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SOM-theme F Interactions between consumers and firms

Abstract

Some authors recommend the use of an additional test for detecting increases in the spread, when using a control chart for individual observations. We examine this recommendation both in a practical situation and theoretically. Both studies show that the additional test gives somewhat more power for detecting a 25% increase of the process variation. For nearly all other deviations from the in-control state the test is more likely to cause confusion. From a practical viewpoint we therefore advise against its use.

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1 Introduction

Control charts for individual observations are frequently used in industry. Such charts are useful because in some applications it may be impossible to collect more than one observation per sample. For example, this is often the case in process industries, where parameters like temperatures and concentrations are monitored. In such cases the $\bar{X}$-R chart (or affiliates) cannot be used since it is impossible to calculate the within-sample variation when the sample size equals one.

A disadvantage of the individuals chart is that every departure from the in-control situation is signalled on only one chart, whereas the $\bar{X}$-R chart monitors changes in the process mean and the process variation separately. Some authors therefore recommend the use of an extra chart, the Moving Range chart (MR-chart), in addition to the individuals chart (X-chart). This suggestion is subject to controversy, however. In a later section we will cite several authors who claim that the additional MR-chart leads to extra confusion, and does not sufficiently improve monitoring performance.

The discussion whether or not to use an additional MR-chart also arose at Philips Semiconductors in Stadskanaal, a QS9000 certified manufacturer of medium power diodes. Before the QS9000 audit Philips was mainly using individuals charts, as advised by Roes and Does (1995). During the QS9000 audit, the auditor urgently advised to 'improve' the power of the SPC system by introducing supplementary Western Electric runs rules. A set of runs rules was selected to oblige the auditor, and as a result the charts gave many additional signals. However, it remained unclear whether these extra signals were caused by either an increased probability of a 'false' signal, or by out-of-control situations that were not detected before (such as changes in the spread). Moreover, the question was raised whether it was useful to add an MR-chart to the existing system for faster detection of changes in the spread.

This article can be divided into two parts. In the first part we investigate whether the MR-chart would be useful in the particular runs-rules context of Philips Semiconductors in Stadskanaal. To this end we evaluate the use of an additional runs rule introduced by Page (1955) that can be considered as a slight modification of the standard MR-chart. Using Page's runs rule, monitoring for a change in the spread fits smoothly within the runs-rules framework, and an additional chart for monitoring the spread is not needed.

We investigate the change in performance when Page's runs rule is added to the existing set of runs rules used at Philips Semiconductors in Stadskanaal. The results indicate that only for specific out-of-control situations – a small increase of the spread

– some minor improvement is attained.

In the second part of this article, we turn from the specific setting of Philips Semiconductors to the more general question whether it is useful to add an *MR*-chart to an individuals chart. We summarize the discussion in recent literature on this point of controversy. Subsequently, we study the change in the Average Run Length (ARL) behavior when an *MR*-chart is added to a standard individuals chart. The results agree with the conclusions obtained for the runs rules setting of Philips Semiconductors Stadskanaal.

Throughout this article, we compare the performance of the different monitoring schemes assuming that they are applied to a process that results in independent and identically distributed normal random variables. Details concerning the computation of the ARL of an individuals chart that is combined with Page's runs rule can be found in Appendix A. For numerical evaluation of the ARL of the combined X-MR-chart, we developed a more flexible version of the integral equation approach of Crowder (1987, 1987a). The technical details can be found in Appendix B.

2 Rules for detection of out-of-control situations

The standard rule of an individuals chart to issue a signal is:

Rule 1: *A control limit is exceeded.*

Usually, the so-called 3σ-limits are taken as the control limits. Three of the many Western Electric runs rules were introduced at Philips Semiconductors Stadskanaal to improve the power of the control chart:

Rule 2: A signal is issued when two out of three measurements are in the same warning zone (the region between a warning limit – usually taken as the so-called 2σ-limits – and the corresponding control limit).

Rule 3: A run of six consecutive measurements either increasing or decreasing.

Rule 4: A run of nine measurements above or below the central line (CL).

The reason for exactly these three runs rules is rather arbitrary. In fact they were selected because it was felt that they were easy to comprehend by operators. This is an important argument because operators may have to stop the process when a signal is given: it certainly helps when they are convinced that there is really something the matter with the process.

An individuals chart is sometimes supplemented with an *MR*-chart, plotting the successive ranges of two consecutive measurements, issuing a signal when the upper control limit is exceeded. The *MR*-chart operates similarly to the following runs rule:

Rule A: Two successive measurements in the opposite warning zones.

Page (1955) discussed Rule A (and related rules) before there was any mention of the *MR*-chart:

> "the occurrence [...] of two near samples outside opposite warning lines points to an increase in the spread of the distribution".

From the results in the second part of this paper, it can be inferred that Rule A is less sensitive to out-of-control situations than the *MR*-chart, but the differences in behavior are small.

Another one of Page's rules was studied by Albin et al. (1997). They investigate the performance of a direct alternative to Rule 2: two out of three measurements in opposite warning zones. However, Rule A is a more direct translation of the *MR*-chart and a really simple one: there is no need for an additional chart, or for a movable transparency, as devised by Adke and Hong (1997). A signal due to Rule A is comparable to exceeding the upper control limit of an *MR*-chart. Theoretically, a lower control limit can be computed for the *MR*-chart as well. However, this limit is not useful, neither in theory (Wieringa (1999) shows that additional power in the one-sided case is larger than in the two-sided case) nor in practice. The lower control limits would be so small that in practice, a signal is issued only when two consecutive measurements are equal, which happens much more often in real life than in theory, due to rounding off of the data.

The purpose of the following section is to evaluate the change in performance of the individuals chart when Rule A is added to the existing set of runs rules. The basic question is whether Rule A helps to identify out-of-control processes, without generating too many false alarms.

3 Performance of the runs rule for variation

In this section, we study the performance of the individuals chart in combination with the aforementioned runs rules through Monte Carlo simulations. The method of Champ and Woodall (1987) for calculating exact run-length probabilities cannot be used in this case because Rule 3 does not fit into the class of runs rules they considered. We computed the control and warning limits assuming an in-control mean of μ_0, and an in-control standard deviation of σ_0. We simulated data from normal distributions, for several combinations of shifts in the mean and shifts in the standard deviation. The size of the shifts in the mean ranges from 0 up to 2.5 units of σ_0. The simulated standard deviations range from $0.25\sigma_0$ up to $3\sigma_0$. We simulated five series of 1,000,000 consecutive measurements for each situation . The resulting ARLs of these five series are presented in Table 1 (the standard errors of these ARLs are generally less than 0.5% of the estimated values except in the upper left-hand corner, where a maximum of 1.6% is attained). We adopted the convention that all rules are reset after a signal has been given – which agrees with the sound practice of bringing the process in control before it is continued.

Table 1: Average Run Lengths without and with Rule A

Without Rule A

σ/σ_0	$(\mu-\mu_0)/\sigma_0$					
	0	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**
0.50	433	23.5	10.1	8.08	4.24	2.17
0.75	417	47.2	13.3	7.10	3.88	2.28
1.00	**151**	43.6	13.0	6.19	3.55	2.29
1.25	38.7	22.3	9.93	5.31	3.30	2.29
1.50	16.0	12.3	7.39	4.59	3.11	2.28
2.00	6.25	5.73	4.65	3.59	2.79	2.23
2.50	3.89	3.75	3.38	2.94	2.51	2.15
3.00	2.93	2.87	2.72	2.50	2.27	2.04

With Rule A

σ/σ_0	$(\mu-\mu_0)/\sigma_0$					
	0	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**
0.50	433	23.5	10.1	8.08	4.24	2.17
0.75	411	47.1	13.3	7.10	3.88	2.28
1.00	**135**	42.5	13.0	6.18	3.55	2.29
1.25	34.2	21.0	9.78	5.29	3.30	2.29
1.50	14.5	11.5	7.18	4.55	3.10	2.28
2.00	5.86	5.42	4.48	3.51	2.76	2.22
2.50	3.72	3.60	3.27	2.87	2.47	2.13
3.00	2.83	2.78	2.64	2.45	2.23	2.02

The in-control ($(\mu-\mu_0)/\sigma_0 = 0,\ \sigma/\sigma_0 = 1$) ARL decreases from 151 to 135 (10.6%) when Rule A is added to the SPC system with runs Rules 1-4. The only situation with a larger decrease in ARL is a slightly larger standard deviation ($(\mu-\mu_0)/\sigma_0 = 0,\ \sigma/\sigma_0 = 1.25$), when the ARL decreases from 38.7 to 34.2 (11.6%). For all other simulated combinations of shifts in μ and σ the decrease in ARL is less than 10%. So the power improvement due to Rule A is maximal when the process variation is slightly larger than in-control; the gain compared to the extra signals when the process is in control is only marginal, however.

In Table 2 the contribution of Rule A to the performance of the SPC system is displayed. For every simulated combination of a shift in the mean and a shift in the standard deviation the percentage of signals from Rule A is given. For evaluation of the effects of the individual runs rules we adopted another convention that each signal will be attributed to only one runs rule: the rule with the shortest window. Hence, Rule 1 gets always first priority, then Rule A, and so on.

Table 2: Percentage of Signals from Rule A.

σ/σ_0	$(\mu-\mu_0)/\sigma_0$					
	0	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**
0.50	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
0.75	1.3%	0.1%	0.0%	0.0%	0.0%	0.0%
1.00	**11.4%**	2.8%	0.4%	0.1%	0.0%	0.0%
1.25	*13.1%*	6.7%	1.9%	0.4%	0.1%	0.0%
1.50	11.4%	8.0%	3.5%	1.3%	0.4%	0.1%
2.00	7.9%	6.8%	4.7%	2.7%	1.4%	0.7%
2.50	5.6%	5.2%	4.2%	3.0%	2.0%	1.2%
3.00	4.0%	3.9%	3.4%	2.7%	2.1%	1.5%

Table 2 confirms the conclusion from Table 1, that the percentage of signals due to Rule A is maximal when the standard deviation is slightly higher than in-control. The gain compared to the in-control percentage (11.4%) is small, however. A signal from Rule A might thus point at a small increase of the standard deviation, but it might just as well mean that the process is still in control!

Trip (2000) investigated the practical problem of how operators should interpret the signals from the different runs rules. He simulated several combinations of shifts in the mean and in the standard deviation, and looked at the number of signals from the different runs rules. Based on this knowledge, he was able to assign the most likely out-of-control situation to a signal from a specific runs rule. He concluded that it is nearly impossible to designate a specific out-of-control situation for which Rule A is the most suitable runs rule. The best conclusion from a signal from Rule A will be that the process mean is still on target; there might be a small increase in variation. This knowledge provides useful information for the design of Out-of-Control Action Plans (OCAP) (see Sandorf and Bassett (1993)). At Philips in Stadskanaal Rule A was not seen as beneficial in this respect, and is therefore not implemented.

4 Discussion of recent literature

In the previous sections, the additional *MR*-chart was discussed from a practical point of view. Instead of the *MR*-chart itself, we examined an alternative runs rule, that behaves similarly (which can be inferred from the results in the next section). In the remaining part of this paper, we perform a theoretical analysis of the value of adding an *MR*-chart, and confirm our earlier conclusions. Before doing so, we start with a discussion of recent literature because there is a controversy on the usefulness of an additional *MR*-chart.

Among authors favoring an additional control chart for the spread are Duncan (1986), Wheeler and Chambers (1992), Wetherill and Brown (1991), and Montgomery (1996). On the other side there is Nelson (1982, 1990), who strongly advises against such a chart based on two arguments. Firstly, he argues that interpretation is complicated by the serial correlation of successive points on the *MR*-chart. His second argument is that the individuals chart itself already contains all the information available.

For the case of individual observations $X_1, X_2, \cdots$ the chart for the spread is usually based on the moving range of two consecutive measurements: $MR_t = |X_t - X_{t-1}|$ $(t \geq 2)$. Roes et al. (1993) computed the conditional probability (assuming independence of the observations) of observing a signal on the *MR*-chart, given that the X-chart itself does not signal. These probabilities are small for the out-of-control situations. Therefore they concluded that the contribution of the *MR*-chart to the power of discovering an out-of-control situation is small. Wieringa (1999) agrees with this conclusion, but not with the argument. A small probability of a signal on the *MR*-chart may be due to a poor design of the chart, e.g. the limits are too wide. What matters are the differences between the probability of signal in the in-control situation and the probabilities of a signal in out-of-control situations. For useful control charts, these differences are large. For the additional *MR*-chart, these differences are small.

Adke and Hong (1997) came to the opposite conclusion. They computed the conditional probability of a signal on the *MR*-chart between $t + 1$ and $t + n$ ($n \geq 2$, given that the X-chart does not signal in that period. They assume that the process is already out of control at time $t + 1$, whereas Roes et al. (1993) conditioned upon the situation that only the last observation is out of control. The probabilities of Adke and Hong (for $n = 2$) are therefore larger than those of Roes et al. Wieringa (1999) shows that Adke and Hong use essentially a one-sided *MR*-chart (not a two-sided one, as they claim).

Rigdon et al. (1994) followed Wetherill and Brown (1991) and examined ranges of not just two, but also of three and four consecutive observations. They selected control limits for the combined X-*MR* chart so that the in-control ARL was the same as for the X-chart alone. Their conclusion was that for shifts in the process mean the X-chart alone is more effective, while this chart is about equally effective in detecting changes in the process variability as the combined X-*MR* chart.

Amin and Ethridge (1998) believe that the relatively poor performance of Rigdon's combined X-MR chart is due to the choice of the chart parameters. But apart from ARL considerations they have additional reasons to recommend the use of the combined X-MR chart. They assert that

> "if only the X-chart is used, the user will not be able to directly distinguish between a shift in the process mean and changes in the process variability".

Furthermore, they state that the X-MR-procedure may be more useful in diagnosing shift(s) than the X-chart alone. Their claims are not proven with data, however.

Albin et al. (1997) investigated the individuals X-chart in combination with runs rules, the MR-chart, and the Exponentially Weighted Moving Average (EWMA) chart. Based on ARL values they recommend the use of X-charts and EWMA-charts without complementary runs rules. Small shifts in the process variation are not well detected, however. If it is critical to detect such shifts, they propose the use of one of the alternative runs rules that we discussed previously.

5 The added value of an additional chart for the spread

In this section it is investigated whether adding either Rule A or the MR-chart increases the power of an individuals chart for detecting out-of-control situations. Since we are mainly interested in the MR-chart and Rule A in combination with the individuals chart, Rule 2, 3 and 4 are left out of consideration. This has the additional advantage that the ARL-computations are simplified so that we do not have to rely on simulation results; exact ARL values are derived instead. Technical details concerning these computations can be found in the Appendices.

Table 3 presents the ARL values of the individuals chart only. As before, a normal distribution is assumed. For the in-control-situation, the mean is assumed to be μ_0, and the standard deviation is assumed to be σ_0. ARL values are computed for several out-of-control situations. The entries in Table 3 are the baseline for determining the usefulness of adding either Rule A or the MR-chart.

Table 4 contains the ARL values of the individuals chart, combined with Rule A. The bracketed percentages express the ARL value as a percentage of the corresponding value in Table 3.

Table 4 shows that all ARL values are smaller if the individuals chart is supplemented with Rule A. From the percentages, we conclude that the decrease in the in-control ARL is larger than the decrease in ARL for all of the out-of-control situations considered. This leads to the conclusion that the improvement in power due to the addition of Rule A is not sufficient to compensate for the increased probability of committing a type I error that is incurred by its use.

Table 3: ARL values of the individuals chart.

σ/σ_0	$(\mu-\mu_0)/\sigma_0$									
	0.00	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**	**2.00**	**2.50**	**3.00**	**4.00**
1.00	370.40	281.15	155.22	81.22	43.89	14.97	6.30	3.24	2.00	1.19
1.25	60.99	53.87	39.52	26.82	18.02	8.68	4.72	2.90	2.00	1.27
1.50	21.98	20.62	17.36	13.70	10.52	6.25	3.95	2.71	2.00	1.34
2.00	7.48	7.32	6.86	6.22	5.51	4.19	3.18	2.47	1.99	1.45
2.50	4.35	4.30	4.18	3.99	3.75	3.22	2.72	2.30	1.97	1.52
3.00	3.15	3.13	3.09	3.01	2.91	2.66	2.40	2.14	1.91	1.56
4.00	2.21	2.20	2.19	2.17	2.14	2.07	1.97	1.87	1.76	1.57

Table 4: ARL values of the Individuals-chart, combined with Rule A.

σ/σ_0	$(\mu-\mu_0)/\sigma_0$									
	0.00	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**	**2.00**	**2.50**	**3.00**	**4.00**
1.00	278.04	227.28	140.00	77.96	43.26	14.95	6.30	3.24	2.00	1.19
	(75.1%)	(80.8%)	(90.2%)	(96.0%)	(98.6%)	(99.9%)	(100.0%)	(100.0%)	(100.0%)	(100.0%)
1.25	48.67	44.35	34.69	24.90	17.34	8.61	4.71	2.90	2.00	1.27
	(79.8%)	(82.3%)	(87.8%)	(92.8%)	(96.2%)	(99.2%)	(99.8%)	(100.0%)	(100.0%)	(100.0%)
1.50	18.43	17.53	15.28	12.53	9.94	6.13	3.93	2.70	2.00	1.34
	(83.8%)	(85.1%)	(88.0%)	(91.5%)	(94.5%)	(98.0%)	(99.4%)	(99.8%)	(100.0%)	(100.0%)
2.00	6.71	6.59	6.24	5.74	5.17	4.04	3.12	2.45	1.99	1.45
	(89.6%)	(90.0%)	(90.9%)	(92.3%)	(93.8%)	(96.5%)	(98.2%)	(99.2%)	(99.7%)	(100.0%)
2.50	4.04	4.01	3.91	3.75	3.56	3.10	2.66	2.27	1.95	1.52
	(93.1%)	(93.2%)	(93.6%)	(94.1%)	(94.8%)	(96.3%)	(97.6%)	(98.6%)	(99.2%)	(99.8%)
3.00	3.00	2.99	2.94	2.88	2.79	2.58	2.34	2.10	1.89	1.55
	(95.2%)	(95.2%)	(95.4%)	(95.6%)	(96.0%)	(96.8%)	(97.6%)	(98.3%)	(98.9%)	(99.6%)
4.00	2.15	2.14	2.13	2.11	2.09	2.02	1.94	1.84	1.74	1.55
	(97.3%)	(97.4%)	(97.4%)	(97.5%)	(97.6%)	(97.8%)	(98.1%)	(98.4%)	(98.8%)	(99.3%)

Table 5 contains the ARL values of the individuals chart combined with a standard one-sided *MR*-chart. Based on the derivation of the three-sigma control limits for the *MR*-chart in appendix C of Roes et al. (1993), we set UCL_{MR}, the upper control limit if the *MR*-chart, equal to $4.65\sigma_0$.

Table 5: ARL values of the individuals chart, combined with the MR-chart.

	$(\mu - \mu_0)/\sigma_0$									
σ/σ_0	**0.00**	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**	**2.00**	**2.50**	**3.00**	**4.00**
1.00	278.23 (75.1%)	226.84 (80.7%)	139.33 (89.8%)	77.59 (95.5%)	43.13 (98.2%)	14.93 (99.8%)	6.30 (100.0%)	3.24 (100.0%)	2.00 (100.0%)	1.19 (100.0%)
1.25	47.66 (78.1%)	43.49 (80.7%)	34.15 (86.4%)	24.62 (91.8%)	17.21 (95.5%)	8.58 (98.9%)	4.71 (99.8%)	2.90 (100.0%)	2.00 (100.0%)	1.27 (100.0%)
1.50	18.00 (81.9%)	17.16 (83.2%)	15.00 (86.4%)	12.35 (90.2%)	9.83 (93.5%)	6.10 (97.6%)	3.92 (99.2%)	2.70 (99.8%)	2.00 (99.9%)	1.34 (100.0%)
2.00	6.59 (88.1%)	6.48 (88.5%)	6.15 (89.6%)	5.67 (91.1%)	5.11 (92.7%)	4.01 (95.8%)	3.11 (97.8%)	2.45 (99.0%)	1.99 (99.6%)	1.44 (99.9%)
2.50	4.00 (92.0%)	3.96 (92.1%)	3.87 (92.5%)	3.71 (93.2%)	3.52 (93.9%)	3.08 (95.6%)	2.64 (97.1%)	2.26 (98.3%)	1.95 (99.0%)	1.52 (99.7%)
3.00	2.97 (94.4%)	2.96 (94.4%)	2.92 (94.6%)	2.86 (94.9%)	2.77 (95.3%)	2.56 (96.2%)	2.33 (97.1%)	2.10 (98.0%)	1.89 (98.6%)	1.55 (99.5%)
4.00	2.14 (96.9%)	2.13 (96.9%)	2.12 (96.9%)	2.10 (97.0%)	2.08 (97.1%)	2.01 (97.4%)	1.93 (97.8%)	1.84 (98.2%)	1.74 (98.5%)	1.55 (99.1%)

Tables 4 and 5 show that the X-*MR*-chart is a little more sensitive for detecting small shifts in the spread than the individuals chart combined with Rule A. The differences are small, however. Similarly to the results for Rule A, adding an *MR*-chart has the disadvantageous property that the decrease in in-control ARL is larger than the ARL-decrease for the out-of-control situations considered. We conclude that adding either Rule A or a standard *MR*-chart does not improve the power of the individuals chart for detecting a shift in the spread.

However, these results may be due to a poor design of the combined procedures. We discussed in the previous section that a proper evaluation requires the combined procedures to have the same in-control ARL as the individuals chart alone. If the added chart adds any power to the individuals chart, this will show up in smaller out-of-control ARL values. Such an evaluation has been done by Rigdon et al. (1994), and was extended by Amin and Ethridge (1998). For the design of the combined procedures, there are two parameters to be determined: one that fixes the control limits for the individuals chart, and one that fixes the limit(s) for the chart for the spread. There is only one restriction: the in-control ARL of the combined procedure should equal 370.4, the in-control ARL of Table 3. Hence, there are many combinations of parameters that lead to the same in-control ARL. For every out-of-control situation, it is possible to determine an optimal combination of parameters: the one that minimizes the ARL for that specific shift in the mean and standard deviation. It is not desirable however, to design a control chart for just one such situation; in most cases it is unknown which out-of-control situation is most likely to occur.

On the other hand, optimizing a combined procedure for a specific out-of-control situation provides insight into the 'best-case' performance for that situation. If we allow for every specific situation the control chart parameters to be different, then a combined procedure with fixed chart parameters cannot produce better ARL values. A 'best-case' evaluation of both combined control charts therefore proceeds as follows. The best combination of chart parameters is determined for various combinations of shifts in the mean and shifts in the standard deviation. In Table 6, the corresponding 'best-case' ARL values are presented.

Table 6: 'Best-case' ARL values of the Individuals-chart, combined with Rule A.

	$(\mu - \mu_0)/\sigma_0$									
σ/σ_0	**0.00**	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**	**2.00**	**2.50**	**3.00**	**4.00**
1.00		281.17 (100.0%) [2.9; 3.000]	155.24 (100.0%) [2.9; 3.000]	81.23 (100.0%) [2.9; 3.000]	43.90 (100.0%) [2.9; 3.000]	14.97 (100.0%) [2.9; 3.000]	6.30 (100.0%) [2.9; 3.000]	3.24 (100.0%) [2.9; 3.000]	2.00 (100.0%) [2.9; 3.000]	1.19 (100.0%) [2.9; 3.000]
1.25	58.61 (96.1%) [2.1; 3.069]	52.67 (97.8%) [2.2; 3.038]	39.37 (99.6%) [2.4; 3.011]	26.82 (100.0%) [2.8; 3.000]	18.02 (100.0%) [2.9; 3.000]	8.68 (100.0%) [2.9; 3.000]	4.72 (100.0%) [2.9; 3.000]	2.90 (100.0%) [2.9; 3.000]	2.00 (100.0%) [2.9; 3.000]	1.27 (100.0%) [2.9; 3.000]
1.50	21.00 (95.6%) [2.1; 3.069]	19.88 (96.4%) [2.2; 3.038]	17.03 (98.1%) [2.3; 3.021]	13.60 (99.3%) [2.4; 3.011]	10.50 (99.8%) [2.5; 3.005]	6.25 (100.0%) [2.9; 3.000]	3.95 (100.0%) [2.9; 3.000]	2.71 (100.0%) [2.9; 3.000]	2.00 (100.0%) [2.9; 3.000]	1.34 (100.0%) [2.9; 3.000]
2.00	7.23 (96.7%) [2.2; 3.038]	7.09 (96.9%) [2.2; 3.038]	6.69 (97.4%) [2.2; 3.038]	6.11 (98.2%) [2.2; 3.038]	5.45 (98.8%) [2.3; 3.021]	4.17 (99.7%) [2.4; 3.011]	3.18 (99.9%) [2.6; 3.002]	2.47 (100.0%) [2.8; 3.000]	1.99 (100.0%) [2.9; 3.000]	1.45 (100.0%) [2.9; 3.000]
2.50	4.25 (97.8%) [2.2; 3.038]	4.21 (97.9%) [2.2; 3.038]	4.10 (98.1%) [2.2; 3.038]	3.92 (98.4%) [2.2; 3.038]	3.70 (98.7%) [2.3; 3.021]	3.20 (99.3%) [2.3; 3.021]	2.71 (99.7%) [2.4; 3.011]	2.30 (99.9%) [2.5; 3.005]	1.97 (100.0%) [2.6; 3.002]	1.52 (100.0%) [2.9; 3.000]
3.00	3.11 (98.6%) [2.2; 3.038]	3.09 (98.6%) [2.2; 3.038]	3.05 (98.7%) [2.2; 3.038]	2.97 (98.8%) [2.3; 3.021]	2.88 (98.9%) [2.3; 3.021]	2.64 (99.3%) [2.3; 3.021]	2.39 (99.6%) [2.4; 3.011]	2.14 (99.8%) [2.4; 3.011]	1.91 (99.9%) [2.5; 3.005]	1.56 (100.0%) [2.7; 3.001]
4.00	2.19 (99.3%) [2.3; 3.021]	2.19 (99.3%) [2.3; 3.021]	2.17 (99.3%) [2.3; 3.021]	2.15 (99.4%) [2.3; 3.021]	2.13 (99.4%) [2.3; 3.021]	2.06 (99.5%) [2.3; 3.021]	1.96 (99.6%) [2.3; 3.021]	1.86 (99.7%) [2.4; 3.011]	1.76 (99.8%) [2.4; 3.011]	1.56 (99.9%) [2.5; 3.005]

The values in this table were obtained as follows. Assume that the warning limits are $\mu_0 \pm WL\,\sigma_0$ and that the control limits of the individuals chart at $\mu_0 \pm M\sigma_0$. For the warning lines parameter WL ranging from 1.8, 1.9, $\cdots$, 2.9 the corresponding individuals chart parameter M is determined, so that the combined procedure has an in-control ARL of 370.4. For each of the resulting twelve (WL, M) combinations, an ARL table like Table 4 is computed. The minimum ARL value of the twelve tables is selected to enter Table 6. The bracketed numbers below express the ARL values as a percentage of the corresponding ARL value of the individuals chart. The selected values of WL and M respectively are also given.

The reason for considering this range of warning lines is the following. For $WL < 1.8$, Rule A is so sensitive that it is not possible to attain an in-control ARL-value of 370.4, no matter how large M is. For $WL \geq 3$, the individuals chart is more sensitive than Rule A.

Table 7 is constructed analogously to Table 6, but then for individuals chart, combined with a one-sided MR-chart with $UCL_{MR} = R\sigma_0$. For the MR-chart parameter we considered the range of $R = 4.2, 4.3, \cdots, 6$.

Table 7: 'Best-case' ARL values of the Individuals-chart, combined with the MR-chart.

	$(\mu - \mu_0)/\sigma_0$									
σ/σ_0	**0.00**	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**	**2.00**	**2.50**	**3.00**	**4.00**
1.00		281.12 (100.0%) [6.0; 3.000]	155.21 (100.0%) [6.0; 3.000]	81.21 (100.0%) [6.0; 3.000]	43.89 (100.0%) [6.0; 3.000]	14.97 (100.0%) [6.0; 3.000]	6.30 (100.0%) [6.0; 3.000]	3.24 (100.0%) [6.0; 3.000]	2.00 (100.0%) [6.0; 3.000]	1.19 (100.0%) [6.0; 3.000]
1.25	57.28 (93.9%) [4.5; 3.127]	52.06 (96.6%) [4.7; 3.057]	39.30 (99.4%) [5.1; 3.011]	26.82 (100.0%) [5.6; 3.001]	18.02 (100.0%) [6.0; 3.000]	8.68 (100.0%) [6.0; 3.000]	4.72 (100.0%) [6.0; 3.000]	2.90 (100.0%) [6.0; 3.000]	2.00 (100.0%) [6.0; 3.000]	1.27 (100.0%) [6.0; 3.000]
1.50	20.53 (93.4%) [4.6; 3.084]	19.52 (94.7%) [4.6; 3.084]	16.88 (97.2%) [4.8; 3.039]	13.56 (99.0%) [5.0; 3.017]	10.50 (99.8%) [5.3; 3.005]	6.25 (100.0%) [5.8; 3.000]	3.95 (100.0%) [6.0; 3.000]	2.71 (100.0%) [6.0; 3.000]	2.00 (100.0%) [6.0; 3.000]	1.34 (100.0%) [6.0; 3.000]
2.00	7.13 (95.3%) [4.6; 3.084]	7.00 (95.6%) [4.6; 3.084]	6.62 (96.4%) [4.7; 3.057]	6.07 (97.5%) [4.8; 3.039]	5.43 (98.4%) [4.9; 3.026]	4.17 (99.6%) [5.1; 3.011]	3.17 (99.9%) [5.4; 3.003]	2.47 (100.0%) [5.7; 3.000]	1.99 (100.0%) [6.0; 3.000]	1.45 (100.0%) [6.0; 3.000]
2.50	4.21 (97.0%) [4.7; 3.057]	4.18 (97.1%) [4.7; 3.057]	4.07 (97.4%) [4.7; 3.057]	3.90 (97.8%) [4.8; 3.039]	3.68 (98.2%) [4.8; 3.039]	3.19 (99.1%) [4.9; 3.026]	2.71 (99.6%) [5.1; 3.011]	2.30 (99.9%) [5.2; 3.007]	1.97 (100.0%) [5.4; 3.003]	1.52 (100.0%) [5.9; 3.000]
3.00	3.09 (98.1%) [4.8; 3.039]	3.08 (98.1%) [4.8; 3.039]	3.03 (98.2%) [4.8; 3.039]	2.96 (98.4%) [4.8; 3.039]	2.87 (98.6%) [4.8; 3.039]	2.64 (99.0%) [4.9; 3.026]	2.38 (99.4%) [5.0; 3.017]	2.13 (99.7%) [5.1; 3.011]	1.91 (99.9%) [5.2; 3.007]	1.56 (100.0%) [5.5; 3.002]
4.00	2.19 (99.1%) [4.9; 3.026]	2.18 (99.1%) [4.9; 3.026]	2.17 (99.1%) [4.9; 3.026]	2.15 (99.2%) [4.9; 3.026]	2.12 (99.2%) [4.9; 3.026]	2.05 (99.3%) [4.9; 3.026]	1.96 (99.5%) [4.9; 3.026]	1.86 (99.6%) [5.0; 3.017]	1.76 (99.7%) [5.1; 3.011]	1.56 (99.9%) [5.2; 3.007]

The results indicate again that Rule A operates quite alike the MR-chart. Not surprisingly, even 'best-case' performance of each of the combined procedures is not faster in detecting shifts in the mean. Furthermore, a little extra sensitivity for detecting shifts in the spread can be gained if the individuals chart is combined with either Rule A or the MR-chart, provided that a shift in the spread is not accompanied with a large shift in the mean.

If the consideration is only based on ARL comparisons, the *MR*-chart is the better of the two alternatives. However, it requires an additional chart (or a transparency-procedure, see Adke and Hong (1997)), whereas Rule A can be read directly from the individuals chart. An overall good choice for the limits of the combined X-*MR* procedure is to choose the upper limit of the *MR*-chart in the range 4.5–4.7, and to choose the limits of the individuals chart accordingly.

However, fixing the control chart parameters for faster detection of pure shifts in the spread leads to an increase of the ARL values for pure shifts in the mean. These effects appear to be the largest for moderate pure shifts in the mean. If $(R, M) = (4.5, 3.127)$ is used, the ARL for detecting a shift in the mean of $1\sigma_0$ increases by 32.6%. Whereas, if $(R, M) = (4.6, 3.084)$ is used, this ARL increases by 20.6%. For $(R, M) = (4.7, 3.057)$, this ARL value is increased by 13.5% (note that these results cannot be inferred from Tables 6 and 7).

6 Conclusions

The results in this paper show that adding either Rule A or the *MR*-chart to an individuals chart provides little extra power to the individuals chart only for very specific out-of-control situations. The disadvantage however is a considerable loss of power for other out-of-control situations. For nearly all practical purposes, this means that the overall performance of the individuals chart (with or without additional runs rules) is not improved. In most out-of-control situations, the inclusion of the additional Rule A or *MR*-chart leads to a less powerful procedure, and adds confusion. The procedure would only be beneficial in cases where a small increase in variation is present. We therefore advise against the use of the additional *MR*-chart or Rule A.

A ARL computation for the X-Rule A chart

The ARL computations of Table 4 are based on the paper of Page (1955). The area between the control limits is split up in three areas according to Figure 1.

Assume that successive realizations of the random variable are statistically independent. If a distribution is assumed for the random variable, it is then possible to define p_0, p_1, and p_2 as the probability that an observation will fall in area 0, 1, 2 respectively. Furthermore, let L_i denote the expected run length of the combined procedure when starting in area i, $(i = 0, 1, 2)$. For L_0, L_1, and L_2 we can write the following system of equations

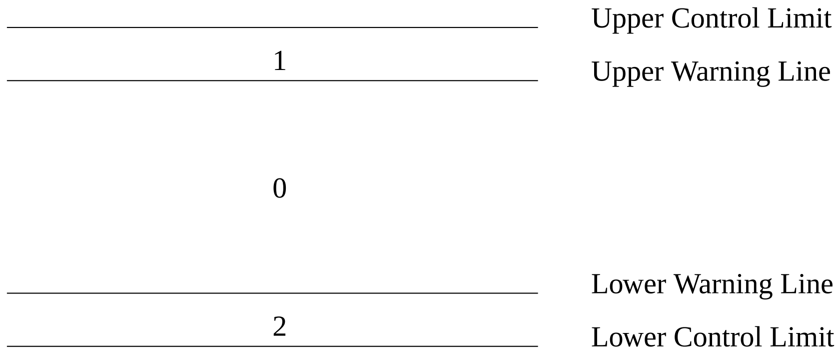

Figure 1: Areas of a control chart

$$
\begin{array}{lcccccccc}
L_0 & = & 1 & + & p_0 L_0 & + & p_1 L_1 & + & p_2 L_2 \\
L_1 & = & 1 & + & p_0 L_0 & + & p_1 L_1 & & \\
L_2 & = & 1 & + & p_0 L_0 & & & + & p_2 L_2.
\end{array}
\tag{1}
$$

Following Page (1955) we solve for L_0, which gives the following expression for the ARL:

$$
\mathrm{ARL} = L_0 = \frac{1 - p_1 p_2}{1 - p_0 - p_1 - p_2 + p_1 p_2 + p_0 p_1 p_2}.
$$

For the computations used in this paper, a normal distribution was assumed so that

$$
\begin{aligned}
p_0 &= \Phi\left(\frac{\mu_0 + WL\,\sigma_0 - \mu}{\sigma}\right) - \Phi\left(\frac{\mu_0 - WL\,\sigma_0 - \mu}{\sigma}\right) \\
p_1 &= \Phi\left(\frac{\mu_0 + M\sigma_0 - \mu}{\sigma}\right) - \Phi\left(\frac{\mu_0 + WL\,\sigma_0 - \mu}{\sigma}\right) \\
p_2 &= \Phi\left(\frac{\mu_0 - WL\,\sigma_0 - \mu}{\sigma}\right) - \Phi\left(\frac{\mu_0 - M\sigma_0 - \mu}{\sigma}\right).
\end{aligned}
$$

B ARL computation of the $X - MR$ chart

For the computation of the ARL of the combined individuals-MR-chart, a modified version of the integral equation approach suggested by Crowder (1987, 1987a) is developed that is more flexible in the choices of M and R. The procedure is as follows.

The ARL of the combined $X - MR$ chart depends on the starting value of the stochastic variabele that is monitored in the X-chart. Let $L(x)$ denote the ARL of the combined $X - MR$ chart that starts in $X = x$, with $x \in [-M, M]$. Let us assume that $R \leq 2M$, otherwise adding the MR-chart does not have any effect on the performance of the X-chart, and let us assume that $M < R$. For $L(x)$ we have:

$$L(x) = 1 + \int_{A(x)} L(y) f(y) \mathrm{d}y \qquad \text{for } x \in [-M, M],$$

where $f(\cdot)$ is the probability density function of X, and $A(x)$ is a region, depending on x, where the combined control chart does not give an out-of-control signal at the next observation y. For $A(x)$ the following holds:

$$A(x) = \{y \in \mathbb{R} | \max(-M, x - R) \leq y \leq \min(M, x + R)\} .$$

Hence, for $L(x)$ we have:

$$L(x) = \begin{cases} 1 + \int_{-M}^{x+R} L(y) f(y) \mathrm{d}y & \text{for } x \in [-M, M - R] \\ 1 + \int_{-M}^{M} L(y) f(y) \mathrm{d}y & \text{for } x \in [M - R, -M + R] \\ 1 + \int_{x-R}^{M} L(y) f(y) \mathrm{d}y & \text{for } x \in [-M + R, M]. \end{cases} \tag{2}$$

In order to be able to numerically solve for the unknown function $L(x)$, the integrals in the equations above need to be replaced by a finite sum, using some integration rule. We will use the so-called trapezium rule (see Atkinson (1989)).

The trapezium rule requires a grid of n points, defined on the interval $[-M, M]$. Let $x_1, x_2, \cdots, x_n$ denote these points. The grid is chosen so that $x_1 = -M$ and $x_n = M$. The distance between two adjacent grid points is denoted by h and satisfies $h = 2M/(n - 1)$.

This grid does not contain some points that are needed for a proper numerical evaluation of $L(x)$. Since R will generally not be a multiple of h, the grid does not contain points like $-M + R$ and $M - R$. The computer program that was presented by Crowder (1987a) is restricted to choices of R that are multiples of h. However, this problem can be elegantly circumvented. Following Crowder (1987), we approximate equation (2) in the grid points. We present the procedure for $x_i \in [-M, M - R]$; the expressions for $x_i \in [M - R, M]$ are derived analogously. For $x_i \in [-M, M - R]$, $L(x_i)$ is approximated by

$$
\begin{aligned}
L(x_i) \quad = \quad 1 \quad + & \quad \sum_{j=1}^{n_i} w_j L(x_j) f(x_j) \quad + \\
+ & \quad \frac{1}{2} \Big(x_i + R - x_{n_i} \Big) \Big(L(x_{n_i}) f(x_{n_i}) + L(x_i + R) f(x_i + R) \Big)
\end{aligned}
\tag{3}
$$

where n_i is defined as

$$
n_i = \max \Big\{ j \in \{1, 2, \cdots, n\} \,\Big|\, x_j < x_i + R \Big\},
$$

and $w_1 = w_{n_i} = h/2$, and $w_j = h$ for $j \in \{2, 3, \cdots, n_i - 1\}$. This leads to a system of n equations in more than n unknowns, unless R is a multiple of h (there is no equation for $L(x_i + R)$ for example). However, since $x_i + R \in [-M + R, M]$, $L(x_i + R)$ can be approximated using grid points only as follows:

$$
L(x_i + R) = 1 + \sum_{j=i}^{n} w'_j L(x_j) f(x_j) \tag{4}
$$

where $w'_i = w'_n = h/2$, and $w'_j = h$ for $j \in \{i + 1, i + 2, \cdots, n - 1\}$. Combining formulas (3) and (4), we are able to approximate $L(x_i)$ in terms of grid points only:

$$
\begin{aligned}
L(x_i) \quad = \quad 1 \quad + & \quad \sum_{j=1}^{n_i} w_j L(x_j) f(x_j) \quad + \\
+ & \quad \frac{1}{2} \Big(x_i + R - x_{n_i} \Big) \times \\
& \times \left(L(x_{n_i}) f(x_{n_i}) + \left[1 + \sum_{j=i}^{n} w'_j L(x_j) f(x_j) \right] f(x_i + R) \right).
\end{aligned}
\tag{5}
$$

Equation (5) can be rewritten as

$$
L(x_i) = v(x_i) \; + \; \sum_{j=1}^{n} w''_{i,j} L(x_j) f(x_j),
$$

where

$$
v(x_i) = 1 \; + \frac{1}{2} \Big(x_i + R - x_{n_i} \Big) \, f(x_i + R),
$$

and

$$w''_{i,j} = \begin{cases} h/2 & \text{for } j = 1 \\ h & \text{for } j = 2, \cdots, i-1 \\ h + \frac{1}{2}\left(x_i + R - x_{n_i}\right) f(x_i + R)h/2 & \text{for } j = i \\ h + \frac{1}{2}\left(x_i + R - x_{n_i}\right) f(x_i + R)h & \text{for } j = i+1, \cdots, n_i - 1 \\ h/2 + \frac{1}{2}\left(x_i + R - x_{n_i}\right)\left(f(x_i + R)h + 1\right) & \text{for } j = n_i \\ \frac{1}{2}\left(x_i + R - x_{n_i}\right) f(x_i + R)h & \text{for } j = n_i + 1, \cdots, n-1 \\ \frac{1}{2}\left(x_i + R - x_{n_i}\right) f(x_i + R)h/2 & \text{for } j = n. \end{cases}$$

Note that for $i = 1$ the first $w''_{i,j}$ differs from these formulas as follows:

$$w''_{1,1} = h/2 + \frac{1}{2}\left(x_i + R - x_{n_i}\right) f(x_i + R)h/2.$$

The roundabout that is used in formula (5) for $x_i \in [-M, M - R]$ can also be used to approximate $L(x_i)$ for $x_i \in [-M + R, M]$. For $x_i \in [M - R, -M + R]$, the formulas for approximating $L(x_i)$ simplify to

$$L(x_i) = 1 + \sum_{j=1}^{n} w_{i,j} L(x_j) f(x_j),$$

where $w_{i,1} = w_{i,n} = h/2$, and $w_{i,j} = h$ for $j = 2, 3, \cdots, n-1$.

The resulting system of n equations in n unknowns is then solved to give the numerical values of $L(x_1), L(x_2), \cdots, L(x_n)$.

The overall ARL is then computed as

$$L = 1 + \sum_{i=1}^{n} w'''_i L(x_i),$$

where $w'''_1 = w'''_n = h/2$, and $w'''_j = h$ for $j \in \{2, 3, \cdots, n-1\}$.

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