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Discrete port-Hamiltonian systems

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Abstract

Either from a control theoretic viewpoint or from an analysis viewpoint it is necessary to convert smooth systems to discrete systems, which can then be implemented on computers for numerical simulations. Discrete models can be obtained either by discretizing a smooth model, or by directly modeling at the discrete level itself. One of the goals of this paper is to *model* port-Hamiltonian systems at the discrete level. We also show that the dynamics of the discrete models we obtain *exactly* correspond to the dynamics obtained via a usual discretization procedure. In this sense we offer an *alternative* to the usual procedure of modeling (at the smooth level) and discretization.

Keywords: Discrete mechanics; Port Hamiltonian systems

1. Introduction

In previous work, see e.g. [6,1,14], it has been shown how port-based network modeling of complex lumped-parameter physical systems naturally leads to a generalized Hamiltonian formulation of the dynamics. In fact, the Hamiltonian is given by the total energy of the energy-storing elements in the system, while the geometric structure, defining together with the Hamiltonian the dynamics of the system, is given by the power-conserving interconnection structure of the system, and is called a Dirac structure. Furthermore, energy-dissipating elements may be added by terminating some of the system ports. The resulting class of open dynamical systems has been called "port-Hamiltonian systems" [14]. The port-Hamiltonian framework offers many fundamental benefits. Firstly, it is instrumental in finding the most convenient representation of the equations of motion of the system; in the format of purely differential equations or of mixed sets of differential and algebraic equations (DAEs). From an analysis point of view it allows to use powerful methods from the theory of Hamiltonian systems. Finally, the Hamiltonian structure may be fruitfully used in control design, e.g. by the explicit use of the energy function and conserved quantities for the construction of a Lyapunov function (possibly after the connection with another port-Hamiltonian controller system), or by directly modifying by feedback the interconnection and dissipation structure and shaping the internal energy. We refer to [8,14] for various work in this direction.

It is well known that for the study of complex physical systems, numerical simulation plays an important role. A fundamental issue in simulation concerns structure preserving algorithms. Numerous studies, for e.g. [2–5], have shown the benefits of preserving the continuous structure at the discrete level. The study of structure preserving algorithms is a well-established field. One fundamental problem concerns the nature of the discretized system. For example, we may discretize a Hamiltonian system and ensure that the discretized system is energy-conserving, and preserves some other conservation laws. But does this mean that the discretized system is Hamiltonian? To answer this we must first of all have a Hamiltonian theory for discrete systems, this has been accomplished in [10], we shall recall some of that paper in the next section. Roughly speaking, [10] defines a discrete differential geometry and *models* physical systems directly at a discrete level in a Hamiltonian framework. Talasila et al. [10] also proves that

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the discrete models exactly coincide (under some conditions) with discretized models of the same system. The goal of this paper is to extend [10] to the port-Hamiltonian setting.

The *basic motivation of this paper* is to formalize the geometric/mathematical structure that port-Hamiltonian systems have at the discrete level. Discrete systems can be obtained in two ways. Either we can discretize a continuous system (there exist a variety of techniques for doing so), or we can directly model the systems at the discrete level. In this paper we proceed along the latter lines, and we show that the discrete models that we obtained (via directly modeling at the discrete level) exactly coincide, under some conditions, with discretized models—thus offering an *alternative approach* towards the modeling and simulation of port-Hamiltonian systems. This paper is an extension of [12].

The outline of the paper is as follows. We briefly recall discrete Hamiltonian mechanics and certain geometrical concepts in Section 2. In Section 3, we discuss the concept of ports in our discrete setting. Discrete Dirac structures, their representations and interconnection properties are introduced in Section 4. The interconnection properties of discrete Dirac structure are derived in Section 5, and discrete port-Hamiltonian systems are defined in Section 6.

2. Geometry and Hamiltonian mechanics on discrete spaces

In this section we briefly recall certain concepts of discrete Hamiltonian mechanics, for more details cf. [10,11]. The first requirement is to choose an appropriate discrete analogue for the reals $\mathbb{R}$. We can use discrete lattices (which have a ring structure), or the space of floating point numbers $\mathbb{F}$ which have a quasi-ring (cf. [10,11]) structure. Since computers uses floating-point numbers, and since our main focus is numerical simulation, $\mathbb{F}$ will be our choice. A *discrete vector* at the point $p \in \mathbb{F}^n$ is a pair (p, q) where $q \in \mathbb{F}^n$. We will denote by $T_p\mathbb{F}^n$ the set defined as the union of all possible vectors defined at the point p, i.e. $T_p\mathbb{F}^n = \{(p, q) \in \mathbb{F}^n \times \mathbb{F}^n\} \sim \mathbb{F}^n$. Unlike in the smooth setting, there are several representations of discrete vectors. *Each representation corresponds to a certain numerical integration technique*. We recall two representations here, the Euler discrete vector and the Runge–Kutta 2 vector. These correspond to the Euler forward difference and the second order Runge–Kutta integration techniques. In [10,11] we have defined others like Runge–Kutta vectors of any order, Leap-Frog vectors, central difference vectors etc. Euler vectors or Runge–Kutta 2 vectors are defined as: $v(f(p)) = (f(p+\varepsilon) - f(p))/h$, where ε is the smallest possible distance from the point p to the next floating point number. The difference between Euler vectors and Runge–Kutta two vectors is of course in the actual value of $f(p+\varepsilon)$. The point we are trying to make is that discrete vectors have the same finite-difference structure, they only differ in the values! A discrete vector[1] does not satisfy the usual Leibniz (or product) rule for derivations, rather it is a linear map $v_i : A_p(\mathbb{F}^n) \to \mathbb{F}$ (where A_p is the algebra of functions defined around the point p) which satisfies the *modified Leibniz rule*: $v(f \cdot g) = v(f) \cdot g(p) + \mathrm{Aut}_v(f(p)) \cdot v(g),\ \forall f, g \in A_p(\mathbb{F}^n)$, where Aut_v is an automorphism which is a linear map $\mathrm{Aut}_v : A_p(\mathbb{F}^n) \to \mathbb{F}$, corresponding to the discrete vector v, defined as: $\mathrm{Aut}_v(f(p)) := f(p+\varepsilon)$, $p \in \mathbb{F}^n$ such that $\mathrm{Aut}_v(f \cdot g) = \mathrm{Aut}_v(f) \cdot \mathrm{Aut}_v(g);\ \forall f, g \in A_p(\mathbb{F}^n)$.

Discrete covectors are defined as mapping pairs of points (i.e. discrete vectors) to a floating point number, i.e. $v^* : (p, q) \to \mathbb{F}$. The set of discrete covectors forms the discrete cotangent space.

Then, we can define *discrete vector fields* as the mapping X which assigns to each point $p \in \mathbb{F}^n$ a discrete vector, i.e. $\forall p \in \mathbb{F}^n$, $X(p) = (p, q)$, $q \in \mathbb{F}^n$. The flow of the discrete vector field X is defined as the sequence of points $p_0, p_1, p_2, \ldots$ in $\mathbb{F}^n$ such that $X(p_i) = (p_i, p_{i+1})$. Likewise we can define discrete one-forms as assigning a discrete covector to each point. A function $f : \mathbb{Z}_k^n \to \mathbb{F}$ is said to be *discrete-differentiable* at $p \in \mathbb{Z}_k^n$ iff there exists a mapping $G : A(\mathbb{Z}_k^n) \to \mathbb{F}^n$ s.t. $(f(p+\varepsilon) - f(p) - G(f(p) \cdot \varepsilon))/\varepsilon = 0$. Note that the above definition does classify discrete functions between those that are discrete differentiable and those which are not. This is easy to see, since we use floating point numbers, the computation—$(f(p+\varepsilon) - f(p) - G(f(p) \cdot \varepsilon))/\varepsilon$ can easily result in a floating point overflow.

The *discrete exterior differential* is a mapping: $\Delta : \bigwedge^k(\mathbb{F}^n) \to \bigwedge^{k+1}(\mathbb{F}^n)$, defined in the following way. Consider, for instance, a function $f \in A(\mathbb{F}^n)$. The function corresponds to the assignment of an element of $\mathbb{F}$ at each point of the discrete space. The definition of a discrete one-form implies that we must construct a covector at each point. We can do that in many different ways, but if we want to preserve at the discrete level the smooth property $X(f) = \langle X, \Delta f \rangle$, the definition of the exterior differential must take into account the type of action that vector fields have on functions. For the forward difference method, this leads us to a definition of the exterior differential such as to define the one-form $\Delta f \in \bigwedge^1(\mathbb{F}^n)$ which for every point $p \in \mathbb{F}^n$ assigns to the one-dimensional hypersurface (i.e. a link) connecting each pair of points (p, q), where the pair of points are defining a discrete vector, the value $f(q) - f(p)$. Hence, in the natural basis, we would obtain as a representation: $\Delta f(p) = \sum_i (f(p + h\varepsilon_i) - f(p))\, \mathrm{d}x^i$, where h is the smallest possible distance from the point p to the next floating point number in the ith direction of the point p, and $\varepsilon_i = [0, \ldots, 1, 0, \ldots]^{\mathrm{T}}$. The concept of *discrete manifolds* has been introduced in [10,11]. Discrete manifolds are those that locally look like $\mathbb{F}^n$, on these we can define the discrete analogues of charts. atlases etc. Since $\mathbb{F}^n$ has a discrete-differentiable structure, this structure can be transferred onto discrete manifolds via chart mappings.

Let us conclude this section with discrete Hamiltonian mechanics. One way to do that would be by defining a discrete Poisson bracket as follows. Let $\mathscr{Z}$ be a discrete manifold and consider the algebra of discrete differentiable functions $A(\mathscr{Z})$ on $\mathscr{Z}$. $\mathscr{Z}$ it is endowed with a Poisson structure if there exists a mapping from the set of discrete functions $A(Z)$ to the set of discrete vector fields $\mathfrak{X}(\mathscr{Z})$ which defines an intrinsic

[1] In [10,11] we have shown that a collection of discrete vectors (Euler vectors, Runge–Kutta vectors etc.) form a 'discrete' tangent space.

operation on the set of discrete functions as follows: $\{f, g\} := X_f(g)$, which satisfies linearity, $X_{f+g} = X_f + X_g$, and $X_f(g) = -X_g(f)$. A discrete manifold $\mathscr{X}$ whose algebra of functions $A(\mathscr{X})$ is endowed with a Poisson bracket is called a discrete Poisson manifold. And then we can define discrete Hamiltonian (Poisson) dynamics as follows: for any $f \in A(\mathscr{X})$: $\Delta f(t)/\Delta t = \{f, H\} \Rightarrow f_{n+\delta} = f_n + \delta X_H(f_n)$. So in the limit as $\delta \to 0$ we recover the definition of dynamics in the smooth case using the smooth Poisson bracket: $\dot{f} = \{f, H\} = X_H(f)$.

3. The concept of ports in a discrete setting

Our approach to defining ports in the discrete setting follows the same mathematical ideas present in the smooth setting, cf. [6,1]. Physical systems can be interconnected to each other by means of *power ports*, essentially meaning that the interconnection is facilitated by an exchange of power between the systems. The constitutive phenomena in network models of physical systems is the storage of energy, defined by the energetic states (or energy variables) of the system denoted by $x \in \mathbb{R}^n$ and by an energy function $H(x)$. In open systems the states can undergo changes with time, this can be described using *port* (or external) variables as follows: $f = \mathrm{d}x/\mathrm{d}t$, $e = \mathrm{d}H(x)$. f, e called the flow and effort variables, are power-conjugated variables, since their duality product is the time variation of the energy function, i.e. the power flow into the energy storage ports is $\langle \partial H/\partial x | \mathrm{d}x/\mathrm{d}t \rangle = \mathrm{d}H/\mathrm{d}t$. Hence these power-conjugated variables can be used to describe the interconnection of physical systems.

The general idea for either the smooth or the discrete setting is basically the same. There are some important technical differences of course, for example the port variables do not belong to vector (and dual vector) spaces, they belong to free quasi-modules (and their duals). In Section 4, we discuss the mathematical details.

But what about the *physical picture*? What do ports mean in the discrete setting? Given two systems with their external powers defined by $P_1 = \langle e_1 | f_1 \rangle$ and $P_2 = \langle e_2 | f_2 \rangle$; for a power conserving interconnection we demand $P_1 = -P_2$. Now a flow (a discrete vector) is defined by a pair of points $f_1 := (p, q)$ and the effort (a discrete covector) is defined as $e_1 : (p, q) \to \mathbb{F}$, and the following definition shows how the physical power is related to this pair of points via the flow and effort. We have:

Definition 1. The power of a discrete physical system defined on a discrete manifold $\mathscr{X}$ at a point p is defined as $P_p := (p, q)_e^f$, and this is to be understood as follows: (p, q) determines the flow f, and the effort e at p is simply the dual of f at p, and the product of this flow and effort is then the power at the point p.

But this is not all, in general there are at least two ways of interpreting the above definition when we consider interconnection of systems. One way is as follows: consider two systems interconnected via the flows $f_1 := (p, q)$, $f_2 := (\bar{p}, \bar{q})$. For defining a formal power-conserving interconnection the interconnecting flows and efforts must be equal to each other. We require $f_1 = -f_2$, and one way to do this could be as follows: the second point 'q' required for defining f_1 is identified with the first point '$\bar{p}$' required for defining f_2, and the second point '$\bar{q}$' required for defining f_2 is identified with the first point 'p' required for defining f_1. So $q \backsim \bar{p}$ and $\bar{q} \backsim p$. Which means that any function takes the same value on p and $\bar{q}$, and likewise for $\bar{p}, q$. And hence the flows (velocities) are the same, with opposite directions of course. The opposite direction is just a sign convention.

Our systems are defined on manifolds with boundary. So another way of equating the flows would be to simply define $p = \bar{q}$ and $q = \bar{p}$. So then we have $(p, q) =: f_1 = -f_2 := (q, p)$.

So then what is the difference between these two ways of equating the flows? In the second case, if we extend the idea to all points in the boundaries of the two (or more) systems, we are essentially dealing with different manifolds having a common boundary. In the first case the manifolds need not have a common boundary. There are physical examples for each of the two ideas. Consider two devices interacting with each other by emitting electromagnetic waves through a lossless media—since the power at the two ports will be the same, an identification of the points of the boundary is enough to describe the port variables. Consider the interconnection of two transmission lines, in this case a common boundary is required. So depending on the type of physical interconnection we need to choose from one of the two ideas of ports.

4. Discrete Dirac structures

In this section we focus on the mathematical formalization of power-conserving interconnections in a discrete setting. The exposition in this section is very similar to the smooth setting, refer to [13,1] for the formulation in the smooth setting. The interconnection of discrete physical systems can be formalized by discrete-Dirac structures, first we consider the special case of constant discrete-Dirac structures. Consider a free quasi-module $\mathbb{F}^n$ and its dual $\mathbb{F}^{n*}$. We call the product space $\mathbb{F}^n \times \mathbb{F}^{n*}$ as the space of *power variables* and on this product space we define the *power* as: $P = \langle e | f \rangle$, $(f, e) \in \mathbb{F}^n \times \mathbb{F}^{n*}$, with P taking values in $\mathbb{F}$. $\mathbb{F}^n$ is called the space of flows, and $\mathbb{F}^{n*}$ the space of efforts. On $\mathbb{F}^n \times \mathbb{F}^{n*}$ there exists a canonically defined bilinear form $\langle\langle , \rangle\rangle$ given by:

$$\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle := \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle, \quad \forall (f_1, e_1), (f_2, e_2) \in \mathbb{F}^n \times \mathbb{F}^{n*}.$$

Definition 2. A constant discrete Dirac structure on a finite-dimensional q-module $\mathbb{F}^n$ is a n-dimensional subspace $\mathscr{D} \subset \mathbb{F}^n \times \mathbb{F}^{n*}$ with the property that

$$\langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle = 0, \quad \forall (f_1, e_1), (f_2, e_2) \in \mathscr{D}, \tag{1}$$

where $\langle | \rangle$ denotes the natural pairing between $\mathbb{F}^n$ and $\mathbb{F}^{n*}$.

Proposition 1. *A constant Dirac structure on $\mathbb{F}^n$ is an n-dimensional subspace $\mathscr{D} \subset \mathbb{F}^n \times \mathbb{F}^{n*}$ with the property that*: $\langle e | f \rangle = 0$, $\forall (f, e) \in \mathscr{D}$.

Proof. Let $(f_1, e_1)=(f_2, e_2)$ then (1) gives $\langle e_1|f_1\rangle+\langle e_1|f_1\rangle=0$ and hence $\langle e|f\rangle = 0$. Conversely, by linearity for all $(f_1, e_1), (f_2, e_2) \in \mathscr{D}$ we have

$$0 = \langle e_1 + e_2|f_1 + f_2\rangle = \langle e_1|f_1\rangle + \langle e_2|f_1\rangle + \langle e_1|f_2\rangle + \langle e_2|f_2\rangle = \langle e_2|f_1\rangle + \langle e_1|f_2\rangle. \quad \square$$

4.1. Representations of Dirac structures

The following representation will be used later on to prove that interconnection of Dirac structures results again in a Dirac structure. Again, the setting is very similar to the smooth setting of [13,1]. Consider an n-dimensional q-module $\mathbb{F}^n$ and its dual n-dimensional q-module $\mathbb{F}^{n*}$. Also consider linear maps $F : \mathbb{F}^n \to W$, $E : \mathbb{F}^{n*} \to W$, with W an n-dimensional q-module. Then define $F + E : \mathbb{F}^n \times \mathbb{F}^{n*} \to W$ as: $(f, e) \in \mathbb{F}^n \times \mathbb{F}^{n*} \overset{F+E}{\longmapsto} F(f) + E(e) \in W$. Then we have:

Proposition 2.

- *Every Dirac structure $\mathscr{D} \subset \mathbb{F}^n \times \mathbb{F}^{n*}$ can be written as $\mathscr{D} = \ker(F + E)$ for linear maps as defined above. Furthermore any such E and F satisfy: $EF^* + FE^* = 0$.*
- *Every n-dimensional subspace $\mathscr{D} = \ker(F + E)$ defined by the above linear maps and satisfying $EF^{\mathrm{T}} + FE^{\mathrm{T}} = 0$, defines a Dirac structure.*
- *$\mathscr{D}$ can be written in an image representation as: $\mathscr{D}=\{(f, e) \in \mathbb{F}^n \times \mathbb{F}^{n*} | f = E^{\mathrm{T}}\lambda, e = F^{\mathrm{T}}\lambda, \lambda \in \mathbb{F}^n\}$.*

Proof. The proof is very similar to that in [13], so we only present one technical detail important for our discrete setting. In the smooth setting, to simplify the proof, [13] identify $\mathbb{F}^n$ with $\mathbb{R}^n$ and also $\mathbb{F}^{n*}$ with $\mathbb{R}^n$, and this was done using the Euclidean inner product on the reals. In our discrete setting we identity $\mathbb{F}^n$ with $\mathbb{F}^n$ and also $\mathbb{F}^{n*}$ with $\mathbb{F}^n$. To do this on our discrete spaces is quite simple—we restrict the Euclidean inner product of the reals to the floating point spaces (giving rise to a restricted metric on $\mathbb{F}^n$). And then the rest of the proof is the same. $\square$

4.2. Non-constant discrete Dirac structures

In many physical systems the power-conserving interconnection is modulated by the energy variables, in which case the definition of a constant Dirac structure, as in Definition 2, has to be generalized to a general Dirac structure on a discrete manifold. Let $\mathscr{Z}$ be a discrete manifold and $A(\mathscr{Z})$ the algebra of functions on $\mathscr{Z}$. Consider the space of discrete vector fields $\mathfrak{X}(A)$ and the space of discrete 1-forms $\Lambda^1(A)$. On $\mathfrak{X}(A) \times \Lambda^1(A)$ consider the following bilinear form: $\langle\langle(X_1, \alpha_1), (X_2, \alpha_2)\rangle\rangle = \langle\alpha_2|X_1\rangle + \langle\alpha_1|X_2\rangle$.

Definition 3. A generalized Dirac structure on an n-dimensional discrete manifold, is a n-dimensional linear subspace $\mathscr{D} \subset \mathfrak{X}(A) \times \Lambda^1(A)$, such that $\mathscr{D} = \mathscr{D}^{\perp}$, where

$$\mathscr{D}^{\perp} = \{(Y, \beta) \in \mathfrak{X}(A) \times \Lambda^1(A) | \langle\alpha|X\rangle + \langle\beta|Y\rangle = 0, \ \forall (X, \alpha) \in \mathscr{D}\}. \tag{2}$$

Since $\mathscr{D} = \mathscr{D}^{\perp}$ it immediately follows, take $(Y, \beta) = (X, \alpha)$ and using (2), that $\langle\alpha|X\rangle = 0, \ \forall (X, \alpha) \in \mathscr{D}$. This implies that a generalized Dirac structure $\mathscr{D}$ on a discrete manifold $\mathscr{Z}$ pointwise defines a constant Dirac structure on $T_l\mathscr{Z}$ for every $l \in \mathscr{Z}$ as defined in Definition 2.

Example 1. Let $J(l) : T_l^*\mathscr{Z} \to T_l\mathscr{Z}$, $l \in \mathscr{Z}$, be a skew-symmetric bilinear map. Then

$$\mathscr{D} = \{(X, \alpha) \in \mathfrak{X}(A) \times \Lambda^1(A) | X_l = J_l\alpha_l, \ \forall l \in \mathscr{Z}\}$$

is a generalized Dirac structure on $\mathscr{Z}$. This corresponds to a generalized discrete Poisson structure $(\mathscr{Z}, \{\cdot, \cdot\})$, where J_l is the structure matrix of the discrete Poisson bracket $\{\cdot, \cdot\} : A(\mathscr{Z}) \times A(\mathscr{Z}) \to A(\mathscr{Z})$, i.e.

$$\{h_1, h_2\}_l = \langle \mathrm{d}h_1|J\,\mathrm{d}h_2\rangle_l, \quad h_1, h_2 \in A(\mathscr{Z}), \ \forall l \in \mathscr{Z}.$$

Using Proposition 2 we immediately obtain the following representation of generalized Dirac structures on discrete manifolds:

Proposition 3. *Let $\mathscr{D}$ be an n-dimensional generalized Dirac structure on an n-dimensional discrete manifold $\mathscr{Z}$. Let $z = (z_1, z_2, \ldots, z_n)$ be local coordinates for $\mathscr{Z}$. Then locally there exist $n \times n$ matrices (more correctly 'module' elements) $E(z)$ and $F(z)$ such that*

$$\mathscr{D}(z) = \{(f, e) \in T_z\mathscr{Z} \times T_z^*\mathscr{Z} | F(z)f + E(z)e = 0\}$$

with

$$E(z)F^*(z) + F(z)E^*(z) = 0.$$

Corresponding to a generalized Dirac structure $\mathscr{D}$ on a discrete manifold $\mathscr{Z}$ we can define certain discrete (co)-distributions, cf. [1] for details in the smooth setting, we present one important co-distribution

$$P_1 = \{\alpha \in \Lambda^1(A) | \exists X \in \mathfrak{X}(A) \text{ such that } (X, \alpha) \in \mathscr{D}\}.$$

We can define a discrete Poisson bracket on so-called *admissible functions*, as follows.

Definition 4. A discrete function $h \in A(\mathscr{Z})$ is said to be admissible if there exists a discrete vector field X such that $(X, \Delta h) \in \mathscr{D}$.

Using the co-distribution we defined above we see that the space of all admissible functions is given by $\mathscr{A}_{\mathscr{D}} = \{h \in A(\mathscr{Z}) | \Delta h \in P_1\}$. Then we have:

Proposition 4. *On $\mathscr{A}_{\mathscr{D}}$ with $(X_1, \Delta h_1), (X_2, \Delta h_2) \in \mathscr{D}$, there is a well-defined Poisson bracket given by $\{h_1, h_2\}_{\mathscr{D}}=\langle\Delta h_1|X_2\rangle$ if and only if $\mathrm{Aut}_{\Delta h_3} = \mathrm{Aut}_{Xh_3}$.*

Proof. $\{h_1, h_2\}_{\mathscr{D}}$ must satisfy the properties of a discrete Poisson bracket. Bilinearity of $\{\cdot,\cdot\}$ follows from the bilinearity of $\langle|\rangle$. Skew-symmetry is due to the following:

$$(X_1, \Delta h_1), (X_2, \Delta h_2) \in \mathscr{D} \Rightarrow \langle \Delta h_1 | X_2 \rangle + \langle \Delta h_2 | X_1 \rangle = 0.$$

Finally, take arbitrary $(X_1, \Delta h_1), (X_2, \Delta h_2), (X_3, \Delta h_3) \in \mathscr{D}$. Then the modified Leibniz rule is

$$\begin{aligned}\{h_1, h_2 h_3\}_{\mathscr{D}} &= -\langle \Delta(h_2 \cdot h_3) | X_1 \rangle \\ &= -\langle (\Delta h_2) \cdot h_3 + \mathrm{Aut}_{\Delta}(h_2) \cdot \Delta h_3 | X_1 \rangle \\ &= h_3 \cdot \{h_1, h_2\}_{\mathscr{D}} + \mathrm{Aut}_{\Delta h_3}(h_2) \cdot \{h_1, h_3\}_{\mathscr{D}} \\ &= h_3 \cdot \{h_1, h_2\}_{\mathscr{D}} + \mathrm{Aut}_{X h_3}(h_2) \cdot \{h_1, h_3\}_{\mathscr{D}}. \qquad \square\end{aligned}$$

5. Interconnection of discrete Dirac structures

In this subsection, we discuss the interconnection properties of discrete Dirac structures. In the smooth setting a fundamental result in the framework of port-Hamiltonian systems is that the interconnection of a number of Dirac structures results again in a Dirac structure. Physically, it is clear that the composition of a number of power-conserving interconnections should result again in a power-conserving interconnection. In the smooth setting this has been formally proved, cf. [13]. The question now is if the same property would hold true for discrete models. We consider the composition of two discrete Dirac structures with partially shared variables as in Fig. 1.

We follow the same sign conventions as in [15] for the power flow corresponding to the power variables $(f_2, e_2) \in \mathscr{D}_b$. Then the interconnection $\mathscr{D}_a \| \mathscr{D}_b$ of the Dirac structures $\mathscr{D}_a$ and $\mathscr{D}_b$ is defined as

$$\begin{aligned}\mathscr{D}_a \| \mathscr{D}_b := \{(f_1, e_1, f_3, e_3) \in \mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_3 \\ \times \mathscr{F}_3^* | \exists (f_2, e_2) \in \mathscr{F}_2 \times \mathscr{F}_2^* \text{ s.t.} \\ (f_1, e_1, f_2, e_2) \in \mathscr{D}_a \text{ and } (-f_2, e_2, f_3, e_3) \in \mathscr{D}_b\}.\end{aligned}$$

First we will need the following result:

Lemma 1. *Given* $\lambda \in \mathbb{F}^{r\times 1}, C \in \mathbb{F}^{l\times r}$

$$(\exists \lambda \text{ s.t. } C\lambda = d) \;\Leftrightarrow\; (\forall \alpha \; s.t. \; \alpha^{\mathrm{T}} C = 0 \Rightarrow \alpha^{\mathrm{T}} d = 0).$$

Proof. Let us prove first from left to right, this is simple.

$$C\lambda = d \Rightarrow \alpha^* C \lambda = \alpha^* d, \;\; \forall \alpha \quad \text{if } \forall \alpha, \;\; \alpha^* C = 0 \Rightarrow \alpha^* d = 0.$$

Now the other way. Suppose $C\lambda \neq d$, then $d \notin span(C_i)$. Define $\hat{C} := \{\alpha | \alpha^* C = 0\}$.

$$\Rightarrow \hat{C}^{\perp} = span(C_i) \Rightarrow d \notin \hat{C}^{\perp} \Rightarrow \exists \alpha^* \in \hat{C} \quad \text{s.t.} \quad \alpha^* d \neq 0.$$

Hence proved. $\square$

With the above result we can now prove the following:

Theorem 1. *Let* $\mathscr{D}_a, \mathscr{D}_b$ *be Dirac structures w.r.t.* $\mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_2 \times \mathscr{F}_2^*$ *and* $\mathscr{F}_2 \times \mathscr{F}_2^* \times \mathscr{F}_3 \times \mathscr{F}_3^*$. *Then* $\mathscr{D}_a \| \mathscr{D}_b$ *is a Dirac structure with respect to the bilinear form on* $\mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_3 \times \mathscr{F}_3^*$.

Proof. The proof presented here follows the same spirit as the proof for the smooth setting, cf. [15]. Consider the Dirac structures $\mathscr{D}_a$ and $\mathscr{D}_b$ defined in matrix (more correctly—'module') kernel representation by

$$\begin{aligned}\mathscr{D}_a = \{(f_1, e_1, f_a, e_a) \in \mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_2 \times \mathscr{F}_2^* | F_1 f_1 \\ + E_1 e_1 + F_{2a} f_a + E_{2a} e_a = 0\}, \\ \mathscr{D}_b = \{(f_b, e_b, f_3, e_3) \in \mathscr{F}_2 \times \mathscr{F}_2^* \times \mathscr{F}_3 \times \mathscr{F}_3^* | F_{2b} f_b \\ + E_{2b} e_b + F_3 f_3 + E_3 e_3 = 0\}.\end{aligned}$$

Using Proposition 2, we can easily see that $\mathscr{D}_a$ and $\mathscr{D}_b$ are alternatively given in the 'matrix' image representation as:

$$\begin{aligned}\mathscr{D}_a &= [E_1^* \;\; F_1^* \;\; E_{2a}^* \;\; F_{2a}^* \;\; 0 \;\; 0]^*, \\ \mathscr{D}_b &= [0 \;\; 0 \;\; E_{2b}^* \;\; F_{2b}^* \;\; E_3^* \;\; F_3^*]^*.\end{aligned}$$

Hence,

$$\begin{aligned}&(f_1, e_1, f_3, e_3) \in \mathscr{D}_a \| \mathscr{D}_b \\ &\quad \Leftrightarrow \exists \lambda_a, \lambda_b \text{ s.t. } [f_1 \;\; e_1 \;\; 0 \;\; 0 \;\; f_3 \;\; e_3]^* \\ &\quad = \begin{bmatrix} E_1^* & F_1^* & E_{2a}^* & F_{2a}^* & 0 & 0 \\ 0 & 0 & E_{2b}^* & -F_{2b}^* & E_3^* & F_3^* \end{bmatrix}^* \begin{bmatrix} \lambda_a \\ \lambda_b \end{bmatrix} \\ &\quad \Leftrightarrow \forall (\beta_1, \alpha_1, \beta_2, \alpha_2, \beta_3, \alpha_3) \text{ s.t. } (\beta_1^* \alpha_1^* \beta_2^* \alpha_2^* \beta_3^* \alpha_3^*) \\ &\qquad \times \begin{bmatrix} E_1^* & F_1^* & E_{2a}^* & F_{2a}^* & 0 & 0 \\ 0 & 0 & E_{2b}^* & -F_{2b}^* & E_3^* & F_3^* \end{bmatrix}^* = 0,\end{aligned}$$

$$\begin{aligned}&\beta_1^* f_1 + \alpha_1^* e_1 + \beta_3^* f_3 + \alpha_3^* e_3 = 0 \\ &\quad \Leftrightarrow \forall (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) \text{ s.t.} \\ &\qquad \times \begin{bmatrix} F_1 & E_1 & F_{2a} & E_{2a} & 0 & 0 \\ 0 & 0 & -F_{2b} & E_{2b} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \alpha_3 \\ \beta_3 \end{bmatrix} = 0,\end{aligned}$$

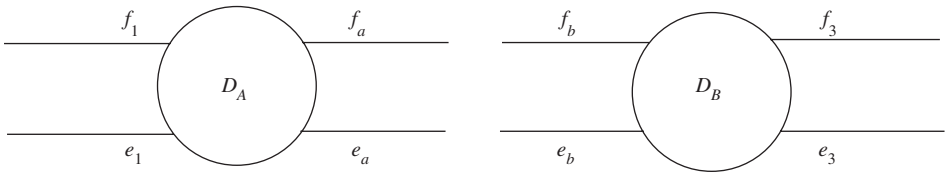

Fig. 1. Composition of Dirac structures.

$$\begin{aligned}&\beta_1^* f_1 + \alpha_1^* e_1 + \beta_3^* f_3 + \alpha_3^* e_3 = 0\\ &\quad\Leftrightarrow \forall(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) \in \mathscr{D}_a \| \mathscr{D}_b,\\ &\qquad \beta_1^* f_1 + \alpha_1^* e_1 + \beta_3^* f_3 + \alpha_3^* e_3 = 0\\ &\quad\Leftrightarrow (f_1, e_1, f_3, e_3) \in (\mathscr{D}_a \| \mathscr{D}_b)^{\perp}.\end{aligned}$$

Thus $\mathscr{D}_a \| \mathscr{D}_b = (\mathscr{D}_a \| \mathscr{D}_b)^{\perp}$, and hence it is a Dirac structure. □

6. Discrete port-Hamiltonian systems

In the modular approach to modeling physical systems the system under consideration is regarded as obtained from the interconnection of simpler sub-systems. These interconnections in general, give rise to algebraic constraints between the state-variables of the systems, thus making the final system model *implicit*. In this section we formalize such implicit physical systems using the concept of discrete Dirac structures.

From a network modeling perspective a finite-dimensional physical system is naturally described by a set of energy-storing elements, a set of energy-dissipating elements and a set of external ports (via which the interaction with the environment can take place)—interconnected to each other by a *power-conserving interconnection* as in Fig. 2. Associated with the energy storing elements are energy variables $z_1, \ldots, z_n$ being coordinates for some n-dimensional state discrete-manifold $\mathscr{Z}$, and a total energy $H : \mathscr{Z} \to \mathbb{F}$. First we formalize the power-conserving interconnection by a constant Dirac structure $\mathscr{D}$ on the finite-dimensional space $\mathscr{F} := \mathscr{F}_S \times \mathscr{F}_R \times \mathscr{F}_P$, with $\mathscr{F}_S$ denoting the space of flows f_S connected to the energy-storing elements, $\mathscr{F}_R$ denoting the space of flows f_R connected to the energy dissipating elements, and $\mathscr{F}_P$ denoting the space of external flows f_P which can be connected to the environment. Dually we write $\mathscr{E} := \mathscr{E}_S \times \mathscr{E}_R \times \mathscr{E}_P$ with the efforts $e_S \in \mathscr{E}_S, e_R \in \mathscr{E}_R, e_P \in \mathscr{E}_P$ being the corresponding dual variables of $f_S \in \mathscr{F}_S, f_R \in \mathscr{F}_R, f_P \in \mathscr{F}_P$, i.e. with $\mathscr{E}_S = \mathscr{F}_S^*, \mathscr{E}_R = \mathscr{F}_R^*, \mathscr{E}_P = \mathscr{F}_P^*$

Definition 5. Let $\mathscr{Z}$ be a discrete n-dimensional manifold of energy variables, and let $H : \mathscr{Z} \to \mathbb{F}$ be a discrete Hamiltonian. Furthermore, let $\mathscr{F}_P$ be the space of external flows f, with $\mathscr{E}_P$ the dual space of external effort e. Consider a Dirac structure on the product space $\mathscr{Z} \times \mathscr{F}_P$, only depending on z.The implicit discrete port-Hamiltonian system corresponding to $\mathscr{Z}$, $\mathscr{D}$, H and $\mathscr{F}_P$ is defined by the specification

$$\left(-\frac{\Delta z}{\Delta t}, f, \overset{\frown}{\partial}_z H(z), e\right) \in \mathscr{D}(z).$$

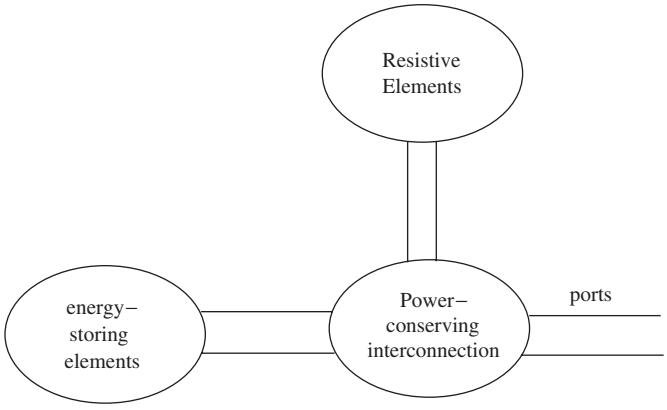

Fig. 2. Network description of physical systems.

Note that the minus sign in front of the flow $\Delta z/\Delta t$ physically means that the ingoing power is positive. The efforts and flows corresponding to the energy-storing elements are given as $f_S = \Delta z/\Delta t$, $e(z) = \overset{\frown}{\partial}_z H(z)$, and then it follows that the physical system is described by the set of *Difference algebraic equations*

$$F_z \frac{\Delta z(t)}{\Delta t} + E_z \overset{\frown}{\partial}_z H(z) + F f(t) + E e(t) = 0.$$

In the smooth setting the next step would be to define the energy balance as follows. For all $(-X, f, \alpha, e) \in \mathscr{D}$ we have: $-\langle \alpha | X \rangle + \langle e | f \rangle = 0$, due to which it follows that an implicit smooth port-Hamiltonian system satisfies the energy balance, cf. [1,13], $\mathrm{d}H/\mathrm{d}t = (\partial H/\partial x)\dot{x} = e^{\mathrm{T}} f$. In the above computation one uses the chain rule for differentiation. The chain rule however does not work in the discrete setting. Let us see this with a very simple example: for instance, consider the action of a derivation on the function $f(x) = x^2$. From the definition of twisted derivation we have: $X(f)(x) = X(x^2) = X(x) \cdot x + \mathrm{Aut}_X(x) \cdot X(x)$. Only if $\mathrm{Aut}_X(x) = x$ for any vector field X, the chain rule is satisfied. So in general we do not have a discrete version of the chain rule. What does this imply?

First of all note that in the discrete case we do have the following: for all $(-X, f, \alpha, e) \in \mathscr{D}$ we have

$$-\langle \alpha | X \rangle + \langle e | f \rangle = 0 \Rightarrow -\overset{\frown}{\partial}_z H(z) \cdot \frac{\Delta z}{\Delta t} + e^{\mathrm{T}} f = 0.$$

However $\Delta H/\Delta t \neq \overset{\frown}{\partial}_z H(z) \Delta z/\Delta t$, since the chain rule is not valid in the discrete setting. And indeed, it is well known that there exist no basic integration techniques (like Euler integration, Runge–Kutta etc.) that preserve the energy balance relation. There exist many special integration techniques that do preserve the energy balance, but these techniques dramatically alter the geometric structure of the Dirac framework. We have discussed these aspects in [10,11] on discrete Hamiltonian systems where we showed how the Poisson structure can get dramatically modified under structure preserving algorithms. Similar analysis also holds for Dirac structures, this will be the subject of future work. In any case, in general we would have an energy relation of the following type:

$$\frac{\Delta H}{\Delta t} = -\overset{\frown}{\partial}_z H(z) \cdot \frac{\Delta z}{\Delta t} + e^{\mathrm{T}} f - \tilde{H} = 0,$$

where $\tilde{H}$ is the extra energy that is *created* in the system as a result of the discrete process. In the continuum limit $\tilde{H} \to 0$.

6.1. Explicit port-Hamiltonian systems as a special case of implicit port-Hamiltonian systems

The following discussion follows the same lines as in the smooth setting, cf. [1]. A discrete Hamiltonian system with

collocated inputs and outputs can be described by

$$\frac{\Delta q}{\Delta t} = \eth_p H(q, p),$$
$$\frac{\Delta p}{\Delta t} = -\eth_q H(q, p) + B(q)u,$$
$$y = B^*(q)\eth_p H(q, p).$$

An important generalization of the class of Hamiltonian systems described above can be given as follows. Let $\mathscr{Z}$ be the discrete manifold of energy variables. Then the following is called an *explicit port-Hamiltonian system*:

$$\frac{\Delta z}{\Delta t} = J(z)\eth_z H(z) + g(z)u, \quad z \in \mathscr{Z}, \ \ u \in \mathbb{F}^m,$$
$$y = g^{\mathrm{T}}(z)\eth_z H(z), \quad y \in \mathbb{F}^m, \tag{3}$$

where $J(z)$ is an $n \times n$ structure matrix corresponding to a discrete Poisson bracket defined on $A(\mathscr{Z})$. Now we show that Definition 5 generalizes the notion of an (explicit) port-Hamiltonian system (3).

Proposition 5. *In the case of* (3) *the Dirac structure* $\mathscr{D}$ *on* $\mathscr{Z} \times \mathscr{F}$ *is given by* $(X, f, \alpha, -e) \in \mathscr{D}$ *iff*

$$X(z) = J(z)\alpha(z) + g(z)f,$$
$$e = g^{\mathrm{T}}(z)\alpha(z), \quad z \in \mathscr{Z}. \tag{4}$$

Proof. Indeed, let $(X, f, \alpha, -e) \in \mathscr{D}^{\perp}$; i.e.

$$\langle \hat{\alpha}|X\rangle + \langle \alpha|\hat{X}\rangle - \langle \hat{e}|f\rangle - \langle e|\hat{f}\rangle = 0 \tag{5}$$

for all $(\hat{X}, \hat{f}, \hat{\alpha}, -\hat{e})$ satisfying (3). By first taking $\hat{f} = 0$ we obtain

$$\hat{\alpha}^{\mathrm{T}}(z)X(z) + \alpha^{\mathrm{T}}(z)J(z)\hat{\alpha}^{\mathrm{T}}(z) - \hat{\alpha}^{\mathrm{T}}(z)g(z)f = 0$$

for all $\hat{\alpha}$, and thus $X(z) = J(z)\alpha(z) + g(z)f$, and substituting this in (5) gives

$$\hat{\alpha}^{\mathrm{T}}(z)g(z)f + \alpha^{\mathrm{T}}(z)g(z)\hat{f} - \hat{\alpha}^{\mathrm{T}}(z)g(z)f - e^{\mathrm{T}}\hat{f} = 0$$

for all $\hat{f}$, implying that $e = g^{\mathrm{T}}(z)\alpha(z)$, and thus $(X, f, \alpha, -e) \in \mathscr{D}$. □

Remark 1 (*Discrete port-Hamiltonian systems with dissipation*). We can generalize (3) to include dissipative elements by modifying the $g(z)u$ and the y terms as follows:

$$[g(z) \ \ g_R(z)]\begin{bmatrix} u \\ u_R \end{bmatrix} = g(z)u + g_R(z)u_R,$$

$$\begin{bmatrix} y \\ y_R \end{bmatrix} = \begin{bmatrix} g^*(z)\eth_z H(z) \\ g_R^*(z)\eth_z H(z) \end{bmatrix}.$$

Here u_R, y_R denote the power variables at the ports which are terminated by static resistive elements:

$$u_R = -F(y_R),$$

where the resistive element $F : \mathbb{F}^{m_r} \to \mathbb{F}^{m_r}$ satisfies:

$$y_R^* F(y_R) \geqslant 0, \quad y_R \in \mathbb{F}^{m_r}.$$

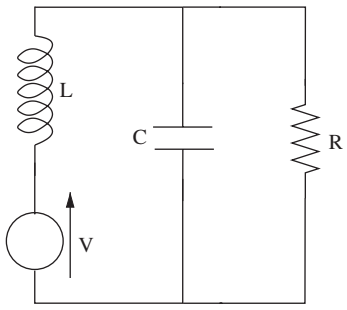

Fig. 3. A driven RLC circuit.

7. Examples

In this section, we present two examples of the modeling and simulation of port-Hamiltonian systems in the discrete setting. We will show that the simulations from our discrete model *exactly* coincide with the simulations that we get via a corresponding discretization technique.

Example 2. Consider the electrical circuit shown in Fig. 3.

For notational simplicity we assume $L = C = 1$. The Hamiltonian function is given by: $H(q, \phi) = \frac{1}{2}(q^2 + \phi^2)$. Then the discrete dynamics are, using the Dirac structure, given by

$$\frac{\Delta q}{\Delta t} = \eth_\phi H - \frac{1}{R}\eth_\phi H,$$
$$\frac{\Delta \phi}{\Delta t} = -\eth_q H + V.$$

Note that $\eth_q H = \eth_q(q^2/2) = ((q + \varepsilon)^2 - q^2)/2\varepsilon = q + \varepsilon/2$. However since ε is extremely small (on the order of $10\times^{-16}$) so, for the examples considered here it does not affect the simulation results, and hence we can safely ignore the ε terms. So $\eth_q H = q$ and $\eth_\phi H = \phi$. Let us use a Runge–Kutta 2 discrete vector, and let us compare the simulation results with the usual second order Runge–Kutta technique. The simulation results in Fig. 4 show an exact matching between the two approaches.

Example 3. Now we model the Van der Pol circuit in our discrete setting. The Hamiltonian is: $H(q, \phi) = \frac{1}{2}(q^2 + \phi^2)$. The discrete dynamics are defined as follows:

$$\frac{\Delta q}{\Delta t} = \eth_\phi H + Ep \cdot q \cdot (1 - \phi^2), \quad \frac{\Delta \phi}{\Delta t} = \eth_q H.$$

The simulation results are shown in Fig. 5 and we have used the Runge–Kutta 2 discrete vector, again the comparison with a second order Runge–Kutta technique shows an exact matching.

8. Conclusions and future work

In this paper, we provided an alternative to the usual two stage process of modeling and discretization of port-Hamiltonian systems—we defined a framework for the discrete modeling of

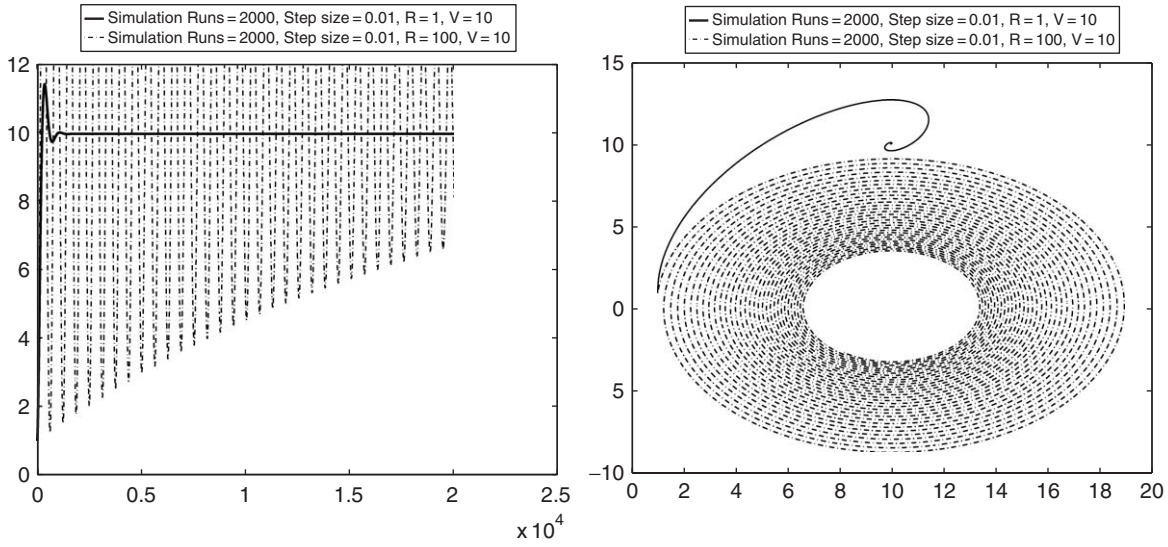

Fig. 4. Discrete dynamics of a driven RLC circuit—left figure is a comparison of the time response, figure on the right is a comparison of the phase plots.

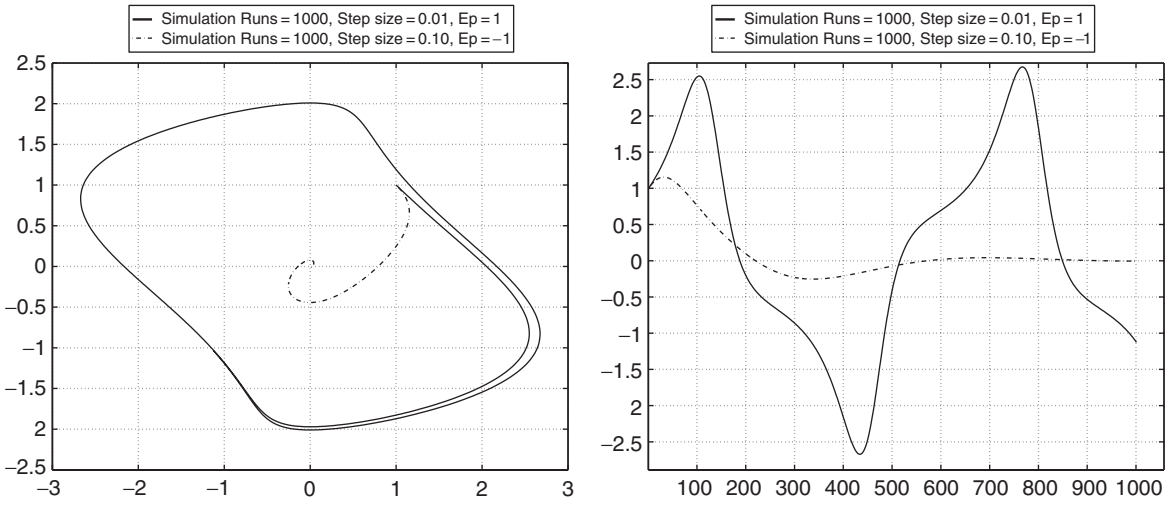

Fig. 5. Hopf bifurcation in the discrete dynamics of the Van der Pol circuit. Comparison of the phase plots and the time responses.

such systems, so as to provide models that are trivially implementable on computers for either numerical simulation or digital control (for other approaches to control of sampled-data systems, refer to [7]). Moreover all of the geometric/mathematical structure, and the corresponding analysis, presented in this paper is also perfectly valid for discretized port-Hamiltonian systems.

This paper is the first stage of the process of formalizing the discrete structure of port-Hamiltonian systems which we would later like to use for *modular simulation*. The port-Hamiltonian framework is suitable for the modular approach to modeling complex physical systems. Regarding simulation, it is well known that basic integration algorithms do not preserve important structure. It is much harder to design structure preserving algorithms (for more details on structure preserving algorithms in a Hamiltonian context, please refer to [9]) for an entire discretized system, than designing structure preserving algorithms for each individual discretized submodel and then interconnecting all these discrete submodels. Of course, to do so we need a formal interconnection theory at the discrete level, which we have provided in this paper. Our future work will concern developing the concept of modular simulations. Another area of interest is in applications like haptics and telemanipulation, wherein we are required to interconnect a smooth port-Hamiltonian system with a discrete one. In this case it can be of benefit to understand the structure of the discrete system, so that the interconnection of a discrete port-Hamiltonian system with a smooth port-Hamiltonian system may be formalized.

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