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Observers as Internal Models for Remote Tracking via Encoded Information

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Summary. In this paper, we consider a servomechanism problem in which the command and control functions are distributed in space, and hence the system consists of different components linked by a communication channel of finite capacity. The desired control goal is achieved by designing appropriate encoders, decoders and internal models of the exogenous signals. As an application, we describe a how the output of a system can be forced to track a reference signal generated by a remotely located nonlinear oscillator.

Keywords: Nonlinear tracking, internal model, encoding, remote control.

1 Introduction

Generally speaking, the problem of tracking and asymptotic disturbance rejection (sometimes also referred to as the generalized *servomechanism problem* or the *output regulation problem*) is to design a controller so as to obtain a closed-loop system in which all trajectories are bounded, and a regulated output asymptotically decays to zero as time tends to infinity. The peculiar aspect of this design problem is the characterization of the class of all possible exogenous inputs (disturbances, commands, uncertain constant parameters) as the set of all possible solutions of a fixed (finite-dimensional) differential equation. In this setting, *any source of uncertainty* (about actual disturbances affecting the system, about actual trajectories that are required to be tracked, about any uncertain constant parameters) is treated as *uncertainty in the initial condition* of a fixed autonomous finite dimensional dynamical system, known as the *exosystem*.

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The body of theoretical results that was developed in this domain over about three decades has scored numerous important successes and has now reached a stage of full maturity. The design of a controller that solves a generalized servomechanism problem is centered around the design of an *internal model*, which is an autonomous dynamical system capable of generating all possible "feed-forward inputs" capable of securing perfect tracking. Even though several different approaches to the design of internal models have been pursued in the literature (see e.g. [13, 3, 19, 8]) it was only recently that the design in question was understood to be based on the very same principles underlying the design of state *observers*. And, of course, in the design of regulators for nonlinear systems, it is the design on *nonlinear observers* that plays a crucial role.

The theory of nonlinear observers has been used, in the design of nonlinear regulators, at different levels of generality. The earliest contribution of this kind is directly related to the pioneering work of Michael Zeitz on the design of nonlinear observers [2]. Professor Zeitz investigated the problem of determining when and how the dynamics of the observation error can be made diffeomorphic to a linear dynamics, a problem that later became known as problem of *linearization by output injection*. As a matter of fact, the same principles inspiring the method of linearization by output injection have been used in [9] for the design of a (special class of) nonlinear internal models. An adaptive version of the method, based on the works of [1, 18], was used later in [10] for the design of a class of *adaptive* nonlinear internal models. Finally, the theory of high-gain nonlinear observers as developed by [12] was used in [6] for the design of a fully general (though not adaptive) nonlinear internal model.

In this paper, we consider a servomechanism problem in which the command and control functions are distributed in space, and hence the system consists of different components linked by communication networks. The simplest case in which this situation may occur is when generation of reference signals and control functions take place at distant locations. The problem addressed is the control of a plant so as to have its output tracking (a family of) reference commands generated at a *remote location* and transmitted through a communication channel of finite capacity. What renders the problem in question different from a conventional tracking problem is that the *tracking error*, that is the difference between the command input and the controlled output, is not available as a *physical entity*, as it is defined as difference between two quantities residing at different (and possibly distant) physical locations. Therefore the tracking error as such cannot be used to drive a feedback controller, as it is the case in a standard tracking problem. As a simple example of application of our method, we describe a how the output of a system can be forced to track a reference signal generated by a remotely located Van der Pol oscillator.

2 Problem Statement

The problem outlined in the introduction can be defined in the following terms. Consider a single-input single-output nonlinear system modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

and suppose its output y is required to asymptotically track the output y_{des} of a remotely located exosystem

$$\begin{aligned} \dot{w} &= s(w) \qquad w \in \mathbb{R}^r \\ y_{\mathrm{des}} &= y_{\mathrm{r}}(w)\,. \end{aligned} \tag{2}$$

The problem is to design a control law of the form

$$\begin{aligned} \dot{\xi} &= \varphi(\xi, y, w_{\mathrm{q}}) \\ u &= \theta(\xi, y, w_{\mathrm{q}}) \end{aligned} \tag{3}$$

in which w_{q} represents a *sampled and quantized* version of the remote exogenous input w, so as to have the tracking error

$$e(t) = y(t) - y_{\mathrm{r}}(w(t)) \tag{4}$$

asymptotically converging to zero as time tends to ∞. Note that the controller in question does not have access to e, which is not physically available, but only to the controlled output and to a sampled and quantized version of the remotely generated command.

We will show in what follows how the theory of output tracking can be enhanced so as to address this interesting design problem. In particular, we will show how, by incorporating in the controller two (appropriate) internal models of the exogenous signals, the desired control goal can be achieved. One internal model is meant to asymptotically reproduce, at the location of the controlled plant, the behavior of the remote command input. The other internal model, as in any tracking scheme, is meant to generate the "feed-forward" input which keeps the tracking error identically at zero.

We begin by describing, in the following section, the role of the first internal model.

3 The Encoder-Decoder Pair

In order to overcome the limitation due to the finite capacity of the communication channel, the control structure proposed here has a decentralized structure consisting of two separate units: one unit, co-located with the command generator, consists of an *encoder* which extracts from the the reference signal the data which are transmitted through the communication channel;

the other unit, co-located with the controlled plant, consists of a *decoder* which processes the encoded received information and of a *regulator* which generates appropriate control input.

The problem at issue will be solved under a number of assumptions most of which are inherited by the literature of output regulation and/or control under quantization. The first assumption, which is a customary condition in the problem of output regulation, is formulated as follows.

(A0) The vector field $s(\cdot)$ in (2) is locally Lipschitz and the initial conditions for (2) are taken in a fixed compact invariant set W_0. $\triangleleft$

The next assumption is, on the contrary, newer and motivated by the specific problem addressed in this paper. In order to formulate rigorously the assumption in question, we need to introduce some notation. In particular let $|x|_S$ denote the distance at a point $x \in \mathbb{R}^n$ from a compact subset $S \subset \mathbb{R}^n$, i.e. the number

$$|x|_S := \max_{y \in S} |x - y|$$

and let

$$L_0 = \max_{\substack{i \in [1,\ldots,r] \\ (x,y) \in W_0 \times W_0}} |x_i - y_i| \,. \tag{5}$$

Furthermore, having denoted by N_b the number of bits characterizing the communication channel constraint, let N be the largest positive integer such that

$$N_b \geq r \lceil \log_2 N \rceil \tag{6}$$

where $\lceil v \rceil$, $v \in \mathbb{R}$, denotes the lowest integer such that $\lceil v \rceil \geq v$.

With this notation in mind, the second assumption can be precisely formulated as follows.

(A1) There exists a compact set $W \supset W_0$ which is invariant for $\dot{w} = s(w)$ and such that

$$\bar{w} \notin W \qquad \Rightarrow \qquad |\bar{w}|_{W_0} \geq \sqrt{r}\frac{L_0}{2N} \,. \quad \triangleleft$$

W being compact and $s(\cdot)$ being locally Lipschitz, it is readily seen that there exists a non decreasing and bounded function $M(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, with $M(0) = 1$, such that for all $w_{10} \in W$ and $w_{20} \in W$ and for all $t \geq 0$

$$|w_1(t) - w_2(t)| \leq M(t)|w_{10} - w_{20}| \tag{7}$$

where $w_1(t)$ and $w_2(t)$ denote the solutions of (2) at time t passing through w_{10} and, respectively, w_{20} at time $t = 0$.

This function, the sampling interval T, the number L_0 defined in (5) and the number N fulfilling (6), determine the parameters of the encoder-decoder pair, which are defined as follows (see [20], [17], [11] for more details).

Encoder dynamics. The encoder dynamics consist of a copy of the exosystem dynamics, whose state is updated at each sampling time kT, $k \in \mathbb{N}$ and determines (depending on the actual state of the exosystem) the centroid of the quantization region, and of an additional discrete-time dynamics which determines the size of the quantization region. Specifically, the encoder is characterized by

$$\begin{aligned} \dot{w}_{\mathrm{e}} &= s(w_{\mathrm{e}}) & w_{\mathrm{e}}(kT) &= w_{\mathrm{e}}(kT^-) + w_{\mathrm{q}}(k)\frac{L(k)}{N} \\ & & w_{\mathrm{e}}(0^-) &\in W_0 \\ L(k+1) &= \sqrt{r}\,\frac{M(T)}{N}L(k) & L(0) &= L_0 \end{aligned}$$

in which w_{q} represents the encoded information given by, for $i = 1, \ldots, r$,

$$w_{\mathrm{q},i}(k) = \mathrm{sgn}(w_i(kT) - w_{e,i}(kT^-)) \cdot \begin{cases} \left\lceil \dfrac{N|w_i(kT) - w_{e,i}(kT^-)|}{L(k)} \right\rceil - \dfrac{1}{2} & N \text{ even} \\ \left\lceil \dfrac{N|w_i(kT) - w_{e,i}(kT^-)|}{L(k)} - \dfrac{1}{2} \right\rceil & N \text{ odd}\,. \end{cases}$$

At each sampling time kT, the vector $w_{\mathrm{q}}(k)$ is transmitted to the controlled plant through the communication channel and then used to update the state of the decoder unit as described in the following. To this regard note that each component of the vector $w_{\mathrm{q}}(k)$ can be described by $\lceil \log_2 N \rceil$ bits and thus the communication channel constraint is fulfilled.

Decoder dynamics The decoder dynamics is a replica of the encoder dynamics and it is given by

$$\begin{aligned} \dot{w}_{\mathrm{d}} &= s(w_{\mathrm{d}}) & w_{\mathrm{d}}(kT) &= w_{\mathrm{d}}(kT^-) + w_{\mathrm{q}}(k)\frac{L(k)}{N} \\ & & w_{\mathrm{d}}(0^-) &= w_{\mathrm{e}}(0^-) \\ L(k+1) &= \sqrt{r}\,\frac{M(T)}{N}L(k) & L(0) &= L_0 \end{aligned} \tag{8}$$

If, ideally, the communication channel does not introduce delays, it turns out that $w_{\mathrm{d}}(t) \equiv w_{\mathrm{e}}(t)$ for all $t \geq 0$. Furthermore, it can be proved that the set W characterized in Assumption (A1) in invariant for the encoder (decoder) dynamics and that the asymptotic behavior of $w_{\mathrm{e}}(t)$ ($w_{\mathrm{d}}(t)$) converges uniformly to the true exosystem state $w(t)$, provided that T is properly chosen with respect to the number N and the function $M(\cdot)$. This is formalized in the next proposition (see [17], [11] for details).

Proposition 1. *Suppose Assumptions (A0)-(A1) hold and that the sampling time T and the number N satisfy*

$$N > \sqrt{r}\,M(T)\,. \tag{9}$$

Then:

(i) for any $w_{\rm d}(0^-) \in W_0$ *and* $w(0) \in W_0$, $w_{\rm d}(t) \in W$ *for all* $t \geq 0$*;*

(ii) for any $w_{\rm d}(0^-) \in W_0$ *and* $w(0) \in W_0$*,*

$$\lim_{t\to\infty} |w(t) - w_{\rm d}(t)| = 0$$

with uniform convergence rate, namely for every $\epsilon > 0$ *there exists* $T^* > 0$ *such that for all initial states* $w_{\rm d}(0^-) \in W_0$*,* $w(0) \in W_0$*, and for all* $t \geq T^*$*,* $|w(t) - w_{\rm d}(t)| \leq \epsilon$*.*

Proof. As W is an invariant set for $\dot{w} = s(w)$, the proof of the first item reduces to show that, for all $k \geq 0$, if $w_{\rm d}(kT^-) \in W$ then necessarily $w_{\rm d}(kT) \in W$. For, note that this is true for $k = 0$. As a matter of fact, since $w_{\rm d}(0^-) \in W_0 \subset W$ and by bearing in mind the definition of $w_{\rm q}$, it turns out that $|w_{\rm d}(0) - w(0)| \leq \sqrt{r}L_0/2N$ which implies, by definition of W in Assumption (A1), that $w_{\rm d}(0) \in W$. For a generic $k > 0$ note that, again by definition of $w_{\rm q}$, it turns out that $|w_{\rm d}(kT) - w(kT)| \leq \sqrt{r}L(k)/2N$. But, by the second of (8) and by condition (9), $L(k) < L(k-1) \leq L_0$ yielding $|w_{\rm d}(kT) - w(kT)| \leq \sqrt{r}L_0/2N$ which implies $w_{\rm d}(kT) \in W$. This completes the proof of the first item. The second item has been proved in [17], [11]. ◁

Remark 1. By composing (6) with (9) it is easy to realize that the number of bits N_b and the sampling interval T are required to satisfy the constraint

$$N_b \geq r \left\lceil \log_2 \left(\sqrt{r}\, M(T) \right) \right\rceil \tag{10}$$

in order to have the encoder-decoder trajectories asymptotically converging to the exosystem trajectories. Since the function $M(\cdot)$ depends on the exosystem dynamics and on the set W_0 of initial conditions for (2), equation (10) can be interpreted as a relation between the bit-rate of the communication channel and the exosystem dynamics which must be satisfied in order to remotely reconstruct the reference signal.

4 The Design of the Regulator

4.1 Standing Hypotheses

As in most of the literature on regulation of nonlinear system, we assume in what follows that the controlled plant has well defined relative degree and normal form. If this is the case and if the initial conditions of the plant are allowed to vary on a fixed (though arbitrarily large) compact set, there is no loss of generality in considering the case in which the controlled plant has relative degree 1 (see for instance [4]). We henceforth suppose that system (1) is expressed in the form

$$\begin{aligned} \dot{z} &= f(z, y, \mu) \qquad && z \in \mathbb{R}^n \\ \dot{y} &= q(z, y, \mu) + u && y \in \mathbb{R} \end{aligned} \tag{11}$$

in which μ is a vector of uncertain parameters ranging in a known compact set P. Initial conditions $(z(0), y(0))$ of (11) are allowed to range on a fixed (but otherwise arbitrary) compact set $Z \times Y \subset \mathbb{R}^n \times \mathbb{R}$.

It is well known that, if the regulation goal is achieved, in steady-state (i.e when the tracking error $e(t)$ is identically zero) the controller must necessarily provide an input of the form

$$u_{\rm ss} = L_s y_{\rm r}(w) - q(z, y_{\rm r}(w), \mu) \tag{12}$$

(where $L_s y_{\rm r}(\cdot)$ stands for the derivative of $y_{\rm r}(\cdot)$ along the vector field $s(\cdot)$) in which w and z obey

$$\begin{aligned} \dot{\mu} &= 0 \\ \dot{w} &= s(w) \\ \dot{z} &= f(z, y_{\rm r}(w), \mu)\,. \end{aligned} \tag{13}$$

As in [5], we assume in what follows that system (13) has a compact attractor, which is also locally exponentially stable. To express this assumption in a concise form, it is convenient to group the components μ, w, z of the state vector of (13) into a single vector $\mathbf{z} = \mathrm{col}(\mu, w, z)$ and rewrite the latter as

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})\,.$$

Consistently, the map (12) is rewritten as

$$u_{\rm ss} = \mathbf{q}_0(\mathbf{z})\,,$$

and it is set $\mathbf{Z} = P \times W \times Z$. The assumption in question is the following one:

(A2) there exists a compact subset $\mathcal{Z}$ of $P \times W \times \mathbb{R}^n$ which contains the positive orbit of the set $\mathbf{Z}$ under the flow of (13) and $\omega(\mathbf{Z})$ is a differential submanifold (with boundary) of $P \times W \times \mathbb{R}^n$. Moreover there exists a number $d_1 > 0$ such that

$$\mathbf{z} \in P \times W \times \mathbb{R}^n\,, \quad |\mathbf{z}|_{\omega(\mathbf{Z})} \le d_1 \qquad \Rightarrow \qquad \mathbf{z} \in \mathbf{Z}\,.$$

Finally, there exist $m \ge 1$, $a > 0$ and $d_2 \le d_1$ such that

$$\mathbf{z}_0 \in P \times W \times \mathbb{R}^n\,, \quad |\mathbf{z}_0|_{\omega(\mathbf{Z})} \le d_2 \qquad \Rightarrow \qquad |\mathbf{z}(t)|_{\omega(\mathbf{Z})} \le m e^{-at} |\mathbf{z}_0|_{\omega(\mathbf{Z})}\,,$$

in which $\mathbf{z}(t)$ denotes the solution of (13) passing through $\mathbf{z}_0$ at time $t = 0$. $\triangleleft$

In what follows, the set $\omega(\mathbf{Z})$ will be simply denoted as $\mathcal{A}_0$. The final assumption is an assumption that allows us to construct an *internal model* of all inputs of the form $u_{\rm ss}(t) = \mathbf{q}_0(\mathbf{z}(t))$, with $\mathbf{z}(t)$ solution of (13) with initial condition in $\mathcal{A}_0$. This assumption, which can be referred to as assumption of *immersion* into a *nonlinear uniformly observable* system, is the following one:

(A3) There exists an integer $d > 0$ and a locally Lipschitz map $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that, for all $\mathbf{z} \in \mathcal{A}_0$, the solution $\mathbf{z}(t)$ of (13) passing through $\mathbf{z}_0$ at $t = 0$ is such that the function $u(t) = \mathbf{q}_0(\mathbf{z}(t))$ satisfies

$$u^{(d)}(t) + \varphi(u(t), u^{(1)}(t), \ldots, u^{(d-1)}(t)) = 0\,. \quad \triangleleft \tag{14}$$

4.2 A Nonlinear Observer as Nonlinear Internal Model

As mentioned in the introduction, nonlinear observers play a fundamental role in the design of nonlinear regulators. To see why this is the case consider a candidate controller having the following structure

$$\begin{aligned} \dot{\xi} &= \Phi(\xi) + \Psi(\xi)v \\ u &= \gamma(\xi) + v \end{aligned} \tag{15}$$

in which $\xi \in \mathbb{R}^\nu$ and v is an additional control, to be determined at a later stage. Controlling the plant (11) by means of (15) yields a system

$$\begin{aligned} \dot{\mu} &= 0 \\ \dot{w} &= s(w) \\ \dot{z} &= f(z, y, \mu) \\ \dot{y} &= q(z, y, \mu) + \gamma(\xi) + v \\ \dot{\xi} &= \Phi(\xi) + \Psi(\xi)v \\ e &= y - y_{\mathrm{r}}(w) \end{aligned}$$

which, regarded as a system with input v and output e, has a well-defined relative degree, equal to one. If the vector field $\Psi(\xi)$ is complete, this system has a globally-defined *normal form* (see e.g. [14, pages 427-432]). Its *zero dynamics* are those of

$$\begin{aligned} \dot{\mu} &= 0 \\ \dot{w} &= s(w) \\ \dot{z} &= f(z, y_{\mathrm{r}}(w), \mu) \\ \dot{\xi} &= \Phi(\xi) + \Psi(\xi)[L_s y_{\mathrm{r}}(w) - q(z, y_{\mathrm{r}}(w), \mu) - \gamma(\xi)]\,, \end{aligned}$$

and these equations, using the concise notation $\mathbf{z} = \mathrm{col}(\mu, w, z)$ introduced above, can be rewritten as

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{\xi} &= \Phi(\xi) + \Psi(\xi)[\mathbf{q}_0(\mathbf{z}) - \gamma(\xi)]\,. \end{aligned} \tag{16}$$

It is known that if a system has relative degree one, a globally defined normal form, and a zero dynamics whose trajectories asymptotically converge to a compact attractor, control by means of *high-gain output feedback* has the effect of keeping trajectories bounded and steering the output itself to zero. In view of this fact, it is reasonable to expected that if the trajectories of (16) converge to a compact attractor, the choice of the additional control v in

(15) as a high-gain feedback on e can be used to solve the problem of output regulation. Leaving aside, for the time being, the fact that the variable e is not – in the present setting – available for feedback, we describe in what follows how the desired asymptotic properties of (16) can be achieved.

Note that the dynamics in question can be viewed as the cascade connection of two subsystems, the upper of which has trajectories which are bounded and attracted by the compact invariant set $\mathcal{A}_0$ (see Assumption (A2)). Thus, the idea is to design $\Phi(\xi), \Psi(\xi), \gamma(\xi)$ so that also in the full system (16) the trajectories are bounded and attracted by a compact invariant set. Looking at how the upper and the lower subsystem of (16) are coupled, it is seen that the coupling takes places through the function $u_{\rm ss}(t) = \mathbf{q}_0(\mathbf{z}(t))$, which is seen as "output" of the upper subsystem and "input" of the lower subsystem. In view of Assumption (A3), as long as $\mathbf{z}_0 \in \mathcal{A}_0$, the function in question can be regarded also as output of the autonomous nonlinear system

$$\begin{aligned}
\dot\zeta_1 &= \zeta_2 \\
\dot\zeta_2 &= \zeta_3 \\
&\cdots \\
\dot\zeta_{d-1} &= \zeta_d \\
\dot\zeta_d &= -\varphi(\zeta_1, \zeta_2, \ldots, \zeta_d) \\
u_{\rm ss} &= \zeta_1
\end{aligned}$$

which is trivially *uniformly completely observable*, in the sense of [12]. Taking advantage of this property, it seems quite natural at this point to choose the lower subsystem of (16) as an *observer* for the set of variables $\zeta_1, \ldots, \zeta_d$ (which is indeed always possible, because the latter possesses the required observability properties). In this way, one is guaranteed that the components of the vector ξ are attracted by a compact set, and the required asymptotic property of (16) is obtained.

The nonlinear observer will be designed according to the so-called "high-gain" construction proposed in [12]. To this end, consider the sequence of functions recursively defined as

$$\tau_1(\mathbf{z}) = \mathbf{q_0}(\mathbf{z}), \quad \ldots, \quad \tau_{i+1}(\mathbf{z}) = \frac{\partial \tau_i}{\partial \mathbf{z}} \mathbf{f}_0(\mathbf{z})$$

for $i = 1, \ldots, d-1$, with d as introduced in assumption (A3), and consider the map

$$\begin{aligned}
\tau : P \times W \times \mathbb{R}^n &\to \mathbb{R}^d \\
(\mu, w, z) \quad &\mapsto \mathrm{col}(\tau_1(\mathbf{z}), \tau_2(\mathbf{z}), \ldots, \tau_d(\mathbf{z}))\,.
\end{aligned}$$

If k, the degree of continuous differentiability of the functions in (11), is large enough, the map τ is well defined and C^1. In particular $\tau(\mathcal{A}_0)$, the image of $\mathcal{A}_0$ under τ is a *compact* subset of $\mathbb{R}^d$, because $\mathcal{A}_0$ is a compact subset of $P \times W \times \mathbb{R}^n$.

Let $\varphi_{\rm c} : \mathbb{R}^d \to \mathbb{R}$ be any locally Lipschitz function of compact support which agrees on $\tau(\mathcal{A}_0)$ with the function φ defined in (A3), i.e. a function such that, for some compact superset $\mathcal{S}$ of $\tau(\mathcal{A}_0)$ satisfies

$$\begin{aligned} \varphi_c(\eta) &= 0 \qquad \text{for all } \eta \notin \mathcal{S} \\ \varphi_c(\eta) &= \varphi(\eta) \text{ for all } \eta \in \tau(\mathcal{A}_0). \end{aligned}$$

With this in mind, consider the system

$$\dot{\xi} = \Phi_c(\xi) + G(u_{ss} - \Gamma\xi) \tag{17}$$

in which

$$\Phi_c(\xi) = \begin{pmatrix} \xi_2 \\ \xi_3 \\ \cdots \\ \xi_d \\ -\varphi_c(\xi_1, \xi_2, \ldots, \xi_d) \end{pmatrix}, \quad G = \begin{pmatrix} \kappa c_{d-1} \\ \kappa^2 c_{d-2} \\ \cdots \\ \kappa^{d-1} c_1 \\ \kappa^d c_o \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 \; 0 \cdots \; 0 \end{pmatrix},$$

the c_i's are such that the polynomial $\lambda^d + c_0\lambda^{d-1} + \cdots + c_{d-1} = 0$ is Hurwitz and κ is a positive number. As shown in [6], if κ is large enough, the state $\xi(t)$ of (17) asymptotically tracks $\tau(\mathbf{z}(t))$, in which $\mathbf{z}(t)$ is the state of system (13). Therefore $\Gamma\xi(t)$ asymptotically reproduces its output (12), i.e. the steady state control $u_{ss}(t)$. As a matter of fact, the following result holds.

Lemma 1. *Suppose assumptions (A1) and (A2) hold. Consider the triangular system*

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{\xi} &= \Phi_c(\xi) + G(\mathbf{q}_0(\mathbf{z}) - \Gamma\xi)\,. \end{aligned} \tag{18}$$

Let the initial conditions for $\mathbf{z}$ *range in the set* $\mathbf{Z}$ *and let* Ξ *be an arbitrarily large compact set of initial condition for* ξ. *There is a number* κ^* *such that, if* $\kappa \geq \kappa^*$, *the trajectories of (18) are bounded and*

$$\text{graph}(\tau|_{\mathcal{A}_0}) = \omega(\mathbf{Z} \times \Xi)\,.$$

In particular $\text{graph}(\tau|_{\mathcal{A}_0})$ *is a compact invariant set which uniformly attracts* $\mathbf{Z} \times \Xi$. *Moreover,* $\text{graph}(\tau|_{\mathcal{A}_0})$ *is also locally exponentially attractive.*

4.3 The Remote Regulator and its Properties

In view of Lemma 1, it would be natural – if the true error variable e were available for feedback purposes – to choose for (11) a control of the form

$$\begin{aligned} \dot{\xi} &= \Phi_c(\xi) - Gke \\ u &= \Gamma\xi - ke\,, \end{aligned} \tag{19}$$

with k a large number. This control, in fact, would solve the problem of output regulation (see [5]). The true error e not being available, we choose instead

$$\hat{e} = y - y_{\rm r}(w_{\rm d}) \tag{20}$$

and the controller accordingly as

$$\begin{aligned} \dot{\xi} &= \Phi_{\rm c}(\xi) - Gk\hat{e} \\ u &= \Gamma\xi - k\hat{e}\,. \end{aligned} \tag{21}$$

The main result which can be established is that there exists $k^* > 0$ such that if $k \geq k^*$ the regulator designed above solves the problem in question (provided that N and T satisfy the condition of Proposition 1).

To this end, it is shown first of all the trajectories of the controlled system, namely those of the system

$$\begin{aligned} \dot{w}_{\rm d} &= s(w_{\rm d}) \qquad\qquad w_{\rm d}(kT) = w_{\rm d}(kT^-) + w_{\rm q}(k)\frac{L(k)}{2N} \\ \dot{z} &= f(z, y, \mu) \\ \dot{y} &= q(z, y, \mu) + \Gamma\xi - k(y - y_{\rm r}(w_{\rm d})) \\ \dot{\xi} &= \Phi_{\rm c}(\xi) - Gk(y - y_{\rm r}(w_{\rm d})) \end{aligned} \tag{22}$$

are bounded. To study trajectories of (22) it is convenient to replace the coordinate y by

$$\hat{e} = y - y_{\rm r}(w_{\rm d})$$

to obtain the system

$$\begin{aligned} \dot{w}_{\rm d} &= s(w_{\rm d}) \\ \dot{z} &= f(z, \hat{e} + y_{\rm r}(w_{\rm d}), \mu) \\ \dot{\xi} &= \Phi_{\rm c}(\xi) - Gk\hat{e} \\ \dot{\hat{e}} &= q(z, \hat{e} + y_{\rm r}(w_{\rm d}), \mu) - L_s y_{\rm r}(w_{\rm d}) + \Gamma\xi - k\hat{e}\,. \end{aligned} \tag{23}$$

This system can be further simplified by changing the state variable ξ into $\tilde{\xi} = \xi - G\hat{e}$ and setting $p = \mathrm{col}(\mu, w_{\rm d}, z, \tilde{\xi})$, so as to obtain a system of the form

$$\begin{aligned} \dot{p} &= F_0(p) + F_1(p, \hat{e})\hat{e} \\ \dot{\hat{e}} &= H_0(p) + H_1(p, \hat{e})\hat{e} - k\hat{e}\,, \end{aligned} \tag{24}$$

in which

$$F_0(p) = \begin{pmatrix} 0 \\ s(w_{\rm d}) \\ f(z, y_{\rm r}(w_{\rm d}), \mu) \\ \Phi(\tilde{\xi}) + G(-q(z, y_{\rm r}(w_{\rm d}), \mu) + L_s y_{\rm r}(w_{\rm d}) - \Gamma\tilde{\xi}) \end{pmatrix}$$

$$H_0(p) = q(z, y_{\rm r}(w_{\rm d}), \mu) - L_s y_{\rm r}(w_{\rm d}) + \Gamma\tilde{\xi}$$

and $F_1(p, \hat{e})$, $H_1(p, \hat{e})$ are suitable continuous functions.

With this notation at hand, it is possible to show that a large value of k succeeds in rendering bounded the trajectories of the switched nonlinear system (24) provided that the sampling interval T is sufficiently large.

Proposition 2. *Consider system (22) with initial conditions in $P \times W \times Z \times Y \times \Xi$. Suppose assumptions (A0)-(A3) hold. Let κ be chosen as indicated in Lemma 1. Then there exist $T^* > 0$ and $k^* > 0$ such that for all sampling intervals $T > T^*$ and all $k \geq k^*$ the trajectories are bounded in positive time.*

Proof. See [16].

Proposition 2 shows that trajectories of the controlled system remain bounded if the time interval T exceeds a minimum number T^* (minimal "dwell-time") which depends on the parameters of the controlled system and on the sets of initial conditions. This, in view of (10), requires N_b to exceed a suitable minimum number N_b^*. [4]

To prove that the tracking error converges to zero, it is useful to observe that, if the coordinate y of (22) is replaced by

$$e = y - y_{\rm r}(w)$$

the system in question can be also rewritten as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(z, e + y_{\rm r}(w), \mu) \\ \dot{\xi} &= \Phi_{\rm c}(\xi) + G(-ke) + G(-k\tilde{e}) \\ \dot{e} &= q(z, e + y_{\rm r}(w), \mu) - L_s y_{\rm r}(w) + \Gamma\xi - ke - k\tilde{e} \end{aligned} \tag{25}$$

having set

$$\tilde{e} = \hat{e} - e\,.$$

The same change of variables used to put (23) in the form (24) yields now a system of the form

$$\begin{aligned} \dot{p} &= F_0(p) + F_1(p, e)e \\ \dot{e} &= H_0(p) + H_1(p, e)e - ke - k\tilde{e}\,, \end{aligned} \tag{26}$$

in which $p = \text{col}(\mu, w, z, \tilde{\xi})$ and $F_0(p), F_1(p, e), H_0(p), H_1(p, e)$ are the same as in (24). This system can be viewed as system

$$\begin{aligned} \dot{p} &= F_0(p) + F_1(p, e)e \\ \dot{e} &= H_0(p) + H_1(p, e)e - ke \end{aligned} \tag{27}$$

forced by a perturbation

$$\tilde{e} = y_{\rm r}(w) - y_{\rm r}(w_{\rm d})$$

which, since $y_{\rm r}(w)$ is continuous, is asymptotically vanishing because of Proposition 1.

[4] If the number N_b is fixed and not compatible with the minimal dwell time T^* determined in the proof of Proposition 2, a more elaborate control structure has to be used, as suggested in [16].

The asymptotic properties of (24) have been investigated in [7]. In particular, the results presented in that paper show that if k is large enough, system (24) is *input-to-state stable*, with restrictions, with respect to a compact subset which is entirely contained in the set $\{(p, e) : e = 0\}$. This property can be exploited to prove the main result of the paper.

Proposition 3. *Consider system (22) with initial conditions in $P \times W \times Z \times \Xi \times Y$. Suppose assumptions (A0)-(A3) hold. Let κ be chosen as indicated in Lemma 1. Then there exist $T^* > 0$ and $k^* > 0$ such that for all sampling intervals $T > T^*$ and all $k \geq k^*$, trajectories are bounded in positive time and*

$$\lim_{t \to \infty} e(t) = 0\,.$$

Proof. See [16].

5 Simulation Results

We consider the problem of synchronizing two oscillators located at remote places through a constrained communication channel. The master oscillator (playing the role of exosystem) is a Van der Pol oscillator described by

$$\begin{aligned} \dot{w}_1 &= w_2 + \epsilon(w_1 - a w_1^3) \\ \dot{w}_2 &= -w_1 \end{aligned} \tag{28}$$

whose output $y_\mathrm{r} = w_2$ must be replied by the output y of a remote system of the form

$$\dot{y} = u\,. \tag{29}$$

Simple computations show that, in this specific case, the steady state control input u_ss coincides with $u_\mathrm{ss} = -w_1$ and the assumption (A3) is satisfied by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -\varphi(\xi_1, \xi_2) \\ u_\mathrm{ss} &= \xi_1 \end{aligned}$$

where $\varphi(\xi_1, \xi_2) = \xi_1 - \epsilon(\xi_2 - 3a\xi_1^2\xi_2)$ through the map

$$\tau(w) = \big(-w_1 \;\; -w_2 + \epsilon\,(w_1 - a w_1^3)\big)^\mathrm{T}$$

We consider a Van der Pol oscillator with $\epsilon = 1.5$ and $a = 1$. The regulator (21) is tuned choosing $\kappa = 3$, $G = (12\ 36)^\mathrm{T}$ and $k = 8$. We consider two different simulative scenarios which differ for the severity of the communication channel constraint. In the first case we suppose that the number of available bits is $N_b = 2$ yielding, according to (6) and to the fact that $r = 2$, $N = 2$. In this case, for a certain set of initial conditions, condition (9) is fulfilled with

$T = 0.15$ s. In the second case the available number of bits is assumed $N_b = 4$ from which (6) and (9) yield a bigger N and T respectively equal to $N = 4$ and $T = 0.5$ s. The simulation results, obtained assuming the exosystem (28) and the system (29) respectively at the initial conditions $w(0) = (1, 0)$ and $y(0) = 5$, are shown in the figure 1 for the first scenario and figure 2 for the second one. In particular figure 1 (respectively 2) shows in the left-half side the phase portrait of the Van der Pol oscillator with overlapped the actual state trajectory of the encoder (decoder) and, in the right-half side, the time behavior of the reference trajectory $y_{\mathrm{r}}(t)$ (dotted line) and of the controlled output $y(t)$ (solid line).

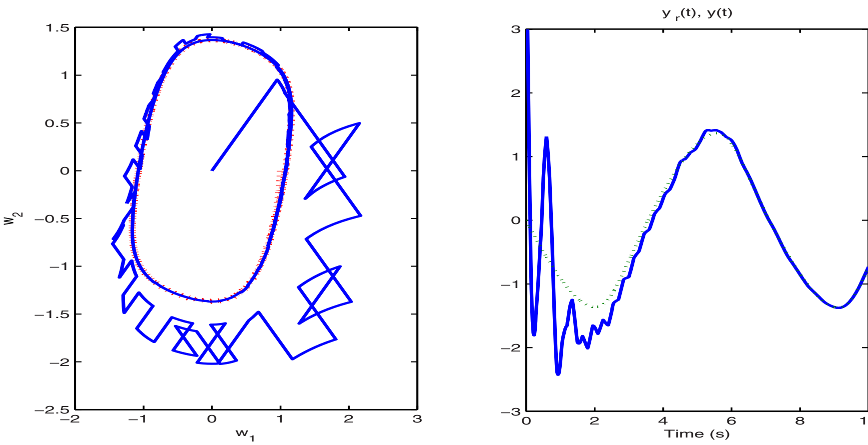

Fig. 1. First control scenario ($N = 2$, $T = 0.15$ s). Left: phase portrait of the exosystem (dotted line) and trajectory $(w_{\mathrm{e}1}, w_{\mathrm{e}2})$ (solid line). Right: time behavior of the reference trajectory $y_{\mathrm{r}}(t)$ (dotted line) and of the controlled output $y(t)$ (solid line).

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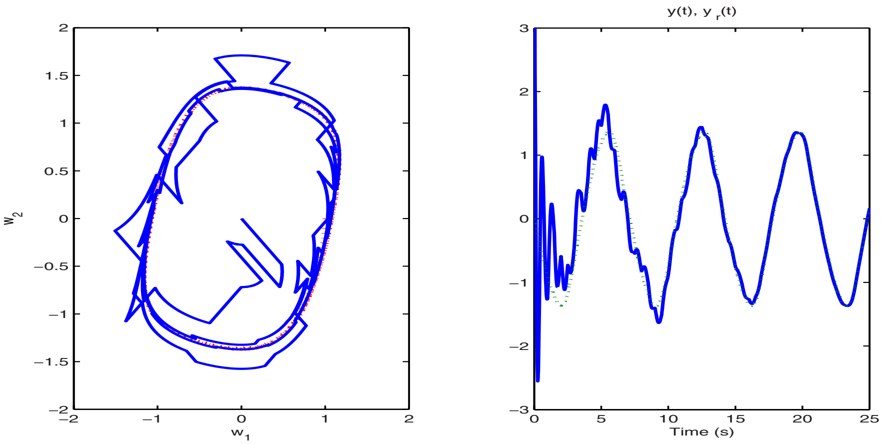

Fig. 2. Second control scenario ($N = 4$, $T = 0.5$ s). Left: phase portrait of the exosystem (dotted line) and trajectory (w_{e1}, w_{e2}) (solid line). Right: time behavior of the reference trajectory $y_{\rm r}(t)$ (dotted line) and of the controlled output $y(t)$ (solid line).

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