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Abstract—We consider the problem of designing encoders, decoders and controllers which stabilize feedforward nonlinear systems over a communication network with finite bandwidth and large delay. The control scheme guarantees minimal data-rate semi-global asymptotic and local exponential stabilization of the closed-loop system. The analysis rests on the stability properties of a class of cascade impulsive time-delay systems.

Index Terms—Nonlinear systems, networked control systems, impulsive systems, quantization, delay systems, packet drop-out.

I. INTRODUCTION

The problem of controlling systems under communication constraints has attracted much interest in the recent years. In particular, many papers have focused on how to cope with the finite bandwidth of the communication channel in the feedback loop. For the case of linear systems (cf. e.g. [6], [2], [12], [7], [25], [27]) the problem has been very well understood, and an elegant characterization of the minimal data rate above which stabilization is always possible is now available. Loosely speaking, the result shows that the minimal data rate is proportional to the inverse of the product of the unstable eigenvalues of the dynamic matrix of the system. Controlling using the minimal data rate is interesting not only from a theoretical point of view, but also from a practical one, even in the presence of communication channels with a large bandwidth. Indeed, having control techniques which employ a small number of bits to encode the feedback information implies for instance that the number of different tasks which can be simultaneously carried out is maximized, results in explicit procedures to convert the analog information provided by the sensors into the digital form which can be transmitted, or improves the performance of the system ([16]). The problem for nonlinear systems has been investigated as well (cf. e.g. [17], [19], [3], [26], [4]). In [17], the author extends the results of [2] to nonlinear systems which are *input-to-state* stabilizable. For the same class of systems, the authors of [19] show that the approach of [27] can be employed also for continuous-time nonlinear systems, although in [19] no attention is paid on the minimal data rate needed to achieve the result. In fact, if the requirement on the data rate in not strict, as it is implicitly assumed in [19], it is shown in [3] that the results of [19] actually hold for the much broader class of *stabilizable* systems. The paper [26], among the other achievements, shows that a minimal data rate theorem for *local* stabilizability of nonlinear systems can be proven by exploiting the linearized system associated to the original one. With the exception of [4], *non* local results for the problem of minimal data rate stabilization of nonlinear systems are missing. The paper [4] has pointed out that, if one restricts the attention to the class of nonlinear feedforward systems, then it is possible to find the infimal data rate above which stabilizability is possible. We recall that feedforward systems represent a very important class of nonlinear systems, which has received much attention in the recent years (see e.g. [28], [24], [13], [21], to cite a few), in which many physical systems fall, and for which it is possible to design stabilizing control laws in spite of saturation on the actuators. A recent paper ([22]) has shown that, when *no* communication channel is present in the feedback loop, any feedforward nonlinear system can be stabilized

regardless of an arbitrarily large delay affecting the control action. In this contribution, exploiting the results of [22], we show that the minimal data rate theorem of [4] holds when an arbitrarily large delay affects the channel (in [4], instantaneous delivery through the channel of the feedback packets was assumed). Note that the communication channel not only introduces delay, but also quantization error and an impulsive behavior, since the packets of bits containing the feedback information are sent only at times when transmission is allowed. Hence, the methods of [22], which are given for continuous-time time-delay systems, must be modified to deal with impulsive time-delay systems in the presence of measurement errors. In addition, our result requires an appropriate redesign, not only of the parameters in the feedback law of [22], but also of the encoder and the decoder of [4]. See [18] for a different approach to the problem of delays and quantization.

In the next section, we present the problem in more detail. The main contribution is stated in Section III. Moreover, building on the coordinate transformations of [29], [22], we introduce a form for the closed-loop system which is suitable for the analysis (Subsection IV-A). The main arguments employed to prove the results are given in Subsection IV-B, although full technical details are omitted. These can be found in [5]. In the conclusions, it is emphasized how the proposed solution is also robust with respect to packet drop-out. As the results of the paper are an outgrowth of [22], [4], our presentation is along the lines of those papers, and only details which differ from those therein are reported. The rest of the section summarizes the notation adopted in the paper.

Notation. Given an integer $1 \leq i \leq \nu$, the vector $(a_i, \ldots, a_\nu) \in \mathbb{R}^{\nu-i+1}$ will be succinctly denoted by the corresponding uppercase letter indexed by i, i.e. A_i. For $i = 1$, we will equivalently use the symbol A_1 or simply a. I_i denotes the $i \times i$ identity matrix. $0_{i \times j}$ (respectively, $1_{i \times j}$) denote an $i \times j$ matrix whose entries are all 0 (respectively, 1). When only one index is present, it is intended that the matrix is a (row or column) vector. If x is a vector, $|x|$ denotes the standard Euclidean norm, i.e. $|x| = \sqrt{x^T x}$, while $|x|_\infty$ denotes the infinity

norm $\max_{1 \leq i \leq n} |x_i|$. The vector $(x^T \ y^T)^T$ will be more simply denoted as (x, y). $\mathbb{Z}_+$ (respectively, $\mathbb{R}_+$) is the set of nonnegative integers (real numbers), $\mathbb{R}^n_+$ is the positive orthant of $\mathbb{R}^n$. A matrix M is said to be Schur stable if all its eigenvalues are strictly inside the unit circle. The sign function $\text{sgn}(x)$, with x a scalar variable, denotes the function which is equal to 1 if $x > 0$, 0 if $x = 0$, and equal to -1 otherwise. If x is an n-dimensional vector, then $\text{sgn}(x)$ is an n-dimensional vector whose ith component is given by $\text{sgn}(x_i)$. Moreover, $\text{diag}(x)$ is an $n \times n$ diagonal matrix whose element (i, i) is x_i. Given a vector-valued function of time $x(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$, the symbol $||x(\cdot)||_\infty$ denotes the supremum norm $||x(\cdot)||_\infty = \sup_{t \in \mathbb{R}_+} |x(t)|$, whereas $||x_t|| = \sup_{-\theta \leq \varsigma \leq 0} |x(t + \varsigma)|$ or $||x_r|| = \sup_{-\tau \leq \sigma \leq 0} |x(r + \sigma)|$, depending on the context. Moreover, $x(\bar{t}^-)$ represents the limit $\lim_{t \to \bar{t}^-} x(t)$. For $i = 1, \ldots, n$, the functions $p_i, q_i : \mathbb{R}^{n-i+1} \to \mathbb{R}$ are defined as [29], [22]

$$
\begin{aligned}
p_i(a_i, \ldots, a_n) &= \sum_{j=i}^{n} \frac{(n-i)! a_j}{(n-j)!(j-i)!} , \\
q_i(a_i, \ldots, a_n) &= \sum_{j=i}^{n} \frac{(-1)^{i+j}(n-i)! a_j}{(n-j)!(j-i)!} ,
\end{aligned}
\tag{1}
$$

with $p_i(q_i(a_i, \ldots, a_n), \ldots, q_n(a_n)) = a_i$, $q_i(p_i(a_i, \ldots, a_n), \ldots, p_n(a_n)) = a_i$. The saturation function [22] $\sigma : \mathbb{R} \to \mathbb{R}$ is an odd C^1 function such that $0 \leq \sigma'(s) \leq 1$ for all $s \in \mathbb{R}$, $\sigma(s) = 1$ for all $s \geq 21/20$, and $\sigma(s) = s$ for all $0 \leq s \leq 19/20$. Furthermore, $\sigma_i(s) = \varepsilon_i \sigma(s/\varepsilon_i)$, with ε_i a positive real number.

II. PRELIMINARIES AND PROBLEM FORMULATION

We are concerned with the problem of controlling a nonlinear system in feedforward form [28], [24], [13], [21], that is

$$
\dot{x}(t) = f(x(t), u(t)) \triangleq \begin{pmatrix} x_2(t) + h_1(X_2(t)) \\ \ldots \\ x_n(t) + h_{n-1}(X_n(t)) \\ u(t) \end{pmatrix}
\tag{2}
$$

where $x_i(t) \in \mathbb{R}$, $X_i(t)$ is the vector of state variables $x_i(t), x_{i+1}(t), \ldots, x_n(t)$, $u(t) \in \mathbb{R}$, each function h_i is C^2, and there exists a positive real

number $M > 0$ such that for all $i = 1, 2, \ldots, n-1$, if $\max_{i \leq j \leq n} |x_j| \leq 1$, then $|h_i(X_{i+1})| \leq M|X_{i+1}|^2$. The communication channel which appears in the feedback loop has finite bandwidth and arbitrarily large constant delay, hereafter denoted as θ. In particular, there exists $\{t_k\}_{k \in \mathbb{Z}_+}$, a sequence of strictly increasing transmission times, satisfying

$$T_m \leq t_{k+1} - t_k \leq T_M , \quad k \in \mathbb{Z}_+ \qquad (3)$$

for some positive and known constants T_m, T_M, at which a packet of $N(t_k)$ bits encoding the feedback information is transmitted, and received at the other end of the channel θ units of time later. The times at which the packets are received are denoted as $\theta_k := t_k + \theta$. As a measure of the data rate employed by the communication scheme we adopt the *average data rate* [27] defined as

$$R_{av} = \limsup_{k \to +\infty} \sum_{j=0}^{k} N(t_j)/(t_k - t_0) ,$$

where $\sum_{j=0}^{k} N(t_j)$ is the total number of bits transmitted in the time interval $[t_0, t_k]$. To deal with the constraints imposed by the channel, the addition of an encoder and a decoder to process the feedback information has proven very effective [27], [25], [19]. As we focus on semi-global results for the sake of simplicity, a bound on the compact set of initial conditions is assumed to be available to both the encoder and the decoder, namely a vector $\bar{\ell} \in \mathbb{R}_+^n$ is known for which

$$|x_i(t_0)| \leq \bar{\ell}_i , \quad i = 1, 2, \ldots, n . \qquad (4)$$

In this paper (cf. [19], [10]), the encoder is an impulsive delay system ([1], [15], [19], [10]):

$$
\begin{aligned}
\dot{\omega}(t) &= f(\omega(t), \alpha(\omega(t-\theta))) \quad t \neq \theta_k \\
\dot{\xi}(t) &= f(\xi(t), \alpha(\omega(t))) \quad t \neq t_k \\
\dot{\ell}(t) &= \mathbf{0}_n \quad t \neq t_k \\
\omega(t) &= \omega(t^-) + g_{\mathcal{E}}(x(t^- - \theta), \xi(t^- - \theta), \\
&\qquad\qquad \ell(t^- - \theta)) \quad t = \theta_k \\
\xi(t) &= \xi(t^-) + g_{\mathcal{E}}(x(t^-), \xi(t^-), \ell(t^-)) \\
&\qquad\qquad\qquad\qquad t = t_k \\
\ell(t) &= \Lambda\ell(t^-) \quad t = t_k \\
y(t) &= \operatorname{sgn}(\Phi(x(t^-) - \xi(t^-))) \quad t = t_k ,
\end{aligned} \qquad (5)
$$

with $(\omega, \xi, \ell) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^n$, and $\Lambda, \Phi \in \mathbb{R}^{n \times n}$, and $L \leq M$, $\kappa \geq 1$ respectively matrices and parameters to design. Moreover,

$$
\begin{aligned}
\alpha(\omega) = &-\frac{L}{M\kappa^n}\sigma_n\left(p_n\left(\kappa^{n-1}\frac{M}{L}\omega_n\right) + \right. \\
&\sigma_{n-1}\left(p_{n-1}\left(\kappa^{n-2}\frac{M}{L}\omega_{n-1}, \kappa^{n-1}\frac{M}{L}\omega_n\right) + \ldots + \right. \\
&\left.\left.\sigma_1\left(p_1\left(\frac{M}{L}\omega_1, \ldots, \kappa^{n-1}\frac{M}{L}\omega_n\right)\right)\ldots\right)\right) ,
\end{aligned}
$$

and $g_{\mathcal{E}}(x, \xi, \ell) = (4\Phi)^{-1}\operatorname{diag}\left[\operatorname{sgn}\left(\Phi(x - \xi)\right)\right]\ell$. The encoder initial conditions are chosen as:

$$||\omega_{\theta_0^-}|| = 0 , \quad \xi(t_0^-) = \mathbf{0}_n , \quad \ell(t_0^-) = 2\Phi\bar{\ell} . \quad (6)$$

A brief discussion on the structure of the encoder is in order. The encoder plays the role of converting the feedback value into data packets $\{y(t_k)\}_{k \in \mathbb{Z}_+}$ of finite length, transmitted through the channel at times $\{t_k\}_{k \in \mathbb{Z}_+}$ set by the communication line. In the present setting, it further processes the data to cope with the transmission delay, and make it sure that the delivered information with delay is still useful to provide a stabilizing control action, as it will be proven in the sequel of the paper. It is well known that the equations of the encoder specify the quantization region at each time. Moreover, such region must be known to the decoder as well, and therefore the state variables for the encoder and the decoder must coincide at any time. This justifies the presence of the delay in the first equation of the encoder, and the reason why its state is reset after θ units of time from the transmission time t_k. Indeed, the packet containing the feedback information is delivered with delay θ, and the delayed input and reset in the decoder equations (see below) reflects this. Nevertheless, encoding must be carried out at the times t_k when the channel allows transmission, and for this reason the ξ equations are requested to work synchronously with the transmission times. As usual, the jump equations take into account the reset of the state due to transmission/reception of a new feedback packet. Finally, the control law α is the well-known nested saturated feedback [28], [24], [13].

Remark. The sampled output $y(t_k)$ takes values in the finite set $\{-1, 0, 1\}^n$, and as such it can be transmitted through the finite data-rate channel by

employing 2 bits per each state component. It is not difficult, although tedious, to modify the function which defines $y(t_k)$ in such a way that the latter actually ranges in the set $\{0,1\}^{N(t_k)}$, $N(t_k)$ being constantly equal to $\lceil \log_2 \sqrt{3n} \rceil = const$. This is omitted for the sake of simplicity. We also notice that, employing a tri-state encoding for each component of the vector, rather than a bi-state encoding as in [4], is only dictated by the need to preserve the null solution as an equilibrium solution for the closed-loop impulsive system, since the stability is established by Lyapunov-based arguments cf. [1], [15].

The sequence $\{y(t_k)\}_{k \in \mathbb{Z}_+}$ generated by the encoder is received at the other end of the communication channel at times θ_k and here processed by the decoder, which loosely speaking constructs an estimate of the state of the process based on the received symbol, and exploits it to provide the control action during the interval $[\theta_k, \theta_{k+1})$. The decoder is an impulsive system as well:

$$
\begin{aligned}
\dot{\psi}(t) &= f(\psi(t), \alpha(\psi(t-\theta))) \\
\dot{\nu}(t) &= \mathbf{0}_n \qquad\qquad\qquad\qquad t \neq \theta_k \\[6pt]
\psi(t) &= \psi(t^-) + g_\mathcal{D}(y(t-\theta), \nu(t^-)) \\
\nu(t) &= \Lambda \nu(t^-) \qquad\qquad\qquad t = \theta_k \\[6pt]
u(t) &= \alpha(\psi(t)) ,
\end{aligned}
\tag{7}
$$

with $g_\mathcal{D}(y, \nu) = (4\Phi)^{-1} \operatorname{diag}(y) n$, and initial conditions set equal to

$$
\|\psi_{\theta_0^-}\| = 0 , \quad \nu(\theta_0^-) = 2\Phi\bar{\ell} .
\tag{8}
$$

Before proceeding, we observe that in the analysis to come it is enough to consider the equations describing the process and the decoder only, as we exactly reconstruct the state of the encoder from the state of the decoder. In fact, we can prove:

Lemma 1: We have:
(i) $\omega(t) = \psi(t)$ for all $t \geq t_0$,
(ii) $\xi(t-\theta) = \psi(t)$ and $\nu(t-\theta) = \ell(t)$ for all $t \geq \theta_0$.

In this paper we design the encoder and the decoder (hence the controller) in such a way that the resulting (delay impulsive) closed-loop system is semi-globally asymptotically and locally exponentially stable, and this is achieved employing an average data rate which is arbitrarily close to the infimal one. The precise formulation of the problem we solve is as follows:

Definition. System (2) is semi-globally asymptotically and locally exponentially stabilizable with an average data rate arbitrarily close to the infimal one if, for any $\bar{\ell} \in \mathbb{R}_+^n$, $\theta > 0$, $\hat{R} > 0$, an encoder (5), (6), and a decoder (7), (8) exist such that for the closed-loop system with state $X := (x, \omega, \xi, \ell, \psi, \nu)$, we have:
(i) There exist a compact set C containing the origin, and $T > 3\theta$, such that $X(t) \in C$ for all $t \geq T$.
(ii) For all $t \geq T$, for some positive real numbers k, δ,

$$
|X(t)| \leq k\|X_T\| \exp(-\delta(t-T)) .
\tag{9}
$$

(iii) $R_{av} < \hat{R}$.

Remark. Item (iii) points out that the average data rate used to achieve the stabilizability result can be made arbitrarily close to zero, which of course is the infimal data rate. This result can be put in relation with the dynamics of the system in the following way. The linearization of the feedforward system at the origin is a chain of integrators, for which the minimal data rate theorem for linear systems draws the same conclusion we have drawn for the original nonlinear system.

In view of the recursive nature of the controller, encoder and decoder design [29], [28], [24], [13], [22], [4], it is convenient to introduce additional notation:

$$
H_i(X_{i+1}(t), u(t)) = \begin{pmatrix} x_{i+1}(t) + h_i(X_{i+1}(t)) \\ \cdots \\ x_n(t) + h_{n-1}(X_n(t)) \\ u(t) \end{pmatrix}
$$
$$
i = 1, 2, \ldots, n .
\tag{10}
$$

III. MAIN RESULT

The main result of the paper is as follows:

Theorem 1: System (2) is semi-globally asymptotically and locally exponentially stable with an average data rate arbitrarily close to the infimal one

Remark. The proof is constructive and explicit values for Λ, Φ and ε_i's, κ_*, L^*, which appear in

the encoder and in the decoder, are determined. For instance, we have

$$1 = 80\varepsilon_n = 80^2\varepsilon_{n-1} = \ldots = 80^n\varepsilon_1 \,, \qquad (11)$$

and κ_*, L^* take the following expressions:

$$\kappa_* = \theta \max \left\{ 4 \cdot (80)^{n+1} n(n+2), \right.$$
$$16n^2(8n(1+n^2)^{n-1}+1)^2,$$
$$\left. 32(1+n^2)^{2n-2}n^4 \right\}$$
$$L^* = \min \left\{ \frac{\kappa}{20 \cdot 80^{n+1} n \cdot n^3(n!)^3}, \right.$$
$$\frac{\kappa}{8(1+n^2)^{n-1}\sqrt{n(n-1)} \cdot n^3(n!)^3},$$
$$\left. \frac{M\kappa}{(n+1)!}, M \right\} \,. \qquad (12)$$

Compared with the analogous expressions in [22], it is seen that, the control law employed here requires an appropriate parameter redesign to cope with the presence of the quantization error and state resets.

IV. SKETCH OF PROOFS

A. Coordinate transformations

Building on the coordinate transformations in [22], [29], we put the closed-loop system in a convenient form, namely equations (13), (17)-(18), (19) below. As far as the equations of the process are concerned, the following linear change of coordinates is very useful. For given positive constants $L \le M$, $\kappa \ge 1$, we define the *non singular positive matrices*[1] Φ_i as:

$$\Phi_i X_i := \begin{bmatrix} p_i\left(\dfrac{M}{L}\kappa^{i-1}x_i, \ldots, \dfrac{M}{L}\kappa^{n-1}x_n\right) \\ \ldots \\ p_n\left(\dfrac{M}{L}\kappa^{n-1}x_n\right) \end{bmatrix},$$
$$i = 1, \ldots, n \,,$$

where the functions p_i are those introduced in (1). Along with the change of time scale $t = \kappa r$, and input coordinate change

$$v(r) = \kappa p_n\left(\frac{M}{L}\kappa^{n-1}u(\kappa r)\right) \,,$$

[1]The matrix Φ_1 will be simply referred to as Φ.

the state coordinate change $Z_i(r) = \Phi_i X_i(\kappa r)$ transforms (10) into [22]

$$\dot{Z}_i(r) = F_i(Z_{i+1}(r), v(r)) :=$$
$$\begin{bmatrix} \sum_{j=i+1}^n z_j(r) + v(r) + f_i(Z_{i+1}(r)) \\ \sum_{j=i+2}^n z_j(r) + v(r) + f_{i+1}(Z_{i+2}(r)) \\ \vdots \\ v(r) \end{bmatrix},$$
$$\qquad (13)$$

where, if $|Z_{i+1}|_\infty \le (M\kappa)/(L(n+1)!)$, then

$$|f_i(Z_{i+1})| \le P|Z_{i+1}|^2 \,, \quad P = n^3(n!)^3 L\kappa^{-1} \,. \qquad (14)$$

Because of the recursive design, in addition to (10), we also consider the following equations associated to the decoder:

$$\dot{\Psi}_i(t) = H_i(\Psi_{i+1}(t), \alpha(\Psi_i(t-\theta)))$$
$$\dot{N}_i(t) = \mathbf{0}_{n-i+1} \quad t \ne \theta_k \,,$$
$$\Psi_i(t) = \Psi_i(t^-) + (4\Phi_i)^{-1}\mathrm{diag}(Y_i(t-\theta))N_i(t^-)$$
$$N_i(t) = \Lambda_i N_i(t^-) \quad t = \theta_k \,,$$

where N_i denotes the components from i to n of ν, and it is convenient to express $\alpha(\Psi_i)$ as

$$\alpha(\Psi_i(t)) = -\frac{L}{M\kappa^n}\sigma_n\left(p_n\left(\kappa^{n-1}\frac{M}{L}\psi_n(t)\right) + \right.$$
$$\sigma_{n-1}\left(p_{n-1}\left(\kappa^{n-2}\frac{M}{L}\psi_{n-1}(t), \kappa^{n-1}\frac{M}{L}\psi_n(t)\right) + \ldots + \right.$$
$$\left. \left. \sigma_i\left(p_i\left(\kappa^{i-1}\frac{M}{L}\psi_i(t), \ldots, \kappa^{n-1}\frac{M}{L}\psi_n(t)\right) + \lambda_{i-1}(t)\right) \ldots \right)\right)$$

with obvious significance of $\lambda_{i-1}(t)$. For $i = 1$, the equations above coincide with (7). In addition to Z_i, new state coordinates are introduced, namely

$$W_i(r) = \Phi_i\Psi_i(\kappa r) \,,$$
$$E_i(r) = \Phi_i(\Psi_i(\kappa r) - X_i(\kappa(r-\tau))) \,, \quad (15)$$
$$P_i(r) = N_i(\kappa r) \,,$$

with $\tau = \theta/\kappa$, and we let $\kappa r_k = t_k$ and $\kappa \rho_k = \theta_k$. In these new coordinates, being $u = \alpha(\psi)$, we have

$$v(r) = \kappa p_n\left(\frac{M}{L}\kappa^{n-1}\alpha(\Phi_i^{-1}W_i(r))\right) =$$
$$-\sigma_n(w_n(r) + \sigma_{n-1}(w_{n-1}(r) + \ldots + \sigma_i(w_i(r) + \hat{\lambda}_{i-1}(r))) \ldots)$$
$$= -\sigma_n(e_n(r) + z_n(r-\tau) + \sigma_{n-1}(e_{n-1}(r) + z_{n-1}(r-\tau) +$$
$$\ldots + \sigma_i(e_i(r) + z_i(r-\tau) + \hat{\lambda}_{i-1}(r)) \ldots)) \,, \qquad (16)$$

with $\hat{\lambda}_{i-1}(r) = \lambda_{i-1}(\kappa r) = \sigma_{i-1}(w_{i-1}(r) + \ldots + \sigma_1(w_1(r)) \ldots)$. Moreover, while the variable Z_i satisfies (13), with $v(r)$ as in (16), the variable E_i

obeys the equation

$$
\dot{E}_i(r) = F_i\left(W_{i+1}(r), v(r-\tau)\right) - \\
F_i\left(Z_{i+1}(r-\tau), v(r-\tau)\right)
$$

$$
= \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} E_i(r) +
$$

$$
\begin{bmatrix} f_i(E_{i+1}(r) + Z_{i+1}(r-\tau)) - f_i(Z_{i+1}(r-\tau)) \\ f_{i+1}(E_{i+2}(r) + Z_{i+2}(r-\tau)) - f_{i+1}(Z_{i+2}(r-\tau)) \\ \vdots \\ f_{n-1}(E_n(r) + Z_n(r-\tau)) - f_{n-1}(Z_n(r-\tau)) \\ 0 \end{bmatrix}
\tag{17}
$$

for $r \neq \rho_k$, and

$$
E_i(r) = E_i(r^-) + 4^{-1}\operatorname{diag}(\operatorname{sgn}(-E_i(r^-)))P_i(r^-) ,
\tag{18}
$$

for $r = \rho_k$, where the variable P_i which appears in the last equality satisfies

$$
\begin{aligned}
\dot{P}_i(r) &= \mathbf{0}_{n-i+1} & r \neq \rho_k \\
P_i(r) &= \Lambda_i P_i(r^-) & r = \rho_k .
\end{aligned}
\tag{19}
$$

B. Sketch of proofs

The proofs consist of a step-by-step construction, where at each step i, we consider the subsystem (13), (16)-(19). In particular, following [4], one can first show that the decoder asymptotically tracks the state of the subsystem, and then that the control law (16) stabilizes the subsystem, despite of the error due to the coding. In this section we only provide the line of reasoning which underlies the proof of the main result, omitting most of the technical details. That the decoder asymptotically tracks the state of the process is proven by the following result:

Lemma 2: Suppose (4) is true. For some index $i = 1, 2, \ldots, n$, if there exists a positive real number Z_{i+1} such that [2] $||Z_{i+1}(\cdot)||_\infty \leq Z_{i+1}$, and, for all $r \geq \rho_0$, $|e_j(r)| \leq p_j(r)/2$, for $j = i+1, i+2, \ldots, n$, with[3] $P_{i+1}(\rho) = \Lambda_{i+1} P_{i+1}(\rho^-)$, for $\rho = \rho_k$, and Λ_{i+1} a Schur stable matrix, then for all $r \geq \rho_0$, $|e_i(r)| \leq p_i(r)/2$, with $p_i(r) = p_i(r^-)/2$, for $r = \rho_k$, if $i = n$,

[2]The conditions are void for $i = n$.

[3]In the statement, the continuous dynamics of the impulsive systems are trivial – the associated vector fields are identically zero – and hence omitted.

and

$$
\begin{bmatrix} p_i(r) \\ P_{i+1}(r) \end{bmatrix} = \begin{bmatrix} 1/2 & * \\ \mathbf{0}_{i-1 \times i} & \Lambda_{i+1} \end{bmatrix} \begin{bmatrix} p_i(r^-) \\ P_{i+1}(r^-) \end{bmatrix}_{r = \rho_k} ,
\tag{20}
$$

if $i \in \{1, 2, \ldots, n-1\}$.

Remark. From the proof of the lemma, it becomes evident that, if $||z(\cdot)||_\infty \leq Z$, for some $Z > 0$, then the evolutions of $e(\cdot)$ and $p(\cdot)$ obey the equations[4]

$$
\begin{aligned}
\dot{e}(r) &= A(r)e(r) \triangleq \\
&\begin{bmatrix} 0 & a_{12}(r) & a_{13}(r) & \ldots & a_{1\,n-1}(r) & a_{1n}(r) \\ 0 & 0 & a_{23}(r) & \ldots & a_{2\,n-1}(r) & a_{2n}(r) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & a_{n-1\,n}(r) \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} e(r) \\
\dot{p}(r) &= \mathbf{0}_n & r \neq \rho_k \\
e(r) &= e(r^-) + 4^{-1}\operatorname{diag}[\operatorname{sgn}(-e(r^-))]p(r^-) \\
p(r) &= \Lambda p(r^-) & r = \rho_k ,
\end{aligned}
\tag{21}
$$

where the off-diagonal components of A rather than being seen as functions of $(r, e(r), z(r-\tau))$, are interpreted as bounded (unknown) functions of r, whose absolute value can be assumed without loss of generality to be upper bounded by a positive constant depending on $Z, \bar{\ell}$ and T_M.

The next lemma points out that, as expected, Lemma 10 in [22] holds even in the presence of a "measurement" disturbance induced by the quantization, which can be possibly large during the transient but it is decaying to zero asymptotically. The lemma is needed to prove that, at each step, the state of the subsystem eventually converges to zero if so does the encoding error.

Lemma 3: Consider the system

$$
\dot{Z}(r) = -\varepsilon\sigma\left[\frac{1}{\varepsilon}(Z(r-\tau) + e(r) + \lambda(r))\right] + \mu(r)
$$

where $Z \in \mathbb{R}$, ε is a positive real number, and additionally:

- $\lambda(\cdot)$ and $\mu(\cdot)$ are continuous functions for which positive real numbers λ_* and μ_* exist such that, respectively, $|\lambda(r)| \leq \lambda_*$, $|\mu(r)| \leq \mu_*$, for all $r \geq r_0$.
- $e(\cdot)$ is a piecewise-continuous function for which a positive time r_* and a positive number e_* exist such that $|e(r)| \leq e_*$, for all $r \geq r_*$.

[4]Again, we adopt the symbol Λ rather than Λ_1.

If

$$\tau \in \left(0, \frac{1}{24}\right] , \quad \lambda_* \in \left(0, \frac{\varepsilon}{80}\right] ,$$

$$e_* \in \left(0, \frac{\varepsilon}{80}\right] , \quad \mu_* \in \left(0, \frac{\varepsilon}{80}\right] ,$$

then there exist positive real numbers Z_* and $R \geq 0$ such that $||Z(\cdot)||_\infty \leq Z_*$, and for all $r \geq R$,

$$|Z(r)| \leq 4(\lambda_* + \mu_* + e_*) .$$

Remark. The upper bounds on λ_*, e_*, μ_* could be lowered to $\varepsilon/40$ and the result would still hold. The more conservative bounds are needed in forthcoming applications of the lemma.

In view of the inductive argument to be used, the following is very useful (cf. [22]):

Inductive Hypothesis There exists $Z_i > 0$ such that $||Z_i(\cdot)|| \leq Z_i$. Moreover, for each $j = i, i+1, \ldots, n$, $|e_j(r)| \leq p_j(r)/2$, for all $r \geq \rho_0$, and there exists $R_i > \tau$ such that for all $r \geq R_i$,

$$|z_j(r)| \leq \frac{1}{4}\varepsilon_j , \quad |e_j(r)| \leq \frac{1}{2n} \cdot \frac{1}{80^{j-i+2}}\varepsilon_j .$$

First step The inductive hypothesis is true for $i = n$, provided that $\tau \leq 1/24$ and the saturation levels ε_i are as in (11). Indeed, consider the system (13), (16)-(19) with $i = n$, and $\Lambda_n = 1/2$. It is a consequence of Lemma 2 that $|e_n(r)| \leq \varepsilon_n/80$ from a certain time R'_n on. Applying Lemma 3 to the z_n sub-system, we conclude that $||z_n(\cdot)||_\infty \leq Z_n$, and there exists a time $R_n > R'_n$ such that $|z_n(r)| \leq \varepsilon_n/4$, and $|e_n(r)| \leq \varepsilon_{n-1}/(n \cdot 160)$ for all $r \geq R_n$, the latter again by Lemma 2.

Inductive step Further we have:

Lemma 4: Let

$$
\begin{aligned}
P &\leq P_m \leq [20 \cdot (80)^n n]^{-1} , \\
\tau &\leq [4 \cdot 80^{n+1} n(n+2)]^{-1} .
\end{aligned}
\tag{22}
$$

If the induction hypothesis is true for some $i \in \{2, \ldots, n\}$, then it is also true for $i-1$.

Applying this lemma repeatedly, one can prove that, after a finite time, the closed-loop system starts evolving according to the equations

$$
\begin{aligned}
\dot{z}(r) &= A_1 z(r) + A_2 z(r-\tau) + A_2 e(r) + \\
&\qquad\qquad\qquad\qquad\qquad f(z(r)) \\
\dot{e}(r) &= A(r)e(r) \\
\dot{p}(r) &= \mathbf{0}_n & r \neq \rho_k \\
z(r) &= z(r^-) \\
e(r^+) &= e(r^-) + 4^{-1}\text{diag}[\text{sgn}(-e(r^-))]p(r^-) \\
p(r) &= \Lambda p(r^-) & r = \rho_k ,
\end{aligned}
\tag{23}
$$

where: (i) A_1, A_2 are matrices for which there exist $q = (1+n^2)^{n-1}$, $a = n$, and $Q = Q^T > 0$ is such that $(A_1+A_2)^T Q + Q(A_1+A_2) \leq -I$, with $||Q|| \leq q$ and $||A_1||, ||A_2|| \leq a$; (ii) There exists $\gamma > 0$ such that $f(z(r)) = [\, f_1(Z_2(r)) \;\; \ldots \;\; f_{n-1}(Z_n(r)) \;\; 0\,]^T$ satisfies $|f(z)| \leq \gamma |z|$; (iii) $A(r)$ is as in (21); (iv) Λ is the Schur stable matrix designed following the proof of Lemma 2.

Concisely rewrite the (e, p) equations of the system above as ([1])

$$
\begin{aligned}
\dot{\epsilon}(r) &= B(r)\epsilon(r) & r \neq \rho_k \\
\epsilon(r^+) &= g_k(\epsilon(r)) & r = \rho_k ,
\end{aligned}
$$

with $\epsilon = (e, p)$, $|g_k(\epsilon)| \geq |\epsilon|/2$, and notice the following consequence of Lemma 2:

Corollary 1: There exists a function $V(r, \epsilon) = V(r, e, p) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that, for all $r \in \mathbb{R}_+$ and for all $\epsilon = (e, p) \in \mathbb{R}^n \times \mathbb{R}^n$ for which $|e| \leq |p|/2$, satisfies

$$c_1 |\epsilon|^2 \leq V(r, \epsilon) \leq c_2 |\epsilon|^2$$

$$\frac{\partial V}{\partial r} + \frac{\partial V}{\partial \epsilon}B(r)\epsilon(r) \leq -c_3 |\epsilon|^2 \quad r \neq \rho_k$$

$$V(r, g_k(\epsilon)) \leq V(r, \epsilon^-) \quad r = \rho_k$$

$$\left|\frac{\partial V(r, \epsilon)}{\partial \epsilon}\right| \leq c_4 |\epsilon| ,$$

for some positive constants c_i, $i = 1, \ldots, 4$.

We now state that the entire cascade impulsive system (23) is exponentially stable.

Lemma 5: Consider system (23). If

$$\gamma \leq \frac{1}{8q} \quad \text{and}$$

$$\tau \leq \min\left\{\frac{1}{16a^2(8aq+1)^2}, \frac{1}{32q^2 a^4}, 2\right\} ,$$

then, for all $r \geq \rho_0$, for some positive real numbers k, δ, we have

$$|(z(r), \epsilon(r))| \leq k||(z, \epsilon)_{\rho_0}|| \exp(-\delta(r - \rho_0)) .$$

We can now state the intermediate result below, where we assume without loss of generality (cf. (12)) that $L \leq (M\kappa)/(n+1)!$

Proposition 1: Consider the closed-loop system (13), (16)-(19) and let $|e_i(\rho_0^-)| \leq p_i(\rho_0^-)/2$ for all $i = 1, 2, \ldots, n$. If (11) holds and

$$0 \leq \tau \leq \tau_m = \begin{aligned}[\max\{4 \cdot (80)^{n+1}n(n+2),\\ 16n^2(8n(1+n^2)^{n-1}+1)^2,\\ 32(1+n^2)^{2(n-1)}n^4\}]^{-1}\end{aligned}$$

$$0 \leq P \leq P_m = \begin{aligned}[\max\{20 \cdot 80^{n+1}n,\\ 8(1+n^2)^{n-1}\sqrt{n(n-1)}\}]^{-1},\end{aligned} \quad (24)$$

then:
(i) For each $j = 1, 2, \ldots, n$, for all $r \geq \rho_0$, $|e_j(r)| \leq p_j(r)/2$, and there exists $R > \tau$ such that, for all $r \geq R$,

$$\begin{aligned}|z_j(r)| &\leq \frac{1}{4}\varepsilon_j, \quad |e_j(r)| \leq \frac{1}{2n} \cdot \frac{1}{80^{j+1}}\varepsilon_j,\\ |p_j(r)| &\leq \frac{1}{n} \cdot \frac{1}{80^{j+1}}\varepsilon_j.\end{aligned}$$

(ii) For all $r \geq R$, for some positive real numbers $\hat{k}, \hat{\delta}$,

$$|(z(r), e(r), p(r))| \leq \hat{k}||(z,e,p)_R|| \exp(-\hat{\delta}(r-R)). \quad (25)$$

The proof of the main result simply amounts to rephrase the statement of the latter proposition in terms of the original system coordinates. To prove the last part of the statement, notice that by definition of R_{av}, $R_{av} < \hat{R}$ provided that $T_m \geq 2n/\hat{R}$. Now, this can always be achieved by discarding feedback packets without affecting the stability property of the closed-loop system. Indeed, the choice of T_m affects the entries of $A(r)$ and Λ, but the exponential stability of the (e, p) equations (and therefore of system (21)) remains true (this is evident from the proof of Lemma 2).

V. CONCLUSION

We have shown that minimal data rate stabilization of nonlinear systems is possible even when the communication channel is affected by an arbitrarily large transmission delay. The system has been modeled as the feedback interconnection of a couple of impulsive nonlinear control systems with the delay affecting the feedback loop. In suitable coordinates, the closed-loop system turns out to be described by a cascade of impulsive delay nonlinear control systems, and semi-global asymptotic plus local exponential stability can been shown. If the encoder is endowed with a device able to detect abrupt changes in the rate of growth of x_n, or if a dedicated channel is available to inform the encoder about the transmission delays, then it is not difficult to derive the same kind of stability result for the case when the delays are time-varying and upper-bounded by θ. Similarly, by adjusting T_M in (3), it is possible to show that the solution proposed in this paper is also robust with respect to packet drop-outs. The same kind of approach appears to be suitable for other problems of control over communication channel with finite data rate, delays and packet drop-out.

REFERENCES

[1] D.D. Bainov, P.S. Simeonov. *Systems with impulse effect. Stability, Theory and Applications.* Ellis Horwood Limited, Chichester, UK, 1989.

[2] R.W. Brockett and D. Liberzon. "Quantized feedback stabilization of linear systems", *IEEE Trans. Automat. Contr.*, vol. 45, pp. 1279-1289, July 2000.

[3] C. De Persis, A. Isidori. "Stabilizability by state feedback implies stabilizability by encoded state feedback", *Systems and Control Letters*, 53, 249-258, 2004.

[4] C. De Persis. "n-bit stabilization of n-dimensional nonlinear systems in feedforward form". *IEEE Transactions on Automatic Control*, 50, 3, 299-311, 2005.

[5] C. De Persis. "Minimal data rate stabilization of nonlinear systems over networks with large delays". Preprint available upon request, November 2006.

[6] N. Elia, S.K. Mitter. "Stabilization of linear systems with limited information, *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1384-1400, Sept. 2001.

[7] F. Fagnani, S. Zampieri. "Stability analysis and synthesis for scalar linear systems with a quantized feedback", *IEEE Transactions on Automatic Control*, 48, 9, 1569–1584, 2003.

[8] A. Halanay. *Differential Equations. Stability, Oscillations, Time Lags.* Academic Press, New York and London, 1966.

[9] J.K. Hale and S.M. Verduyn Lunel. *Introduction to functional differential equations*, Springer-Verlag, New York, 1993.

[10] J.P. Hespanha, D. Liberzon, A.R. Teel. "On input-to-state stability of impulsive systems", in *Proc. 44th IEEE Conf. on Decision and Control*, 2005.

[11] J.P. Hespanha, P. Naghshtabrizi, Y. Xu. "A survey of recent results in networked control systems", Submitted, 2006.

[12] H. Ishii, B.A. Francis. "Stabilizing a linear system by switching control with dwell time", *IEEE Transactions on Automatic Control*, vol. 47, no. 12, 1962-1973, December 2002.

[13] M. Jankovic, R. Sepulchre, P. Kokotovic. "Constructive Lyapunov stabilization of nonlinear cascade systems", *IEEE Trans. Autom. Control*, vol. 41, no. 12, pp. 1723l735, Dec. 1996.

[14] N.N. Krasovskii. *Stability of Motion.* Stanford University Press, Stanford, CA, 1963.

[15] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov. *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

[16] M.D. Lemmon and R. Sun. "Performance-rate functions for dynamically quantized feedback systems". In *Proc. 45th IEEE Conf. Dec. Contr.*, San Diego, CA, 5513–5518, 2006.

[17] D. Liberzon. "Hybrid feedback stabilization of systems with quantized signals". *Automatica*, **39**, 1543-1554, 2003.

[18] D. Liberzon. "Quantization, time delays, and nonlinear stabilization", *IEEE Transactions on Automatic Control*, **51**, 7, 1190–1195, 2006.

[19] D. Liberzon, J.P. Hespanha. "Stabilization of nonlinear systems with limited information feedback". *IEEE Transactions on Automatic Control*, 50(6), 910915, 2005.

[20] X. Liu. "Stability of impulsive control systems with time delay", *Mathematical and Computer Modeling*, 39, 511-519, 2004.

[21] L. Marconi, A. Isidori. "Robust global stabilization of a class of uncertain feedforward nonlinear systems", *Syst. Control Lett.*, vol. 41, pp. 281290, 2000.

[22] F. Mazenc, S. Mondié, R. Francisco. "Global asymptotic stabilization of feedforward systems with delay in the input", *IEEE Transactions on Automatic Control*, 49, 5, 844-850, 2004.

[23] F. Mazenc, S. Mondié, S. Niculescu, "Global asymptotic stabilization for chains of integrators with a delay in the input", *IEEE Trans. Automat. Contr.*, vol. 48, pp. 5763, Jan. 2003.

[24] F. Mazenc, L. Praly. "Adding integrations, saturated controls and stabilization for feedforward systems", *IEEE Trans. Autom. Control*, vol. 41, no. 11, pp. 15571559, Nov. 1996.

[25] G.N. Nair, R.J. Evans. "Exponential stabilisability of finite-dimensional linear systems with limited data rates", *Automatica*, vol. 39, pp. 585593, 2003.

[26] G. Nair, R. Evans, I. Mareels, W. Moran. "Topological feedback entropy and nonlinear stabilization", *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 15851597, Sep. 2004.

[27] S. Tatikonda and S. Mitter. "Control under communication constraints", *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 10561068, Jul. 2004.

[28] A. Teel. "A nonlinear small gain theorem for the analysis of control systems with saturations", *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 12561270, Sep. 1996.

[29] A. Teel. "Global stabilization and restricted tracking for multiple integrators with bounded controls", *Syst. Control Lett.*, vol. 18, pp. 165171, 1992.