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Nonlinear Switched Systems with State Dependent Dwell-Time[1]

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Abstract

The asymptotic convergence of nonlinear switched systems in the presence of disturbances is studied in this paper. The system switches among a family of integral input-to-state stable systems. The time between two consecutive switchings is not less than a value τ_D. This dwell-time τ_D is allowed to take different values according to a function whose argument is the state of the system at the switching times. We propose a dwell-time function which depends on the comparison functions which characterize the integral input-to-state stability and guarantees the state of the switched system to converge to zero under the action of disturbances with "bounded energy". The main feature of the analysis is that it does not rely on the property that the switching stops in finite time. The two important cases of locally exponentially stable and feedforward systems are analyzed in detail.

1 Introduction

A successful strategy to deal with a *largely uncertain* system, whose model is unknown but belongs to one of many families of nominal (and known) models, is to design a controller for each family of possible models, and then let a supervision logic decide from time to time what is the better controller to place in the feedback loop. This control strategy is typically known as *switching* or *logic-based* or *supervisory* control ([14], [15], [16], [12], [7], [10], [8], [11]). The study of a logic-based supervisory control system usually boils down to the analysis of the "convergence" properties[1] of a *switched* system, which depends on the piece-wise constant switching signal σ generated by the supervision logic and is driven by disturbance signals which typically are in $\mathcal{L}_p$, $p \in [1, \infty]$. If the switched system is *linear*, then the convergence property is proven by testing whether or not the linear switched system is asymptotically stable ([15]). If the system is not linear then things are more complicated because, as it is well-known, the simple asymptotic stability doesn't guarantee any "bounded-input bounded-state" or "convergent-input convergent-state" property. In [10] it was shown that, if it is possible to design the supervisory control system in such a way that each system of the family which defines the switched system is input-to-state stable (ISS) ([17]) with respect to the disturbance signal and if the switching stops in finite time, then the state of the switched system exhibits the desired convergence property. However, it is not always possible to design a control law which guarantees the input-to-state stability of a closed-loop system. A weaker property than input-to-state stability is the *integral* input-to-state stability, introduced in [18]. There are various examples of systems which cannot be rendered ISS but can be made integral input-to-state stable (iISS). The supervision of integral input-to-state stabilizing controllers has been analyzed in [8]. In that paper, the authors examined the case in which the switching among a family of integral input-to-state systems is orchestrated by the "scale-independent hysteresis" switching logic ([9]). As already noted in [11], the analysis of the behavior of a control systems supervised by a hysteresis switching logic depends on the property that under certain conditions the switching stops in finite time. This is an idealistic situation which is unlikely to occur in the presence of noise and exogenous disturbance. So in order to deal with more realistic scenarios, one is urged to take in consideration different switching logics, such as the dwell-time switching logic. This kind of switching logic has been used e.g. in [15] to infer stability properties of a switched linear system. Indeed, there are simple examples of stable linear systems whose state response can diverge to infinity if the switching among them occurs with an "improper timing". A way to remedy this situation is to constrain the switching signal to "dwell" at each value for at least a *constant* amount of time, usually called "dwell-time". Despite of what happens in the linear case, however, a constant dwell-time can lead to an unsatisfactory behavior for nonlinear systems. It is then important to consider the case of nonlinear switched systems with a

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[1]Convergence to zero of (nonlinear time-varying) systems with disturbances is a fundamental issue which arises not only in (switching) adaptive control, but also in many other control problems. See [20] for other results in this regard.

dwell-time which can change as time goes by. An example of this kind of analysis has been provided in [11], where it has been shown that a system *without disturbances*, which switches among globally asymptotically stable systems with a dwell-time which is "constant only on the average", preserves its stability property, provided that the Lyapunov functions which characterize the asymptotic stability are "similar" in a suitable sense (cf. [11]).
In this paper, we introduce a different kind of "time-varying" or "adaptive" dwell-time to deal with the presence of disturbances. The analysis is not based on Lyapunov-like arguments, and this allows to avoid unnecessary conditions on the Lyapunov functions. We consider the case in which the value assumed by the dwell-time depends on the value of the state at which the system is. This dependence is described by means of a suitable function whose expression is explicitly determined. After introducing some preliminary definitions in Section 2, we show in Section 3 that the state of the switched system, in the presence of a disturbance with bounded energy, globally converges to zero, provided that the dwell-time function is suitably chosen. This result is then particularized in Section 4 to the case of locally exponentially stable systems and in Section 5 to systems in feedforward form. The latter case is particularly important to address the problem of supervisory control of systems in the presence of control constraints. For lack of space, we do not analyze the behavior of supervisory control systems with state dependent dwell-time switching logic and its application to the control of largely uncertain systems in the presence of constraints, for which we refer the reader to [3], [5]. Concise comments on these subjects are given in Section 6.
Most of the proofs have been omitted from the paper and can be found in [4].

2 Preliminaries

We consider the family of nonlinear maps $\mathcal{F} := \{A_p(x,d), p \in \mathcal{P}\}$ indexed by the parameter p, taking on values in the set $\mathcal{P}$. Each map $A_p : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ of $\mathcal{F}$ is assumed to be locally Lipschitz. For each piece-wise constant signal $\sigma : [0,\infty) \to \mathcal{P}$, we can define the switched nonlinear system

$$\dot{x} = A_\sigma(x,d) \,, \tag{1}$$

where $d : [0,\infty) \to \mathbb{R}^p$ is a measurable and locally essentially bounded function. We additionally assume the following

Assumption 1 *For each $p \in \mathcal{P}$ system*

$$\dot{x} = A_p(x,d) \,, \tag{2}$$

is integral input-to-state stable (iISS), that is to say (cf. [18], [1])[2] *there exist class-$\mathcal{K}_\infty$ functions $\alpha(\cdot), \tilde{\theta}_1(\cdot), \bar{\theta}_2(\cdot)$, and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that the solution $x(t)$ of (2) from the initial condition $x(t_0) = x_0$ under the input $d(\cdot)$ exists for all $t \geq t_0$ and satisfies*

$$\alpha(|x(t)|) \leq \tilde{\theta}_1(\bar{\theta}_2(|x_0|)e^{-(t-t_0)}) + \int_{t_0}^{t} \gamma(|d(\tau)|)d\tau \,, \tag{3}$$

for all $t \geq t_0 \geq 0$, all $x_0 \in \mathbb{R}^n$ and all $d(\cdot)$.

The inputs $d(\cdot)$ in (1) are required to satisfy

$$\int_0^\infty \gamma(|d(\tau)|)d\tau < \infty \tag{4}$$

where $\gamma(\cdot)$ is the function which appears in (3).
Finally, we define the class $\mathcal{S}$ of admissible switching signals $\sigma : [0,\infty) \to \mathcal{P}$ as that set of piece-wise constant signals which:

(i) Exhibit an infinite number of switching times in the interval $[0,\infty)$;

and such that:

(ii) Denoted with $\mathcal{T} := \{0 =: t_0, t_1, \ldots, t_j, \ldots\}$ the infinite and strictly increasing sequence of switching times, and with $\tau_D : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ a continuous function, the difference $t_{i+1} - t_i$ is greater than or equal to $\tau_D(|x(t_i)|)$ for each $i = 0, 1, \ldots$, where $x(t_i)$ is the solution of (1) at time t_i.

Our aim is to find a function $\tau_D(\cdot)$ such that the state of the system (1) asymptotically converges to zero as time goes to infinity. Property (i) is introduced to prevent the problem to become trivial. Indeed, if $\sigma(t) = q$ for $t \geq T$ on, then the switched system becomes $\dot{x} = A_q(x,d)$ for $t \geq T$, and asymptotic convergence trivially descends from the iISS property and (4) (cf. Proposition 6 in [18]). Property (ii) says that a switching signal is admissible if it dwells at a certain $q \in \mathcal{P}$ for an amount of time which is not less than a prescribed value before switching to a new parameter in $\mathcal{P}$. The time which σ dwells at the parameter $\sigma(t_i)$ depends on the value of the state of the switched system at the switching time t_i, and is computed according to the function $\tau_D(\cdot)$.
Before proceeding, we note that in the present setting no finite escape time exists for the solutions of (1) (cf. [18]).

[2] $\mathcal{K}$ is the class of functions $[0,\infty) \to [0,\infty)$ which are zero at zero, strictly increasing and continuous, $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ consisting of all those functions that are unbounded.

Fact 1 *If Assumption 1 holds, then for each $\sigma \in \mathcal{S}$, for each initial condition $x(0)$, and for each input $d(\cdot)$, the response of system (1) exists for all $t \in [0, \infty)$.*

Indeed, let the response of system (1) exist for all $t \in [0, T)$, with $T \leq \infty$. Let t_i be the largest switching time such that $t_i < T$. Then for all $t \in [t_i, T)$, the response of (1) satisfies (3), that is

$$\alpha(|x(t)|) \leq \tilde{\theta}_1(\tilde{\theta}_2(|x(t_i)|)e^{-(t-t_i)}) + \int_{t_i}^{t} \gamma(|d(s)|)ds \; ,$$

for $t \in [t_i, T)$. The two terms on the left-hand side of the previous inequality are both bounded on the interval $[t_i, T)$ and therefore $|x(t)|$ is bounded on $[t_i, T)$. This implies that necessarily $T = \infty$.

3 State dependent dwell-time function and convergence property

In this section, we show how to choose the function $\tau_{\mathrm{D}}(\cdot)$ to guarantee asymptotic convergence for the switched system. To this end we introduce the following notation.

Let $\alpha(\cdot)$, $\tilde{\theta}_1(\cdot)$, $\tilde{\theta}_2(\cdot) \in \mathcal{K}_\infty$ be as in (3). Define the functions

$$\theta_1(r) := \tilde{\theta}_1^{-1}\left(\frac{1}{2}\alpha\left(\frac{1}{3}r\right)\right) \; , \quad \theta_2(r) := \tilde{\theta}_2(r) \; , \qquad (5)$$

and set[3]

$$\tau_\Delta(r) := \ln \frac{\theta_2(r)}{\theta_1(r)} \; , \quad r > 0 \; . \qquad (6)$$

Before introducing the main convergence result, we state an intermediate result. The result ensures that the *sequence of the states $\{x(t_i)\}$ at the switching times* of the switched system converge to zero.

Lemma 1 *Consider system (1), assume that Assumption 1 holds and that there exists $\lim_{r \to 0^+} \tau_\Delta(r) =: \bar{\tau}_\Delta$, with $\bar{\tau}_\Delta < +\infty$. Let $\sigma \in \mathcal{S}$ with function $\tau_{\mathrm{D}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ satisfying*

$$\tau_{\mathrm{D}}(r) \geq \begin{cases} \tau_\Delta(r) \; , & r > 0 \\ \bar{\tau}_\Delta \; , & r = 0 \; , \end{cases} \qquad (7)$$

$\tau_\Delta(\cdot)$ being defined as in (6). Then, for each $x_0 \in \mathbb{R}^n$, for each input $d(\cdot)$ fulfilling (4), the solution $x(t)$ of (1) starting from the initial condition $x(0) = x_0$ and under the input $d(\cdot)$ is such that

$$\lim_{i \to \infty} |x(t_i)| = 0 \; .$$

[3]Note that $\tau_\Delta(r) > 0$ for all $r > 0$. Indeed, setting $t = t_0$ and replacing $|x(t_0)|$ with r in (3) we obtain that $\alpha(r) \leq \tilde{\theta}_1(\tilde{\theta}_2(r))$. As $\alpha(\cdot)$ is a strictly increasing function, the previous inequality also implies, for $r > 0$, that $\alpha(r/3)/2 < \tilde{\theta}_1(\tilde{\theta}_2(r))$. Therefore, $\tilde{\theta}_1^{-1}(\alpha(r/3)/2) =: \theta_1(r) < \tilde{\theta}_2(r) =: \theta_2(r)$, and $\theta_2(r)/\theta_1(r) > 1$ for all $r > 0$.

Using Lemma 1 we can then prove that if the function $\tau_{\mathrm{D}}(\cdot)$ satisfies (7) the *state* $x(\cdot)$ itself converges to zero.

Theorem 1 *Consider system (1), assume that Assumption 1 is true and that there exists $\lim_{r \to 0^+} \tau_\Delta(r) =: \bar{\tau}_\Delta$, with $\bar{\tau}_\Delta < +\infty$. Let $\sigma \in \mathcal{S}$ with function $\tau_{\mathrm{D}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ satisfying (7). Then, for each $x_0 \in \mathbb{R}^n$, for each input $d(\cdot)$ fulfilling (4), the solution $x(t)$ of (1) starting from the initial condition $x(0) = x_0$ and under the input $d(\cdot)$ is such that*

$$\lim_{t \to \infty} |x(t)| = 0 \; .$$

Remark. In the case system (1) is linear, the function $\tau_\Delta(r)$ in (6) is equal to a constant for all $r \geq 0$. As a matter of fact, it is well-known (see [18]) that for linear systems the iISS property is equivalent to asymptotic stability, and the inequality (3) can be written for instance as

$$|x(t)| \leq ((c|x(t_0)|)^{1/k} e^{-(t-t_0)})^k + \bar{\gamma} \int_{t_0}^{t} |d(\tau)|^2 d\tau \; ,$$

for suitable positive real numbers $c, k, \bar{\gamma}$. Therefore, $\tilde{\theta}_1(r) = r^k$, $\tilde{\theta}_2(r) = (cr)^{1/k}$, $\alpha(r) = r$, and from (5) and (6) we have $\tau_\Delta(r) = \ln(6c)/k$ (cf. e.g. (44) in [15]). $\triangleleft$

If Assumption 1 holds, the functions $\theta_1(\cdot), \theta_2(\cdot)$ which appear in the dwell-time function (7) are class-$\mathcal{K}_\infty$ functions. In general, this is not enough to guarantee that the condition $\lim_{r \to 0^+} \tau_\Delta(r) < +\infty$ is fulfilled. One may look for a different set of functions $\alpha(\cdot), \tilde{\theta}_1(\cdot), \tilde{\theta}_2(\cdot)$ for which both the assumptions of Theorem 1 hold. In many cases, as for instance those considered in the next two sections, it is more convenient to use the following result which does not require $\tau_\Delta(r)$ to be defined for $r = 0$. It states that, if the dwell-time is set equal to a suitable *constant* value when the state of the system enters a neighborhood of the origin, then the state will remain arbitrarily close to the origin, although it does not necessarily converge.

Lemma 2 *Consider system (1) and assume that Assumption 1 hold. Given any $\delta > 0$, let $\bar{r}$ be a positive real number such that $\bar{r} = \theta_2^{-1}(\theta_1(3\delta))$. Let $\sigma \in \mathcal{S}$ with function $\tau_{\mathrm{D}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ satisfying*

$$\tau_{\mathrm{D}}(r) \geq \begin{cases} \tau_\Delta(r) \; , & r \geq \bar{r} \\ \tau_\Delta(\bar{r}) \; , & r < \bar{r} \; , \end{cases} \qquad (8)$$

$\tau_\Delta(\cdot)$ being defined as in (6). Then, for each $x_0 \in \mathbb{R}^n$, for each input $d(\cdot)$ fulfilling (4), there exists a time T_δ such that the solution $x(t)$ of (1) starting from the initial condition $x(0) = x_0$ and under the input $d(\cdot)$ satisfies

$$|x(t)| < \delta \; ,$$

for all times $t \geq T_\delta$.

Lemma 2 shows that if the dwell-time is set equal to a constant value when the state is in a neighborhood of the origin, then we obtain regulation of the state *about* the origin, but not asymptotic convergence. The switched system, however, can converge asymptotically to zero – even with a dwell-time function which is constant in a neighborhood of the origin – provided that each system (2) has an additional property, namely that of being locally exponentially stable. The aim of the next section is to explain this in detail.

4 Switching among locally exponentially stable systems

Assumption 1 implies that for each $p \in \mathcal{P}$ the zero-input system associated with (2) is globally asymptotically stable. We strengthen here this requirement asking that for each $p \in \mathcal{P}$ it is also locally exponentially stable, namely (see e.g. [13])

Assumption 2 *For each $p \in \mathcal{P}$, system*

$$\dot{x} = A_p(x, 0) \tag{9}$$

is globally asymptotically stable and locally exponentially stable, i.e. there exist class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, positive real numbers a_1, a_2, a_3, $\bar{s}$ and a smooth function $W_p(x) : \mathbb{R}^n \to \mathbb{R}$, such that

$$\begin{aligned} &\alpha_1(|x|) \le W_p(x) \le \alpha_2(|x|) \\ &\frac{\partial W_p}{\partial x} A_p(x,0) \le -\alpha_3(|x|) \,, \end{aligned} \tag{10}$$

for all $x \in \mathbb{R}^n$ and

$$\alpha_i(s) = a_i s^2, \quad i = 1, 2, 3 \,, \tag{11}$$

for all $s \in [0, \bar{s}]$.

The following is a technical lemma which eases the proof of Theorem 2 below.

Lemma 3 *Consider system (1), and assume that Assumption 2 holds and there exists a switching time t_i such that the solution $x(t)$ of (1) satisfies $|x(t)| \le \bar{s}$ for all $t \ge t_i$. Let k be a constant such that $k \in (0, \frac{a_3}{2a_2}]$ and $\sigma \in \mathcal{S}$ with*

$$\tau_{\mathrm{D}}(r) \ge \frac{1}{k} \ln \frac{a_2}{a_1} \,, \quad r \in [0, \bar{s}] \,. \tag{12}$$

Then there exists a class-$\mathcal{K}$ function $\delta(\cdot)$ such that

$$a_1 |x(t)|^2 \le a_2 |x(t_i)|^2 \cdot \mathrm{e}^{-k(t-t_i)} + \frac{a_2}{a_1} \int_{t_i}^{t} \delta(|d(\tau)|) d\tau \,, \tag{13}$$

for all $t \ge t_i$.

If an additional assumption on the maps $A_p(x, d)$ holds, we can give an expression to the function $\delta(\cdot)$.

Assumption 3 *There exists a continuous nonnegative function $\varphi : \mathbb{R}^n \to [0, \infty)$, such that $|A_p(x,d) - A_p(x,0)| \le \varphi(x)|d|$ for each $p \in \mathcal{P}$, for all x and d.*

Corollary 1 *In the hypotheses of Lemma 3:*

- *If Assumption 3 is fulfilled and $k = \frac{a_3}{2a_2}$, then (13) holds with $\delta(r) = \bar{\delta} r$, for some $\bar{\delta} > 0$.*
- *If Assumption 3 is fulfilled, $k = \frac{a_3}{3a_2}$, and for each $p \in \mathcal{P}$, $A_p(x, 0)$ is continuously differentiable for all $|x| \in [0, \bar{s}]$, then (13) holds with $\delta(r) = \bar{\delta} r^2$, for some $\bar{\delta} > 0$.*

We can now introduce the main result of this section. The result asserts that the state of a system which switches among iISS, locally exponentially stable systems, converges to zero even when the dwell-time of the switching signal is set equal to a constant in a neighborhood of the origin.

Theorem 2 *Consider system (1), and assume that Assumptions 1 and 2 hold. Let $\delta(\cdot)$ and k be the function and, respectively, the positive constant introduced in Lemma 3, and $\gamma(\cdot)$ the function for which (3) holds. Define $\theta_1(\cdot), \theta_2(\cdot)$ as in (5) and $\tau_\Delta(\cdot)$ as in (6) and set*

$$\bar{r} := \theta_2^{-1}(\theta_1(3\bar{s})) \,, \tag{14}$$

with $\bar{s}$ as in Assumption 2. Let $\sigma \in \mathcal{S}$ with $\tau_{\mathrm{D}} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ satisfying

$$\tau_{\mathrm{D}}(r) \ge \begin{cases} \tau_\Delta(r) \,, & r \ge \bar{r} \\ \max\{\tau_\Delta(\bar{r}), \frac{1}{k} \ln \frac{a_2}{a_1}\} \,, & r < \bar{r} \end{cases} \tag{15}$$

where a_1, a_2 are as in (11). If there exists a finite $c > 0$ such that $\delta(r) \le c\gamma(r)$ for all $r \ge 0$, then, for each $x_0 \in \mathbb{R}^n$, for each input $d(\cdot)$ fulfilling (4), the solution $x(t)$ of (1) starting from the initial condition $x(0) = x_0$ and under the input $d(\cdot)$ is such that

$$\lim_{t \to \infty} |x(t)| = 0 \,.$$

Proof: Since Assumption 1 holds, the function $\tau_{\mathrm{D}}(\cdot)$ satisfies (15) and $\bar{r} = \theta_2^{-1}(\theta_1(3\bar{s}))$, we know from Lemma 2 that there exists a switching time $T_{\bar{s}}$ such that $|x(t)| \le \bar{s}$ for all $t \ge T_{\bar{s}}$. We also know from the proof of Lemma 2 (see [4]) that, for each switching time $t_k \ge T_{\bar{s}}$, we have $|x(t_k)| < \bar{r}$. Hence, from (15), $\tau_{\mathrm{D}}(|x(t_k)|) \ge 1/k \ln(a_2/a_1)$ for each $t_k \ge T_{\bar{s}}$ and using

Lemma 3 we can say that there exist a class-$\mathcal{K}$ function $\delta(\cdot)$ and a $k > 0$ such that (13) holds for $t \geq T_{\bar{s}}$.
Moreover, $d(\cdot)$ fulfilling (4) and $\delta(r) \leq c\gamma(r)$ for all $r \geq 0$, imply that

$$\int_0^\infty \delta(|d(\tau)|)d\tau < \infty$$

and the convergence to zero of the state of the switched system follows from (13) in view of Proposition 6 of [18]. ■

In the next section, we examine an important class of systems for which the results of the theorem above applies.

5 The Case of Feedforward Systems

Consider the class of nonlinear feedforward systems of the form

$$\begin{array}{rcl} \dot{x}_1 & = & A_1 x_1 + f_1(x_2, \ldots, x_{\nu-1}, x_\nu, u, d) \\ & \vdots & \\ \dot{x}_{\nu-1} & = & A_{\nu-1}x_{\nu-1} + f_{\nu-1}(x_\nu, u, d) \\ \dot{x}_\nu & = & A_\nu x_\nu + f_\nu(u, d) \,, \end{array} \tag{16}$$

with $x = (x_1^T \ldots x_\nu^T)^T \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$, satisfying the assumption (cf. Assumption 2.1 in [19])

Assumption 4 *The f_i's are locally Lipschitz for all x, u, d; $f_i(0, \ldots, 0) = 0$; The linearization at the origin with $d = 0$ is stabilizable; The matrices A_i's are critically stable; There exist continuous and nondecreasing functions $\varphi_i(|(x_{i+1}, \ldots, x_\nu, u)|)$, such that*

$$|f_i(x_{i+1}, \ldots, x_\nu, u, d) - f_i(x_{i+1}, \ldots, x_\nu, u, 0)| \leq \varphi_i(|(x_{i+1}, \ldots, x_\nu, u)|)|d| \,.$$

For this class of system, the following result is well known (cf. [19], Theorem 2.2).

Theorem 3 ([19]) *For system (16) satisfying Assumption 4 there exists a feedback law $u = \chi(x)$ such that, for any initial condition $x_0 \in \mathbb{R}^n$ and any disturbance $d(\cdot) \in \mathcal{L}_2$, the trajectory of the closed-loop system satisfies*

$$\|x\|_\infty \leq \max\{\gamma_\infty^0(|x_0|), \gamma_\infty^d(\|d(\cdot)\|_2)\} \,, \tag{17}$$

where the class-$\mathcal{K}_\infty$ functions $\gamma_\infty^0(\cdot), \gamma_\infty^d(\cdot)$ are independent of x_0 and $d(\cdot)$. Moreover, when $d = 0$, the closed-loop system is globally asymptotically stable and locally exponentially stable.

Remark. The control law proposed in [19] has the form

$$u = \mathrm{sat}(K_1 x + \mathrm{sat}(K_2 x + \mathrm{sat}(\ldots + \mathrm{sat}(K_\nu x)))) \,, \tag{18}$$

where $K_i \in \mathbb{R}^{m \times n}$ and $\mathrm{sat}(\cdot)$ is a generalized saturation function, i.e. a function which is differentiable at the origin and satisfies $|\mathrm{sat}(u+v) - \mathrm{sat}(u)| \leq \min\{a|v|, b\}$ and $|\mathrm{sat}(u) - u| \leq a u^T \mathrm{sat}(u)$ for some $a > 0$ and $b > 0$ and for all $u, v \in \mathbb{R}^m$.
If the functions $f_i(\cdot)$ in (16) are continuously differentiable in a neighborhood of $(x, u, d) = (0, 0, 0)$, then choosing a function $\mathrm{sat}(\cdot)$ which is continuously differentiable in a neighborhood of the origin as well, yields that the map which defines the closed-loop system is continuously differentiable in a neighborhood of $(x, d) = (0, 0)$. ◁

The following lemma, which is a consequence of the results of [19] and [2], points out that the feedback $u = \chi(x)$ renders the system (16) integral input-to-state stable with a *quadratic* gain function.

Lemma 4 *For system (16) satisfying Assumption 4 there exists a feedback law (18) such that the closed-loop system (16), (18) is iISS with respect to the disturbance $d(\cdot)$. In particular, (3) holds for some $\alpha(\cdot)$, $\tilde{\theta}_1(\cdot)$, $\tilde{\theta}_2(\cdot) \in \mathcal{K}_\infty$ and for $\gamma(r) = r^2$.*

Remark. It is possible to prove that the integral input-to-state stability of the closed-loop system (16),(18) holds with different gain functions $\gamma(\cdot)$. However, the importance of having a quadratic function $\gamma(r) = r^2$ stems from the fact that in the context of supervisory control this allows to deal with more general classes of uncertain systems (see [15], [5], [6] and Section 6). ◁

The statement implies that if, for each $p \in \mathcal{P}$, the systems (2) which define the switched system $\dot{x} = A_\sigma(x, d)$ are obtained from a feedforward system of the form (16) satisfying Assumption 4 in closed-loop with a controller of the form (18), then Assumptions 1 and 2 are trivially satisfied, the former in particular with a function $\gamma(r) = r^2$. Moreover, if Assumption 4 holds and the functions $f_i(\cdot)$, $i = 1, \ldots, \nu$, and $\mathrm{sat}(\cdot)$ are continuously differentiable in a neighborhood of the origin, then (cf. Remark after Theorem 3) the hypotheses of Corollary 1 are satisfied and $\delta(r) = \tilde{\delta} r^2$. Therefore we have $\gamma(r) \geq c\delta(r)$, for any $c \in (0, 1/\tilde{\delta}]$. Hence, Theorem 2 yields the following consequence.

Theorem 4 *Consider system (1) and assume that for each $p \in \mathcal{P}$ system (2) is obtained from a feedforward system of the form (16) satisfying Assumption 4 in closed-loop with a controller of the form (18). Assume also that the functions $f_i(\cdot)$, $i = 1, \ldots, \nu$, and* $\mathrm{sat}(\cdot)$

are continuously differentiable in a neighborhood of the origin. Let $\sigma \in \mathcal{S}$ with $\tau_D(\cdot)$ satisfying (15). Then, for each $x_0 \in \mathbb{R}^\nu$, for each input $d(\cdot)$ fulfilling (4), with $\gamma(r) = r^2$, the solution $x(t)$ of (1) starting from the initial condition $x(0) = x_0$ and under the input $d(\cdot)$ is such that $\lim_{t\to\infty} |x(t)| = 0$.

6 Conclusive Comments

In this paper we have presented results about the convergence of the state of nonlinear switched systems in the presence of disturbances. They generalize to nonlinear systems the results available in [15] for linear systems and are instrumental to design supervisory control architectures for nonlinear and largely uncertain systems. In particular, if the unknown process belongs to a family of nominal models $\mathcal{F}$, indexed by a parameter p which takes values in the set $\mathcal{P}$, the results presented in this paper allow to introduce a state dependent dwell-time switching logic which orchestrates the switching among a family of candidate controllers, one for each component of the family $\mathcal{F}$, in such a way to guarantee global regulation to zero of the state of the process and global boundedness of all the continuous states (see [3], [5]). A particularly important class of systems for which such a design is possible is that of linear plants which are open-loop unstable but not exponentially unstable and whose control input is subject to input constraint (cf. Sections 4 and 5). Observe also that, since we allow in Theorem 4 square integrable disturbances, our method is suitable to deal with both cases in which $\mathcal{P}$ is a continuum of points (cf. [6]) and a finite set (see [15], Section III, for more comments on this observation).

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