Seven-branes and instantons in type IIB supergravity

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Chapter 2

The one-half BPS branes of type IIB supergravity

2.1 Introduction

The closed IIB superstring theory can be extended by including Dirichlet p-branes or Dp-branes. These are objects with a $(p+1)$-dimensional world-volume on which an open fundamental string is ending. The open fundamental string, called an F-string, appears in type IIB supergravity as an infinitely long and straight string solution preserving one-half, that is, 16 supersymmetries. The fundamental string can be defined as that 1-brane that couples to the NSNS 2-form B_2 and that preserves 16 supersymmetries. The Dp-branes of type IIB string theory have $p = -1, 1, 3, 5, 7, 9$.

When the fundamental string ends on a Dp-brane the charges at the endpoints of the F-string produce a vector on the world-volume of the Dp-brane. This leads to a $U(1)$ gauge field, a Born–Infeld vector (see section 2.3), on the world-volume of the Dp-brane. When the D1-brane, also called the D-string, is placed in the region of IIB moduli space where the string coupling g_s is small and in which the RR axion is non-vanishing, it carries F-string charges on its world-sheet. Such a string will be referred to as an $(n, 1)$ string [29] where n denotes the F-string charge and the one reflects the fact that there is only one D-string. Alternatively, it is possible to interpret the $(n, 1)$ string as the fundamental string (with no strings ending on it[1].) of some $SL(2,\mathbb{Z})$ transformed theory.

The Dirichlet branes of the $SL(2,\mathbb{Z})$ transformed theory are objects on which a fundamental string is ending that is the $SL(2,\mathbb{Z})$ transformed version of the F-string.

[1] A string world-sheet theory with vectors is classically equivalent to a world-sheet theory without vectors because a vector in 1+1 dimensions is a non-propagating field that can be integrated out [30, 31].

For example a D-string with no F-string charge on it, i.e. a $(0,1)$ string is the S-dual of the fundamental string and ends on the S-dual of the Dp-branes. Due to the $SL(2,\mathbb{Z})$ duality of the theory there exists an infinite number of fundamental strings and associated Dirichlet branes all of which are $SL(2,\mathbb{Z})$ transformations of the F-string – Dp-brane system.

The one-half BPS objects of the type IIB theory can in most cases be characterized by saying that they form the object on which a string is ending that relates to the F-string by some $SL(2,\mathbb{Z})$ transformation. However, there are some one-half BPS objects for which such an interpretation does not exist (or is at least not very apparent). These objects arise in the cases $p=-1,7,9$. The 9-branes will be briefly discussed in section 2.5 and the case $p=7$ will be briefly discussed in section 2.4 and more elaborately in chapter 3. The instanton case that has $p=-1$ will be discussed in chapter 4.

2.2 One-half BPS projectors, tensions and gauge potentials

In order to study the one-half BPS p-branes of type IIB supergravity it is useful to first consider the p-brane as a probe brane in 10-dimensional Minkowski space-time. In this setting one can use the IIB super Poincaré algebra to classify the one-half BPS states.

A fermionic operator Q^i is introduced with $i=1,2$ an $SO(2)$ (R-symmetry group) index. The Q^i are taken to be Majorana–Weyl spinors with chirality given by $\gamma_{11}Q^i = Q^i$. A global IIB supersymmetry transformation on Minkowski space-time is then generated by $\delta = \bar{\epsilon}_1 Q^1 + \bar{\epsilon}_2 Q^2$. The anti-commutator of two fermionic generators Q^i is given by

$$\begin{aligned}\{Q^i, Q^j\} \;=\;& \delta^{ij} P_+ \gamma^\mu C^{-1} P_\mu + P_+ \gamma^\mu C^{-1} Z^{ij}_\mu + \frac{1}{3!}\epsilon^{ij} P_+ \gamma^{\mu_1\mu_2\mu_3} C^{-1} Z_{\mu_1\mu_2\mu_3} \\ & + \frac{1}{5!}\delta^{ij} P_+ \gamma^{\mu_1\cdots\mu_5} C^{-1} Z^+_{\mu_1\cdots\mu_5} + \frac{1}{5!} P_+ \gamma^{\mu_1\cdots\mu_5} C^{-1} Z^{+\,ij}_{\mu_1\cdots\mu_5}\,, \qquad (2.2.1)\end{aligned}$$

with P_μ the momentum operator and the Z's represent central charges. The central charges Z^{ij}_μ and $Z^{+\,ij}_{\mu_1\cdots\mu_5}$ are symmetric in i and j and traceless, i.e. $\delta_{ij}Z^{ij}=0$ and $\delta_{ij}Z^{+\,ij}_{\mu_1\cdots\mu_5}=0$. One can thus write

$$\begin{aligned} Z^{11}_\mu = -Z^{22}_\mu \equiv Z^{(1)}_\mu\,, &\quad Z^{12}_\mu = Z^{21}_\mu \equiv Z^{(2)}_\mu\,, \\ Z^{+\,11}_{\mu_1\cdots\mu_5} = -Z^{+\,22}_{\mu_1\cdots\mu_5} \equiv Z^{+\,(1)}_{\mu_1\cdots\mu_5}\,, &\quad Z^{+\,12}_{\mu_1\cdots\mu_5} = Z^{+\,21}_{\mu_1\cdots\mu_5} \equiv Z^{+\,(2)}_{\mu_1\cdots\mu_5}\,. \qquad (2.2.2)\end{aligned}$$

The plus on the 5-form central charges means to indicate that these 5-forms are self-dual. The operator P_+ is the chirality operator, $P_+ = \frac{1}{2}(1+\gamma_{11})$. It can be checked, by using formulae (A.2.10) and (A.2.11), that the right hand side of (2.2.1) has the same symmetry properties as the left hand side. Further, the number of independent components on both sides equals 528.

The R-symmetry group of the entire super Poincaré IIB supersymmetry algebra is $SO(2)$. The central charges in (2.2.1) form representations of the R-symmetry group. The 1-form central charges Z_μ^{ij} as well as the self-dual 5-form central charges $Z^{+\,ij}_{\mu_1\ldots\mu_5}$ form doublets under $SO(2)$. The remaining central charges $Z_{\mu_1\mu_2\mu_3}$ and $Z^{+}_{\mu_1\ldots\mu_5}$ form singlets under $SO(2)$.

From the anti-commutator of two supersymmetry generators defined on Minkowski space-time (2.2.1) the structure of the one-half supersymmetry projector can be obtained. An important role is played by the central charges of the IIB algebra on Minkowski space-time. In the local supersymmetry algebra these projectors take the same form with the only difference that the central charges in the projectors of the probe branes on Minkowksi space-time become functions of the two real scalars of the IIB theory. In the local theory the p-brane solutions can be constructed as fully back-reacted solutions and in this case the analysis of the central charges has been performed in [32].

Suppose the vacuum is a massive string (treated as a probe-brane in Minkowski space-time) lying in the 1-direction and preserving half of the 32 supersymmetries. In the rest frame, $P_\mu = M\delta_\mu^0$ and $Z_\mu^{ij} = Z^{ij}\delta_\mu^1$, of the string one then has

$$\{Q^i, Q^j\} = P_+\gamma^{\underline{0}}\left(\delta^{ij}M + \gamma_{\underline{0}}\gamma_{\underline{1}}Z^{ij}\right)C^{-1}\,, \tag{2.2.3}$$

in which the central charge Z^{ij} is

$$Z^{ij} = \sigma_3^{ij}Z^{(1)} + \sigma_1^{ij}Z^{(2)}\,, \tag{2.2.4}$$

where σ_3 and σ_1 are the standard Pauli matrices symmetric in i and j. The one-half BPS projector P in that $\{Q^i, Q^j\}$ annihilates the vacuum consisting of a massive string in the 1-direction is then

$$P = \frac{1}{2}\left(\mathbb{1} \pm \gamma_{\underline{01}}\frac{Z^{(1)}\sigma_3 + Z^{(2)}\sigma_1}{\sqrt{(Z^{(1)})^2 + (Z^{(2)})^2}}\right)\,, \tag{2.2.5}$$

where the mass M is related to the central charges via

$$M = \sqrt{(Z^{(1)})^2 + (Z^{(2)})^2}\,. \tag{2.2.6}$$

This relation follows from the condition that $P^2 = P$.

One could also consider the time index on the central charge Z_μ^{ij}. This component can be dualized to a 9-form central charge with nine spatial indices corresponding to a probe 9-brane in the background Minkowski space-time. This gives rise to the projector

$$P = \frac{1}{2}\left(\mathbb{1} \pm \frac{\tilde{Z}^{(1)}\sigma_3 + \tilde{Z}^{(2)}\sigma_1}{\sqrt{(\tilde{Z}^{(1)})^2 + (\tilde{Z}^{(2)})^2}}\right) , \tag{2.2.7}$$

where $\tilde{Z}^{(1)}$ and $\tilde{Z}^{(2)}$ are defined via

$$\epsilon_{01\dots 9} Z_0^{ij} = Z_{1\dots 9}^{ij} = \tilde{Z}^{(1)}\sigma_3^{ij} + \tilde{Z}^{(2)}\sigma_1^{ij} \, . \tag{2.2.8}$$

It can be concluded that the projectors for the strings and the 9-branes are doublet representations of $SO(2)$. The 3-brane projector and by dualization of the 3-brane central charge, the 7-brane projectors form singlets under $SO(2)$. There are further in eq. (2.2.1) three self-dual 5-form central charges: a doublet and a singlet. The doublet provides the projectors for the 5-branes of the theory whereas the singlet forms the central charge of a Kaluza–Klein monopole [33].

Generalizing the one-half BPS projectors for the massive branes to the locally supersymmetric theory the central charges appearing in the projectors become functions of the scalars of the theory. Other than that the structure of the projectors is left unaltered. In the local theory the action for a massive brane (not being the Kaluza–Klein monopole) takes the following general form (in Einstein frame)

$$\int_{\Sigma_{p+1}} d^{p+1}\sigma \, |f| \sqrt{-g_{p+1}} + \int_{\Sigma_{p+1}} X_{p+1} \, , \tag{2.2.9}$$

where $|f|$ is the tension, a function of τ and $\bar{\tau}$, g_{p+1} is the determinant of the metric g_{AB} on the p-brane world-volume Σ_{p+1} and X_{p+1} is the gauge potential that couples electrically to the p-brane. Both $X_{A_1\dots A_{p+1}}$ and g_{AB} are pull-backs of the target space-time metric $g_{\mu\nu}$ and potential $X_{\mu_1\dots\mu_{p+1}}$. These pull-backs are

$$g_{AB} = \frac{\partial X^\mu}{\partial \sigma^A}\frac{\partial X^\nu}{\partial \sigma^B} g_{\mu\nu} \, , \tag{2.2.10}$$

$$X_{A_1\dots A_{p+1}} = \frac{\partial X^{\mu_1}}{\partial \sigma^{A_1}} \cdots \frac{\partial X^{\mu_{p+1}}}{\partial \sigma^{A_{p+1}}} X_{\mu_1\dots\mu_{p+1}} \, , \tag{2.2.11}$$

in which X^μ are the embedding coordinates and σ^A the world-volume coordinates. The static gauge corresponds to taking

$$\frac{\partial X^\mu}{\partial \sigma^A} = \delta^\mu_A \, . \tag{2.2.12}$$

The kinetic terms for the $9-p$ world-volume scalars X^{p+1} to X^9, describing the motion of the brane in the target space-time, is provided by $\sqrt{-g_{p+1}}$. In order for the bosonic degrees of freedom of the world-volume theory of the p-brane to correspond to the bosonic part of some supermultiplet it is necessary that besides the $9-p$ scalars there is one massless vector on the $(p+1)$-dimensional world-volume theory. The origin of the vector lies in requiring the coupling to X_{p+1} to be gauge invariant (see section 2.3). For the case $p=1$ such a vector can always be integrated out and the world-sheet theory has eight scalars. For the case $p=9$ there are no world-volume scalars and world-volume supersymmetry only requires vectors.

Requiring that (2.2.9) vanishes under a supersymmetry variation gives rise to one-half BPS projectors in the local theory. Knowing that the structure of these projectors is already fixed up to a scalar dependent function by the global algebra it is possible to deduce the $SL(2,\mathbb{R})$ or $SU(1,1)$ representations of the $(p+1)$-form potentials. The supersymmetry variation of a $(p+1)$-form potential must contain spinor bilinears that have zero $U(1)$ charge with respect to the $U(1)$ connection of the coset $SU(1,1)/U(1)$ (see chapter 1). These spinor bilinears involve either the gravitini ψ_μ with the local supersymmetry parameter ϵ or the dilatini λ together with ϵ. Since ψ_μ, λ and ϵ all carry a $U(1)$ charge the $U(1)$ charge of the spinor bilinear (when necessary) must be neutralized by scalars in the form of $V_\pm^\alpha$. From the way the $V_\pm^\alpha$ objects appear in the supersymmetry transformation of the $(p+1)$-forms (after closure of the commutator of two supersymmetry transformations acting on the $(p+1)$-forms into the universal supersymmetry algebra has been verified) the representation of that potential under $SU(1,1)$ follows. The result is that there exists a doublet A_2^α of 2-forms, a 4-form singlet, a doublet A_6^α of 6-forms, a triplet $A_8^{\alpha\beta}$ of 8-forms, a quadruplet $A_{10}^{\alpha\beta\gamma}$ of 10-forms and a doublet A_{10}^α of 10-forms. The massive branes of type IIB supergravity are listed in table 2.2.1. The brane actions can be obtained by using formula (2.2.9) together with the entries of table 2.2.1.

For completeness table 2.2.1 also includes projectors, tensions and electric couplings that apply to space-time instantons. Projectors such as $\epsilon=0$ do not make sense in classical Lorentzian type IIB supergravity as it would imply the vanishing of ϵ_C. Instantons, however should be interpreted as approximations to Euclidean path integrals in which case $\epsilon=0$ does not imply the vanishing of ϵ_C. The instanton case will be treated separately in chapter 4.

One way to deduce what type of string can end on a brane is by constructing gauge invariant Wess–Zumino terms. The gauge transformations for the potentials are given at the end of subsection (1.1.3) and in subsection 1.1.5 for the 10-forms. The Wess–Zumino (WZ) term of a p-brane is of the form $\int_{\Sigma_{p+1}} \mathrm{WZ}_{p+1}$. If the surface Σ_{p+1} has no boundary, as will always be assumed, then the gauge transformation of WZ_{p+1} is allowed to be the exterior derivative of some p-form. In most cases the WZ term can only be made gauge invariant by inclusion of a Born–Infeld (BI) vector. This will be illustrated in the next section by considering the case of the 3-brane. The introduction of a Born–Infeld vector for the 7-brane case is discussed in section 2.4.

	$\lvert f\rvert$	$X_{(p+1)}$	Projector
$p=-1$	$\frac{1}{4\sqrt{\det Q}}\log\frac{T+2\sqrt{\det Q}}{T-2\sqrt{\det Q}}$ $\det Q>0$	$\pm i\chi'$ $\det Q>0$	$\epsilon'=0$ (upper sign) $\epsilon'_C=0$ (lower sign)
$p=-1$	$\frac{1}{T}$ $\det Q=0$	$\pm i\,\chi'$ $\det Q=0$	$\epsilon=0$ (upper sign) $\epsilon_C=0$ (lower sign)
$p=1$	$\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}$	$\pm\frac{1}{2}q_\alpha A_2^\alpha$	$\frac{q_\alpha V_-^\alpha}{\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}}\,\epsilon\mp\gamma_{\underline{01}}\epsilon_C=0$
$p=3$	1	$\pm 4A_4$	$\frac{1}{2}\left(1\mp i\gamma_{\underline{0123}}\right)\epsilon=0$
$p=5$	$\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}$	$\pm\frac{1}{2}q_\alpha A_6^\alpha$	$\frac{q_\alpha V_-^\alpha}{\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}}\,\epsilon\mp i\gamma_{\underline{01}\cdots\underline{5}}\epsilon_C=0$
$p=7$	$q_{\alpha\beta}V_+^\alpha V_-^\beta$	$\pm q_{\alpha\beta}A_8^{\alpha\beta}$	$\frac{1}{2}\left(1\mp i\gamma_{\underline{01}\cdots\underline{7}}\right)\epsilon=0$
$p=9$	$\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{3/2}$	$\pm\frac{3}{2}q_\alpha q_\beta q_\gamma A_{10}^{\alpha\beta\gamma}$	$\frac{q_\alpha V_-^\alpha}{\left(q_\beta q_\gamma V_+^\beta V_-^\gamma\right)^{1/2}}\,\epsilon\mp i\epsilon_C=0$
$p=9$	$\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}$	$\mp\frac{1}{2}q_\alpha A_{10}^\alpha$	$\frac{q_\alpha V_-^\alpha}{\left(q_\beta q_\gamma V_+^\beta V_-^\gamma\right)^{1/2}}\,\epsilon\mp\epsilon_C=0$

Table 2.2.1: Tensions $\lvert f\rvert$, gauge potentials X_{p+1} and 1/2 BPS projectors for instantons (first two rows), p-branes (third to eighth row). The plus/minus signs refer to branes/anti-branes.

2.3 Three-brane

The emergence of a BI vector through the requirement of a gauge invariant WZ term will be discussed explicitly for the case $p = 3$. The 4-form gauge transformation is given in eq. (1.1.43). Using eq. (1.2.6) it follows that

$$\delta\left(A_4 - \frac{1}{16}q_\alpha A_2^\alpha \wedge \tilde{q}_\beta A_2^\beta\right) = d\Sigma_3 - \frac{1}{8}\tilde{q}_\alpha A_2^\alpha \wedge \delta q_\beta A_2^\beta\,. \tag{2.3.1}$$

To cancel the second term in (2.3.1) one must add a new degree of freedom to the 3-brane world-volume theory (there are already 6 embedding scalars in static gauge describing the position of the 3-brane in the target space-time) in the form of a BI vector $q_\beta V_1^\beta$ whose gauge transformation is such that

$$\delta q_\alpha F_2^\alpha = \delta q_\alpha\left(A_2^\alpha + dV_1^\alpha\right) = 0\,, \tag{2.3.2}$$

where the BI field strength $F_2^\alpha = A_2^\alpha + dV_1^\alpha$ has been introduced. It follows that the gauge transformation of V_1^α is

$$\delta V_1^\alpha = -\Sigma_1^\alpha + d\Sigma^\alpha\,, \tag{2.3.3}$$

where Σ^α is a doublet of world-volume scalar gauge transformation parameters. The gauge invariant WZ term is then

$$4A_4 + \frac{1}{4}q_\alpha A_2^\alpha \wedge \tilde{q}_\beta A_2^\beta - \frac{1}{2}\tilde{q}_\alpha A_2^\alpha \wedge q_\beta F_2^\beta\,. \tag{2.3.4}$$

In terms of q_α and $\tilde{q}_\alpha$ taken as in (1.3.78) with the definitions of subsection 1.3.4 the WZ term becomes

$$4A_4 - \frac{1}{2}B_2 \wedge C_2 + C_2 \wedge F_2\,, \tag{2.3.5}$$

where $F_2 = B_2 + dV_1$ is defined such that $q_\alpha F_2^\alpha = 2\sqrt{p}\,F_2$. In the choice (1.3.78) the BI vector is associated with the NSNS 2-form B_2. Hence, it can be interpreted as coming from the open fundamental string ending on the 3-brane, which has thus turned into a D3-brane. The WZ term (2.3.5) is however not yet the full answer. One should consider the D3-brane in the regime where the string coupling g_s is small. In the quantum IIB moduli space $SO(2)\backslash PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$ (see figure 3.9.1 on page 81) this is at $\tau_0 = i\infty$, i.e. at infinity reached along the imaginary τ axis. The point $\tau_0 = i\infty$ is a fixed point of the transformations (1.3.90) to (1.3.93). Applying these transformations to (2.3.5) one finds

$$4A_4 - \frac{1}{2}B_2 \wedge C_2 + C_2 \wedge F_2 \to 4A_4 - \frac{1}{2}B_2 \wedge C_2 + C_2 \wedge F_2 - \frac{1}{2}F_2 \wedge F_2\,, \tag{2.3.6}$$

where an irrelevant total derivative, $-\frac{1}{2}dV_1 \wedge dV_1$, has been added to the right hand side of (2.3.6). Since the D3-brane is defined in the region near $\tau_0 = i\infty$ one must require it to be invariant under the transformations (1.3.90) to (1.3.93) that leave $i\infty$ invariant. This can be achieved by adding to (2.3.5) the gauge invariant term $+\frac{1}{2}\chi' F_2 \wedge F_2$. The resulting D3-brane WZ term is then

$$\mathrm{WZ}_{\mathrm{D3}} = C_4 + C_2 \wedge F_2 + \frac{1}{2}\chi F_2 \wedge F_2 \,, \tag{2.3.7}$$

where $C_4 = 4A_4 - \frac{1}{2}B_2 \wedge C_2$.

Instead of choosing q_α and $\tilde{q}_\alpha$ as in (1.3.78) they could have been kept arbitrary. In that case the BI vector would be $q_\alpha V_1^\alpha$ that is tied to the 2-form $q_\alpha A_2^\alpha$. Using the definitions of the NSNS and RR 2-forms, eqs. (1.3.44) and (1.3.45), one can write $q_\alpha A_2^\alpha$ as

$$q_\alpha A_2^\alpha = p' B_2 - q' C_2 \,, \tag{2.3.8}$$

where $p' = q_1 + q_2$ and $q' = i(q_1 - q_2)$. The parameters p' and q' are taken to be positive. The string charged with respect to (2.3.8) can be called a (p', q') string that can be considered to be a bound state of p' fundamental strings and q' D-strings [29]. The 3-brane with $q_\alpha V_1^\alpha$ as its BI vector can be called a (p', q') 3-brane since it has a (p', q') string ending on it. The WZ term of a (p', q') 3-brane is

$$\mathrm{WZ}_{(p',\,q')\ \text{3-brane}} = 4A_4 + \frac{1}{4}q_\alpha A_2^\alpha \wedge \tilde{q}_\beta A_2^\beta - \frac{1}{2}\tilde{q}_\alpha A_2^\alpha \wedge q_\beta F_2^\beta + \frac{1}{8}\chi' q_\alpha F_2^\alpha \wedge q_\beta F_2^\beta \,, \tag{2.3.9}$$

where χ' is defined in eq. (1.2.10).

The NSNS and RR 2-forms transform under $SL(2,\mathbb{Z})$ as in (1.3.51). This implies that q' and p' transform as

$$\begin{pmatrix} -q' \\ p' \end{pmatrix} \to (\Lambda^{-1})^T \begin{pmatrix} -q' \\ p' \end{pmatrix} . \tag{2.3.10}$$

When the numbers p' and q' are integers and relatively prime the (p', q') string is related to the fundamental $(1, 0)$ string by an $SL(2,\mathbb{Z})$ transformation, say Λ_0. A (p', q') string with p' and q' relatively prime can be considered to be the fundamental string of the Λ_0 transformed IIB string theory whose coupling constant is the Λ_0 transformed version of e^ϕ and whose perturbative region in moduli space is the region near the point that is the Λ_0 transform of $i\infty$ [34].

The world-volume degrees of freedom of a (p', q') 3-brane with p' and q' relatively prime consist of six embedding scalars and one BI vector, that together form the bosonic part of an $N = 4$, $d = 4$ vector supermultiplet. The kinetic terms for the embedding scalars and for the BI vector are contained in the Dirac–Born–Infeld (DBI) part of the (p', q') 3-brane action. The DBI action and the WZ term are related by kappa symmetry [26]. The type IIB $SL(2,\mathbb{Z})$ transformations acting on the background fields to which the 3-brane couples manifest themselves, on the world-volume of the 3-brane, as electric-magnetic duality transformations.

So far, the 3-brane world-volume action, since the 3-brane was put near $\tau = i\infty$ or $\Lambda i\infty$ for some $\Lambda \in SL(2,\mathbb{Z})$, was required to be invariant under the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$ or after quantization under integers shifts of χ' defined in (1.2.10). The 3-brane couples to a 4-form that is a singlet under $SL(2,\mathbb{Z})$ and so the 3-brane can move around in the moduli space, figure 3.9.1 on page 81, of the IIB theory. Is there are a supergravity description of the 3-brane world-volume theory when the 3-brane is placed near any of the points $\tau = i$ or $\tau = \rho$ of the quantum moduli space (3.9.1)?

Classically, it is possible to write down a 3-brane action that is invariant under the full $SL(2,\mathbb{R})$ duality group. This can be done by writing down an action that depends on two BI vectors transforming as a doublet under $SL(2,\mathbb{R})$. These two BI vectors are non-locally related in that their fields strengths are dual to each other. Such actions have been written down in [35] and [36] where in [36] the Pasti–Sorokin–Tonin (PST) formalism [37, 38] was used. These actions are kappa symmetric and unique. Because they are $SL(2,\mathbb{R})$ invariant they can be placed at any point of the moduli space, in particular at $\tau = i, \rho$. Since these points are fixed points of certain $SO(2)$ transformations that belong to the $SL(2,\mathbb{R})$ group and since $SO(2)$ rotations always rotate both the BI vectors, it is not expected that the actions in [35, 36] near $\tau = i, \rho$ can be reduced to an action containing a single BI vector. A 3-brane near $\tau = i, \rho$ will be called a Q3-brane. Near the point $\tau = i\infty$ it is possible to eliminate one of the two BI vectors with the remaining BI vector being related to the NSNS 2-form that is invariant under the shift of the RR axion of which $\tau = i\infty$ is a fixed point.

Finally, one should distinguish between the case with two coinciding $(1,0)$ 3-branes, say, that would give rise to a non-Abelian world-volume theory with gauge group $U(2)$, and a $(2,0)$ 3-brane that is defined to be one 3-brane on which is ending a $(2,0)$ string. The number of 3-branes is determined by the parameter in the WZ term multiplying the 4-form, which here was taken to be one.

2.4 Seven-branes

The 8-form potential that couples to a 7-brane has the general form $q_{\alpha\beta}A_8^{\alpha\beta}$. Consider the case $q_{\alpha\beta} = q_\alpha q_\beta$, i.e. $\det Q = 0$. As shown in [27] the gauge invariant and χ' shift symmetry invariant WZ term contains one BI vector, viz. $q_\alpha V_1^\alpha$. The full expression for the WZ term will not be needed. The 8-form potential $q_\alpha q_\beta A_8^{\alpha\beta}$ can be written as (see subsection 1.3.3) $pC_8 + qB_8 + \frac{r}{2}D_8$. Using that the string charges p' and q' are given by $p' = \frac{1}{2}(q_1 + q_2)$ and $q' = \frac{i}{2}(q_1 - q_2)$ it can be concluded that the 7-brane parameters p, q, r and the string parameters p' and q' are related via

$$p = p'^2\,, \quad q = q'^2\,, \quad r = \pm 2p'q'\,, \tag{2.4.1}$$

in which p' and q' are taken positive.

In order that one is dealing with a single 7-brane on which a fundamental string is ending it must be that p' and q' are relatively prime. Just as for the case of the 3-brane one can refer to such a 7-brane as a (p', q') 7-brane. When they are not relatively prime the 7-brane is formed out of a coincident set of single (p', q') 7-branes and for such a system of identical coincident 7-branes the world-volume theory is non-Abelian. The non-Abelian nature of the world-volume theory of two coincident identical (p', q') 7-branes comes from the presence of two additional massless vectors that are associated with open strings that have their endpoints on different branes. The gauge invariance of the 8-form WZ term, $\int_{\Sigma_8} q_\alpha q_\beta A_8^{\alpha\beta}$, when one considers only the center of mass motion of the coincident branes, does not require the non-Abelian BI vectors. If the relative motion is taken into consideration the world-volume scalars become non-Abelian and correspondingly non-Abelian BI vectors are needed to make the coupling to the 8-form $q_\alpha q_\beta A_8^{\alpha\beta}$ gauge invariant, see e.g. [39, 40].

Consider next the case $\det Q > 0$, which will be referred as a Q7-brane. In this case $q_{\alpha\beta}$ can be parameterized as in (1.2.5) and it follows that in order to make the WZ term $\int_{\Sigma_8} q_{\alpha\beta} A_8^{\alpha\beta}$ gauge invariant under the gauge transformations (1.1.45) two BI vectors are needed [17]. It will be argued in section 3.11 that the properties of Q7-brane solutions can be understood in terms of certain F-theory 7-branes becoming coincident. F-theory will be discussed in section 2.6 and F-theory 7-branes are (p', q') 7-branes with p' and q' relatively prime. This means that in the limit in which one only considers the center of mass motion of the Q7-brane one needs two Abelian BI vectors to produce a gauge invariant WZ term. More will be said about this in section 3.11 where it will be argued that a Q7-branes consists of two or more (p', q') 7-branes such that (at least) two out of the full set of 7-branes making up the Q7-brane differ in their values for p' and q'. For example a particular Q7-brane could be formed out of one $(1, 0)$ 7-brane and one $(1, 1)$ 7-brane or out of two $(1, 0)$ 7-branes and one $(1, 1)$ 7-brane.

Instead of referring to a 7-brane as a (p', q') 7-brane it will prove more convenient to refer to a generic 7-brane as a (p, q, r) 7-brane since this captures all possible 7-branes including those that are formed by taking different (p', q') 7-branes coincident.

2.5 Nine-branes

This section is meant to address the possible role of the doublet of 10-form potentials in the IIB theory. Some of the results presented below are non conclusive and could be considered to be work in progress.

The IIB supersymmetry algebra has two $\mathbb{Z}_2$ automorphisms that are often denoted by $(-1)^{F_L}$ and Ω. The action of $(-1)^{F_L}$ and Ω on the fields of the local supersymmetry algebra can be deduced from the supersymmetry transformations. The result for the fields of subsection (1.3.4) is

$$\begin{aligned}
(-1)^{F_L}: &\quad \epsilon \to \epsilon_C\,, \quad \psi_\mu \to \psi_{C\,\mu}\,, \quad \lambda \to \lambda_C\,,\\
(-1)^{F_L}: &\quad \phi \to \phi\,, \quad \chi \to -\chi \quad ,\\
(-1)^{F_L}: &\quad B_2 \to B_2\,, \quad C_2 \to -C_2\,, \quad A_4 \to -A_4\,, \quad C_6 \to -C_6\,,\\
(-1)^{F_L}: &\quad B_6 \to B_6\,, \quad C_8 \to -C_8\,, \quad B_8 \to -B_8\,, \quad D_8 \to D_8\,. \qquad (2.5.1)
\end{aligned}$$

and

$$\begin{aligned}
\Omega: &\quad \epsilon \to i\epsilon_C\,, \quad \psi_\mu \to i\psi_{C\,\mu}\,, \quad \lambda \to i\lambda_C\,,\\
\Omega: &\quad \phi \to \phi\,, \quad \chi \to -\chi \quad ,\\
\Omega: &\quad B_2 \to -B_2\,, \quad C_2 \to C_2\,, \quad A_4 \to -A_4\,, \quad C_6 \to C_6\,,\\
\Omega: &\quad B_6 \to -B_6\,, \quad C_8 \to -C_8\,, \quad B_8 \to -B_8\,, \quad D_8 \to D_8\,. \qquad (2.5.2)
\end{aligned}$$

Using the transformations (1.3.103), (1.3.39), (1.3.51), (1.3.52), (1.3.53) it can be shown that $S(-1)^{F_L}S^{-1} = \Omega$ where S is the S-duality transformation with $a = d = 0$ and $b = -c = 1$.

Nine-branes couple to 10-form potentials. As shown in [22] there are two sets of 10-form potentials, a quadruplet $A_{10}^{\alpha\beta\gamma}$ and a doublet A_{10}^{α} representation of $SU(1,1)$ whose properties are summarized in subsection 1.1.5. Consider first the quadruplet. A generic 10-form would take the form $q_{\alpha\beta\gamma}A^{\alpha\beta\gamma}$ where $q_{\alpha\beta\gamma}$ is a rank 3 symmetric tensor (a generalization of $q_{\alpha\beta}$ introduced in section 1.2). One could then ask the question: does the potential $q_{\alpha\beta\gamma}A^{\alpha\beta\gamma}$ couple to a 9-brane? As shown in [27, 41] the answer is affirmative if $q_{\alpha\beta\gamma} = q_\alpha q_\beta q_\gamma$. At leading order, i.e. for zero Born–Infeld field strengths the 1/2 BPS 9-brane action in Einstein frame is given by

$$S = \int_{\mathcal{M}_{10}} d^{10}x \left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{3/2} \sqrt{-g} \pm \frac{3}{2}\int_{\mathcal{M}_{10}} q_\alpha q_\beta q_\gamma A_{10}^{\alpha\beta\gamma}\,, \qquad (2.5.3)$$

where the integration is over the entire 10-dimensional space-time. The 1/2 BPS supersymmetry projector is

$$\frac{q_\alpha V_-^\alpha}{\left(q_\beta q_\gamma V_+^\beta V_-^\gamma\right)^{1/2}}\,\epsilon \mp i\epsilon_C = 0\,. \qquad (2.5.4)$$

Employing the $U(1)$ gauge choice, eq. (1.2.11), the projector reduces to

$$\epsilon \mp i\epsilon_C = 0\,. \qquad (2.5.5)$$

This projector can be interpreted as

$$\epsilon = \mp\Omega\epsilon\,, \qquad (2.5.6)$$

where the action of Ω on the supersymmetry transformation parameter ϵ is given in the first line of eq. (2.5.2). The projector (2.5.6) is the starting point of the type I truncation of type IIB supergravity down to the $N=1$ supergravity multiplet of type I supergravity.

Next, consider the doublet of 10-form potentials. Here, one can again ask the question does $q_\alpha A_{10}^\alpha$, i.e. the generic member of the 10-form doublet, couple to a 9-brane? The answer is also again yes it does. The 1/2 BPS 9-brane action is given by

$$S=\int_{\mathcal{M}_{10}} d^{10}x\,\left(q_\alpha q_\beta V_+^\alpha V_-^\beta\right)^{1/2}\sqrt{-g}\pm\frac{1}{2}\int_{\mathcal{M}_{10}} q_\alpha A_{10}^\alpha\,. \tag{2.5.7}$$

The 1/2 BPS supersymmetry projector is

$$\frac{q_\alpha V_-^\alpha}{\left(q_\beta q_\gamma V_+^\beta V_-^\gamma\right)^{1/2}}\,\epsilon\pm\epsilon_C=0\,. \tag{2.5.8}$$

Using the same $U(1)$ gauge as in the quadruplet case the projector can be written as

$$\epsilon=\mp(-1)^{F_L}\epsilon\,, \tag{2.5.9}$$

where the transformation $(-1)^{F_L}$ is given in (2.5.1). The projector (2.5.9) is the starting point for the truncation of type IIB supergravity down to the $N=1$ supergravity multiplet of heterotic supergravity.

In the $U(1)$ gauge choice with the parameter q_α as taken in (1.3.78) the brane action (2.5.3) corresponds to the D9-brane. The S-dual of the D9-brane has a 1/2 BPS projector that depends on τ. This is because S-duality acts on spinors as a local $U(1)$ transformation depending on the axidilaton field τ, see eq. (1.3.103). Another way to see this is to replace in formulae (2.5.3) and (2.5.4) q_α with $\tilde{q}_\alpha$. This can be done as the calculations do not depend on the properties of q_α. The supersymmetry projector is then (2.5.4) with q_α replaced by $\tilde{q}_\alpha$. Using the parametrization (1.3.78) the projector for the S-dual D9-brane is obtained. If in the projector of the S-dual D9-brane the axion is put to zero then the resulting projector agrees with (2.5.9), but there is a priori no reason to put the axion equal to zero. The projector of the S-dual D9-brane can be written as

$$iS\epsilon\mp S\epsilon_C=0\,, \tag{2.5.10}$$

where $S\epsilon$ is the S-dual transformed version of ϵ. Using that $S\Omega S^{-1}=(-1)^{F_L}$ eq. (2.5.10) is also

$$S\epsilon=\pm(-1)^{F_L}S\epsilon\,. \tag{2.5.11}$$

Hence, in the S-dual transformed basis Ω acts as $(-1)^{F_L}$. Comparing eq. (2.5.11) with eq. (2.5.9) it is clear that the two projectors are inequivalent. As it will be argued next, this inequivalence is lifted when one truncates the type IIB theory (or divides out the IIB theory) using Ω or $(-1)^{F_L}$.

The full truncations are obtained by working out what (2.5.6) and (2.5.9) imply for the fields of IIB supergravity by requiring that the projectors are consistent with the supersymmetry rules. This must be done since the projectors are statements about supersymmetry in the entire 10-dimensional space-time. The answer can be read off from the transformations of Ω and $(-1)^{F_L}$ given in (2.5.2) and (2.5.1). The projectors (2.5.6) and (2.5.9) can be interpreted as saying that a given type IIB configuration of fields must be identified with its Ω, respectively, $(-1)^{F_L}$ transformed versions. Hence, fields that are odd under Ω, respectively, $(-1)^{F_L}$ will be truncated from the spectrum. It can be observed that the set of bosonic fields that survive the Ω truncation are the S-dual partners of the bosonic fields that survive the $(-1)^{F_L}$ truncation. For the fermions this is also the case when the axion χ is zero, which it is in both truncations.

The process of dividing out the IIB theory by Ω is known as orientifolding the theory. It corresponds not only to the insertion of a D9-brane, but at the same time of an O9-plane. Tadpole cancelation conditions, that is requiring that the force between the O9-plane and the D9-brane cancel, forces the D9-brane charges to be 32 in units of 1 for each D9-brane in order to cancel the charge of a single O9-plane. The open string sector for a theory with 32 D9-branes and one O9-plane has gauge group $SO(32)$. In [33] the idea is advocated that the SO(32) heterotic string theory can be obtained from the IIB theory by applying an S-duality transformation on the system of 32 D9-branes plus one O9-plane. The S-duality transforms Ω into $S\Omega S^{-1}$ which should be interpreted as world-sheet parity on the world-sheet of the D1-string that is the fundamental string ending on the S-dual D9-brane.

Further, it might be relevant to remark that the 1/2 BPS supersymmetry projector of the system of 32 S-dual D9-branes orientifolded by $S\Omega S^{-1}$ preserves the same supersymmetries as the 9-brane charged under $q_\alpha A_{10}^\alpha$. Therefore from the point of view of supersymmetry it is possible to add such a 9-brane to a system of 32 S-dual D9-branes orientifolded by $S\Omega S^{-1}$ since $S\Omega S^{-1} = (-1)^{F_L}$. Similarly, if one adds to a system of 32 D9-branes orientifolded by Ω the 9-brane that couples to $\tilde{q}_\alpha A_{10}^\alpha$ supersymmetry is preserved. Could it be that the combined systems of 9-branes coupled to $q_\alpha q_\beta q_\gamma A_{10}^{\alpha\beta\gamma}$ and $\tilde{q}_\alpha A_{10}^\alpha$ and of 9-branes coupled to $\tilde{q}_\alpha \tilde{q}_\beta \tilde{q}_\gamma A_{10}^{\alpha\beta\gamma}$ and $q_\alpha A_{10}^\alpha$ has an effective orientifold interpretation[2]?

[2]The gauge transformations for A_{10}^α, when its supersymmetry transformation is given by (1.1.60), are trivial, eq. (1.1.61), the 9-branes that couple to A_{10}^α are not expected to interfere with the Yang–Mills fields coming from the open string sectors. It is mentioned that results from superspace, obtained in [26] suggest a rather different form for the gauge transformation of A_{10}^α. If a 10-form with such a gauge structure exist it would imply that the supersymmetry variation for them differs from (1.1.60) and so they would not lead to the 1/2 BPS 9-branes (2.5.7) (see also the discussion in subsection 1.1.5).

2.6 F-theory

2.6.1 Duality between type I on T^2 and type IIB on $T^2/\mathbb{Z}_2$

Consider type I string theory as a system of 32 D9-branes plus one O9-plane. This configuration is T-dual to type IIA string theory on $S^1/\mathbb{Z}_2$. The orbifold $S^1/\mathbb{Z}_2$ is defined as $x^9 \sim x^9 + 1$ and $x^9 \sim -x^9$ where x^9 is the coordinate along S^1. The background $\mathcal{M}_{1,8} \times S^1/\mathbb{Z}_2$ with $\mathcal{M}_{1,8}$ some 9-dimensional space-time can be turned into an orientifold background of the IIA theory by taking $\mathbb{Z}_2 = \{1, I_9\Omega\}$ where I_9 is the inversion operation of the 9th coordinate, i.e. $I_9 : x^9 \to -x^9$ [42]. The orbifold $S^1/\mathbb{Z}_2$ has two fixed points and at each one of them an orientifold 8-plane is placed. With each O8-plane there are eight D8-branes coinciding. In total there are thus 16 D8-branes. The number of D8-branes is 16 and not 32 because the action I_9 forces the D8-branes at the fixed points of $S^1/\mathbb{Z}_2$ to be paired. Because of this pairing one O8-plane with 8 D8-branes is effectively a stack of 16 D8-branes whose world-volume theory is truncated by Ω leading to the gauge group $SO(16)$. Gauge groups for stacks of D-branes plus an orientifold were studied in [43]. Therefore, type IIA on $S^1/\mathbb{Z}_2$, also known as type I' string theory, has gauge group $SO(16)\times SO(16)$. Performing a second T-duality transformation (reduction over a world-volume direction and uplifting over a transverse space direction) relates the above-mentioned IIA configuration on $S^1/\mathbb{Z}_2$ with two O8-planes to a IIB configuration on $T^2/\mathbb{Z}_2$ which is an orbifold with four fixed points. At each fixed point there is an O7-plane coincident with four D7-branes. The system of four D7-branes and one O7-plane has gauge group $SO(8)$.

The relation between type I on T^2 and type IIB on T^2/Z_2 can be made more explicit by considering the reductions of these theories. The field content of the type I supergravity multiplet is: a metric $g_{\mu\nu}$, a 2-form $C_{\mu\nu}$ (the type IIB RR 2-form) and the dilaton ϕ. This is the set of IIB fields that are even under Ω, see eq. (2.5.2). Taking $\mu = (a, 8, 9)$ with $a = 0, 1, \ldots, 7$ and reducing the 10-dimensional metric on T^2 gives in eight dimensions a metric g_{ab}, two Kaluza–Klein vectors A^1_a and A^2_a and three real scalars that can be organized into one complex scalar τ, the complex structure of the 2-torus and $\tilde{\varphi}$, describing the size of the 2-torus. The reduction of the 2-form $C_{\mu\nu}$ on T^2 gives in eight dimensions a 2-form C_{ab}, two vectors A^3_a and A^4_a and one real scalar C_{89}. The two real scalars $\tilde{\varphi}$ and C_{89} can be organized into one complex scalar $\sigma = C_{89} + ie^{-\tilde{\varphi}}$, which is the complex Kähler modulus of the 2-torus. Both τ and σ transform under $PSL(2,\mathbb{Z})$. Of course, there is also the real 8-dimensional dilaton denoted by φ.

Consider next the reduction of the type IIB theory over $T^2/\mathbb{Z}_2$. The 2-torus T^2 is described by the complex coordinate $z = x^8 + ix^9$ that is such that $z \sim z + 1$ and $z \sim z + \sigma$ for some complex structure modulus σ. The orbifold $T^2/\mathbb{Z}_2$ is obtained by identifying z with $-z$. The space $\mathcal{M}_{1,7} \times T^2/\mathbb{Z}_2$ in which $\mathcal{M}_{1,7}$ is some 8-dimensional space-time, can be turned into an orientifold background of type IIB by combing the $\mathbb{Z}_2$ symmetry $z \to -z$ with the perturbative world-sheet symmetry $(-1)^{F_L}\Omega$. One has $(-1)^{F_L}\Omega = -\mathbb{1}$ with $-\mathbb{1}$ an element of $SL(2,\mathbb{Z})$ as follows from eqs. (2.5.1) and (2.5.2). Hence, when reducing type IIB over $T^2/\mathbb{Z}_2$ the $\mathbb{Z}_2$ group is taken to be $\{1, I_{89}(-1)^{F_L}\Omega\}$, where I_{89} means $I_{89}: z \to -z$. When reducing the 10-dimensional metric over $T^2/\mathbb{Z}_2$ one obtains an 8-dimensional metric, g_{ab}, together with three real scalars, a complex structure scalar σ and one real scalar φ describing the size of $T^2/\mathbb{Z}_2$. The two Kaluza–Klein vectors are truncated since they are even under $(-1)^{F_L}\Omega$ but odd under I_{89}. The NSNS and RR 2-forms $B_{\mu\nu}$ and $C_{\mu\nu}$, respectively are both odd under $(-1)^{F_L}\Omega$, therefore only those components of $B_{\mu\nu}$ and $C_{\mu\nu}$ that are odd under I_{89} remain. These are the components with one leg in the space $T^2/\mathbb{Z}_2$ and one leg in the space $\mathcal{M}_{1,7}$. This leads to a total of four vectors denoted by A^1_a, A^2_a, A^3_a and A^4_a. The self-dual 4-form is even under $(-1)^{F_L}\Omega = -\mathbb{1}$ and hence only those components that are even under I_{89} remain. These are the components with two or no legs in $T^2/\mathbb{Z}_2$, i.e. A_{ab89} and A_{abcd}. The 4-form A_{abcd} can be dualized in eight dimensions to a 2-form and this 2-form together with A_{ab89}, due to the self-duality constraint in ten dimensions, produce one 2-form in eight dimensions, denoted by C_{ab}. The complex axidilaton τ reduces to a complex axidilaton in eight dimensions that is again denoted by τ.

The identification of the type I supergravity multiplet reduced over T^2 and the type IIB supergravity multiplet reduced over $T^2/\mathbb{Z}_2$ is obtained by identifying the 8-dimensional fields. The dilaton in type I on T^2 was denoted by φ and this field is identified with the size of the orbifold $T^2/\mathbb{Z}_2$, that is equally denoted by φ, such that weakly coupled type I on T^2 corresponds to a large $T^2/\mathbb{Z}_2$. The complex structure and Kähler modulus of the 2-torus on the type I side is identified with the complex axidilaton field and the complex structure modulus of $T^2/\mathbb{Z}_2$, respectively, on the IIB side. The type I 2-form C_{ab} on the type IIB side originates from the reduction of the 4-form. Finally, the metric and the four 1-forms A^1_a to A^4_a coming from the type I and IIB sides are identified. The reduction of the supergravity multiplets thus leads to a $(U(1))^4$ gauge group in eight dimensions. The total number of degrees of freedom in eight dimensions form an $N=1$, $d=8$ supergravity multiplet containing 48 bosonic and 48 fermionic degrees of freedom plus two $N=1$, $d=8$ vector supermultiplets each containing 8 bosonic and 8 fermionic degrees of freedom[3].

The open string sector of type I string theory has gauge group $SO(32)$. On the other hand the orientifold background $\mathcal{M}_{1,7} \times T^2/\mathbb{Z}_2$ contains four O7-planes placed at the fixed points of the orbifold $T^2/\mathbb{Z}_2$ with $\mathbb{Z}_2$ denoting $z \to -z$. These fixed points are $z = 0, \frac{1}{2}, \frac{\sigma}{2}, \frac{\sigma+1}{2}$. Each of the O7-planes is coincident with a stack of four D7-branes. Hence, on the IIB side the open string gauge group is $(SO(8))^4$. The reduction of type I over T^2 needs to employ Wilson lines that break $SO(32)$ down to $(SO(8))^4$.

[3] The number N counts the number of irreducible supercharges. In eight dimensions an irreducible spinor is Majorana and has 16 real components. The 48+48 N=1, $d=8$ supergravity multiplet contains the graviton, a 2-form, two vectors and one real scalar (see for example [44]).

2.6.2 Type IIB on $T^2/\mathbb{Z}_2$ is F-theory on $T^4/\mathbb{Z}_2$

The orientifolding obtained by taking $T^2/\{1, I_{89}(-1)^{F_L}\Omega\}$ can be given a purely geometrical interpretation. The action $(-1)^{F_L}\Omega$ is equal to the $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$. This means that when going around any of the four fixed points of $T^2/\{1, I_{89}\}$ the type IIB fields transform under the $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$. The $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$ can be geometrically interpreted as follows. Introduce a complex coordinate w describing a 2-torus with modular parameter τ, so that $w \sim w+1$ and $w \sim w+\tau$ with τ the axidilaton field. To each point of the transverse space of the 7-brane configuration one adds the 2-torus described by w. Identifying the IIB theory when going around a fixed point of $T^2/\{1, I_{89}\}$ means that one considers w and $-w$ equivalent as the transformation $w \to -w$ corresponds to the $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$. Hence, the 2-torus $\{w \in \mathbb{C} \mid w \sim w+1, w \sim w+\tau, w \sim -w\}$ is the orbifold $T^2/\mathbb{Z}_2$. In this way one engineers a description in which both the metric and τ are treated fully geometrically. The orientifold background $T^2/\{1, I_{89}(-1)^{F_L}\Omega\}$ can now be described as $T^4/\mathbb{Z}_2$ where

$$T^4/\mathbb{Z}_2 = \{(z,w) \in \mathbb{C}^2 \mid z \sim z+1, z \sim z+\sigma, z \sim -z, w \sim w+1, w \sim w+\tau, w \sim -1\}\,. \tag{2.6.1}$$

The modular parameters σ and τ are arbitrary complex constants. The orbifold $T^4/\mathbb{Z}_2$ forms the starting point of the orbifold definition of the surface K3 (see for example [45]). The interpretation of the type IIB orientifold background $T^2/\{1, I_{89}(-1)^{F_L}\Omega\}$ as the orbifold limit of the K3 surface was first considered in [46].

F-theory is a 12-dimensional interpretation of certain type IIB backgrounds that was first introduced in [13]. The complex axidilaton field can be interpreted as the complex modulus of a 2-torus that is added to the 10-dimensional IIB theory. Solutions describing 7-branes can in this language be constructed by fibering the 2-torus over a 2-dimensional base manifold that is the transverse space of the 7-branes. The F-theory approach to 7-brane solutions does not try to answer the question whether or not there really exists a 12-dimensional origin of the IIB theory, instead it is just a convenient mathematical tool for analyzing 7-brane solutions. $N = 1$ supersymmetry in eight dimensions is realized by requiring the T^2 fibration over the 7-brane transverse space to form a Calabi–Yau 2-fold, i.e. a K3 surface. The K3 surface is the unique compact Calabi–Yau 2-fold.

One of the main arguments[4] in support of a 12-dimensional theory is the statement that when a $U(1)$ gauge field is introduced on the world-sheet of some (p', q') string with p' and q' relatively prime (so that it can be considered the fundamental string of the (p', q') transformed perturbative IIB string theory) the dynamics of such a string appears to have a 12-dimensional character [13, 30]. In [13] it is argued that (off-shell) the quantization of such a string would require a background space-time with signature $(10, 2)$ ([13] also gives arguments why the second timelike direction is

[4] For more arguments in support of a 12-dimensional F-theory see [47].

not visible in the construction of 7-brane solutions) and in [30] an action for a (p', q') string with two Born–Infeld vectors is constructed that seems to have a 12-dimensional interpretation. Both the arguments of [13, 30] are valid off-shell.

2.6.3 F-theory on K3

In the previous subsection it was shown that it is possible to interpret the 7-brane orientifold background geometrically using a 12-dimensional approach in which τ becomes the complex structure of a 2-torus. The orbifold $T^4/\mathbb{Z}_2$ can be obtained as a special limit of the K3 surface.

The K3 surface is the unique compact Calabi–Yau 2-fold and can be described as an elliptic fibration of T^2 over a base manifold. For the moment this will be just some 2-dimensional manifold. Later when 7-branes are included the base manifold is identified with the transverse space of the 7-branes. The axidilaton is identified with the complex structure modulus or the modular parameter of the elliptically fibered T^2. For the K3 surface the base manifold is a 2-sphere.

An elliptically fibered 2-torus is described by the following elliptic curve

$$y^2 = x^3 + P(z)x + Q(z)\,, \tag{2.6.2}$$

in which x, y, z are complex variables defined on the Riemann sphere and in which P and Q are polynomials in z. The z coordinate is the complex coordinate of the base manifold. As will be shown in section 3.11 in order for (2.6.2) to describe the K3 surface the polynomials P and Q must be of order 8 and 12, respectively. The axidilaton field τ, the modular parameter of the 2-torus, defined via (2.6.2), is given through

$$j(\tau) = \frac{4P^3}{4P^3 + 27Q^2}\,, \tag{2.6.3}$$

where $j(\tau)$ is Klein's modular j-function. Conversely, any $\tau(z)$ that does not satisfy (2.6.3) is not the modular parameter of an elliptically fibered 2-torus. Hence, functions $\tau(z)$ not satisfying (2.6.3) do not correspond to a Calabi–Yau 2-fold. The elliptic curve and τ have the scale symmetry

$$P \to \lambda^4 P\,, \quad Q \to \lambda^6 Q\,, \quad x \to \lambda^2 x\,, \quad y \to \lambda^3 y\,. \tag{2.6.4}$$

The K3 surface is without any singularities provided that $4P^3 + 27Q^2 \neq 0$. For more explicit details about K3 see for example [45, 48].

The F-theory 7-branes are located at those points on the base manifold where $4P^3 + 27Q^2$ vanishes making the fibre singular at that point. The polynomial $4P^3 + 27Q^2$ is of order 24, so that there are in total 24 F-theory 7-branes. When going around a zero of $4P^3 + 27Q^2$ the axidilaton τ will undergo the $PSL(2, \mathbb{Z})$ transformation corresponding to a particular F-theory 7-brane. These and related issues will be discussed in full detail in the next chapter. The type of singularity depends on the details of the zeros of $4P^3 + 27Q^2$, i.e. whether or not the zero of $4P^3 + 27Q^2$ is also a zero of either P and/or Q and what the orders of the zeros of P, Q and $4P^3+27Q^2$ are. The singularities of an elliptically fibered 2-torus have been classified by Kodaira (see for example [49]) and the relation between the singularity type of the singular fibre with the order of the zeros of P, Q and $4P^3+27Q^2$ follows from applying Tate's algorithm [50]. The possible singularities are listed in table 2.6.1 which has been adopted from [51]. Table 2.6.1 is useful in determining the non-Abelian parts of the 7-brane gauge groups. An A_{n-1} singularity for $n \geq 2$ leads to a gauge group $SU(n)$, a D_{n+4} singularity to gauge group $SO(2(n+4))$ and the E_6, E_7 and E_8 sigularities lead to the exceptional gauge groups E_6, E_7 and E_8. The third to fifth rows of table 2.6.1 correspond to the Argyres–Douglas singularities [52,53] denoted by H_0, H_1 and H_2. When the singularity type in table 2.6.1 is referred to as 'none' then the 7-brane gauge group is trivial when it concerns the first row (there is no 7-brane since the order of the zero of $4P^3 + 27Q^2$ is zero) and Abelian when it concerns the third row of table 2.6.1. The non-Abelian gauge groups may be enlarged by $U(1)$ factors.

In [13] it has been argued that the above-mentioned F-theory on K3 is dual to heterotic string theory compactified on T^2 with appropriately chosen Wilson lines to break the heterotic gauge groups $SO(32)$ and $E_8 \times E_8$ down to the 7-brane gauge groups. These 7-brane gauge groups depend on the positions of the 7-branes, i.e. whether they are coincident with other 7-branes or not.

Here the duality with heterotic string theory will be discussed for the generic situation in which none of the 7-branes are coincident. In order to argue for the duality the type IIB theory will be reduced over a 2-sphere with 24 7-branes and the result will be compared with the heterotic theory reduced over T^2 with Wilson lines. The reduction is not as explicit as in subsection 2.6.1 but can still be done qualitatively. The following discussion is similar to the one in [13].

The number of complex parameters describing the polynomials $P(z)$ and $Q(z)$ is 22 since P is an order 8 and Q is an order 12 polynomial. One complex parameter can be fixed using the scale symmetry (2.6.4) and three more complex parameters can be fixed using the $SL(2,\mathbb{C})$ reparametrization invariance of the Riemann sphere on which z is defined. Thus the number of free parameters characterizing F-theory on K3 with 24 7-branes is 18. These 18 moduli will appear as complex scalars in the 8-dimensional reduced theory. The 18 moduli describe the relative positions of the 24 7-branes and thus there will be a $(U(1))^{18}$ 8-dimensional gauge group. Since the reduction is over a 2-sphere not all the components of the 10-dimensional metric can be consistently turned on. The reduction of the 10-dimensional metric over S^2 will give, besides an 8-dimensional metric, two Kaluza–Klein vectors and one real scalar describing the size of the 2-sphere. There is no complex structure scalar. The 2-forms are truncated from the spectrum due to the fact that the j-function is invariant under $PSL(2,\mathbb{Z})$ and so τ and $PSL(2,\mathbb{Z})$ transformations of τ are declared equivalent. This implies

Order zero P	Order zero Q	Order zero $4P^3+27Q^2$	Singularity Type
≥ 0	≥ 0	0	none
0	0	n	A_{n-1}
≥ 1	1	2	none (H_0)
1	≥ 2	3	A_1 (H_1)
≥ 2	2	4	A_2 (H_2)
2	≥ 3	$n+6$	D_{n+4}
≥ 2	3	$n+6$	D_{n+4}
≥ 3	4	8	E_6
3	≥ 5	9	E_7
≥ 4	5	10	E_8

Table 2.6.1: The Kodaira classification of singular fibres of an elliptically fibered 2-torus. The relation between the orders of the zeros of P, Q, $4P^3+27Q^2$ and the singularity type follows from Tate's algorithm. When the singularity in the last column is called 'none' it means that the group contains no non-Abelian part.

that one must in fact declare equivalent all IIB field configurations that are related by $SL(2,\mathbb{Z})$. This truncates all the p-forms with $p \neq 0$ from the IIB spectrum[5]. The reduction of the 4-form that can either have two or no legs in the 2-sphere gives rise to one 2-form in eight dimensions. The resulting 8-dimensional fields comprise one $N=1$, $d=8$ supergravity multiplet and 18 $N=1$, $d=8$ vector multiplets. There are in total 20 vectors (there are two vectors in the $N=1$, $d=8$ supergravity multiplet) giving rise to the gauge group $(U(1))^{20}$.

The reduction of the $N=1$, $d=10$ heterotic supergravity multiplet over a 2-

[5]One may wonder about the 18 $U(1)$'s coming from the 7-branes since these are associated with Born–Infeld vectors that arise as the zero modes of the 2-forms. The statement is that one cannot define any 2-form globally, but since the 2-form zero modes will be formed out of functions of the solution they can be globally well-defined.

torus gives the $N = 1$, $d = 8$ supergravity multiplet coupled to two $N = 1$, $d = 8$ vector multiplets (the analysis is analogous to the reduction of the type I supergravity multiplet over T^2). The heterotic gauge groups $SO(32)$ and $E_8 \times E_8$ are of rank 16 and thus have a $(U(1))^{16}$ subgroup. Hence, one must add Wilson lines on the T^2 that break $SO(32)$ or $E_8 \times E_8$ down to $(U(1))^{16}$. Therefore, also the reduction of heterotic supergravity over T^2 can give rise to one $N = 1$, $d = 8$ supergravity multiplet and 18 $N = 1$, $d = 8$ vector multiplets. In the duality the heterotic dilaton is identified with the real Kähler modulus describing the size of the 2-sphere. Weakly coupled heterotic string theory corresponds to a small S^2.

The orbifold limit in which the K3 with 24 7-branes reduces to $T^4/\mathbb{Z}_2$ of the previous subsection is obtained by taking $P^3 = cQ^2$ in which c is some nonzero complex number [46]. This implies that $P = R^2$ and $Q = c^{-1/2}R^3$ where R is an arbitrary polynomial of order four. The scaling symmetry (2.6.4) and the freedom to perform $SL(2,\mathbb{C})$ transformations imply that R has one unfixed complex parameter. The one free complex parameter is the complex structure modulus σ of the base manifold $T^2/\mathbb{Z}_2$. The 7-branes are located at the zeros of $4P^3 + 27Q^2$ which in the limit becomes equal to R^6. Hence, there are in total four zeros at each of which six $\det Q = 0$ (p, q, r) 7-branes have come together. This admits an interpretation in terms of 4 D7-branes plus one O7-plane as will be explicitly shown in section 3.11. The F-theory gauge symmetry $(U(1))^{20}$ in the orbifold limit is enhanced to $(SO(8))^4 \times (U(1))^4$ which is T-dual to type I as argued 2.6.1. This can be understood from the heterotic side as follows. The orbifold limit corresponds to an enhancement of the fully broken $(U(1))^{16}$ gauge group, coming from the Yang–Mills sector reduced over the 2-torus, to $(SO(8))^4$ (which is both a subgroup of $SO(32)$ and $E_8 \times E_8$). If on the F-theory side one considers a large sized base manifold $T^2/\mathbb{Z}_2$ then the heterotic theory becomes strongly coupled. Applying an S-duality to the strongly coupled heterotic theory then gives weakly coupled type I on T^2 with gauge group $(SO(8))^4$. The emergence of the gauge group $(SO(8))^4$ on the F-theory side can be understood by taking $P = R^2$ and $Q = c^{-1/2}R^3$ and looking in table 2.6.1. Since R has four different zeros, and each zero of R is a second order zero of P, a third order zero of Q and sixth order zero of $4P^3 + 27Q^2$ the gauge group, according to table 2.6.1, at each zero of R should be $SO(8)$ and hence a gauge group $(SO(8))^4$ appears.

In [54] it is shown that there exist more orbifold limits of $K3$ that give rise to new gauge symmetry enhancements involving the exceptional groups E_8, E_7 and E_6. These F-theory orbifolds are $T^4/\mathbb{Z}_n$ with $n = 3, 4, 6$ and require that either the polynomial P or the polynomial Q are equal to zero, so that τ is fixed to be either ρ or i, respectively.

Finally, it is mentioned here that there also exists a fiberwise T-duality between F-theory on K3 and M-theory on a T^2 fibered over some 9-dimensional manifold. This is explained for example in [15]. This last point is mentioned to underline the richness of the F-theory moduli space. One can construct dualities between F-theory on K3 with type I on T^2, with both the heterotic theories on T^2 and with M-theory on a fibered T^2.