String effective actions, dualities, and generating solutions

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Chapter 3

Heterotic Supergravity, Chern-Simons Terms and Field Redefinitions

In this chapter we will introduce the supergravity action as the low-energy effective action of superstring theories. We shall also outline various approaches that have been used to construct such an action and the corresponding higher derivative corrections. Field redefinitions and equivalent effective actions will be studied for the heterotic string to order α', having the Chern-Simons terms included. Also some comments on higher order terms in α' will be made.

3.1 Supergravity Theory

3.1.1 Preliminary

Supergravity theories were first presented as extensions of general relativity with fermionic and bosonic matter fields [77]. Such extensions have been performed in a way that the theory has a local supersymmetry which can be considered as an extension of Poincaré symmetry of general relativity. The Poincaré Lie algebra is formulated as a semi direct product of spacetime translations with generators P_m and Lorentz rotations J_{mn} such that

$$[P_m, P_n] = 0, \qquad [P_m, J_{nk}] = \eta_{mn} P_k - \eta_{mk} P_n, \tag{3.1.1a}$$

$$[J_{mn}, J_{kl}] = \eta_{nk} J_{ml} - \eta_{mk} J_{nl} - \eta_{nl} J_{mk} + \eta_{ml} J_{nk}. \tag{3.1.1b}$$

Adding extra fermionic generators Q_α to the Poincaré algebra leads to the well-known super Poincaré algebra. The generators Q_α are fermionic in the sense that they transform in spinor representations of the Lorentz group where α is the spinor index. Therefore the super Poincaré algebra must contain anti-commutation relations along with the commutation relations. We consider the example of the minimal super algebra[1]

$$[J_{mn}, Q_\alpha] = -\frac{1}{4}(\gamma_{mn})_\alpha{}^\beta Q_\beta, \qquad [Q_\alpha, P_m] = 0, \tag{3.1.2a}$$

$$\{Q_\alpha, Q_\beta\} \sim \gamma^m_{\alpha\beta} P_m. \tag{3.1.2b}$$

It is clear from these relations that the job of the Q_α-generator is to rotate fermion and boson fields to each other.

Local supersymetric invariant equations of motion and a set of fields that lie in irreducible representations of a super algebra lead to a supergravity theory. In order to construct the supergravity multiplet one has to associate to every generator a vector field. In other words, the gauge fields that correspond to P_m are the vielbein e^m_μ and for J_{mn} the gauge fields are the spin connection 1-form ω^{mn}_μ which due to the equations of motion is considered as a variable dependent on the vielbein, i.e. $\omega^{mn}_\mu(e)$. On the other hand, the gauge field corresponding to a supersymmetry generator is the gravitino denoted by ψ^α_μ, spin 3/2-field. The supergravity multiplet is then defined as the smallest set of fields involving the vielbein and the gravitino that form an irreducible representation of the super algebra[2]. Note that the number of boson and fermion degrees of freedom in any multiplet should be equal. This can be seen from the fact that $Q|Boson\rangle = |Fermion\rangle$. Acting again with the operator Q, one finds from the algebra that $Q^2|Boson\rangle \sim P|Boson\rangle = |Boson'\rangle$, where $|Boson'\rangle$ is a translated boson. Now if translations are invertible the dimension of the bosonic space is equal that of the fermionic.

Besides the supergravity multiplet one can out of the representations of super algebra construct multiplets that do not describe gravity, namely do not contain graviton and gravitino. These multiplets are representations of rigid supersymmetry, and are called the scalar, vector and tensor multiplets. It is worth noting that a rigid supersymmetry can be converted to local by coupling the multiplet to the supergravity multiplet.

So far we have just mentioned what we have called minimal super algebra, i.e., super algebra with one spinor Q_α. One can also generalize this to more supersymmetry generators Q^I_α, with I runs from $1 \cdots N$. The supergravity theories that have been

[1] The Gamma matrices $(\gamma_m)_\alpha{}^\beta$ obey the relation $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$. The matrices γ_{mn} are the antisymmetric products $\gamma_{mn} = \gamma_{[m}\gamma_{n]}$. The charge conjugation matrix is playing the role of the metric on spinor space; It satisfies the relations $C^T = \kappa C$ and $\gamma^T_m = \epsilon C \gamma_m C^{-1}$ with κ and ϵ take the values +1 or -1.

[2] Often one has to add scalars such as dilaton, vectors, e.g. graviphoton, and fermions (dilatini) etc...

Superstring theory	Low-energy approximation
Type IIA	$N = 2$ type IIA supergravity
Type IIB	$N = 2$ type IIB supergravity
Type I	$N = 1$ supergravity coupled to SO(32) YM multiplet
Heterotic SO(32)	$N = 1$ supergravity coupled to SO(32) YM multiplet
Heterotic $E_8 \times E_8$	$N = 1$ supergravity coupled to $E_8 \times E_8$ YM multiplet

Table 3.1.1: Superstring theories and their low-energy limits.

found so far are labelled by the number of supersymmetries N and the spacetime dimension D where they live. The number of components of the irreducible spinors Q^I_α is known as supercharges. The maximal number of supercharges that a field theory (theory that does not contain fields with spin higher than two) can have is 32 or less. For example in $D = 4$, a spinor has four real components, then the maximal number of supersymmetries is $N = 8$, e.g. maximal supergravity in $D = 4$. One special example is 11 dimensional supergravity where the spinor has 32 supercharges and hence $N = 1$. In 11 dimensions, the supergravity theory is unique and there is only one supergravity multiplet consisting out of the 11-bein e^m_μ, gravitino ψ^α_μ and the 3-form gauge potential $A_{\mu\nu\rho}$.

3.1.2 Supergravity Effective Actions

Although supergravity theory was not shown to be a finite perturbation theory to all orders, their effective actions are still crucial for many applications, especially because of the remarkable fact that they turned out to describe the low-energy effective behavior of superstring theories see table 3.1.1. Several different methods can be used to formulate supergravity theories and their derivative corrections.

One straightforward approach is to directly gauge the supersymmetry algebra the way we described above. In addition, most of the methods that have been pursued to construct the low-energy effective actions $\mathcal{L}_{\text{eff}}$ of superstring theories containing closed strings- supergravity actions with derivative corrections- are to some extent the same approaches mentioned in chapter two for constructing the open superstring effective actions. The first method, already outlined in chapter 2, is to simply construct the first quantized string theory in a background field (see for example [78–82]). The consistency requirements on the string theory then lead to constraints on the background fields, which can be promoted to be the equations of motion. In other words a consistent string theory can be constructed whenever the corresponding σ-model[3]

[3] To remind the reader, the nonlinear sigma model is a scalar field theory in which the scalar field takes values in some non-trivial manifold M, the target space.

is conformally invariant and the requirement of conformal invariance yields the classical equations of motion- which are identified with the β function, renormalization group coefficients- for the background fields. This method has the advantage that the 10-dimensional symmetries of superstring theory can be made explicitly and that one may have the ability to get results that are valid to all orders in perturbation theory. This method has the drawback of demanding an n-loop computation of the β functions in order to obtain the nth order term in the effective action $\mathcal{L}_{\text{eff}}$.

We have also seen in chapter 2 that there exists another method, which is in practice somewhat simpler, for constructing the effective action for open superstring theory. Here one can also use closed string theory to calculate the scattering amplitudes of its massless particles in the tree-level approximation. One then constructs an effective Lagrangian which reproduces the closed superstring S-matrix [83, 84]. Practically, S-matrix method, can be implemented as well in a perturbative fashion (in analogy with open string case) in a sense that one first constructs a 2-point function $\mathcal{L}_2$ that encodes the massless free particle of the closed superstring theory. We then incorporate cubic terms, i.e. the 3-point interactions, thus yielding $\mathcal{L}_3$. The 4-point function string scattering amplitudes can then be added[4]. The pole corresponding to the intermediate massive particles having no singularities for small values of momentum and can therefore be expanded in a power in α'. On the other hand, each term in this expansion can be reproduced by the local vertex operator, defined in section 2.1.4, namely the 4-point vertex operator V_4 which actually starts out quartic in the massless fields. Thus the 4-point sector $\mathcal{L}_4$ is constructed, the effective action for theories with closed superstring correct through quartic order. This machinery can be repeated for higher point amplitude, e.g. five, six and so forth, thereby yielding, in principle to all orders. In fact, by exploiting the local and global symmetries of the theory, the task of constructing the effective action $\mathcal{L}_{\text{eff}}$ can be greatly simplified. Roughly speaking, these symmetries help with generating terms at a given order that must appear in higher orders as a result of such symmetries.

3.1.3 Field Redefinitions Ambiguity

The effective action constructed this way, namely following either of the methods outlined above, will not be *unique*. That is because the scattering amplitude is unaffected by a field redefinition. In other words if we construct an action $\mathcal{L}[\Phi_a]$ to yield the S-matrix for particles represented by the fields Φ_a, the Lagrangian

$$\mathcal{L}[\Phi_a(\Phi')] \equiv \mathcal{L}'[\Phi'_a] \tag{3.1.3}$$

[4] Through unitarity one might guarantee that the massless poles will be those follow from the tree diagrams of $\mathcal{L}_3$.

will give the same S-matrix. Field redefinition can be performed order by order in perturbation theory provided that the field redefinition transformation

$$\Phi' \to \Phi(\Phi') = \Phi' + a_2\Phi'^2 + a_3\Phi'^3 + \cdots \tag{3.1.4}$$

is nonsingular.

To illustrate the field redefinition ambiguity in S-matrix method, let us imagine that we have calculated the 3-point sector $\mathcal{L}_3$ for one of the string theories. Now, we wish to find $\mathcal{L}_4$. This will involve new 4-point terms, to account for the pieces of the 4-point function which are not implied by $\mathcal{L}_3$. In order to obtain these we first denote by $\mathcal{L}_s$ the Lagrangian which reproduces all string theory 4-point amplitudes to desired order, i.e., $\mathcal{L}_s$ encodes a set of quartic terms. Then one can find a similar set of terms which we call $\mathcal{L}_f$, reproducing all the 4-point amplitudes coming from $\mathcal{L}_3$. Subtracting $\mathcal{L}_f$ from $\mathcal{L}_s$, one then obtain the terms that should be added to $\mathcal{L}_3$ to yield $\mathcal{L}_4$. For the sake of simplicity, let's calculate $\mathcal{L}_4$ for a toy model having $\mathcal{L}_s = 0$ and

$$\mathcal{L}_3 = -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \kappa(\partial_\mu\partial_\nu\Phi\partial^\mu\Phi\partial^\nu\Phi). \tag{3.1.5}$$

The only contribution to the 4-point function following from $\mathcal{L}_3$ is the diagram consisting of the exchange of Φ. Therefore the vertex to which Φ couples can be obtained by varying the action $\mathcal{L}_3$ w.r.t the field, i.e.,

$$V_2 = \kappa\frac{\delta}{\delta\Phi}(\partial_\mu\partial_\nu\Phi\partial^\mu\Phi\partial^\nu\Phi) = -\kappa(\partial_\mu\partial_\nu\Phi\partial^\mu\partial^\nu\Phi) + \kappa(\partial^2\Phi\partial^2\Phi), \tag{3.1.6}$$

where we have made use of momentum conservation to move the derivatives from the Φ of intermediate state (virtual) to the physical ones. The expression 3.1.6 is evaluated on-shell. Therefore it is allowed to add terms to 3.1.6 that vanish on-shell. The term that we should add is

$$V_2' = -\kappa\left(2\partial^\mu\Phi\partial_\mu\partial^2\Phi + \frac{3}{2}(\partial^2\Phi)^2 + \frac{1}{2}\Phi(\partial^2)^2\Phi\right). \tag{3.1.7}$$

The expression 3.1.6 becomes

$$V_2 = -\frac{1}{4}\kappa(\partial^2)^2(\Phi^2). \tag{3.1.8}$$

We have done this in order to have the momenta in the vertex operator V_2 emerge in the form of an inverse of propagator, that cancel with the propagators to which it is attached.

The Lagrangian $\mathcal{L}_f$ representing the scattering amplitude behaves as

$$\begin{aligned}\mathcal{L}_f &= V_2 P V_2\\ &= -\frac{1}{8}\kappa^2[(\partial^2)^2(\Phi^2)]\frac{-1}{\partial^2}[(\partial^2)^2(\Phi^2)],\end{aligned} \tag{3.1.9}$$

where P is the Φ propagator. Again, one can add terms involving $\partial^2\Phi$, that vanish on-shell, to obtain the following expression[5]

$$\mathcal{L}_f = -\frac{1}{4}\kappa^2\partial^\rho\partial^\sigma\Phi\partial_\rho\partial_\sigma\Phi\partial^\delta\Phi\partial_\delta\Phi. \tag{3.1.10}$$

Since $\mathcal{L}_s$ vanishes, we have $\mathcal{L}_4 = \mathcal{L}_3 - \mathcal{L}_f$. Then the 4-point effective Lagrangian, after integrating by parts, takes on the form

$$\begin{aligned}\mathcal{L}_4 = &-\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi + \kappa\partial^\mu\partial^\nu\Phi\partial_\mu\Phi\partial_\nu\Phi \\ &-\frac{1}{2}\kappa^2(\partial^\rho\partial^\sigma\Phi\partial_\rho\Phi\partial^\delta\partial_\sigma\Phi\partial_\delta\Phi).\end{aligned} \tag{3.1.11}$$

Performing the following field redefinition

$$\Phi' = \Phi - \frac{1}{2}\kappa\partial^\mu\Phi\partial_\mu\Phi, \tag{3.1.12}$$

one can then realize that 3.1.11 is *equivalent* to a free theory with

$$\mathcal{L}_4 = -\frac{1}{2}\partial^\mu\Phi'\partial_\mu\Phi'. \tag{3.1.13}$$

This agrees with the fact that the 3 and 4-point scattering amplitudes for our model and for the free field theory are identical; they are all zero, and that the S-matrix does not change under field redefinitions.

The same ambiguity exists in the previously discussed σ-model approach to the string equations of motion. Indeed, the β-functions of a renormalizable field theory with couplings Φ_a are not *unique*. They depend upon the definition of the coupling constant and the renormalization prescription. Using the definition 3.1.3 and the transformation 3.1.4, we find that the equations of motion have the same content since the extrema of $\mathcal{L}$ and $\mathcal{L}'$ are equal

$$\frac{\delta\mathcal{L}[\Phi]}{\delta\Phi_a} = \frac{\delta}{\delta\Phi_a}\mathcal{L}'[\Phi'(\Phi)] = \sum_b \frac{\delta\mathcal{L}'}{\delta\Phi'_b}\frac{\delta\Phi'_b}{\delta\Phi_a}, \tag{3.1.14}$$

as long as the Jacobian $\delta\Phi'_b/\delta\Phi_a$, is nonsingular. Now, if we redefine the couplings, namely the fields, $\Phi \to \Phi(\Phi')$, the β-functions

$$\beta_a(\Phi) = \mu\left(\frac{\partial\Phi}{\partial\mu}\right), \tag{3.1.15}$$

[5]Naively $\mathcal{L}_f$ seemed to have a pole due to the propagator, however this pole cancelled by the inverse propagator in the vertex, leaving a contact term.

transform under a field redefinition as

$$\beta_a(\Phi) = \mu \frac{\partial}{\partial \mu} \Phi_a(\Phi') = \beta'_a(\Phi') \frac{\partial \Phi_a}{\partial \Phi'_b}. \tag{3.1.16}$$

Nonetheless, The zeroes of β_a, identified with the equations of motion, are invariant under a non-singular field redefinition. In order to avoid having the fields over-constrained, both sets of equations β_a and $\frac{\delta \mathcal{L}}{\delta \Phi_a} = 0$ have to be satisfied and coincide. The properties of 3.1.4 suggests that they are related by a metric in the field space

$$\beta_a(\Phi) = G_{ab} \frac{\delta \mathcal{L}}{\delta \Phi_b}. \tag{3.1.17}$$

A direct connection between the β functions and the equations of motion is argued for in [85].

3.2 Strings in Background Fields: Nonlinear Sigma Model

Let's now make use of the σ-model approach and derive the bosonic sector of the supergravity action. We restrict ourselves to the bososnic string and try to describe a string moving in a more general spacetime than the Minkowski space we have considered in chapter 2. The most general covariant action we can write down with two worldsheet derivatives and appropriate symmetries, i.e. gauge invariance and local Weyl invariance, is the *nonlinear sigma model action*

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma d\tau d\sigma (\sqrt{-h} h^{ab} G_{\mu\nu}(X) - \varepsilon^{ab} B_{\mu\nu}(X)) \partial_a X^\mu \partial_b X^\nu + S[\Phi], \tag{3.2.1}$$

where ε^{ab} is the fully antisymmetric tensor in two dimensions, and the integral[6] is over the worldsheet Σ.

Actually, one can think of this action as a string moving in coherent backgrounds, $G_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ called an axion, and a scalar field Φ, i.e. the dilaton. The sector $S[\Phi]$ of the action represents the coupling of string to the dilaton

$$S[\Phi] = -\frac{1}{4\pi} \int_\Sigma d\sigma \mathrm{d}\tau \sqrt{-h} \mathcal{R}^{(2)} \Phi(X) - \frac{1}{2\pi} \int_{\partial\Sigma} ds \mathcal{K} \Phi(X), \tag{3.2.2}$$

Where $\mathcal{R}^{(2)}$ is the two-dimensional Ricci scalar of the two-dimensional worldsheet metric h_{ab}, and $\mathcal{K}$ is an extrinsic curvature and is added to cancel the total derivative

[6]The integral over Σ reflects the fact that closed string vertex operators are inserted in the bulk of Σ.

that is obtained by varying $\mathcal{R}^{(2)}$ [86]. Note that by setting the dilaton to a constant mode Φ_0, the first term of 3.2.2 is proportional to a topological invariant quantity from the worldsheet viewpoint, the Euler characteristic

$$\chi = \frac{1}{4\pi}\int_\Sigma d\sigma d\tau \sqrt{-h}\mathcal{R}^{(2)} = 2 - 2b, \tag{3.2.3}$$

where b is the genus, i.e. number of holes, of the Riemann surface Σ. This means that the first term in 3.2.2 provides us with information about the number of loops in string S-matrix. Therefore one can easily notice that the different topologies in the path integral representation of Euclideanized version of S are weighted with g_s^{2-2b} where the string coupling constant identified with the vev value of e^Φ. We know that the symmetries of the free field theory action 2.1.2 are crucial in obtaining a consistent quantization of the string since they are actually responsible for the decoupling of unphysical degrees of freedom. However, now we are dealing with an interacting field theory which does not turn into the Polyakov action in the conformal gauge $h_{ab} = \Lambda\eta_{ab}$, which makes it a non-trivial 2-dimensional field theory. As a result, if we want to do quantum calculations we are forced to a perturbation expansion in α'. In other words, the Weyl symmetry is ruined at quantum level unless the renormalization group β-functions for the field dependent couplings $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ vanish. At first non-trivial order in α' and tree-level in the loop expansion one obtains

$$\begin{aligned}
\beta^G_{\mu\nu} &= R_{\mu\nu} - 2\nabla_\mu\partial_\nu\Phi + \frac{9}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \mathcal{O}(\alpha') = 0\\
\beta^B_{\mu\nu} &= \nabla_\rho H^\rho{}_{\mu\nu} - 2H^\rho{}_{\mu\nu}\partial_\rho\Phi + \mathcal{O}(\alpha') = 0\\
\beta^\Phi &= (D-26) + 3\alpha'(R + 4(\partial\Phi)^2 - 4\nabla^2\Phi + \frac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}) + \mathcal{O}(\alpha') = 0,
\end{aligned} \tag{3.2.4}$$

where $R_{\mu\nu}$ and R are respectively the Ricci tensor and Ricci scalar associated to the background metric $G_{\mu\nu}$, and ∇_μ is the spacetime covariant derivative. $H_{\mu\nu\rho}$ is the field strength of the Kalb-Ramond background $B_{\mu\nu}$ defined by

$$H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]} = \frac{1}{3}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}). \tag{3.2.5}$$

Note that H is invariant under the gauge transformation assigned to $B_{\mu\nu}$.

The constraints 3.2.4 can be interpreted as target spacetime equations of motion and one can wonder whether they could be derived from an action principle. Indeed, the action has been found to be

$$S = \frac{1}{2}\int d^Dx\sqrt{G}e^{-2\Phi}\Big[-\frac{(D-26)}{3\alpha'} - R + 4(\partial\Phi)^2 - \frac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big] + \mathcal{O}(\alpha'). \tag{3.2.6}$$

The action 3.2.6 is known as the low-energy effective action; it describes the massless modes of slowly varying embedding coordinates X^μ in the target space in which the string moves. In the critical dimensions, i.e. $D = 26$, the dimension dependent terms in 3.2.4 and 3.2.6 drop out.

The appearance of $G_{\mu\nu}$ in the low-energy action 3.2.6 adds additional evidence to the argument that string theory would be the theory of quantum gravity. Of course, one can alternatively view the action 3.2.6 differently; it can be seen as the action of 26-dimensional gravity coupled to tensor and scalar fields. The action 3.2.6 can receive higher order terms in α' or string loops, namely stringy corrections to general relativity. This is a good place to point out to the reader that two coupling constants for two entirely different quantum theories were actually introduced:

- α' coupling constant: it controls the auxiliary two dimensional theory living on the worldsheet. For some special backgrounds, flat and some curved ones, the 2-dimensional theory can be found completely to all orders in α'. While for a generic background this is not plausible anymore. One has to evaluate β-function order by order in α'. This gives rise to higher derivative terms in the effective action.

- The string coupling g_s : dictates the loop expansion in the underlying target space theory.

The analysis of higher orders corrections in α', particularly to heterotic string, will come later.

This mechanism of calculating the low-energy effective action might be applied as well for supersymmetric string, i.e. superstring theory. Indeed, it has been shown that the low-energy approximation of superstring is the 10-dimensional supergravity, locally supersymmetric quantum field theory. As mentioned in chapter 2, the $N = 1$ worldsheet supersymmetry induces $N = 2$ spacetime supersymmetry. The way that these supersymmetries enter in the theory determines the types of superstring theories and the corresponding low-energy effective actions see table 3.1.1.

Type I

Although this theory is a theory of open strings, closed strings are also involved in type I and that is due to the fact that a closed string can split up into two interacting open strings. For such a theory, the boundary conditions of open string break the original $N = 2$ to $N = 1$ supersymmetry. We recall from chapter 2 that there is a non-abelian group (Yang-Mills) with charges attached at the endpoints of open string. The gauge group which is allowed by the consistency at the quantum level, is SO(32). According to [87–89] the bosonic part of $N = 1$, $D = 10$ supergravity reads

$$S_{\text{type I}} = \frac{1}{2}\int d^{10}x\sqrt{G}\Big[e^{-2\Phi}(-R + 4(\partial\Phi)^2) - \frac{3}{4}H_{(3)}^2 + \frac{1}{4}e^{-\Phi}F_2^I F_2^I\Big], \qquad (3.2.7)$$

where the subscripts (3) and (2) are to indicate the rank of the field strength. The gauge field associated to the gauge group SO(32) is represented by the field strength of the vector field which lies in the adjoint representation of SO(32).

Type IIA

The type IIA theory contains only closed strings. The theory is non-chiral in the sense that the two spacetime supersymmetries of the theory show up with two opposite chiralities. Contrary to type I, type IIA does not have a gauge group. The bosonic field content of this theory comprises, besides the metric, axion and dilaton of type I, a one-form field $C_{(1)}$ and 3-form gauge field $C_{(3)}$ (see table 2.1.3 in chapter 2). The type IIA supergravity action [90–92] behaves as

$$S_{type\,IIA} = \frac{1}{2}\int d^{10}x\sqrt{G}\Big[e^{-2\Phi}\left(-R+4(\partial\Phi)^2-\frac{3}{4}H^2_{(3)}\right)$$
$$\frac{1}{4}G^2_{(2)}+\frac{3}{4}G^2_{(4)}+\frac{1}{64}(G)^{-\frac{1}{2}}\epsilon_{10}\partial C_3\partial C_3 B_{(2)}\Big], \tag{3.2.8}$$

with $G_{(2)}$ and $G_{(4)}$ are the field strengths of the R-R gauge fields $C_{(1)}$ and $C_{(3)}$ respectively, and ϵ_{10} is the 10-dimensional fully antisymmetric tensor. Notice that the fields of NS-NS sector have an explicit dilaton coupling via the factor $e^{-2\Phi}$, whereas the R-R fields are not multiplied by this factor. The appearance of the coupling as such reflects the fact that R-R fields correspond to a higher order in string coupling constant. The existence of R-R fields, bosonic fields, in type IIA action 3.2.8 is necessitated by the extension of supersymmetry from $N = 1$ to $N = 2$. It is worth recalling that the solutions-p-brane- that couple to these R-R fields belong to non-perturbative spectrum.

Type IIB

The type IIB theory is a theory of closed strings as well, having $N = 2$ supersymmetry, though for this theory the two-supersymmetries have the same chirality, i.e., it is a chiral theory. Similarly to type IIA, there is no possibility for non-abelian gauge groups, and besides the NS-NS fields, one has R-R sector consisting of a scalar C_0, 2-form field $C_{(2)}$ and a selfdual 4-form gauge field $C^{+}_{\nu\mu\rho\lambda}$, table 2.1.3. The selfduality property of the 4-form prohibits writing down an effective action of type IIB in a covariant way. An action has been found in [93] wherein there has not been made use of the selduality condition, but is added as an extra condition on the 4-form. The type IIB supergravity action is

$$\begin{aligned} S_{\text{type}\,IIB} = \frac{1}{2}\int d^{10}x\sqrt{G}\Big[e^{-2\Phi}\left(-R + 4(\partial\Phi)^2 - \frac{3}{4}H_{(3)}^2\right) \\ - \frac{1}{2}(\partial C_0)^2 - \frac{3}{4}(G_{(3)} - C_0 H_{(3)})^2 - \frac{5}{6}G_{(5)}^2 \\ - \frac{\epsilon_{10}}{96\sqrt{G}} C_{(4)} \wedge G_{(3)} \wedge H_{(3)}\Big], \end{aligned} \tag{3.2.9}$$

with

$$H_{(3)} = dB_{(2)}, \qquad G_{(1)} = dC_0, \qquad G_{(3)} = dC_{(2)}. \tag{3.2.10}$$

The above IIB action is called the non-selfdual action, as we pointed out before the selfduality condition of 4-form does not follow from the action. The equations of motion have to be supplemented by

$$G_{(5)\,\mu_1\cdots\mu_5} = \frac{1}{5!\sqrt{G}}\epsilon_{\mu_1\cdots\mu_{10}} G_{(5)}^{\mu_6\cdots\mu_{10}}. \tag{3.2.11}$$

Heterotic String

The structure of heterotic string theory rests upon the fact that closed strings which form this theory have independent the right and left moving sectors. In heterotic string, one sector is supersymmetric, namely the theory has $N = 1$ supersymmetry (which is enough to remove the tachyon from the spectrum). This can be seen from the fact that the left moving sector can coincide with a purely bosonic strings, contrasting with a right moving sector which consists of modes of a superstring. In heterotic string theory we do have a non-abelian (Yang-Mills) gauge theory which results from the compactification of the bosonic sector on a 16-dimensional compact internal space, yielding 10-dimensional superstring theory. Due to quantum consistency, the gauge group turns out to be SO(32) or $E_8 \times E_8$. Therefore the bosonic part of the low-energy effective action, i.e., the bosonic sector of heterotic supergravity [94] is written as

$$S_{\text{Het}} = \frac{1}{2}\int d^{10}x\sqrt{G}e^{-2\Phi}\Big[-R + 4(\partial\Phi)^2 - \frac{3}{4}H_{(3)}^2 + \frac{1}{4}F_{(2)}^I F_{(2)\,I}\Big]. \tag{3.2.12}$$

Note that the metric G, the dilaton Φ and the axion B appear in the same way in all string theories, except type I. This has been referred to as the *common sector* in supergravity.

11-dimensional Supergravity

We pointed out in chapter 2 that in spite of the fact that superstring theory lives in $D = 10$, there is also a supergravity theory living in 11 dimensions. Despite the intimate relation between superstring and supergravity theories, the 11-dimensional supergravity does not follow from a low-energy effective action of superstring theory. However, 11-dimensional supergravity is still interesting by itself. It plays a crucial role in unifying the above five superstring theories. It is well-known that the higher number of dimensions that supergravity can live in is eleven[7]. Therefore 11-dimensional supergravity is a unique theory with $N = 1$ supersymmetry. The bosonic sector of 11-dimensional supergravity action [96] is expressed as

$$S_{11-\text{sup.}} = \frac{1}{2}\int d^{11}x\sqrt{G}\Big[-R + \frac{3}{4}G_{(4)}^2 + \frac{1}{384\sqrt{G}}\epsilon_{11}C\partial C\partial C\Big], \tag{3.2.13}$$

where the field contents of eleven dimensional supergravity are the metric G and the 3-form gauge field $C_{\mu\nu\rho}$ with $G_{(4)} = dC$. $\epsilon_{(11)}$ is a fully anti-symmetric tensor in 11 dimensions.

3.3 String Effective Action and Chern-Simons Terms

The low-energy effective action of string theory often involves Chern-Simons forms, which are totally antisymmetric tensors $\mathcal{O}_{\mu_1\cdots\mu_n}$. They depend on one or more lower rank gauge fields or spin connections/Christoffel symbols rather than just the field strength. Consequently, $\mathcal{O}$ is not invariant under the gauge transformation associated with these lower rank gauge fields. However $\mathcal{O}$ has a peculiar property that the variations of $\mathcal{O}$ under various gauge transformations are exact forms:

$$\delta\mathcal{O}_{\mu_1\cdots\mu_n} = \partial_{[\mu_1}\varphi_{\mu_2\cdots\mu_n]}, \tag{3.3.1}$$

for some quantity φ. Therefore the curvature $\partial_{[\mu_1}\mathcal{O}_{\mu_2\cdots\mu_{n+1}]}$ is a covariant tensor. Let us give an example of such a Chern-Simons term. Assume the theory has a r-form gauge field $B_{(1)}^{\mu_1\cdots\mu_r}$ and a s-form gauge field $B_{(2)}^{\mu_1\cdots\mu_s}$ with associated gauge transformations of the form

$$\delta A_{\mu_1\cdots\mu_r} = \partial_{[\mu_1}\beta_{\mu_2\cdots\mu_r]}, \qquad \delta B_{\mu_1\cdots\mu_s} = \partial_{[\mu_1}\gamma_{\mu_2\cdots\mu_s]}. \tag{3.3.2}$$

Then the $r+s+1$-form

$$\mathcal{O}_{\mu_1\cdots\mu_{r+s+1}} = A_{[\mu_1\cdots\mu_r}\partial_{\mu_{r+1}}B_{\mu_{r+2}\cdots\mu_{r+s+1}]} \tag{3.3.3}$$

transforms by a total derivative of the form 3.3.1 under the gauge transformation induced by β. Thus $\mathcal{O}_{\mu_1\cdots\mu_{r+s+1}}$ defined in 3.3.3 is a Chern-Simons $(r+s+1)$-form.

It may happen that the Chern-Simons terms show up in the expression of low-energy effective action of string theory in two different ways:

[7] For supergravity theories in dimensions higher than eleven, fields with spin greater than two appear [95], and it is not clear how to deal with these higher spin fields in an adequate way

- The action itself might contain a Chern-Simons term of the form

$$\int d^D x \epsilon^{\mu_1 \cdots \mu_D} \mathcal{O}_{\mu_1 \cdots \mu_D}. \tag{3.3.4}$$

  Since $\delta\mathcal{O}$ is total derivative, an action of this form is gauge invariant up to surface terms.

- In some theories the gauge invariant field strength associated with an antisymmetric tensor field $B_{\mu_1 \cdots \mu_{n-1}}$ is given by

$$H_{\mu_1 \cdots \mu_n} = \partial_{[\mu_1} B_{\mu_2 \cdots \mu_n]} + \mathcal{O}_{\mu_1 \cdots \mu_n} \tag{3.3.5}$$

  for some Chern-Simons n-form $\mathcal{O}$ constructed out of lower dimensional gauge fields and spin connection. Under the gauge transformation 3.3.1, $B_{\mu_1 \cdots \mu_{n-1}}$ is assigned the transformation

$$\delta B_{\mu_1 \cdots \mu_{n-1}} = -\varphi_{\mu_1 \cdots \mu_{n-1}}, \tag{3.3.6}$$

  such that

$$\delta H_{\mu_1 \cdots \mu_n} = 0. \tag{3.3.7}$$

  A typical example of such a term is the 3-form field strength associated with the NS sector 2-form gauge field of heterotic string theory. The definition of the three form field strength comprises both gauge and Lorentz Chern-Simons LCS 3-forms. In such cases the low-energy effective action being a function of $H_{\mu_1\mu_2\mu_3}$ is invariant under the gauge transformation 3.3.1 and 3.3.6 for $n = 3$.

3.4 α'-Corrections to Heterotic Supergravity

The heterotic supergravity action defined above has received higher curvature corrections as it is the low-energy effective actions of heterotic superstring theory. In this section we are going to clarify the relation between two formulations of the order α' heterotic string effective action. One formulation follows from the methods discussed in section 3.1, namely the string S-matrix calculations [84, 97] and the requirement of conformal symmetry of the corresponding sigma model to the appropriate order [97, 98], the other formulation [99, 100] is based on the supersymmetrization of Lorentz Chern-Simon forms. In [C] it has been argued that the bosonic expression for the order α' corrections constructed in [97] has to be part of a supersymmetric invariant. It has been proved a long time ago [99] that the heterotic string effective action is supersymmetric through order α'. A few months later, in [100], the supersymmetry of the action has been established to order α'^2 and α'^3. In [C] we have shown that to order α' [99] agrees with [97], demonstrating in a direct way that the action of [97] is indeed part of a supersymmetric invariant. The field redefinitions required to establish this correspondence generate additional terms at higher orders in α'.

In what follows we will try to establish that the two actions are equivalent [C]. We relegate the reader to appendix B for necessary material and conventions. Then we discuss the terms of order α'^2 and α'^3.

The heterotic string effective action to order α', as found in [97], is

$$\begin{aligned}\mathcal{L}_{\mathrm{MT}} &= -\frac{2}{\kappa^2}ee^{-2\Phi}\Big[R(\Gamma) - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi \qquad (3.4.1)\\ &\quad +\frac{1}{8}\alpha'\{R_{\mu\nu ab}(\Gamma)R^{\mu\nu ab}(\Gamma) - \frac{1}{2}R_{\mu\nu ab}(\Gamma)H^{\mu\nu c}H^{abc}\\ &\quad -\frac{1}{8}(H^2)_{ab}(H^2)^{ab} + \frac{1}{24}H^4\}\Big], \qquad (3.4.2)\end{aligned}$$

where we have

$$\begin{aligned}H_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]}, \qquad H^2 = H_{abc}H^{abc},\\ (H^2)_{ab} &= H_{acd}H_b{}^{cd}, \qquad H^4 = H^{abc}H_a{}^{df}H_b{}^{ef}H_c{}^{de}, \qquad (3.4.3)\end{aligned}$$

normalisations are as in [97].

On the other hand there is the result of supersymmetrising the LCS of [99,100]. In this section we only discuss the bosonic contributions to the effective action. Fermionic contributions can be found in [100]. Thus the bosonic terms take on the form

$$\begin{aligned}\mathcal{L}_{\mathrm{BR}} &= \frac{1}{2}ee^{-2\Phi}[\{-R(\omega) - \frac{1}{12}\widetilde{H}_{\mu\nu\rho}\widetilde{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\} \qquad (3.4.4)\\ &\quad -\frac{1}{2}\alpha R_{\mu\nu ab}(\Omega_-)R^{\mu\nu ab}(\Omega_-)]. \qquad (3.4.5)\end{aligned}$$

With respect to [100] we have redefined the dilaton and the normalisation of $B_{\mu\nu}$ (see Appendix B.1). In 3.4.4 $\widetilde{H}$ contains the LCS terms with H-torsion:

$$\begin{aligned}\widetilde{H}_{\mu\nu\rho} &= H_{\mu\nu\rho} - 6\alpha\mathcal{O}_{\mu\nu\rho}(\Omega_-), \qquad (3.4.6)\\ \mathcal{O}_{3\,\mu\nu\rho}(\Omega_-) &= \Omega_{-[\mu}ab\partial_\nu\Omega_{-\rho]}{}^{ab} - \frac{2}{3}\Omega_{-[\mu}{}^{ab}\Omega_{-\nu}{}^{ac}\Omega_{-\rho]}{}^{cb}, \qquad (3.4.7)\\ \Omega_{-\mu}{}^{ab} &= \omega_\mu{}^{ab} - \frac{1}{2}\widetilde{H}_\mu{}^{ab}. \qquad (3.4.8)\end{aligned}$$

The coefficient α is proportional to α', notice that the relative normalization between the LCS term and the R^2 action is fixed.

In order to show that the two actions (3.4.1, 3.4.2) and (3.4.4, 3.4.5) are equivalent we expand $R(\Omega_-)$ in 3.4.5, perform the required field redefinitions and fix the normalisations.

To start with, we have

$$R_{\mu\nu}{}^{ab}(\Omega_-) = R_{\mu\nu}{}^{ab}(\omega) - \frac{1}{2}(\mathcal{D}_\mu \widetilde{H}_\nu{}^{ab} - \mathcal{D}_\nu \widetilde{H}_\mu{}^{ab}) - \frac{1}{8}(\widetilde{H}_\mu{}^{ac}\widetilde{H}_\nu{}^{cb} - \widetilde{H}_\nu{}^{ac}\widetilde{H}_\mu{}^{cb}). \quad (3.4.9)$$

where the derivatives $\mathcal{D}$ are covariant with respect to local Lorentz transformations. Obviously the substitution of 3.4.9 in 3.4.5 gives terms similar to those in 3.4.2, additional terms come from expanding $\widetilde{H}$ (see Appendix B.3) in 3.4.4. The effect of these substitutions is, to order α :

$$\begin{aligned}
\mathcal{L}_{\mathrm{BR}} \;=\; & \frac{1}{2}ee^{-2\Phi}[-R(\omega) - \frac{1}{12}\bar{H}_{\mu\nu\rho}\bar{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi \\
& +\alpha\{\frac{1}{2}H^{\mu\nu\rho}\partial_\mu(\omega_\nu{}^{ab}H_\rho{}^{ab}) - \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H_\rho{}^{ab}H^{\mu\nu\rho} + \frac{1}{4}H^{\mu\nu\rho}H_\mu{}^{ab}\mathcal{D}_\nu H_\rho{}^{ab} \\
& -\frac{1}{12}H^4\} && (3.4.10) \\
& -\frac{1}{2}\alpha\{R_{\mu\nu}{}^{ab}(\omega)R^{\mu\nu ab}(\omega) && (3.4.11) \\
& -2R^{\mu\nu ab}(\omega)\mathcal{D}_\mu H_{\nu ab} && (3.4.12) \\
& +\frac{1}{2}(\mathcal{D}_\mu H_\nu{}^{ab} - \mathcal{D}_\nu H_\mu{}^{ab})\mathcal{D}^\mu H^{\nu ab} && (3.4.13) \\
& -R_{\mu\nu}{}^{ab}(\omega)H^{\mu ac}H^{\nu cb} && (3.4.14) \\
& +\frac{1}{2}(\mathcal{D}_\mu H_\nu{}^{ab} - \mathcal{D}_\nu H_\mu{}^{ab})H^{\mu ac}H^{\nu cb} && (3.4.15) \\
& +\frac{1}{8}((H^2)_{ab}(H^2)^{ab} - H^4)\}]. && (3.4.16)
\end{aligned}$$

Here $\bar{H}$ contains the LCS term without H-torsion:

$$\bar{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - 6\alpha\mathcal{O}_{3\,\mu\nu\rho}(\omega). \quad (3.4.17)$$

We now rewrite the terms (3.4.11-3.4.16) in $\mathcal{L}_{\mathrm{BR}}$, see Appendix B.4 for details. The result, keeping only contributions to order α, is

$$\begin{aligned}
\mathcal{L}_{\mathrm{BR}} \;=\; & \frac{1}{2}ee^{-2\Phi}[-R(\omega) - \frac{1}{12}\bar{H}_{\mu\nu\rho}\bar{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi \\
& -\frac{1}{2}\alpha\{R_{\mu\nu}{}^{ab}(\omega)R^{\mu\nu ab}(\omega) + \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H_\rho{}^{ab}H^{\mu\nu\rho} \\
& +\frac{1}{8}(H^2)_{ab}(H^2)^{ab} + \frac{1}{24}H^4\} && (3.4.18) \\
& -\frac{1}{2}\alpha\{R_\mu{}^c(\omega)H^{\mu ab}H_{abc} + e^\mu{}_c e^\nu{}_d\mathcal{D}_\nu H_{abd}\mathcal{D}_\mu H_{abc} \\
& +2\partial_c\Phi H_{abd}\mathcal{D}_d H_{abc} - 2\partial_d\Phi H_{abd}\mathcal{D}_c H_{abc}\}]. && (3.4.19)
\end{aligned}$$

The term proportional to the Ricci tensor in 3.4.19 contributes through a field redefinition to the terms quartic in H, and gives an additional contribution involving derivatives of Φ (see B.2.4). Making use of B.2.2 and integrating by parts all remaining terms can be made to cancel.

The final result is then

$$\begin{aligned}\mathcal{L}_{\rm BR} &= \frac{1}{2}ee^{-2\Phi}[-R(\omega) - \frac{1}{12}\bar{H}_{\mu\nu\rho}\bar{H}^{\mu\nu\rho} + 4\partial_\mu\phi\partial^\mu\Phi \\ &\quad -\frac{1}{2}\alpha\{R_{\mu\nu}{}^{ab}(\omega)R^{\mu\nu ab}(\omega) + \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H_\rho{}^{ab}H^{\mu\nu\rho} \\ &\quad -\frac{1}{8}(H^2)_{ab}(H^2)^{ab} + \frac{1}{24}H^4\}], \end{aligned} \tag{3.4.20}$$

in agreement with [97] if we set $R(\Gamma) = -R(\omega)$ and $\alpha = -\frac{1}{4}\alpha'$, and adjust the overall normalisation. Of course [97] also includes the LCS term in H^2 for the heterotic string effective action, see the footnote in [97], page 400.

3.4.1 Higher Orders and Field Redefinitions

It has been shown in [100] that the effective action to order α^2 consists of terms which are bilinear in the fermions (3.4.4, 3.4.5). This is no longer true when the effective action at order α is in the form 3.4.20.

Since the steps to go from (3.4.4, 3.4.5) to 3.4.20 have all been explicitly determined, the effective action at order α^2 can in principle be constructed. Let us identify the sources of bosonic terms of order α^2 that we have encountered:

1. From the action 3.4.4 there are contributions outlined in Appendix B.3. We should now expand $\widetilde{H}$ to order α^2, which means that in $\mathcal{A}$ B.3.2 also terms of order α should be considered. Then one should calculate $\widetilde{H}^2$.

2. $\bar{H}$ contains the LCS term of order α. These should now also be kept in the higher order contributions.

3. In a number of places we have used the identity B.4.1, the resulting R^2 terms contribute to order α^2.

4. We have used field redefinitions to modify the effective action at order α. A field redefinition is of the form

$$e_\mu{}^a \rightarrow e_\mu{}^a + \alpha\Delta_\mu^a, \tag{3.4.21}$$

and is applied to the order α^0 action. This has the effect of giving an extra contribution

$$\alpha\Delta_\mu^a\mathcal{E}^\mu{}_a \tag{3.4.22}$$

to the action, where $\mathcal{E}^{\mu}{}_{a}$ is the Einstein equation at order α^0. Thus one can eliminate a term

$$-\alpha\Delta^a_\mu\mathcal{E}^{\mu}{}_{a}, \tag{3.4.23}$$

at order α. Contributions of order α^2 arise because the transformation should also be applied to the order α action.

Accordingly, the bosonic part of terms with six derivatives in the effective action at order α^2, corresponding to the order α action 3.4.2, can be obtained, including the complete dependence on H.

At order α^3 the situation is different. In [100] an invariant related to the supersymmetrisation of the LCS terms was constructed. The status of R^4 invariants was discussed in [101], with extensive reference to the earlier work.

3.5 Conclusion

We have devoted this chapter to introduce a supergravity action as the low-energy effective action of a superstring theory, outlining the most powerful methods that have been pursued for constructing such an action and the derivative corrections (α' corrections) contributions to them. We found out that the heterotic string actions (with Chern-Simons forms) which follow from the σ-model approach and the string S-matrix calculation- note that it has been established in [97] that those two actions are equivalent to order α' modulo field redefinitions- are equivalent to order α' to the heterotic string action constructed in [100], i.e. through the supersymmetrisation of LCS. Actually, our interest in the relation between these results was triggered by a remark in a paper of Sahoo and Sen [102] in which the entropy of a supersymmetric black hole was obtained using the method of [103], with [97] for the derivative corrections to the action. The result was found to agree with that obtained by several other methods, which was taken by [102] as an indirect indication that the bosonic expression for the order α' corrections given [97] must be part of a supersymmetric invariant.