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On flat systems behaviors and observable image representations

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Abstract

In this short paper we propose a definition of *flatness* for systems not necessarily given in input/state/output representation. A flat system is a system for which there exists a mapping such that the manifest system behavior is equal to the image of this mapping, and such that the latent variable appearing in this image representation can be written as a function of the manifest variable and its derivatives up to some order. For linear differential systems, flatness is equivalent to controllability. We will generalize the main theorem of Levine and Nguyen (Systems Control Lett. 48 (2003) 69) to general linear differential systems.

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1. Introduction

In recent papers [1–3], differentially flat systems have been studied, and their relevance in control problems has been outlined. In particular, in [3], linear flat systems represented in terms of polynomial matrices have been considered, and for a particular class of such systems flat outputs have been characterized in terms of their so-called defining matrices. The aim of this short paper is to explain how the notion of flatness fits naturally into a behavioral perspective to systems. In fact, flat systems are systems whose system behavior admit an image representation in which the latent variable is observable from the manifest system variable. In this short paper we will generalize the main theorem of [3] to general linear differential systems.

2. Flat systems

In this section we will briefly review the notion of differentially flat system as was introduced in [1–3]. In the sequel, $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ will denote the space of infinitely often differentiable functions from \mathbb{R} to \mathbb{R}^n . Let $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a given function, and consider the system

$$\frac{d}{dt}x(t) = f(x(t), u(t))$$

with $x(t) \in \mathbb{R}^n$ being the state and $u(t) \in \mathbb{R}^m$ the input. This system is called *differentially flat*, or just *flat*, if there exists a set of independent variables (to be called a *flat output* of the system) such that both the system variables x and u are functions of this flat output and a finite number of its successive derivatives. To be precise, if there exist nonnegative integers p and q , and functions $\alpha: \mathbb{R}^{(p+1)m} \rightarrow \mathbb{R}^n$, $\beta: \mathbb{R}^{(p+1)m} \rightarrow \mathbb{R}^m$, and $h: \mathbb{R}^n \times \mathbb{R}^{(q+1)m} \rightarrow \mathbb{R}^m$ such that the following two

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conditions hold:

1. $(x, u) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^m)$ satisfies the differential equation $(d/dt)x = f(x, u)$ if and only if there exists a function $y \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ such that

$$x = \alpha(y, y^{(1)}, y^{(2)}, \dots, y^{(p)}),$$

$$u = \beta(y, y^{(1)}, y^{(2)}, \dots, y^{(p)}),$$

2. for all $y \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ we have: $x = \alpha(y, y^{(1)}, y^{(2)}, \dots, y^{(p)})$ and $u = \beta(y, y^{(1)}, y^{(2)}, \dots, y^{(p)})$ implies $y = h(x, u, u^{(1)}, \dots, u^{(q)})$.

In effect, flatness of the system $(d/dt)x = f(x, u)$ means that the space of solutions (x, u) of the differential equation can be represented as the *image* of some mapping $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$,

$$y \mapsto \begin{pmatrix} \alpha(y, y^{(1)}, y^{(2)}, \dots, y^{(p)}) \\ \beta(y, y^{(1)}, y^{(2)}, \dots, y^{(p)}) \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}.$$

In addition, to any given (x, u) corresponds a unique y , which can be obtained as $y = h(x, u, u^{(1)}, \dots, u^{(q)})$. In [3], any such y is called a *flat output* of the system. Note that, by definition, the number of flat output components is equal to m , the number of input components of the system.

It was shown in [2] that the linear state-space system $(d/dt)x = Ax + Bu$ (with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$) is flat if and only if the pair (A, B) is controllable.

In [3] linear systems of the form $A(d/dt)x = Bu$ were considered, with $A(\xi)$ a given real polynomial matrix, B a given real constant matrix, and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$. Such system was called *linearly flat* if it is flat, and if the functions α , β and h can be chosen to be linear. Hence, the system $A(d/dt)x = Bu$ is linearly flat if and only if there exist real polynomial matrices $P(\xi)$, $Q(\xi)$ and $L(\xi)$, respectively, of dimensions $n \times m$, $m \times m$ and $m \times (n + m)$, such that $(x, u) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^m)$ satisfies the differential equation $A(d/dt)x = Bu$ if and only if there exists a function $y \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ such that

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} P \left(\frac{d}{dt} \right) \\ Q \left(\frac{d}{dt} \right) \end{pmatrix} y,$$

and such that for all $y \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ we have: $x = P(d/dt)y$ and $u = Q(d/dt)y$ implies $y = L(d/dt)\begin{pmatrix} x \\ u \end{pmatrix}$.

3. Flat system behaviors

In this section we will extend the notion of flatness to more general systems, not necessarily in input/state/output representation.

Let q , r and s be given nonnegative integers, let $f: \mathbb{R}^{(s+1)q} \rightarrow \mathbb{R}^r$ be a given function, and consider the higher order nonlinear differential equation in the unknown $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$:

$$f(w(t), w^{(1)}(t), \dots, w^{(s)}(t)) = 0. \quad (3.1)$$

The subset $\mathfrak{B} \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ of all solutions to the differential equation (3.1) is called the *behavior* of the differential system represented by the differential equation (3.1), w is called the manifest variable of the system. The system behavior \mathfrak{B} will be called flat if there exist nonnegative integers k , l , p , and functions $g: \mathbb{R}^{(k+1)l} \rightarrow \mathbb{R}^q$ and $h: \mathbb{R}^{(p+1)q} \rightarrow \mathbb{R}^l$ such that the following two conditions hold:

1. $\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \text{there exists } \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l) \text{ such that } w = g(\ell, \ell^{(1)}, \dots, \ell^{(k)})\}$,
2. for all $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l)$ we have: $w = g(\ell, \ell^{(1)}, \dots, \ell^{(k)})$ implies $\ell = h(w, w^{(1)}, \dots, w^{(p)})$.

In other words, the system behavior \mathfrak{B} represented by (3.1) is called flat if \mathfrak{B} can be represented as the image of some map $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, $\ell \mapsto g(\ell, \ell^{(1)}, \dots, \ell^{(k)})$, with the property that ℓ can be recovered from the given manifest variable trajectory w by $\ell = h(w, w^{(1)}, \dots, w^{(p)})$. In the terminology of the behavioral approach, the variable ℓ in the above is called a *latent variable*, and the representation $w = g(\ell, \ell^{(1)}, \dots, \ell^{(k)})$ is called an image representation of \mathfrak{B} . The image representation is *observable* in the sense that the latent variable trajectory ℓ is uniquely determined by the manifest variable trajectory w through $\ell = h(w, w^{(1)}, \dots, w^{(p)})$. Note that, in contrast to the definition of flatness in [3], in our definition the number of components of the *flat latent variable* ℓ is not required to be equal to the number of inputs of the system.

Clearly, by taking $w = (x, u)$, we see that flatness of the system $(d/dt)x = f(x, u)$ in the sense [3] implies flatness in the sense of our definition.

We will now turn attention to linear differential systems. A linear differential system is a system in

which the function f is linear. In that case the representing differential equation is of the form $R_0 w(t) + R_1 w^{(1)}(t) + \dots + R_s w^{(s)}(t) = 0$ for given real $r \times q$ matrices R_i . After introducing a real $r \times q$ polynomial matrix $R(\xi) := R_0 + R_1 \xi + \dots + R_s \xi^s$, this differential equation can be written as

$$R \left(\frac{d}{dt} \right) w = 0. \quad (3.2)$$

The subspace $\mathfrak{B} \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ of all solutions to (3.2) is called the behavior of the linear differential system represented by $R(d/dt)w = 0$. The system behavior \mathfrak{B} will be called linearly flat if there exists a nonnegative integer l and real polynomial matrices $M(\xi)$ and $L(\xi)$ of sizes $q \times l$ and $l \times q$, respectively, such that the following two conditions hold:

- (1) $\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \text{there exists } \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l) \text{ such that } w = M(d/dt)\ell\}$,
- (2) for all $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l)$ we have: $w = M(d/dt)\ell$ implies $\ell = L(d/dt)w$.

If (1) and (2) above hold, then $\ell = L(d/dt)w$ is called a *linear flat latent variable* for \mathfrak{B} . Clearly, if the linear differential system $A(d/dt)x = Bu$ studied in [3] is linearly flat in the sense of [3] then it is flat in the sense of our definition.

4. Controllable behaviors and image representations

We will now quickly review the notions of controllability and observable image representations in a behavioral framework. Let $R(\xi)$ be a real $r \times q$ polynomial matrix and consider the linear differential system behavior \mathfrak{B} represented by $R(d/dt)w = 0$. \mathfrak{B} is called controllable if for any two trajectories w_1 and w_2 in \mathfrak{B} there exists $T \geq 0$ and a trajectory w in \mathfrak{B} such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t)$ for $t \geq T$. It is well known, see for example [5], that \mathfrak{B} is controllable if and only if the following condition on the polynomial matrix $R(\xi)$ holds:

$$\text{rank}(R(\lambda)) = \text{rank}(R) \quad \text{for all } \lambda \in \mathbb{C}.$$

In other words, if for any complex number λ , the rank of the complex matrix $R(\lambda)$ is equal to the rank of the polynomial matrix $R(\xi)$.

Whereas a linear differential system behavior is *defined* as the space of solutions \mathfrak{B} of a differential equation of the form $R(d/dt)w = 0$, it can have other representations as well. One of these is the image representation. Let $M(\xi)$ be a real polynomial matrix with q rows and, say, l columns. If

$$\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \text{there exists } \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l) \text{ such that } w = M(d/dt)\ell\},$$

then we call $w = M(d/dt)\ell$ an image representation of the system behavior \mathfrak{B} . The image representation is called an observable image representation if from $w = M(d/dt)\ell_1 = M(d/dt)\ell_2$ it follows that $\ell_1 = \ell_2$, in other words, if any w in the behavior is the image of exactly one latent variable trajectory ℓ .

Not all linear differential systems behaviors admit an image representation. In fact, the linear differential system behavior \mathfrak{B} admits an image representation if and only if it is controllable. In that case it also admits an observable image representation. Furthermore, observability of an image representation $w = M(d/dt)\ell$, with ℓ taking its values in \mathbb{R}^l , can be tested in terms of the $q \times l$ polynomial matrix $M(\xi)$: the image representation $w = M(d/dt)\ell$ is observable if and only if

$$\text{rank}(M(\lambda)) = l \quad \text{for all } \lambda \in \mathbb{C}.$$

Another result that we will use is the following. Suppose $M_1(\xi)$ and $M_2(\xi)$ are $q \times l$ polynomial matrices. Then $w = M_1(d/dt)\ell$ and $w = M_2(d/dt)\ell'$ are observable image representations of the same linear differential behavior \mathfrak{B} if and only if there exists a unimodular $l \times l$ polynomial matrix $W(\xi)$ such that $M_1(\xi) = M_2(\xi)W(\xi)$.

A detailed discussion of these standard result in the behavioral theory to linear systems can be found in [5], or in [7].

We now briefly recall the concept of input cardinality of a linear differential system. Given a linear differential system behavior \mathfrak{B} with manifest variable w , the condition $w \in \mathfrak{B}$ leaves some of the components of w free, in the sense that these components can be chosen to be arbitrary functions in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. The number of these free components is equal to the number of inputs of the behavior \mathfrak{B} , and is denoted by $m(\mathfrak{B})$, called the input cardinality of \mathfrak{B} . This number can be computed in terms of the polynomial matrices $R(\xi)$ in any kernel representation $R(d/dt)w = 0$,

and in terms of the polynomial matrix $M(\xi)$ in any image representation $w = M(d/dt)\ell$ of \mathfrak{B} . In fact,

$$m(\mathfrak{B}) = q - \text{rank}(R) = \text{rank}(M).$$

In particular, the input cardinality of \mathfrak{B} is equal to the number of latent variable components in any observable image representation of \mathfrak{B} .

Finally, we state the following useful well-known result. Suppose \mathfrak{B}_1 and \mathfrak{B}_2 are controllable linear differential systems such that $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$. Then $m(\mathfrak{B}_1) = m(\mathfrak{B}_2)$ (equality of the input cardinalities) implies that $\mathfrak{B}_1 = \mathfrak{B}_2$, see [7].

5. Flat system behaviors and observable image representations

In this section we will formulate and prove our main result, which generalizes [3, Theorem 1].

Theorem 5.1. *Let $R(\xi)$ be a real $r \times q$ polynomial matrix, and let \mathfrak{B} be the linear differential system represented by $R(d/dt)w = 0$. Then the following statements are equivalent:*

1. \mathfrak{B} is a linearly flat system,
2. \mathfrak{B} is controllable,
3. \mathfrak{B} admits an image representation,
4. \mathfrak{B} admits an observable image representation.

In the rest of this theorem statement, assume that any these equivalent conditions on \mathfrak{B} hold. Then the number of components of any linear flat latent variable ℓ for \mathfrak{B} is equal to $m(\mathfrak{B}) = q - \text{rank}(R)$, the input cardinality of \mathfrak{B} .

Furthermore, polynomial matrices $M(\xi)$ and $L(\xi)$ defining an (observable) image representation $w = M(d/dt)\ell$ together with a linear flat latent variable $\ell = L(d/dt)w$ for \mathfrak{B} can be obtained from the polynomial matrix $R(\xi)$ as follows:

- put $l := q - \text{rank}(R)$,
- for $M(\xi)$ take any $q \times l$ polynomial matrix such that $R(\xi)M(\xi) = 0$ and such that $\text{rank}(M(\lambda)) = l$ for all $\lambda \in \mathbb{C}$,
- for $L(\xi)$ take any polynomial left inverse of $M(\xi)$, i.e., any $l \times q$ polynomial matrix such that $L(\xi)M(\xi) = I_{l \times l}$.

Finally, for any $q \times l$ polynomial matrix $M'(\xi)$ we have: $w = M'(d/dt)\ell'$ is an observable image representation of \mathfrak{B} if and only if there exist a $l \times l$ unimodular polynomial matrix $W(\xi)$ such that $M'(\xi) = M(\xi)W(\xi)$. In that case, $\ell' = L'(d/dt)w$, with $L'(\xi) := W^{-1}(\xi)L(\xi)$, is the flat latent variable corresponding to the image representation $w = M'(d/dt)\ell'$.

Proof. The equivalence of statements (2)–(4) is standard in the behavioral approach, see for example [5]. The implication (1) \Rightarrow (3) follows from the definition of linearly flat system. We prove the implication (4) \Rightarrow (1). Let $w = M(d/dt)\ell$ be an observable image representation of \mathfrak{B} . Then $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, so the Smith form of the polynomial matrix $M(\xi)$ equals $(I \ 0)^T$. Consequently $M(\xi)$ has a polynomial left inverse, say $L(\xi)$. Clearly $w = M(d/dt)\ell$ then implies $L(d/dt)w = L(d/dt)M(d/dt)\ell = \ell$. The statement about the number of components of any flat latent variable follows from the fact that the number of latent variables in any observable image representation of \mathfrak{B} is equal to $m(\mathfrak{B})$.

For completeness, we also prove the remaining statements on the computation of $M(\xi)$ and $L(\xi)$. Assume $M(\xi)$ is a $q \times l$ polynomial matrix such that $R(\xi)M(\xi) = 0$. Define

$$\mathfrak{B}' := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \text{there exists } \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l) \text{ such that } w = M(d/dt)\ell\}.$$

Then it is clear that $\mathfrak{B}' \subseteq \mathfrak{B}$. Since $M(\lambda)$ has full column rank l for all $\lambda \in \mathbb{C}$, $M(\xi)$ also has full column rank l as a polynomial matrix. Hence we have $m(\mathfrak{B}') = l$. Since this is equal to $m(\mathfrak{B})$, and since both \mathfrak{B}' and \mathfrak{B} are controllable, we in fact have $\mathfrak{B}' = \mathfrak{B}$. We conclude that $w = M(d/dt)\ell$ is indeed an image representation of \mathfrak{B} . Since $\text{rank}(M(\lambda)) = l$ for all λ , it is observable. Finally, as already explained above, by taking a polynomial left inverse $L(\xi)$ of $M(\xi)$ we get a linear flat latent variable $\ell = L(d/dt)w$ for \mathfrak{B} .

The remaining statements follow from the fact that for any two $q \times l$ polynomial matrices M_1 and M_2 , defining observable image representations of one and the same behavior \mathfrak{B} , there exists a unimodular matrix W such that $M_1 = M_2W$. \square

The actual computation of one suitable pair of polynomial matrices $M(\xi)$ and $L(\xi)$, starting from the representing polynomial matrix $R(\xi)$ can be made more concrete as follows. First note that, by controllability, $\text{rank}(R(\lambda)) = \text{rank}(R) = : r$ for all $\lambda \in \mathbb{C}$. Hence the nonzero polynomials in the Smith form of $R(\xi)$ are all equal to 1, so there exist unimodular matrices $U(\xi)$ and $V(\xi)$ such that

$$U(\xi)R(\xi)V(\xi) = \begin{pmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{pmatrix}.$$

Define then

$$M(\xi) := V(\xi) \begin{pmatrix} 0 \\ I_{(q-r) \times (q-r)} \end{pmatrix},$$

$$L(\xi) := (0 \quad I_{(q-r) \times (q-r)}) V^{-1}(\xi).$$

Remark 5.2. The question arises as to what extend the result of Theorem 5.1 can be extended to more general classes of systems. In general, nonlinear systems, even controllable ones, have an image representation only in exceptional cases. Another class of systems of interest is the class of ND-systems, i.e. systems represented by a finite number of linear, constant coefficient partial differential equations. It has been shown in [4] for ND-systems that the existence of an image representation is equivalent to controllability. However, only in exceptional cases a controllable ND-system admits an observable image representation. In [6], for ND-systems the property of admitting an

observable image representation has been shown to be equivalent to the property of *rectifiability*, which, in turn, has been shown to be equivalent to *strong controllability*.

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