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Results of ISS Type for Hysteretic Lur'e Systems: a Differential Inclusions Approach*

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1 Introduction

The paper comprises a study of absolute stability, input-to-state stability, and boundedness properties of a feedback interconnection of a finite-dimensional, linear, m-input, m-output system (A, B, C) and a set-valued nonlinearity Φ. With reference to Figure 1, we assume that D is a set-valued map in which input or disturbance signals are embedded. The analytical framework is of sufficient generality

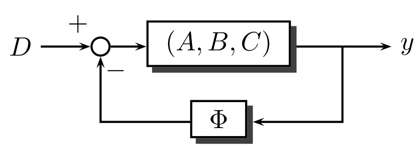

Figure 1. *Feedback interconnection of linear system* (A, B, C) *and nonlinearity* Φ

to encompass feedback systems with hysteresis operators (that is, a causal rate-independent operator) in the feedback loop. To illustrate this, let F be a causal operator from $\mathrm{dom}(F) \subset L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ to $L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$, where $\mathbb{R}_+ := [0, \infty)$, and consider the feedback system (structurally of Lur'e type), with input $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$,

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given by the functional differential equation

$$\dot{x}(t) = Ax(t) + B\big(d(t) - (F(Cx))(t)\big). \tag{1}$$

Assume that F can be embedded in a set-valued map Φ in the sense that

$$y \in \mathrm{dom}(F) \quad \Longrightarrow \quad (F(y))(t) \in \Phi(y(t)) \ \text{ for a.a. } t \in \mathbb{R}_+ \,.$$

If the input d is such that $d(t) \in D(t)$ for almost all t, then any solution of (1) is *a fortiori* a solution of the feedback interconnection in Figure 1. In this sense, properties of solutions of the feedback interconnection are inherited by solutions of (1). Under particular regularity assumptions on D and Φ, generalized sector conditions on Φ, and positive-real conditions related to the linear component (A, B, C), we establish input-to-state stability (in the sense of [10], but extended to differential inclusions) and boundedness properties of solutions of the system in Figure 1.

2 Set-valued nonlinearities and differential inclusions

A set-valued map $y \mapsto \Phi(y) \subset \mathbb{R}^m$, with non-empty values and defined on $\mathbb{R}^m$, is said to be *upper semicontinuous at* $y \in \mathbb{R}^m$ if, for every open set U containing $\Phi(y)$, there exists an open neighbourhood Y of y such that $\Phi(Y) := \cup_{z\in Y}\Phi(z) \subset U$; the map Φ is said to be *upper semicontinuous* if it is upper semicontinuous at every $y \in \mathbb{R}^m$. The set of upper semicontinuous compact-convex-valued maps

$$\Phi : \mathbb{R}^m \to \{S \subset \mathbb{R}^m \mid S \text{ non-empty, compact and convex}\}$$

is denoted by $\mathcal{U}$. Let $D : \mathbb{R}_+ \to \{S \subset \mathbb{R}^m \,|\, S \neq \emptyset\}$ be a set-valued map. The map D is said to be *measurable* if the preimage $D^{-1}(U) := \{t \in \mathbb{R}_+ \,|\, D(t) \cap U \neq \emptyset\}$ of every open set $U \subset \mathbb{R}^m$ is Lebesgue measurable; D is said to be *locally essentially bounded* if D is measurable and the function $t \mapsto |D(t)|$ is in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$. The set of all locally essentially bounded set-valued maps $\mathbb{R}_+ \to \{S \subset \mathbb{R}^m \,|\, S \neq \emptyset\}$ is denoted by $\mathcal{B}$. For $D \in \mathcal{B}$, $I \subset \mathbb{R}_+$ an interval and $1 \le p \le \infty$, the L^p-norm of the restriction of the function $t \mapsto |D(t)|$ to the interval I is denoted by $\|D\|_{L^p(I)}$.

The feedback system shown in Figure 1 corresponds to the initial-value problem

$$\dot{x}(t) - Ax(t) \in B\left(D(t) - \Phi(Cx(t))\right), \quad x(0) = x^0 \in \mathbb{R}^n, \ \ D \in \mathcal{B}\,, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $\Phi \in \mathcal{U}$. By a solution of (2) we mean an absolutely continuous function $x : [0,\omega) \to \mathbb{R}^n$, $0 < \omega \le \infty$, such that $x(0) = x^0$ and the differential inclusion in (2) is satisfied almost everywhere on $[0,\omega)$; a solution is *maximal* if it has no proper right extension that is also a solution; a solution is *global* if it exists on $[0,\infty)$. We record the following existence result (a consequence of, for example, [3, Corollary 5.2]).

Lemma 1. *Let $\Phi \in \mathcal{U}$. For each $x^0 \in \mathbb{R}^n$ and each $D \in \mathcal{B}$, the initial-value problem (2) has a solution. Moreover, every solution can be extended to a maximal solution $x : [0,\omega) \to \mathbb{R}^n$ and, if x is bounded, then x is global.*

3 Input-to-state stability: the main results

In the context of the differential inclusion (2), the transfer-function matrix of the linear system given by (A, B, C) is denoted by G, i.e., $G(s) = C(sI - A)^{-1}B$.

We assemble the following hypotheses which will be variously invoked in the theory developed below. Recall that $\mathcal{K}_\infty$ is the set of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ that are continuous, strictly-increasing and unbounded with $\varphi(0) = 0$; $\mathcal{KL}$ is the set of all functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \in \mathbb{R}_+$ and, for each $r \in \mathbb{R}_+$, $\beta(r, t) \downarrow 0$ as $t \to \infty$.

(H1) There exist numbers $a < b$ and $\delta > 0$ such that

$$\langle ay - v, by - v\rangle \le 0 \quad \forall\, v \in \Phi(y),\ \forall\, y \in \mathbb{R}^m\,, \tag{3}$$

$G(I + aG)^{-1} \in H^\infty$ and $(I + bG)(I + aG)^{-1} - \delta I$ is positive real.

(H2) $\Phi(0) = \{0\}$ and there exist numbers $a > 0$, $\delta \in [0, 1)$ and $\theta \ge 0$ such that

$$a\|y\|^2 \le \langle y, v\rangle \quad \forall\, v \in \Phi(y),\ \forall\, y \in \mathbb{R}^m\,, \tag{4}$$

$$\|v - a\delta y\| \le \langle y, v - a\delta y\rangle \quad \forall\, v \in \Phi(y),\ \forall\, y \in \mathbb{R}^m \text{ with } \|y\| \ge \theta \tag{5}$$

and $G(I + \delta aG)^{-1}$ is positive real.

(H3) There exist $\varphi \in \mathcal{K}_\infty$ and numbers $b > 0$ and $\delta \in [0, 1)$ such that

$$\max\left\{\varphi(\|y\|)\|y\|, \|v\|^2/b\right\} \le \langle y, v\rangle \quad \forall\, v \in \Phi(y),\ \forall\, y \in \mathbb{R}^m \tag{6}$$

and $(\delta/b)I + G$ is positive real.

(H4) $\Phi(0) = \{0\}$ and there exist $\varphi \in \mathcal{K}_\infty$ and a number $\theta \ge 0$ such that

$$\varphi(\|y\|)\|y\| \le \langle y, v\rangle \quad \forall\, v \in \Phi(y),\ \forall\, y \in \mathbb{R}^m\,, \tag{7}$$

$$\|v\| \le \langle y, v\rangle \quad \forall\, v \in \Phi(y),\ \forall\, y \in \mathbb{R}^m \text{ with } \|y\| \ge \theta \tag{8}$$

and G is positive real.

Remark 2. (a) (H1) is a set-valued version of the familiar multivariable sector condition.

(b) If $m = 1$ (the single-input, single-output case), then the combined frequency-domain assumptions in (H1) admit a graphical characterization in terms of the Nyquist diagram of G (see, e.g., [5, pp. 268]).

(c) Conditions (4) and (7) can be viewed as the limits of (3) and (6), respectively, as $b \to \infty$.

(d) A sufficient condition for (6) to hold is the "nonlinear" sector condition

$$\left\langle \varphi(y)\|y\|^{-1}y - v\,,\, by - v\right\rangle \le 0 \quad \forall\, v \in \Phi(y),\ \forall\, y \in \mathbb{R}^m, \tag{9}$$

(e) If $m = 1$ and (4) holds, then (5) is trivially satisfied for any $\theta \ge 1$ and any $\delta \in [0, 1)$. Similarly, if $m = 1$ and (7) holds, then (8) is satisfied for every $\theta \ge 1$.

(f) If (6) holds for some $\varphi \in \mathcal{K}_\infty$ and for some $b > 0$, then $\Phi(0) = \{0\}$ and, furthermore, (8) is satisfied for any $\theta > 0$ satisfying $\varphi(\theta) \geq b$.

Definition 3. *System* (2) *is said to be* input-to-state stable with bias $c \geq 0$ *if every maximal solution of* (2) *is global, and there exist* $\beta_1 \in \mathcal{KL}$ *and* $\beta_2 \in \mathcal{K}_\infty$ *such that, for all* $x^0 \in \mathbb{R}^n$ *and all* $D \in \mathcal{B}$*, every global solution* x *satisfies*

$$\|x(t)\| \leq \max\left\{\beta_1(\|x^0\|, t)\,,\, \beta_2(\|D\|_{L^\infty[0,t]} + c)\right\} \quad \forall\, t \in \mathbb{R}_+\,. \tag{10}$$

System (2) *is* input-to-state stable *if it is input-to-state stable with bias* 0*.*

System (2) has the *converging-input-converging-state property* if, for all $x^0 \in \mathbb{R}^n$ and all $D \in \mathcal{B}$ with $\|D\|_{L^\infty[t,\infty)} \to 0$ as $t \to \infty$, every maximal solution x of (2) is global and satisfies $x(t) \to 0$ as $t \to \infty$. The following lemma shows in particular that if system (2) is input-to-state stable, then it has the converging-input-converging-state property.

Lemma 4. *Assume that system* (2) *is input-to-state stable with bias* $c \geq 0$ *and let* β_1 *and* β_2 *be as in* Definition 3*. Then, for all* $x^0 \in \mathbb{R}^n$ *and all* $D \in \mathcal{B}$*, every global solution* x *of* (2) *satisfies*

$$\limsup_{t\to\infty} \|x(t)\| \leq \limsup_{t\to\infty} \beta_2(\|D\|_{L^\infty[t,2t]} + c)\,.$$

We now arrive at the main results on input-to-state stability (proofs of which can be found in [4]).

Theorem 5. *Let the linear system* (A, B, C) *be stabilizable and detectable. Assume that* (H1) *holds. Then, every maximal solution of* (2) *is global and there exist positive constants* c_1*,* c_2 *and* ε *such that, for all* $x^0 \in \mathbb{R}^n$ *and* $D \in \mathcal{B}$*, every global solution* x *satisfies*

$$\|x(t)\| \leq c_1 e^{-\varepsilon t}\|x^0\| + c_2\|D\|_{L^\infty[0,t]} \quad \forall\, t \in \mathbb{R}_+\,.$$

In particular, system (2) *is input-to-state stable.*

Theorem 6. *Let the linear system* (A, B, C) *be minimal. Assume that at least one of hypotheses* (H2)*,* (H3) *or* (H4) *holds. Then system* (2) *is input-to-state stable.*

In [1] it is has been proved, for single-valued Φ and D, that, if (A3) holds, then (2) is input-to-state stable. Therefore, Theorem 6 can be considered as a generalization of the main result in [1].

In the following corollaries (to Theorems 5 and 6, respectively), we will consider not only nonlinearities satisfying at least one of the conditions (3), (4), (6) and (7) *for all* arguments $y \in \mathbb{R}^m$, but also nonlinearities $\Phi \in \mathcal{U}$ with the property that there exist a set-valued map $\tilde{\Phi} \in \mathcal{U}$ satisfying at least one of the conditions (3), (4), (6) and (7) and a compact set $K \subset \mathbb{R}^m$ such that

$$y \in \mathbb{R}^m \backslash K \quad \Longrightarrow \quad \Phi(y) \subset \tilde{\Phi}(y)\,. \tag{11}$$

In particular, single-input, single-output hysteretic elements can be subsumed by this set-valued formulation provided that the "characteristic diagram" of the hysteresis is contained in the graph of some $\Phi \in \mathcal{U}$.

Corollary 7. *Let the linear system (A, B, C) be stabilizable and detectable. Let $\Phi \in \mathcal{U}$ be such that there exist a set-valued map $\tilde{\Phi} \in \mathcal{U}$ and a compact set $K \subset \mathbb{R}^m$ such that* (11) *holds. Assume that* (H1) *holds with Φ replaced by $\tilde{\Phi}$. Then, every maximal solution of* (2) *is global and there exist positive constants c_1, c_2 and ε such that, for all $x^0 \in \mathbb{R}^n$ and $D \in \mathcal{B}$, every global solution x satisfies*

$$\|x(t)\| \le c_1 e^{-\varepsilon t}\|x^0\| + c_2(\|D\|_{L^\infty[0,t]} + E) \quad \forall\, t \in \mathbb{R}_+\,,$$

where

$$E := \sup_{y\in K} \sup_{v\in\Phi(y)} \inf_{\tilde{v}\in\tilde{\Phi}(y)} \|v - \tilde{v}\|\,. \tag{12}$$

Corollary 8. *Let the linear system (A, B, C) be minimal and let $\Phi \in \mathcal{U}$ be such that there exist a set-valued map $\tilde{\Phi} \in \mathcal{U}$ and a compact set $K \subset \mathbb{R}^m$ such that* (11) *holds. Assume that at least one of the hypotheses* (A1), (A2) *or* (A3) *holds with Φ replaced by $\tilde{\Phi}$. Then system* (2) *is input-to-state stable with bias E given by* (12).

4 Hysteretic feedback systems

We return to the feedback interconnection of Figure 1, but now in a single-input, single-output setting and with a hysteresis operator F in the feedback path. An operator $F : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a *hysteresis operator* if it is causal and rate independent. Here *rate independence* means that $F(y \circ \zeta) = (Fy) \circ \zeta$ for every $y \in C(\mathbb{R}_+)$ and every time transformation ζ, where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *time transformation* if it is continuous, non-decreasing and surjective. Conditions on F which ensure well-posedness of the feedback interconnection (existence and uniqueness of solutions of the associated initial-value problem) are expounded in, for example, [8] and [9]. The so-called Preisach operators are among the most general and most important hysteresis operators: in particular, they can model complex hysteresis effects such as nested loops in input-output characteristics. Therefore, and for clarity of presentation, we focus on the class of Preisach operators.

A basic building block for these operators is the *backlash* operator. A discussion of the *backlash* operator (also called *play* operator) can be found in a number of references, see for example [2], [6] and [7]. Let $\sigma \in \mathbb{R}_+$ and introduce the function $b_\sigma : \mathbb{R}^2 \to \mathbb{R}$ given by

$$b_\sigma(v_1, v_2) := \max\big\{v_1 - \sigma\,,\, \min\{v_1 + \sigma, v_2\}\big\} = \begin{cases} v_1 - \sigma, & \text{if } v_2 < v_1 - \sigma \\ v_2, & \text{if } v_2 \in [v_1 - \sigma, v_1 + \sigma] \\ v_1 + \sigma, & \text{if } v_2 > v_1 + \sigma\,. \end{cases}$$

Let $C_{\rm pm}(\mathbb{R}_+)$ denote the space of continuous piecewise monotone functions defined on $\mathbb{R}_+$. For all $\sigma \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$, define the operator $\mathcal{B}_{\sigma,\,\xi} : C_{\rm pm}(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by

$$\mathcal{B}_{\sigma,\,\xi}(y)(t) = \begin{cases} b_\sigma(y(0), \xi) & \text{for } t = 0\,, \\ b_\sigma(y(t), (\mathcal{B}_{\sigma,\,\xi}(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i = 0, 1, 2, \dots\,, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \dots$, $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval $[t_i, t_{i+1}]$. We remark that ξ plays the role of an "initial state". It is not difficult to show that the definition is independent of the choice of the partition (t_i). Figure 2 illustrates how $\mathcal{B}_{\sigma,\,\xi}$ acts. It is well-known that $\mathcal{B}_{\sigma,\,\xi}$ extends to a

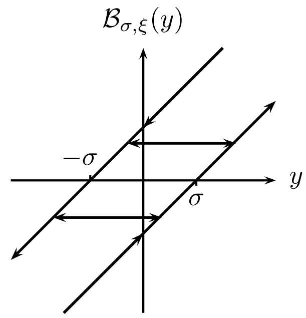

Figure 2. *Backlash hysteresis*

Lipschitz continuous operator on $C(\mathbb{R}_+)$ (with Lipschitz constant $L = 1$), the so-called backlash operator, which we shall denote by the same symbol $\mathcal{B}_{\sigma,\,\xi}$. It is well-known that $\mathcal{B}_{\sigma,\,\xi}$ is a hysteresis operator.

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let μ be a signed Borel measure on $\mathbb{R}_+$ such that $|\mu|(K) < \infty$ for all compact sets $K \subset \mathbb{R}_+$, where $|\mu|$ denotes the total variation of μ. Denoting Lebesgue measure on $\mathbb{R}$ by μ_L, let $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $w_0 \in \mathbb{R}$. The operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(\mathcal{P}_\xi(y))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{\sigma,\,\xi(\sigma)}(y))(t)} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) + w_0 \forall\, u \in C(\mathbb{R}_+)\,,\ \ \forall\, t \in \mathbb{R}_+\,, \tag{13}$$

is called a *Preisach* operator, cf. [2, p. 55]. It is well-known that $\mathcal{P}_\xi$ is a hysteresis operator (this follows from the fact that $\mathcal{B}_{\sigma,\,\xi(\sigma)}$ is a hysteresis operator for every $\sigma \ge 0$). Under the assumption that the measure μ is finite and w is essentially bounded, the operator $\mathcal{P}_\xi$ is Lipschitz continuous with Lipschitz constant $L = |\mu|(\mathbb{R}_+)\|w\|_\infty$ (see [7]) in the sense that

$$\sup_{t\in\mathbb{R}_+} |\mathcal{P}_\xi(y_1)(t) - \mathcal{P}_\xi(y_2)(t)| \le L \sup_{t\in\mathbb{R}_+} |y_1(t) - y_2(t)| \quad \forall\, y_1, y_2 \in C(\mathbb{R}_+).$$

This property ensures the well-posedness of the feedback interconnection.

Setting $w(\cdot,\cdot) = 1$ and $w_0 = 0$ in (13), we obtain the *Prandtl* operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$\mathcal{P}_\xi(y)(t) = \int_0^\infty (\mathcal{B}_{\sigma,\,\xi(\sigma)}(y))(t)\mu(\mathrm{d}\sigma) \quad \forall\, u \in C(\mathbb{R}_+)\,,\ \ \forall\, t \in \mathbb{R}_+\,. \tag{14}$$

For $\xi \equiv 0$ and μ given by $\mu(E) = \int_E \chi_{[0,5]}(\sigma)\mathrm{d}\sigma$ (where $\chi_{[0,5]}$ denotes the indicator function of the interval $[0,5]$), the Prandtl operator is illustrated in Figure 3. The

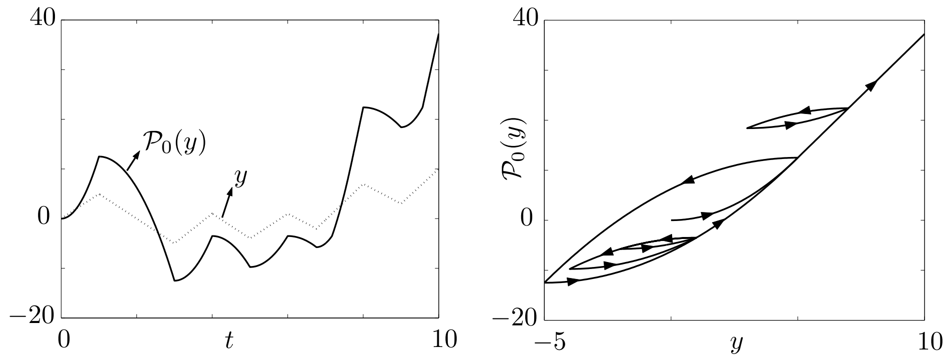

Figure 3. *Example of Prandtl hysteresis*

next proposition identifies conditions under which the Preisach operator (13) satisfies a generalized sector bound. For simplicity, we assume that the measure μ and the function w are non-negative (an important case in applications), although the proposition can be extended to signed measures μ and sign-indefinite functions w.

Proposition 9. *Let $\mathcal{P}_\xi$ be the Preisach operator defined in* (13)*. Assume that the measure μ is non-negative, $a_1 := \mu(\mathbb{R}_+) < \infty$ and $a_2 := \int_0^\infty \sigma\mu(\mathrm{d}\sigma) < \infty$. Furthermore, assume that*

$$b_1 := \operatorname{ess\,inf}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} w(s,\sigma) \geq 0\,, \quad b_2 := \operatorname{ess\,sup}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} w(s,\sigma) < \infty$$

and set

$$a_{\mathcal{P}} := a_1 b_1, \quad b_{\mathcal{P}} := a_1 b_2\,, \quad c_{\mathcal{P}} := a_2 b_2 + |w_0|\,. \tag{15}$$

Then

$$\forall\, y \in C(\mathbb{R}_+)\ \forall\, t \in \mathbb{R}_+\,,\quad y(t) \geq 0 \implies a_{\mathcal{P}} y(t) - c_{\mathcal{P}} \leq (\mathcal{P}_\xi(y))(t) \leq b_{\mathcal{P}} y(t) + c_{\mathcal{P}}\,, \tag{16}$$

$$\forall\, y \in C(\mathbb{R}_+)\ \forall\, t \in \mathbb{R}_+\,,\quad y(t) \leq 0 \implies b_{\mathcal{P}} y(t) - c_{\mathcal{P}} \leq (\mathcal{P}_\xi(y))(t) \leq a_{\mathcal{P}} y(t) + c_{\mathcal{P}}\,, \tag{17}$$

and, for every $\eta > 0$,

$$\forall\, y \in C(\mathbb{R}_+)\ \forall\, t \in \mathbb{R}_+\,,\quad |y(t)| \geq c_{\mathcal{P}}/\eta \implies (a_{\mathcal{P}} - \eta) y^2(t) \leq (\mathcal{P}_\xi(y))(t) y(t) \leq (b_{\mathcal{P}} + \eta) y^2(t)\,. \tag{18}$$

Let $\mathcal{P}_\xi$ be a Preisach operator satisfying the hypotheses of Proposition 9. Let $a_{\mathcal{P}}$, $b_{\mathcal{P}}$ and $c_{\mathcal{P}}$ be given by (15) and define Φ, $\tilde{\Phi} \in \mathcal{U}$ by

$$\Phi(y) := \begin{cases} \{v \in \mathbb{R} \mid a_{\mathcal{P}} y - c_{\mathcal{P}} \leq v \leq b_{\mathcal{P}} y + c_{\mathcal{P}}\}, & y \geq 0 \\ \{v \in \mathbb{R} \mid b_{\mathcal{P}} y - c_{\mathcal{P}} \leq v \leq a_{\mathcal{P}} y + c_{\mathcal{P}}\}, & y < 0. \end{cases}$$

$$\tilde{\Phi}(y) := \{v \in \mathbb{R} \mid (a_{\mathcal{P}} - \eta)y^2 \le vy \le (b_{\mathcal{P}} + \eta)y^2\},$$

where $\eta > 0$. In view of (16) and (17),

$$y \in C(\mathbb{R}_+) \quad \Longrightarrow \quad (\mathcal{P}_\xi(y))(t) \in \Phi(y(t)) \;\; \forall\, t \in \mathbb{R}_+.$$

Moreover, writing $K := [-c_{\mathcal{P}}/\eta\,,\, c_{\mathcal{P}}/\eta]$, we have

$$\Phi(y) \subset \tilde{\Phi}(y) \quad \forall\, y \in \mathbb{R}\backslash K \;\; \text{and} \;\; E := \sup_{y \in K} \sup_{v \in \Phi(y)} \inf_{\tilde{v} \in \tilde{\Phi}(y)} |v - \tilde{v}| = c_{\mathcal{P}}.$$

Let the linear system (A, B, C) (with transfer function G) be stabilizable and detectable. Write $a := a_{\mathcal{P}} - \eta$, $b := b_{\mathcal{P}} + \eta$ and assume that $G/(1 + aG) \in H^\infty$ and, for some $\delta > 0$, $(1 + bG)/(1 + aG) - \delta$ is positive real. Then hypothesis (H1) holds with $m = 1$ and $\tilde{\Phi}$ replacing Φ. We are now in a position to invoke Corollary 7 to conclude properties of solutions of the single-input, single-output, functional differential equation

$$\dot{x}(t) = Ax(t) + B\big[d(t) - (\mathcal{P}_\xi(Cx))(t)\big], \quad x(0) = x^0. \tag{19}$$

We reiterate that, for each $x^0 \in \mathbb{R}^n$ and $d \in L^\infty_{\text{loc}}(\mathbb{R}_+)$, (19) has unique global solution. An application of Corollary 7 (with $D(t) = \{d(t)\}$ for all $t \in \mathbb{R}_+$) yields the existence of constants $\varepsilon, c_1, c_2 > 0$ such that, for every global solution x,

$$\|x(t)\| \le c_1 e^{-\varepsilon t}\|x^0\| + c_2 \left(\|d\|_{L^\infty[0,t]} + c_{\mathcal{P}}\right) \quad \forall\, t \in \mathbb{R}_+, \tag{20}$$

showing in particular that (19) is input-to-state stable with bias $c_{\mathcal{P}}$. Furthermore, by Lemma 4,

$$\lim_{t\to\infty} d(t) = 0 \quad \Longrightarrow \quad \limsup_{t\to\infty} \|x(t)\| \le c_2 c_{\mathcal{P}}\,. \tag{21}$$

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