Lie symmetries and differential galois groups of linear equations

Oudshoorn, W.R.; Put, M. van der

MATHEMATICS OF COMPUTATION
Volume 71, Number 237, Pages 349–361
S 0025-5718(01)01397-7
Article electronically published on October 4, 2001

LIE SYMMETRIES AND DIFFERENTIAL GALOIS GROUPS OF LINEAR EQUATIONS

W. R. OUDSHOORN AND M. VAN DER PUT

Abstract. For a linear ordinary differential equation the Lie algebra of its infinitesimal Lie symmetries is compared with its differential Galois group. For this purpose an algebraic formulation of Lie symmetries is developed. It turns out that there is no direct relation between the two above objects. In connection with this a new algorithm for computing the Lie symmetries of a linear ordinary differential equation is presented.

1. Introduction

Some ordinary differential equations (ode's) can be solved in an algebraic or symbolic way. Algorithms for testing ode's for the existence of symbolic solutions and actually producing such solutions have been developed in two directions. The oldest one is Lie's theory of symmetries of an ode. Its goal is to compute the Lie algebra of the infinitesimal symmetries of a given ode. If this Lie algebra is sufficiently large, then the ode can be solved or at least be transformed into a simpler form. The second theory concerns only linear ode's and is called differential Galois theory. It attaches a linear algebraic group G to a linear ode, which gives precise information about symbolic solutions. This theory and the corresponding algorithms are highly developed.

For the case of linear ode's we will compare the two theories and the algorithms. In particular, we present a new algorithm which computes the Lie symmetries of a *linear* ode. In order to compare the two theories, we have to formalize the notion of Lie symmetry. It is known (see [KM, ML] et al.) that the dimension of the Lie algebra of symmetries of a linear ode of order n is equal to 8 for $n = 2$ and is equal to $n + 1$, $n + 2$ or $n + 4$ for $n > 2$. The structure of the Lie algebra is known in all cases. We will prove the same result, using our formalization, and moreover give a complete classification of the equations for the cases $n + 2$ and $n + 4$. It turns out that the exceptional cases $n+2$ and $n+4$ lead to exceptional differential Galois groups $G \subset \mathrm{GL}(n)$, which are also computed. However, there is apparently no direct connection between G and the Lie algebra of symmetries. Moreover, any algebraic subgroup of $\mathrm{GL}(n)$ does occur as a differential Galois group for a linear ode belonging to the "class $n + 1$".

Recently, a differential Galois theory for nonlinear ode's has been proposed. There is not enough information available to investigate a possible relation with the Lie algebra of symmetries of ode's.

Received by the editor October 13, 1999 and, in revised form, January 24, 2000.
2000 *Mathematics Subject Classification.* Primary 34A30, 34G34, 34Mxx, 65L99.

2. Definition of Lie symmetries

In the literature various definitions of Lie symmetries can be found. None of these definitions are suitable for Picard-Vessiot theory and differential Galois groups. Therefore we develop an algebraic context for differential equations and their Lie symmetries which allows Picard-Vessiot theory. This algebraic context includes most of the other definitions. For differential rings, Picard-Vessiot theory, and differential Galois groups, we refer to [K, P].

Definition 2.1. The ring R of functions in x and y.

Let C be a field of characteristic 0 and R a commutative C algebra with a unit $1 \neq 0$. The ring R is equipped with two commuting derivations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, and is supposed to have the following properties:

1) The subring $\{f \in R| \ \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\}$ is equal to C. (The elements of C will be called constants).

2) R contains two elements x and y such that $\frac{\partial x}{\partial x} = \frac{\partial y}{\partial y} = 1$ and $\frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0$. (The expressions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ will be abbreviated to f_x and f_y. Other expressions, like f_{xyx}, then have an obvious meaning).

Let $R^{\frac{\partial}{\partial y}}$ denote the subring of R consisting of the elements f with $f_y = 0$. This ring is invariant under $\frac{\partial}{\partial x}$, and we will write d/dx for the restriction of $\frac{\partial}{\partial x}$ to $R^{\frac{\partial}{\partial y}}$. Intuitively, we think of the above R as a ring of functions in x and y and of $R^{\frac{\partial}{\partial y}}$ as the subring of functions depending only on x. The following examples will make this clear.

Examples 2.2.

(1) The case for which the Lie symmetries are most often defined concerns the ring R of C^∞-functions on the plane. That is, we take the two-dimensional vector space $\mathbf{R}^2$ with coordinate functions x and y and let R be $C^\infty(\mathbf{R}^2)$. On this ring we have canonical derivations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. It is obvious that R together with $x, y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $C = \mathbf{R}$ satisfies Definition 2.1.

(2) One can consider variations on example (1) by taking $R = C^\infty(U)$ or $R = C^\infty_p$, where U is an open connected subset of $\mathbf{R}^2$. In the second expression p is a point of $\mathbf{R}^2$ and R is the ring of C^∞-germs at p. Further, one can consider complex-valued functions instead of real-valued ones.

(3) One can consider an open connected $U \subset \mathbf{C}^2$ and $R = \mathcal{O}(U)$ the ring of holomorphic functions on U. Also here there are many variations possible.

(4) R is one of the rings $\mathbf{C}[x, y]$, $\mathbf{C}(x)[y]$, $\mathbf{C}[[x]][y]$, et cetera.

Definition 2.3. The ring A of polynomial differential equations over R.

The differential equations that we are interested in are polynomials in the derivatives of y and those that have coefficients in R. We introduce variables $y_1, y_2, \ldots$ to denote the derivatives of y and write $y_0 = y$ for notational convenience. More precisely, let $\{y_i\}_{i\geq 1}$ denote a countable set of variables. Then $A := R[\{y_i\}]$ is the free polynomial ring over R equipped with a differentiation d/dx (the total derivative) defined by

1) $(d/dx)f = f_x + y_1 f_y$ for $f \in R$,

2) $(d/dx)y_k = y_{k+1}$ for $k \geq 0$.

By definition a (polynomial) differential equation is an element $\omega \in A$. One considers an extension of differential rings $E \supset R^{\frac{\partial}{\partial y}}$ and writes again d/dx for the derivation on E. A *solution* $f \in E$ of ω has the following meaning:

1) A homomorphism of differential rings $\phi : A \to E$, i.e., $\phi \circ d/dx = d/dx \circ \phi$, such that ϕ is the identity on $R^{\frac{\partial}{\partial y}}$ and such that

2) $\phi(y) = f$ and $\phi(\omega) = 0$.

Let $\langle\omega\rangle \subset A$ denote the ideal generated by ω and all its derivatives $(d/dx)^m\omega$. The kernel of the above ϕ clearly contains $\langle\omega\rangle$.

The differential equation ω is called *linear homogeneous* if it has the form $\sum_{0\leq j\leq n} a_j y_j$ with all $a_j \in R^{\frac{\partial}{\partial y}}$ and *linear* if it has the form $a + \sum_{0\leq j\leq n} a_j y_j$ with $a, a_0, \ldots, a_n \in R^{\frac{\partial}{\partial y}}$.

The classical idea of Lie symmetry of a differential equation ω is a C^∞-automorphism ϕ of (say) $\mathbf{R}^2$, which preserves the collection of integrals (graphs of solutions in $\mathbf{R}^2$) of ω. One is not interested in a single ϕ but in a differentiable one-parameter group $\{\phi_t\}_{t\in\mathbf{R}}$ of such automorphisms. The derivative at $t = 0$ of this family is a vector field $\nabla := \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$ for certain C^∞-functions ξ, η. The ϕ_t and ∇ also transform the derivatives y_k and the differential equation ω. Lemma 2.4 produces the natural formulas for ∇y_k. One supposes, for convenience, that the integrals for ω determine ω, or more precisely determine the ideal $\langle\omega\rangle$. The automorphism ϕ_t transforms the integrals of ω to the integrals of $\phi_t\omega$. Thus the collection of integrals is stable under ϕ_t precisely if the ideal $\langle\omega\rangle$ is stable under ϕ_t. Suppose that this holds for all t, then also $\nabla(\langle\omega\rangle) \subset \langle\omega\rangle$. In this way our formal definitions of vector field and Lie symmetry of ω are justified. More generally, one can consider a differential ideal $I \subset A$, i.e., an ideal such that $d/dx(I) \subset I$. One defines the Lie symmetries of I as the Lie algebra of the ∇ satisfying $\nabla(I) \subset I$.

A *vector field* ∇ or $\nabla_{\xi,\eta}$ is a derivation of R of the form $\nabla(f) = \xi f_x + \eta f_y$ with (fixed) $\xi, \eta \in R$.

Lemma 2.4. *For any $\xi, \eta \in R$, the vector field $\nabla_{\xi,\eta}$ has a unique extension to a derivation of A (with the same name) such that the Lie bracket $[d/dx, \nabla_{\xi,\eta}]$ is a multiple of d/dx.*

Proof. We note that extending $\nabla_{\xi,\eta}$ to a derivation of A amounts to choosing $\nabla_{\xi,\eta} y_k \in A$ for all $k \geq 1$. Any choice is valid and determines an extension. The relation $[d/dx, \nabla_{\xi,\eta}] = h(d/dx)$ applied to x implies $h = d\xi/dx$. The same relation applied to y_k for $k \geq 0$ yields $\nabla_{\xi,\eta} y_{k+1} = d(\nabla_{\xi,\eta} y_k)/dx - d\xi/dx \cdot y_{k+1}$. This determines the extension of $\nabla_{\xi,\eta}$ to A. The expression $D := [d/dx, \nabla_{\xi,\eta}] - d\xi/dx \cdot (d/dx)$ is again a derivation of A. By construction $D(y_k) = 0$ for $k \geq 1$. For $f \in R$, one easily calculates that $D(f) = 0$. This shows that $D = 0$. $\square$

The unique extension of the vector field $\nabla = \nabla_{\xi,\eta}$ to A is called the *prolongation* of ∇. In the sequel we will identify a vector field with its prolongation to A. The collection of all vector fields forms a Lie algebra $\mathcal{L}$ with respect to the Lie brackets defined by $[\nabla_1, \nabla_2] = \nabla_1\nabla_2 - \nabla_2\nabla_1$.

Definition 2.5. Let $\omega \in A$ be a differential equation. The Lie algebra of (point) symmetries $\mathcal{L}_\omega$ of ω is the Lie algebra consisting of the elements $\nabla \in \mathcal{L}$ satisfying $\nabla(\langle\omega\rangle) \subset \langle\omega\rangle$. We note that the last condition is equivalent to $\nabla(\omega) \in \langle\omega\rangle$.

Lemma 2.6. *Let $n \geq 2$ and suppose that $\omega \in A$ has the form y_n+ terms involving only y_k with $k < n$. Then $\nabla = \nabla_{\xi,\eta} \in \mathcal{L}$ is a Lie symmetry for ω if and only if $\nabla(\omega) = (\xi_x + \eta_y - (n+1)d\xi/dx)\omega$.*

Proof. The expressions $(d/dx)^m\omega$ have the form $y_{n+m}+$ terms involving only y_k for $k < n+m$. They generate $\langle\omega\rangle$. Any $f \in R[y_1, \dots, y_{n+m}]$ can be written uniquely as $f = q\cdot(d/dx)^m\omega + r$ with $q \in R[y_1, \dots, y_{n+m}]$ and $r \in R[y_1, \dots, y_{n+m-1}]$. It follows easily from this that $\langle\omega\rangle \cap R[\{y_k\}_{k<n}] = 0$. By induction on N, one easily shows that for $N \geq 2$, the expression ∇y_N is equal to $(\xi_x + \eta_y - (N+1)d\xi/dx))y_N+$ terms involving only y_k for $k < N$. We conclude that $\nabla(\omega) = (\xi_x + \eta_y - (n+1)d\xi/dx)y_n+$ terms involving only y_k with $k < n$. Then $\nabla(\omega) - (\xi_x + \eta_y - (n+1)d\xi/dx)\omega \in R[\{y_k\}_{k<n}]$. Now the statement follows. □

3. Linear equations and their symmetries

In the sequel, we will assume that $K := R^{\frac{\partial}{\partial y}}$ has one of the following forms:

(i) a differential field with an algebraically closed field of constants C.

(ii) Some ring of real or complex valued C^∞ functions. If this ring of functions is defined on some open connected subset U of $\mathbf{R}$, then it turns out that the Lie symmetries of an equation may change if U is replaced by some open connected subset of U (see Remark 4.3 (3)). We prefer, therefore, to work locally on $\mathbf{R}$ and consider C_0^∞, the ring of germs at $0 \in \mathbf{R}$ of complex valued C^∞-functions. This ring has zero divisors which may be not welcome in some calculations. Occasionally, we will make the assumption that some leading coefficient is invertible.

In order to guarantee that certain linear equations do have enough solutions, we have to extend K. In case (i) this means that K is replaced by some Picard-Vessiot extension. In case (ii) no extension is needed.

We consider only linear equations of the form $\omega = y_n + a_{n-1}y_{n-1} + \cdots + a_1y_1 + a_0y + a$ with $a_{n-1}, \dots, a_0, a \in K$ and leading coefficient 1. In case (i) the condition about the leading coefficient is superfluous. In case (ii) this is essential since a leading coefficient, which has a zero of a certain order or even has all its derivatives 0 at the point 0, will generate new and different phenomena.

The first step is a reduction to the homogeneous linear situation, i.e., $a = 0$. After extending K we may assume that there is a solution $f \in K$ of ω. We define new "variables" and derivations by

$$X = x, \quad Y = y - f, \quad \frac{\partial}{\partial X} = \frac{\partial}{\partial x} + (df/dx)\cdot\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial y}.$$

Further the Y_k are defined as $(d/dX)^kY$. These formulas are not seen as an automorphism of R and A. It just means that we have introduced new elements X, Y in the ring R and new elements Y_k in A, satisfying the same rules. In particular, ω has the same Lie symmetries as before. Now ω is homogeneous with respect to the $\{Y_k\}$ and has again leading coefficient 1. In the sequel we will consider only homogeneous equations.

3.1. **Equations of order 2 and symmetric powers.** We start with the linear homogeneous equation $\omega = y_2$. Its Lie algebra of symmetries consists of the $\nabla = \nabla_{\xi,\eta}$ satisfying $\nabla y_2 \in \langle y_2\rangle$. The formula for ∇y_2 reads

$$(\eta_y - 2\xi_x - 3\xi_y y_1)y_2 - \xi_{yy}y_1^3 + (\eta_{yy} - 2\xi_{xy})y_1^2 + (2\eta_{xy} - \xi_{xx})y_1 + \eta_{xx}.$$

This leads to the equations

$$\xi_{yy} = 0, \quad \eta_{yy} = 2\xi_{xy}, \quad 2\eta_{xy} = \xi_{xx}, \quad \eta_{xx} = 0.$$

It follows that ξ, η are elements of $R^{\frac{\partial}{\partial y}}[y]$ of degrees ≤ 2 and ≤ 3 with respect to the variable y. A straightforward calculation shows that the solution space has dimension 8 over C. Finally one can verify that the Lie algebra $\mathcal{L}_{y_2}$ is isomorphic to $\underline{\mathrm{sl}}(3)_C$.

A geometric interpretation can be given as follows. The integrals for y_2 are the lines $ax + by + c = 0$ in the plane. We can also see those lines as lines in the projective plane $\mathbf{P}^2_C$. The group of automorphisms of $\mathbf{P}^2_C$ is the projective linear group $\mathrm{PGL}(3)_C$. This group preserves the collection of lines. The Lie algebra of this group is $\underline{\mathrm{sl}}(3)_C$ and coincides, according to the formal calculation above, with $\mathcal{L}_{y_2}$.

The next case that we consider is $\omega = y_2 + ay_1 + by$ with $a, b \in K$. According to Lemma 2.6, $\nabla = \nabla_{\xi,\eta} \in \mathcal{L}_\omega$ satisfies $E := \nabla\omega - (\eta_y - 2\xi_x - 3\xi_y y_1)\omega$ is zero. A straightforward calculation yields $E = -\xi_{yy}y_1^3 + (\eta_{yy} - 2\xi_{xy} + 3a\xi_y)y_1^2 + *y_1 + *$ for some $* \in R$. We conclude that $\xi_{yy} = \eta_{yyy} = 0$, and so $\xi, \eta \in K[y]$. Thus for the calculation of $\mathcal{L}_\omega$ we may replace R by $K[y]$. This simplifies the situation. The coefficients of ξ, η with respect to the variable y are solutions of a set of linear differential equations with respect to the variable x. Instead of solving the equations, we will choose new "variables" and derivations such that ω reads as y_2 for the new variables.

We assume that there is an invertible element f, solution of ω. In case (i) one has to replace K by a Picard-Vessiot extension. In case (ii) a solution f with $f(0) = 1$ exists. Then one makes the following change of variables and derivations:

$$X = x, \quad Y = f^{-1}y, \quad \frac{\partial}{\partial X} = \frac{\partial}{\partial x} + \frac{f'}{f}y\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial Y} = f\frac{\partial}{\partial y}, \quad Y_k = (d/dX)^k Y.$$

For the new variables the function 1 is a solution of ω. Equivalently, we assume to begin with that the solution space of ω is $C1 + Cg$. In case (ii) we may suppose that $g(0) = 0$. This makes g into a local coordinate. Again we make a change of variables and derivations:

$$X = g, \quad Y = y, \quad \frac{\partial}{\partial X} = \frac{1}{g'}\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial y}, \quad Y_k = (d/dX)^k Y.$$

After this change we write x, y, etc., for the new variables and derivations. Now ω has the solution space $C1 + Cx$. This implies that $\omega = ay_2$ with $a \in K$ invertible. Thus we have reduced the calculation of the Lie symmetries of ω to the case y_2.

Proposition 3.1. *The algebra of Lie symmetries of $y_2 + ay_1 + by$ is, after a possible extension of K, isomorphic to $\underline{\mathrm{sl}}(3)_C$.*

We remark that for the case that we have avoided, namely $K = C^\infty(U)$ with $U \subset \mathbf{R}$, open and connected, the proposition remains valid. Indeed, the differential equations for the coordinates of ξ and η with respect to y are linear and have invertible leading coefficients. Locally on U there is a solution space of dimension 8 and locally the Lie algebra is isomorphic to $\underline{\mathrm{sl}}(3)$. By continuation of this system of solutions, this also holds globally on U.

We will also be interested in a slightly different kind of symmetry. Let K be as above. Then $K[\partial]$ denotes the skew ring of differential operators in which ∂ stands for the operator $\frac{d}{dx}$. The multiplication in this skew ring is given by $\partial a = a\partial + a'$, with $a' = \frac{da}{dx}$. For any operator $L \in K[\partial], L \neq 0$, with an invertible leading coefficient, one considers the operators $b\partial + a$ with $a, b \in K$, such that $L(b\partial + a)$ is a left multiple of L. We will call an element $b\partial + a$ with this property a *symmetry of the operator* L. The C-vector space of the symmetries of the operator L forms a Lie algebra with respect to the Lie-brackets given by

$$[b_1\partial + a_1, b_2\partial + a_2] = (b_1\partial + a_1)(b_2\partial + a_2) - (b_2\partial + a_2)(b_1\partial + a_1).$$

By comparing degrees and the leading coefficients, one sees that $b\partial+a$ is a symmetry if and only if there is an $\tilde{a} \in K$ with $L(b\partial + a) = (b\partial + \tilde{a})L$. We make a few observations:

(1) The case where L has degree 0, i.e., $L = A \in K^*$, is not so interesting, since any $b\partial + a$ is a symmetry. We note that $\tilde{a} = a - b\frac{A'}{A}$.

(2) Let the degree of L be $n > 0$ and write $L = A(\partial^n + \cdots)$. The solution space V of L, either in a Picard-Vessiot extension of K or in C_0^∞, is a vector space of dimension n over C. This solution space determines L up to its leading coefficient. If $b\partial + a$ is a symmetry of L, then clearly $b\partial + a$ maps V into V. On the other hand, suppose that $b\partial + a$ maps V into V. One writes $(b\partial + a)L = (b\partial + \tilde{a})L + R$ with R an operator of lower degree than L. Since R vanishes on V, one has $R = 0$. Thus $b\partial + a$ is a symmetry of L if and only if the operator $b\partial + a$ maps V into itself. Further $\tilde{a} = a + nb' - b\frac{A'}{A}$.

(3) The symmetries $b\partial + a$ of $a_1(\partial + a_0)$ are given by the equation $a = a_0 b + c$ for any $b \in K$ and $c \in C$.

(4) The symmetries $b\partial + a$ of $a_2(\partial^2 + a_1\partial + a_0)$ are given by the equations $2a' = -b'' + a_1 b' + a_1' b$ and $b''' + (4a_0 - 2a_1' - a_1^2)b' + (2a_0' - a_1 a_1' - a_1'')b = 0$. The solution space for the $b\partial + a$ clearly has dimension 4 over C.

Let L be the differential operator $\partial^2 + a_1\partial + a_0$. The $(n-1)$-st *symmetric power* L_n of L can be defined as follows. Let f_1, f_2 denote a basis over C of the solution space of L. Then L_n is the monic operator of degree n such that its space of solutions has basis $\{f_1^d f_2^e |\ 0 \leq d, e;\ d + e = n - 1\}$. There is another description of the operator L_n. Consider a nonzero solution e_0 of L. Then L_n is the monic operator, of smallest degree, with $L_n e_0^{n-1} = 0$.

Proposition 3.2. *Let L be the differential operator $\partial^2 + a_1\partial + a_0$.*

(a) *The $(n-1)$-st symmetric power L_n of L has the form*

$$\partial^n + b_n a_1 \partial^{n-1} + (c_n a_1^2 + d_n a_1' + e_n a_0)\partial^{n-2} + \cdots,$$

where $b_n = \frac{n(n-1)}{2}$, $c_n = \frac{n(n-1)(n-2)(3n-1)}{24}$, $d_n = \frac{n(n-1)(n-2)}{6}$, $e_n = \frac{(n+1)n(n-1)}{6}$ are positive integers.

(b) *L_n has the same symmetries as L.*

(c) *Suppose that $L_n(b\partial + a) - (b\partial + \tilde{a})L_n$, with $\tilde{a} = a + nb'$, has degree $\leq n - 3$. Then $L_n(b\partial + a) = (b\partial + \tilde{a})L_n$.*

Proof. (a) We describe an algorithm for the calculation of L_n, which will also prove the required formula. One considers a nonzero solution e_0 of L and its derivative e_1. By assumption, $e_0' = e_1$ and $e_1' = -a_0 e_0 - a_1 e_1$. One defines monic operators M_i of degree i for $i = 1, \ldots, n-1$, by $M_i e_0^{n-1} = (n-1)\cdots(n-i)e_0^{n-i-1}e_1^i$. Thus $M_1 = \partial$ and $M_{i+1} = (\partial + *a_1)M_i + *a_0 M_{i-1}$ holds for $i \leq n-2$ with certain integers $*$. For certain integers $*$, the operator $T := (\partial + *a_1)M_{n-1} + *a_0 M_{n-2}$ maps e_0^{n-1} to 0. Hence $L_n = T$.

By induction one proves that M_i has the form

$$\partial^i + *a_1\partial^{i-1} + (*a_1^2 + *a_1' + *a_0)\partial^{i-2} + \cdots$$

with integers $*$. The same formula holds then for L_n. The integers $*$ in the formula for L_n can be calculated by choosing special cases for a_1 and a_0.

(b) and (c). Again let f_1, f_2 be a basis of the solution space of L. If $b\partial + a$ is a symmetry for L, then $bf_i' + af_i$, for $i = 1, 2$, are in $Cf_1 + Cf_2$. Then $b\partial + a$ maps the solution space of L_n into itself and thus $b\partial + a$ is a symmetry for L_n.

One easily calculates that the C-vector space S of the $b\partial + a$ such that the degree of $L_n(b\partial + a) - (b\partial + a + nb')L_n$ is $\leq n - 3$, has dimension 4. S contains the space of symmetries of L, and therefore S is equal to the space of symmetries of L. □

3.2. **Equations of order** ≥ 3**.** We start by indicating a computation (see also [KM]).

Lemma 3.3. *Let $\nabla = \nabla_{\xi,\eta}$ be a Lie symmetry of the homogeneous linear element $\omega = y_n + a_{n-1}y_{n-1} + \cdots + a_1y_1 + a_0y \in A$ with all $a_j \in K$. If $n > 2$, then $\xi_y = 0$ and $\eta_{yy} = 0$.*

Proof. According to Lemma 2.6, we have that $\nabla(\omega) = (\xi_x + \eta_y - (n+1)d\xi/dx)\omega$. We prefer to work with the operator $M = \nabla - \xi(d/dx)$. This operator has the properties $[d/dx, M] = 0$ and $M(f) = (\eta - \xi y_1)f_y$ for $f \in R$. Thus M is K-linear and $M(y_k) = (d/dx)^k M(y) = (d/dx)^k(\eta - \xi y_1)$. The condition is now that the expression $E(\omega) := M(\omega) - (\xi_x + \eta_y - (n+1)d\xi/dx)\omega + \xi(d/dx)(\omega)$ is 0. We want to calculate the coefficients of the monomials y_1y_{n-1} and y_2y_{n-1} in $E(\omega)$. Write $\omega = y_n + \tilde{\omega}$. Then $E(\omega) = E(y_n) + E(\tilde{\omega})$. It can be seen that $E(\tilde{\omega})$ does not contain the terms y_2y_{n-1} and y_1y_{n-1}. Thus we have to compute those coefficients in the expression $E(y_n) = (d/dx)^n(\eta - \xi y_1) - (\xi_x + \eta_y - (n+1)d\xi/dx)y_n + \xi y_{n+1}$. The coefficient of y_2y_{n-1} turns out to be $-\binom{n+1}{2}\xi_y$; hence, $\xi_y = 0$. Using this, one computes that the coefficient of y_1y_{n-1} is $n\eta_{yy}$. Therefore, $\eta_{yy} = 0$. □

As before, we conclude that for the determination of Lie symmetries of ω we may replace R by $K[y]$ and A by $K[y, y_1, y_2, \ldots]$. In the sequel we will consider only vector fields $\nabla_{\xi,\eta}$ with $\xi_y = \eta_{yy} = 0$.

We want to identify the homogeneous elements of A with the operators $K[\partial]$. This will simplify the calculation and give some theoretical insight into the properties of Lie symmetries. Let Linear $\subset A$ denote the set of the linear expressions $a_ny_n + a_{n-1}y_{n-1} + \cdots + a_1y_1 + a_0y + a$ with $n \geq 0$ and $a_n, \ldots, a_0, a \in K$. The homogeneous expressions will be identified with Linear$/K \subset A/K$. One considers the K-linear isomorphism $\phi : K[\partial] \to$ Linear$/K$ given by $\phi(\sum_{i=0}^n a_i\partial^i) = \sum_{i=0}^n a_iy_i$ modulo K (here y_0 stands for y).

The operator d/dx acts on Linear$/K$ and corresponds with multiplication on the left by ∂ in $K[\partial]$. In formula $\phi(\partial \cdot L) = d/dx(\phi(L))$. Consider ξ, η with $\xi_y = \eta_{yy} = 0$ and the operator $M = \nabla_{\xi,\eta} - \xi(d/dx)$. This operator M on A commutes with d/dx, is K-linear, $M(1) = 0$, $M(y) = \eta - \xi y_1$, $M(y_k) = (d/dx)^k(M(y))$. We note that on Linear$/K$, the formula for $M(y)$ reads $\eta_y y - \xi y_1$. This operator corresponds with $\tilde{M}$ acting upon $K[\partial]$ and given by $L \mapsto L \cdot (-\xi\partial + \eta_y)$. The operator $\nabla = \nabla_{\xi,\eta}$ also acts on Linear$/K$ and corresponds with the operator $\tilde{\nabla}_{\xi,\eta}$ on $K[\partial]$, given by $L \mapsto L(-\xi\partial + \eta_y) + \xi\partial L$.

Proposition 3.4. *Fix a monic $L \in K[\partial]$ of degree $n > 2$ and let $\omega = \phi(L)$. The map $\nabla_{\xi,\eta} \mapsto \xi\partial - \eta_y$ provides a surjective homomorphism of $\mathcal{L}_\omega$, the algebra of Lie symmetries of ω, to the Lie algebra of the symmetries of the operator L. The kernel of this map consists of the Lie symmetries in $\mathcal{L}_\omega$ of the form $\nabla_{0,\eta}$ with η in the solution space of ω (or L).*

Proof. $\nabla_{\xi,\eta}$ belongs to $\mathcal{L}_\omega$ if and only if $\nabla_{\xi,\eta}\omega = (\eta_y - n\xi')\omega$ (see Lemma 2.6). This implies $\tilde{\nabla}_{\xi,\eta}L = (\eta_y - n\xi')L$ or equivalently $L(-\xi\partial + \eta_y) = (-\xi\partial + \eta_y - n\xi')L$. In other words, $\xi\partial - \eta_y$ is a symmetry for the operator L. A straightforward calculation shows that the map of the propostion is a homomorphism from the Lie algebra $\mathcal{L}_\omega$ to the Lie algebra of the symmetries of L. The kernel of this map consists of the $\nabla_{0,\eta} \in \mathcal{L}_\omega$ with $\eta_y = 0$. It is easily seen that the η's with $\eta_y = 0$ and $\nabla_{0,\eta} \in \mathcal{L}_\omega$ are just the elements in K (or in a Picard-Vessiot extension of K) with $L(\eta) = 0$.

Now let $b\partial + a$ be a symmetry of L. Define $\xi = b$ and $\eta = \eta_0 - ay$ with, for the moment, an unknown $\eta_0 \in K$. By construction, the required equation $\nabla_{b,-ay}\omega = (\eta_y - n\xi')\omega$ holds modulo K. Thus we have to choose η_0 such that $\nabla_{0,\eta_0}\omega = f := (\eta_y - n\xi')\omega - \nabla_{b,-ay}\omega \in K$. This is the differential equation $L(\eta_0) = f$ and has a solution (after taking a Picard-Vessiot extension of K). This shows that the map is surjective. □

Theorem 3.5. *Let $L \in K[\partial]$ be a monic operator of degree $n \geq 3$ and let $\omega = \phi(L)$. Then:*

(a) *The dimension of $\mathcal{L}_\omega$ can only be $n+1, n+2$ or $n+4$.*

(b) *The dimension is $n+4$ if and only if L is the $(n-1)$-st symmetric power of a monic operator of degree 2.*

(c) *The dimension is $n+2$ if and only if L is not an $(n-1)$-st symmetric power of an operator of degree 2 and moreover there is a $\Delta := b\partial + a$ with b invertible and there are constants $c_i \in C$ such that $c_n = 1$ and $L = b^{-n}\sum_{i=0}^{n} c_i\Delta^i$. In the case where K is a differential field, b and a belong to some Picard-Vessiot extension of K.*

Proof. (a) and (b). By Proposition 3.4 we have to show that the dimension of the Lie algebra of the symmetries of L can only be $1, 2$ or 4. Of course, any constant $c \in C$ is a symmetry for L and this dimension is at least 1. Using Proposition 3.2, there is a unique monic operator L_2 of degree 2 such that $L = L_n + R$, where L_n is the $(n-1)$-st symmetric power of L_2 and R is an operator of degree $m \leq n-3$. If $b\partial + a$ is a symmetry of L, then $L_n(b\partial + a) - (b\partial + \tilde{a})L_n$, with $\tilde{a} = a + nb'$, has degree $\leq n-3$. Thus $(b\partial + a)$ is a symmetry of both L_n and R. If $R = 0$, then the dimension of the space of symmetries of L is 4. If $R \neq 0$ then $R = A\partial^m + \cdots$ and $A\tilde{a} = A(a + mb') - bA'$ holds. Together with $\tilde{a} = a + nb'$ this implies $b^{n-m}A$ is a nonzero constant. This property can at most be valid for a single b (up to a multiple). Therefore, the dimension of the symmetries of L is at most 2.

(c) If L has the form of the statement, then clearly Δ is a symmetry of L and the dimension of the Lie symmetries of ω is $\geq n+2$. On the other hand, suppose that L has a symmetry $\Delta := b\partial + a$ with b invertible (b and a are allowed to lie in a Picard-Vessiot extension of K). Then we can of course write $L = b^{-n}M$ with $M := \sum_{i=0}^{n} c_i\Delta^i$, where $c_n = 1$ (and all c_i in some Picard-Vessiot extension of K). The condition $L\Delta = (\Delta + nb')L$ translates into $M\Delta = \Delta M$, and this easily implies that all c_i are constants. □

Corollary 3.6. *The operator* $L = b^{-n}\sum_{i=0}^{n} c_i(b\partial + a)^i$, *with* $c_i \in C$ *and* $c_n = 1$, *has a four-dimensional space of symmetries if and only if there are constants* $\lambda_1, \lambda_2 \in C$ *such that all the roots, counted with multiplicity, of the polynomial* $\sum c_i T^i$ *are* $\{m\lambda_1 + (n-1-m)\lambda_2 |\ m = 0, \ldots, n-1\}$.

Proof. If L is the $(n-1)$-st symmetric power of a monic operator L_2, then the symmetry $b\partial + a$ of L is also a symmetry of L_2. Thus $L_2 = b^{-2}(\Delta^2 + d_1\Delta + d_0)$, with $\Delta = b\partial + a$ and $d_1, d_0 \in C$. Write $L_2 = b^{-2}(\Delta - \lambda_1)(\Delta - \lambda_2)$ with $\lambda_i \in C$. It is easily seen that the $(n-1)$-st symmetric power of L_2 is

$$b^{-n}\prod_{m=0}^{n-1}(\Delta - (m\lambda_1 + (n-1-m)\lambda_2)).$$

The proof follows from this formula. □

4. Galois theory and Lie symmetries

In this section we suppose that K is a differential field with an algebraically closed field of constants C of characteristic 0.

Theorem 4.1. *Let* $L \in K[\partial]$ *be a monic operator of degree* $n > 2$ *such that the homogeneous equation* $\phi(L)$ *has an* $n+4$*-dimensional space of Lie symmetries (over a Picard-Vessiot extension of* K*). By Theorem 3.5,* L *is the* $(n-1)$*-st symmetric power of a monic operator* $L_2 \in K[\partial]$ *of degree* 2*. Then the following holds*:

(a) *The differential Galois group of* L *is isomorphic to* G/N*, where* $G \subset \mathrm{GL}(2)_C$ *is the differential Galois group of* L_2 *and* N *is the finite normal subgroup*

$$N = G \cap \{\lambda \cdot id |\ \lambda^{n-1} = 1\}.$$

(b) *Let* $\tilde{K} \supset K$ *be the Picard-Vessiot field for* L_2*. Then the Lie symmetries* $\nabla_{\xi,\eta}$ *of* $\phi(L)$ *are defined over* $\tilde{K}$*, i.e.,* $\xi, \eta \in \tilde{K}[y]$.

Proof. Let V be the solution space of L_2 with its G-action. Then $\mathrm{sym}^{n-1} V$ is the solution space for L and the induced action of G on this space is the differential Galois group of L. This proves (a). For (b), it suffices to show that a symmetry $b\partial + a$ of L has coordinates $a, b \in \tilde{K}$. Let f_1, f_2 be a basis of V. Then $bf_1' + af_1, bf_2' + af_2 \in V \subset \tilde{K}$. Hence $a, b \in \tilde{K}$. □

We now take a closer look at part (c) of Theorem 3.5. Let $\tilde{K} \supset K$ be a Picard-Vessiot field which contains the coordinates a and b of $\Delta = b\partial + a$. The relation between a and b is given by the formula $2a' = -b'' + a_1b' + a_1'b$ of subsection 3.1, where the operator $\partial^2 + a_1\partial + a_0 \in K[\partial]$ is such that the degree of $L - \mathrm{sym}^{n-1}(\partial^2 + a_1\partial + a_0)$ is $\leq n-3$. After shifting Δ over a constant, we may therefore suppose that $2a = -b' + a_1b$.

Let σ be a differential automorphism of $\tilde{K}/K$. Then $\sigma(\Delta) = \sigma(b)\partial + \sigma(a)$ is also a symmetry of $L \in K[\partial]$. By assumption we are in the "$n+2$ case". Thus $\sigma(\Delta) = \lambda\Delta + \mu$ with $\lambda \in C^*, \mu \in C$. Then $\sigma(b) = \lambda b$ and, by our choice of a, also $\sigma(a) = \lambda a$. We conclude that $\mu = 0$.

Since $L \in K[\partial]$, we have $\sigma(L) = L$ and thus $\lambda^{-n}b^{-n}\sum c_i\lambda^i\Delta^i = b^{-n}\sum c_i\Delta^i$. We conclude that λ is a root of unity, since L is not in the "$n+4$ case". Therefore $K(b) \supset K$ is a cyclic extension of some degree d and $\frac{a}{b} \in K$. Further, L can be written as $b^{-n}\sum_{i=0}^{[n/d]} d_i\Delta^{n-id}$ with all $d_i \in C$ and $d_0 = 1$.

Theorem 4.2. *Let $L \in K[\partial]$ be a monic operator of degree $n > 2$, and suppose that the homogeneous equation $\phi(L)$ has, over a Picard-Vessiot extension, an $n+2$-dimensional space of symmetries. Let $b\partial + a$ with $b \neq 0$ be a symmetry of L. Then $K(b) \supset K$ is a finite cyclic extension. The differential Galois group of L over the field $K(b)$ is an algebraic subgroup of the commutative algebraic group $\mathbf{G}_{m,C}^s \times \mathbf{G}_{a,C} = (C^*)^s \times C^+$ for some $s \leq n$.*

Proof. The only new statement is the one about the differential Galois group. In proving this we may suppose that $b \in K$ and $L = b^{-n} \prod_{i=1}^{s} (\Delta - \lambda_i)^{m_i}$ with distinct $\lambda_i \in C$ and $\sum m_i = n$. Let us first consider the case where all m_i are 1. The Picard-Vessiot field $\tilde{K} \supset K$ contains nonzero elements f_i for $i = 1, \ldots, n$ satisfying $(\Delta - \lambda_i)(f_i) = 0$. The solution space of L is $Cf_1 \oplus \cdots \oplus Cf_n \subset \tilde{K}$. Any differential automorphism σ of $\tilde{K}/K$ satisfies $\sigma(f_i) = c_i f_i$ for some $c_i \in C^*$. Thus the differential Galois group G is an algebraic subgroup of the torus $(C^*)^n$.

Suppose now that, say, $m_1 > 1$. Define the nonzero elements $f_1, \ldots, f_s \in \tilde{K}$ by $(\Delta - \lambda_i) f_i = 0$. Let hf_1 satisfy $(\Delta - \lambda_1)(hf_1) = f_1$. Then hf_1 is in the solution space of L. Therefore $h \in \tilde{K}$. A straightforward calculation shows that $h' = b^{-1}$. The solution space is now seen to have the basis $\{f_i h^d \mid i = 1, \ldots, s \text{ and } d = 0, \ldots m_i - 1\}$. The action of a differential automorphism σ of $\tilde{K}/K$ is seen to have the form $\sigma(f_i) = c_i f_i$ with $c_i \in C^*$ and $\sigma(h) = h + c$ with $c \in C$. Thus the differential Galois group is an algebraic subgroup of $(C^*)^s \times C$. □

Remarks 4.3.

(1) We have seen that the monic operators L of degree $n \geq 3$ such that $\phi(L)$ has a Lie algebra of symmetries of dimension $n+2$ or $n+4$ have a rather special form and as a consequence their differential Galois groups are also special. The converse is not true as we will indicate.

Consider the case $K = \mathbf{C}(z)$. It is well known that any algebraic subgroup $G \subset \mathrm{GL}(n, \mathbf{C})$ is the differential Galois group of a differential module M over K of dimension n. We want to produce a cyclic vector $e \in M$ such that the monic operator $L \in K[\partial]$ of degree n, defined by $Le = 0$, admits only constants as symmetries. This corresponds with the "$n+1$ case" for the Lie algebra of symmetries for $\phi(L)$.

Consider the special case where the differential module M has a trivial differential Galois group. It is not difficult to produce a $\mathbf{C}$-vector space $V \subset K$ with dimension n such that the only b, a satisfying $bf' + af \in V$ for all $f \in V$ are $b = 0$ and $a \in \mathbf{C}$. The monic differential operator $L \in K[\partial]$ with kernel V corresponds with a cyclic vector for M and has only constants as symmetries. Explicit calculations for the assertion in the general case are rather awkward; instead, we will give an intuitive reasoning for the assertion that the set of the $e \in M$, such that either e is not cyclic or e is cyclic but the corresponding operator L has more symmetries, is a "thin" subset of M and in particular $\neq M$.

Let S denote the union of the differential submodules $N \subset M$ with $N \neq M$. A vector $e \in M$ is cyclic if and only if $e \notin S$. It is well known that $S \neq M$, i.e., there exists cyclic vectors. An alternative way to see this is to observe that there are relatively few proper differential submodules of M, since those submodules correspond to proper $\mathbf{C}$-linear, G-invariant subspaces of the solution space $\ker(\partial, \tilde{K} \otimes_K M)$, where $\tilde{K} \supset K$ is the Picard-Vessiot field for M. Thus the proper differential submodules of M can be parametrized by finitely many algebraic varieties over $\mathbf{C}$. Then S is "thin".

Now we consider a cyclic $e \in M$ such that L happens to be in the "$n+4$ case". Then L is the $(n-1)$-st symmetric power of an operator of degree 2. Let N denote the corresponding differential module of dimension 2. One can reformulate the above as follows: There is an isomorphism $\psi : \mathrm{sym}^{n-1} N \to M$ and a cyclic vector f for N such that e is the image of $f \otimes \cdots \otimes f$. We fix a basis of N and we fix the isomorphism ψ. This determines a basis of M over K and the coordinates of the element e, with respect to the induced basis of M, satisfy some homogeneous equations over K. The module N is unique up to tensoring by a suitable one-dimensional module. The choice of ψ is not unique, the collection of all ψ's is parametrized by an algebraic subgroup of $\mathrm{GL}(n, \mathbf{C})$. Thus the set of e's, such that the corresponding operator L is in the "$n+4$ case", is the union of a family of proper algebraic subsets (over K) of M, parametrized by an algebraic variety over $\mathbf{C}$. This defines a "thin" subset of M.

Suppose that L happens to be in the "$n+2$ case". With the notation of Theorem 3.5, we have that $L = b^{-n} \sum_{i=0}^{n} c_i \Delta^i$, with $c_i \in \mathbf{C}$ and $\Delta = b\partial + a$. For a fixed L of this form the collection of e's with $Le = 0$ is a finite-dimensional subspace over $\mathbf{C}$. One can vary the operator L by varying b, a and the $c_i \in \mathbf{C}$. The union of all the elements $e \in M$ with $Le = 0$ for some L of the above form is again a "thin" subset of M.

(2) For an operator L (say monic of degree n), one can identify the endomorphisms of the differential module $K[\partial]/K[\partial]L$ with the operators M of degree $< n$ such that LM lies in $K[\partial]L$. Over a suitable Picard-Vessiot extension of K the space of all endomorphisms is isomorphic with the Lie algebra $\mathrm{gl}(n)$ of the matrices of size $n \times n$. The formalism of Lie symmetries asks for those endomorphisms M which have degree ≤ 1. This is due to the restriction to infinitesimal transformation (or vector fields) in the plane. If one would allow vector fields involving $x, y, y', \ldots, y^{(n-1)}$, then the Lie algebra of symmetries would be (for every L) isomorphic to the semidirect sum $V \rtimes \mathrm{End}(V)$, where V is the solution space of L. The elements of this space will be written as pairs (v, A) with $v \in V$ and $A \in \mathrm{End}(V)$. The Lie algebra structure is given by $[(v_1, A_1), (v_2, A_2)] = (A_1 v_2 - A_2 v_1, [A_1, A_2])$. The restriction to endomorphisms represented by operators M of degree ≤ 1 replaces the part $\mathrm{End}(V)$ by a Lie subalgebra E, containing the identity, and with dimension 1, 2 or 4. Using Proposition 3.4, one easily determines this Lie subalgebra E:

(a) In the "$n+4$ case", the operator L is the $(n-1)$-st symmetric power of an operator L_2 of degree 2. The solution space V can be identified with $\mathrm{sym}^{n-1} W$, where W is the solution space of L_2. The symmetries of L_2 are identified with $\mathrm{End}(W)$ (acting upon W). The injective homomorphism of Lie algebras $\mathrm{End}(W) \to \mathrm{End}(V)$ has image E. The structure of the Lie algebra of symmetries does not depend on L in this case.

(b) In the "$n+2$ case", the Lie subalgebra E is the two-dimensional space generated by the identity and the expression $b\partial + a$ in the notation of Theorem 3.5(c). In this case the Lie algebra depends on the operator L, since the action of $b\partial + a$ on the solution space V depends on L.

(3) The situation where $K = C^{\infty}(U)$ with U an open connected subset of $\mathbf{R}$ is slightly more complicated than C_0^{∞}. Indeed, for a monic operator $L \in K[\partial]$ of

degree $n \geq 3$, the dimension of the space of Lie symmetries may vary from point to point on U. This is due to the presence of C^∞ functions on U with compact support. An example is $\partial^n + f,\ n > 2$, with $f \in C^\infty(\mathbf{R}),\ f \neq 0$, which has support in $[0,1]$. For an open connected set which has empty intersection with $[0,1]$, the dimension of the space of Lie symmetries is $n+4$. For other connected open sets, the dimension is at most $n+2$.

5. An algorithm for the symmetries

Let $L \in K[\partial]$ be a monic operator of degree n and let $\phi(L) \in A$ denote its equation. We present here a new algorithm which computes the Lie symmetries of $\phi(L)$. The Lie symmetries of $\phi(L)$ are easily deduced from the symmetries of the operator. For $n > 2$ this follows from the formula $-\xi\partial + \eta_y = b\partial + a$, given in the proof of Proposition 3.4. What follows is an algorithm computing the symmetries of the operator L.

For $n = 1, 2$, the explicit solutions for the symmetries $b\partial + a$ are given in Section 3. For $n > 2$, we will use the symmetric powers of an operator $\partial^2 + a_1\partial + a_0$ of degree 2. Symmetric powers of operators have been implemented in "DEtools" of "MAPLE". For the special case, which is needed here, the proof of part (a) of Proposition 3.2 provides another algorithm. We will now give the steps of the algorithm.

(1) L has degree $n > 2$ and can, by using part (a) of Proposition 3.2, be written as $\mathrm{sym}^{n-1}(\partial^2 + a_1\partial + a_0) + R$, where R has degree $\leq n-3$. If $R = 0$, then L and $\partial^2 + a_1\partial + a_0$ have the same symmetries. They are given by the solutions of the equations $2a' = -b'' + a_1 b' + a_1' b$ and $b''' + (4a_0 - 2a_1' - a_1^2)b' + (2a_0' - a_1 a_1' - a_1'')b = 0$. This is "class $n+4$".

(2) If $R \neq 0$, then we write $R = AL_1$, where L_1 is monic of degree $n_1 \geq 0$. The only possibility for $b \neq 0$ is (a nonzero multiple of) $A^{-1/(n-n_1)}$. We will give b the value $A^{-1/(n-n_1)}$ and have to verify that $b\partial + a$, with $2a' = -b'' + a_1 b' + a_1' b$, is a symmetry of both $\partial^2 + a_1\partial + a_0$ and L_1. For the first operator the verification is obvious. We can then continue with L_1.

(3) For $n_1 = 0, 1, 2$, the verification follows from the explicit formulas in Section 3. If $n_1 > 2$, then we write again $L_1 = \mathrm{sym}^{n_1-1}(\partial^2 + b_1\partial + b_0) + R_1$ such that the degree of R_1 is $\leq n_1 - 3$. We have to verify that $b\partial + a$ is a symmetry of $\partial^2 + b_1\partial + b_0$. If $R_1 \neq 0$ has degree n_2, then we write $R_1 = b^{n_2 - n_1} L_2$. The condition is now that the operator L_2 has a constant leading coefficient and admits $b\partial + a$ as a symmetry. Recursion completes the algorithm.

References

[K] I. Kaplansky, *An introduction to differential algebra*, Second edition, Hermann, Paris, 1976. MR **57:**297

[KM] J. Krause and L. Michel, *Équations différentielles linéaires d'ordre* $n > 2$ *ayant une algèbre de Lie de symétrie de dimension* $n+4$, C.R. Acad. Sci. Paris Sér. I Math. **307** (1988), 905–910. MR **89m:**58183

[L1] S. Lie, *Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen*, Bearbeitet und herausgegeben von Dr. G. Scheffers, Teubner, Leipzig, 1893.

[L2] S. Lie, *Klassifikation und Integration von gewöhnlicher Differentialgleichungen zwischen* x, y, *die eine Gruppe von Transformationen gestetten* I, Math. Ann. **22** (1888), 213–253.

[ML] F.M. Mahomed and P.G.L. Leach, *Symmetry Lie algebras of* n*th order ordinary differential equations*, J. Math. Anal. Appl. **151** (1990), 80–107.

[O1] P.J. Olver, *Applications of Lie groups to differential equations*, Springer Verlag, New York, 1986. MR **88f:**58161

[O2] P.J. Olver, *Equivalence, invariants and symmetry.* Cambridge University Press, Cambridge, 1995. MR **96i:**58005

[P] M. van der Put, *Galois theory of differential equations, algebraic groups and Lie algebras. Differential algebra and differential equations.* J. Symbolic Comput. **28** (1999), 441–472. MR **2001b:**12012

[St] H. Stephani, *Differential equations. Their solution using symmetries*, Cambridge University Press, Cambridge, 1989. MR **91a:**58001

Prinsengracht 275 Den Haag, The Netherlands
E-mail address: `woudshoo@sctcorp.com`

Department of Mathematics, P.O. Box 800, 9700 AV, Groningen, The Netherlands
E-mail address: `mvdput@math.rug.nl`