mathbf{C}[z^{1/p}], \ M_q = E(q) \otimes \tilde{M}_q.$$

$E(q)$ is the one-dimensional module with basis e_q and action $\partial(e_q) = -q' e_q$. $\tilde{M}_q$ is a regular singular module over $\hat{\mathcal{P}}$; in particular one can write $\tilde{M}_q = \hat{\mathcal{P}} \otimes_{\mathbf{C}} V_q$, where V_q is a vector space over $\mathbf{C}$. The action of $\delta = z\partial$ on V_q is given by a $\mathbf{C}$-linear map A_q on V_q and by $z(d/dz)$ on $\hat{\mathcal{P}}$. Thus, in total $\partial(f \otimes v) = f' \otimes v + z^{-1} f \otimes A_q(v)$.

The above decomposition is unique. The operator τ commutes with ∂ and thus, we have $\tau(M_q) = M_{\phi(q)^+}$. Recall that by $\phi(q)^+$ we denote the element $\phi(q)$ truncated such that it belongs to the set $\mathcal{Q}$. Since the decomposition is finite, we must have that the set $\{\phi^j(q)^+ \mid j \in \mathbf{Z}\}$ is finite, for any q with $M_q \neq 0$. An easy calculation shows that this condition on q is equivalent to $q \in \mathcal{Q}_{\leq 1}$. Then every M_q in fact is invariant under τ.

We conclude that the differential module M is mild. Let τ_0 be as above. Then one can write $\tau = C\tau_0$, for some map $C: M \to M$. It follows from the properties of τ, τ_0, ∂ that C is a $\mathcal{P}$-linear (resp. $\hat{\mathcal{P}}$-linear) bijective map such that $\partial C = C\partial$. In other words, C is an automorphism of the differential module M. This brings us to the conclusion:

PROPOSITION 33 *Every (τ, ∂)-module over $\mathcal{P}$ (or $\hat{\mathcal{P}}$) is, seen as a differential module, mild. Moreover $\tau = C\tau_0$, where $\tau_0 = \sum_{m \geq 0} \partial^m / m!$ is the canonical difference operator on M and where C is an automorphism of the differential module M.*

4.2 Symbolic Solutions

Let M be a (τ, ∂)-module over $\mathcal{P}$ (or $\hat{\mathcal{P}}$). We write $\tau = \tau_M$ and $\tau = C\tau_0$, with $\tau_0 = \sum_{m \geq 0} \partial^m / m!$ and C an automorphism of M viewed as differential module. Let S denote a $\hat{\mathcal{P}}$-algebra of symbolic elements. We assume that both d/dz and ϕ act upon S. Then the set of symbolic solutions of M in S is by definition the set of $\mathcal{P}$-linear (or $\hat{\mathcal{P}}$-linear) maps $l: M \to S$, which commute with the actions of τ and ∂. If one translates the (τ, ∂)-module M in terms of the (commuting) matrix equations of order n:

$$y' = Ay \text{ and } \phi(y) = By,$$

then the set of symbolic solutions over S is the set of the $y \in S^n$, satisfying both equations.

We take for S the ring $R^+ := \hat{\mathcal{P}}[\{e(q)\}, \{Z^a\}, l] \cong R_{\partial, \leq 1}$, where the $q \in \mathcal{Q}_{\leq 1}$, and the $a \in \mathbf{C}$, and the actions of ∂ and τ as defined earlier. The set of symbolic solutions for the *differential* module M is a vector space V of the correct dimension n. From the definition of τ_0 it follows that τ_0 acts as the identity on V. Therefore τ acts as an invertible $\mathbf{C}$-linear map, again called C on V. The set of symbolic solutions for the (τ, ∂)-module is therefore equal to $\{V \in V \mid C(v) = v\}$. If C is not the identity, then the dimension of this vector space is strictly smaller than n. In other words there are not enough symbolic solutions. We will give a simple example of this situation.

Example 34 $M = \hat{\mathcal{P}}f$ with $\partial f = af$ and $\tau f = bf$, $a, b \in \mathbf{C}$ and $b \neq 0$. The map τ_0 is easily seen to be $\tau_0(f) = e^a f$. In the case that $\tau = \tau_0$ (i.e., $b = e^a$), there is a symbolic solution $F = e(az)$.

In the case that $b \neq e^a$ we try to find another symbolic solution. Consider first the equation $\phi(F) = bF$. This has many symbolic solutions, namely $F = e^{a_0 z} G(t)$, where a_0 satisfies $e^{a_0} = b$ and, where G is a "function" of $t = \mathrm{e}^{2\pi \mathrm{i} z}$. The equation $F' = aF$ leads to a differential equation for G namely:

$$\frac{G'(t)}{G(t)} = \frac{a - a_0}{2\pi \mathrm{i} t}.$$

A solution to this equation would be $G = t((a - a_0)/2\pi \mathrm{i})$. It is not clear that this makes sense. In particular, one may not give this expression for G the interpretation $e^{(a-a_0)z}$, since the latter is not ϕ-invariant.

Suppose however that a symbolic solution F of the above equations could be defined. Then a lift $\tilde{F}$ of this symbolic solution to a sector at ∞ is a meromorphic function satisfying the equations $\tilde{F}' = a\tilde{F}$ and $\tilde{F}(z+1) = b\tilde{F}(z)$. It is clear that $\tilde{F} \neq 0$ does not exist.

4.3 Conclusions

Let us call a (τ, ∂)-module M *realizable* if $\tau = \tau_0$. The category of the mild differential modules over $\mathcal{P}$ (or $\hat{\mathcal{P}}$) is equivalent to the category of the realizable (τ, ∂)-modules over $\mathcal{P}$ (or over $\hat{\mathcal{P}}$). If the realizable module M is defined over $\mathcal{P}$, then the asymptotic theory for the realizable module M, i.e., the theory of lifting symbolic solutions to solutions on sectors at ∞, is identical with the asymptotic theory for the mild differential module M.

One can make this more precise as follows. Suppose that a symbolic solution $v \in V$ with coordinates in R^+ is given. Let a sector at ∞ be given and let $\tilde{v}$ be a lift of v such that $\tilde{v}$ satisfies the differential equation. Since $\tau_0 = \sum_{m\geq 0} \partial^m/m!$, one finds that $\tau_0(\tilde{v}) = \tilde{v}$. Thus $\tilde{v}$ is also a lift of v for the (τ, ∂)-module.

For non-realizable modules M there seems no possibility to obtain enough symbolic solutions.

4.4 Examples

Example 35 Let M be a two-dimensional (τ, ∂)-module. Suppose we can choose a basis $\{e_1, e_2\}$, such that M corresponds with the equations

$$y' = Ay, \quad A = \begin{pmatrix} a & 0 \\ 0 & z^{-1} \end{pmatrix}, \quad \text{for some } a \in \mathbf{C},$$

and

$$\phi(y) = By, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \quad b_i \in \hat{\mathcal{P}},\ i = 1, \ldots 4,\ b_1b_4 - b_2b_3 \neq 0.$$

Because of the commutativity of ∂ and τ we must have $\phi(A)B = B' + BA$. If we write this out, we get four equations that we can explicitly solve for the b_i's. Under the condition that the b_i's are in $\hat{\mathcal{P}}$ we find

$$b_1 = \alpha, \quad b_2 = b_3 = 0, \quad b_4 = \beta\frac{z+1}{z},$$

for some $\alpha, \beta \in \mathbf{C}^*$.

As $\partial e_1 = -ae_1$, we have $\tau_0 e_1 = e^{-a}e_1$. From $\partial e_2 = -z^{-1}e_2$ easily follows that $\partial^n e_2 = (-1)^n n! z^{-n} e_2$ and thus it follows that $\tau_0 e_2 = (z/(z+1))e_2$. So the module is realizable (i.e. $\tau = \tau_0$), if and only if $\alpha = e^a$, $\beta = 1$. (Recall that one gets the system of difference equations $\phi(y) = By$, from solving $\tau(y) = y$ with $\tau(y) = B^{-1}\phi(y)$.)

On the other hand, a $\hat{\mathcal{P}}$-linear operator C on M is a differential automorphism if and only if $\det C \neq 0$ and $C' = AC - CA$ (where we also used C to denote the matrix representation of C with respect to the basis $\{e_1, e_2\}$). Indeed we find (under the condition that C has its entries in $\hat{\mathcal{P}}$)

$$C = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \quad \gamma_1, \gamma_2 \in \mathbf{C}^*.$$

That is, any differential automorphism of M is of the form $B^{-1}B_0$, where B_0 corresponds with τ_0 (that is, $\tau_0(y) = B_0^{-1}\phi(y)$). This corresponds with the fact that $\tau = C\tau_0$, for some ∂-automorphism of M.

Note that $v := e_1 + e_2$ is a cyclic vector for M. We have

$$\partial^2 v = -f_1\partial v - f_2 v, \quad \text{with } f_1 = \frac{a^2z^2}{z(az-1)},\ f_2 = a(f_1 - a),$$

$$\tau v = -g_1\partial v - g_2 v, \quad g_1 = \frac{\beta z(z+1) - \alpha z^2}{\alpha\beta(z+1)(az-1)},\ g_2 = ag_1 - \frac{1}{\alpha}$$

(α and β as above) and we obtain

$$M \cong \hat{A}/\hat{A}L_1 + \hat{A}L_2,$$

with $L_1 = \partial^2 + f_1\partial + f_2$ and $L_2 = \tau + g_1\partial + g_2$.

Since both f_1 and f_2 have valuation equal to zero, the Newton polygon $\mathcal{N}(L_1)$ has only finite slope 1. This shows that M as ∂-module is mild, as should be the case.

Example 36 Let M again be a two-dimensional (τ, ∂)-module, but now suppose we can choose a basis $\{e_1, e_2\}$, such that M corresponds with the equations

$$y' = Ay,\ A = \begin{pmatrix} a & z^{-1} \\ 0 & a \end{pmatrix}, \quad \text{for some } a \in \hat{\mathcal{P}}^+ = \cup_{p\geq 1}\mathbf{C}[[z^{-1/p}]],$$

and

$$\phi(y) = By, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix},\ b_i \in \hat{\mathcal{P}},\ i = 1, \ldots, 4,\ b_1b_4 - b_2b_3 \neq 0.$$

Because of the commutativity of ∂ and τ we must have $\phi(A)B = \mathrm{B}' + BA$. Thus we obtain the relations

$$(\phi(a) - a)b_3 = b_3', \quad (\phi(a) - a)b_1 = b_1' - (z+1)^{-1}b_3,$$
$$(\phi(a) - a)b_4 = b_4' + z^{-1}b_3,$$
$$(\phi(a) - a)b_2 = b_2' + z^{-1}b_1 - (z+1)^{-1}b_4.$$

Note that $a \in \hat{\mathcal{P}}^+$ (i.e. $y' = Ay$ a mild equation) also follows from the condition that $b_3 \in \hat{\mathcal{P}}$.

The differential automorphisms of M are given by the non-singular matrices C, with coefficients in $\hat{\mathcal{P}}$, that satisfy $C' = AC - CA$. This yields

$$C = \begin{pmatrix} \gamma & \eta \\ 0 & \gamma \end{pmatrix}, \quad \gamma, \eta \in \mathbf{C}, \gamma \neq 0.$$

Since $\partial e_1 = -ae_1$ we find $\tau_0 e_1 = fe_1$ for some $f \in \hat{\mathcal{P}}$, $f \neq 0$. From $\partial e_2 = -z^{-1}e_1 - ae_2$ we get $\tau_0 e_2 = ge_1 + fe_2$, for some $g \in \hat{\mathcal{P}}$. From these observations, together with the commutativity relations, we deduce that $b_3 = 0$, $b_1 = b_4 = 1/f$ and $b_2 = -\log(1 + z^{-1})/f$, i.e.

$$B_0 = \frac{1}{f}\begin{pmatrix} 1 & -\log(1 + z^{-1}) \\ 0 & 1 \end{pmatrix}.$$

For general admissible B holds that $B = B_0 C^{-1}$.

5 DIFFERENTIAL – q-DIFFERENCE EQUATIONS

In this section we will make plausible that for differential–q-difference equations similar results hold as those proved in Sections 3 and 4, and that they could be proved with similar techniques.

We suppose that the complex number q satisfies $0 < |q| < 1$ and we write $q = e^{2\pi i t}$ with $t \in \mathbf{C}$ and $\mathrm{Im}(t) > 0$. We consider the q-difference operator σ, defined by $\sigma(z) = qz$. The action of σ on $\hat{\mathcal{P}}$ is defined by the formula $\sigma(z^\lambda) = e^{2\pi i t\lambda} z^\lambda$. The ring of differential–q-difference operators $\hat{\mathcal{P}}[\sigma, \sigma^{-1}, \delta]$ is a skew ring defined by the rules $\sigma\delta = \delta\sigma$, $\sigma f = \sigma(f)\sigma$ and $\delta f = f\delta + \delta(f)$ for any $f \in \hat{\mathcal{P}}$. We recall that $\delta(f) := z\,(\mathrm{d}f/\mathrm{d}z)$ for $f \in \hat{\mathcal{P}}$.

Along the lines of the proof of Proposition 18, one can show that $\hat{\mathcal{P}}[\sigma, \sigma^{-1}, \delta]$ has only trivial two-sided ideals. This implies that the ring homomorphism $\hat{\mathcal{P}}[\sigma, \sigma^{-1}, \delta] \to \hat{\mathcal{E}}$, given by $\sigma \mapsto \sum_{n \geq 0} (2\pi i t\delta)^n / n!$ and $\delta \mapsto \delta$, is an embedding.

The analogue of Proposition 20 holds, i.e., the left ideal in $\hat{\mathcal{P}}[\sigma, \sigma^{-1}, \delta]$ generated by a single element L has in general a countable codimension as vector space over $\hat{\mathcal{P}}$. Again one would like to produce a space of symbolic solutions for $L(y) = 0$ of countable dimension over $\mathbf{C}$. One can combine the universal ring of symbols R_∂ for differential equations with the universal ring of symbols for q-difference equations studied in [vdPS97]. For some operators one finds in this way many symbolic solutions. We will give details in the example following this discussion. The second option is to consider left modules M over $\hat{\mathcal{P}}[\sigma, \sigma^{-1}, \delta]$ of finite dimension over $\hat{\mathcal{P}}$. In other words, M is a finite dimensional vector space over $\hat{\mathcal{P}}$ equipped with two operations σ_M,δ_M respecting the rules:

* σ_M is additive, bijective and $\sigma_M(fm) = \sigma(f)\sigma_M(m)$ for $f \in \hat{\mathcal{P}}$ and $m \in M$.
* δ_M is additive and $\delta_M(fm) = \delta(f)m + f\delta_M(m)$ for any $f \in \hat{\mathcal{P}}$ and $m \in M$.
* σ_M and δ_M commute.

Let us call such an object a (σ, δ)-module. As in Section 4.1, the question is: what are those (σ, δ)-modules?

Now M viewed as a differential module can be classified as $M = \oplus M_q$ with $q \in \mathcal{Q} = \cup_{p \geq 1} z^{1/p}\mathbf{C}[z^{1/p}]$ and $M_q = E(q) \otimes \tilde{M}_q$ (see Section 4.1 for the definitions). We note that q has now two meanings, however from the context it will be clear which one is meant. The action of σ_M on M commutes with δ_M and we conclude that $\sigma_M(M_q) = M_{\sigma(q)}$. If $q \neq 0$ then the set $\{\sigma^n(q) \mid n \in \mathbf{Z}\}$ is infinite. We conclude that only $q = 0$ occurs in the classification of the differential module M. In other words, M is a regular singular differential module.

Conversely, let M be a regular singular differential module over $\hat{\mathcal{P}}$. Then there is a finitely generated, free submodule M_0 over $\hat{\mathcal{P}}^+$ with $M_0 \subset M$, $\delta(M_0) \subset M_0$ and $\hat{\mathcal{P}} \otimes M_0 \to M$ an isomorphism. Define σ_0 on M by the formula $\sigma_0 = \exp(2\pi i t \delta_M)$. One sees that this infinite expression converges on M_0 for the pointwise convergence of coefficients. Therefore σ_0 is well defined on M. This makes M into a (σ, δ)-module for $\sigma_M = \sigma_0$. Any other (σ, δ)-module structure on M is given by $\sigma_M = C\sigma_0$, where C is an automorphism of the differential module M. As in the Sections 4.1, 4.2 and so on, one concludes that the (realizable) (σ, δ)-modules are essentially the same as the regular singular differential modules over $\hat{\mathcal{P}}$.

Example 37 Let $L \in \mathbf{C}[[z^{-1}]][\delta, \sigma, \sigma^{-1}]$ be an operator involving σ (and with coefficients in the subring $\mathbf{C}[[z^{-1}]]$ of $\hat{\mathcal{P}}$). Then L, as element of $\hat{\mathcal{E}}$, has a horizontal line as Newton polygon. Write L as a differential operator of infinite order $\sum_{n\geq 0} z^{-n} l_n$. For convenience we will suppose that $l_0 \neq 0$. Then $\bar{L} = l_0$. This element lies in the image of $\mathbf{C}[\delta, \sigma, \sigma^{-1}]$ in $\hat{\mathcal{E}}^{+}_{1,0}/z^{-1} = \text{Ent}$. Let us write the elements of Ent as entire functions of the variable ξ. Then $\delta, \sigma, \sigma^{-1}$ are mapped to the functions $\xi, e^{2\pi i t\xi}$, $e^{-2\pi i t\xi}$. We identify $\mathbf{C}[\delta, \sigma, \sigma^{-1}]$ with the subring $\mathbf{C}[\xi, e^{2\pi i t\xi}, e^{-2\pi i t\xi}]$ of Ent. This identification has the property $\bar{L}(Z^a) = \bar{L}(a) Z^a$ for all $Z^a \in R_{\partial,\leq 0}$. We are interested in the kernel of L operating on $R_{\partial \leq 0}$. If the order of $\bar{L}$ is finite, i.e., $\bar{L}(\xi)$ has finitely many zeros, then by Corollary 9, we can factor L and find that the kernel of L on $R_{\partial,\leq 0}$ has dimension equal to $\text{ord}_0(L)$. The more interesting case is:

Suppose that $\bar{L}$ does not have the form $c(\xi)e^{2n\pi i t\xi}$ (with $c(\xi)$ a polynomial and $n \in \mathbf{Z}$). Then $\bar{L}$ has infinitely many zeros on $\mathbf{C}$. *Moreover, for every $\xi_0 \in \mathbf{C}$ the number of zeros of $\bar{L}$ in the set $\xi_0 + \mathbf{Z}$ is finite. Finally, there is a natural bijection between the set of zeros of $\bar{L}$ and a basis of the kernel of L on $R_{\partial,\leq 0} = \hat{\mathcal{P}}[\{Z^a\}_{a\in\mathbf{C}}, l]$.*

Proof By assumption $\bar{L} = \sum_{n=n_0}^{n_1} c_n(\xi) e^{2n\pi i t\xi}$ with $n_0 < n_1$, the $c_n(\xi)$ polynomials in ξ and $c_{n_0} \neq 0$, $c_{n_1} \neq 0$. Suppose that $\bar{L}(\xi)$ has only finitely many zeros on $\mathbf{C}$. Then there exist a polynomial a and an entire function b such that $\bar{L} = a(\xi) e^{b(\xi)}$. Since the growth of the function $\bar{L}(\xi)$ for $\xi \to \infty$ is bounded by some $e^{N|\xi|}$, it follows that b also is a polynomial.

Now $\bar{L}$ satisfies the differential equation $a\bar{L}' = (a' + ab')\bar{L}$ (where $'$ means $d/d\xi$). Explicitly

$$\sum_n (a(c_n' + 2n\pi i t c_n)) e^{2n\pi i t\xi} = \sum_n ((a' + ab')c_n) e^{2n\pi i t\xi}.$$

Using that ξ and $e^{2\pi i t\xi}$ are algebraically independent functions over $\mathbf{C}$, one finds the contradiction that the degree of $(n_1 - n_0)2\pi i t a$ is strictly less than the degree of a. This proves the first assertion.

The function $\bar{L}$ on the set $\xi_0 + \mathbf{Z}$ can be written as $\sum_n c_n(\xi_0 + m) \times e^{2n\pi i t\xi_0} q^{nm}$. Using that $0 < |q| < 1$ and that the c_n are polynomials, one finds that $\bar{L}$ has at most a finite set of zeros on $\xi_0 + \mathbf{Z}$. Using Proposition 16, part (2), one finds for every zero of $\bar{L}(\xi)$ (counted with multiplicity) a solution $y \in \mathrm{R}_{\partial,\leq 0}$ of $L(y) = 0$. □

In the following explicit examples we suppose that the coefficients of L are convergent, i.e., lie in $\mathbf{C}\{z^{-1}\}$, which allows us to discuss the possible convergence of the solutions y with $L(y)=0$.

(i) Consider $L=z^{-1}+\alpha+\sigma$, $\alpha\neq 0$. As differential operator of infinite order we have $L=z^{-1}+\alpha+\exp(2\pi i t\delta)$, hence $F_{L,0}(\xi)=\alpha+\exp(2\pi i t\xi)$. This function has infinitely many (simple) zeros $\xi_m=\xi_0+m/t$ with $m\in\mathbf{Z}$ and one obtains infinitely many formal (symbolic) solutions $z^{\xi_m}\hat{f}_m$ of $L(y)=0$, where $\hat{f}_m\in 1+z^{-1}\mathbf{C}[[z^{-1}]]$ is uniquely determined. A calculation shows that the $\hat{f}_m$ are actually convergent. As q_n difference operator L is of the first order, and one would expect a one dimensional solution space over the field of functions f, satisfying $f(z)=f(qz)$. Indeed, the infinitely many solutions we found above, are dependent over this field (i.e., $(qz)^{\xi_m}=z^{\xi_{m+1}}$).

(ii) Now let $L=\alpha+\sigma+z^{-1}\sigma^2$ (with $\alpha\neq 0$). One finds again infinitely many symbolic solutions of the form $z^{\xi_m}\hat{f}_m(z)$ with $\hat{f}_m\in 1+z^{-1}\mathbf{C}[[z^{-1}]]$. In this case, the power series $\hat{f}_m$ is highly divergent, in the sense that it is not even of some Gevrey-order and cannot be summed by the method of multisummability (see e.g. [Mal95] or [Bal94]). But it is "summable" with the method developed over the recent years by Ramis, Bézivin, Zhang, *et al.* (cf. [Ram92, Béz93, Zha, MZ]). This method is based on another definition of the Newton polygon of a q-difference equation and a new set of "elementary" transcendental functions, that play a similar role as the classical exponential functions in the theory for ordinary differential equations.

(iii) Let $L=\delta+z^{-1}\sigma$. The 0-characteristic is $F_{L,0}(\xi)=\xi$, so $L(y)=0$ has a unique formal power series solution $\hat{y}(z)\in 1+z^{-1}\mathbf{C}[[z^{-1}]]$, namely $\sum_{h\geq 0}\frac{1}{h!}q^{-(1/2)h(h-1)}z^{-h}$. This series is divergent since $0<|q|<1$. The Newton polygon for q-difference operators, as defined in for example [Zha], does not seem to apply to this differential–q-difference operator. Consequently, we have no method to sum the series $\hat{y}$.

(iv) We note the decomposition of Corollary 9 produces $L=E(\delta+d(z))$ for some $d(z)\in\mathbf{C}[[z^{-1}]]$ and E an invertible element of $\hat{\mathcal{E}}_1$. Hence, solving L in R_0 is equivalent to solving the formal first order differential equation $zy'(z) + d(z)y(z) = 0$. One easily checks that this equation has $\hat{y}$ as solution in R_0.

(v) Let $L = \delta + \sigma + z^{-1}$. Clearly, $F_{L,0}(\xi) = \xi + e^{2\pi i \xi}$. This function has countably many zeros, all of them simple. Let μ be a zero. Then $L(y) = 0$ has a unique symbolic solution of the form $y_\mu = Z^\mu(1 + \cdots) \in Z^\mu \mathbf{C}[[z^{-1}]]$. The formal power series part of this solution is again "highly" divergent.

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