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Abstract— **The robustness of quantized continuous-time nonlinear systems with respect to the discrepancy (mismatch) between the ranges of the encoder and the decoder quantizers is investigated. A condition which guarantees asymptotic stability and which describes the interplay between quantization density and mismatch is derived.**

I. INTRODUCTION

In a quantized control system, control inputs and/or measurement outputs are quantized, i.e. they are processed by a quantizer, which is a discontinuous map from the state space to a finite set of values. As a result quantized controls or measurements are piece-wise constant signals which take value in a finite set. These signals can be transmitted over finite bandwidth communication channels, and in such a case a quantizer will be present at the coder side, and another one at the decoder side.

Early results on quantized nonlinear systems have been established in [9] for the class of systems which are input to state stable (ISS) with respect to perturbations, and considering a general class of quantizers. More specific examples of quantizers are the uniform quantizers, the logarithmic quantizers ([10]), etc. Since in many cases it is difficult to guarantee a robustness property like ISS for nonlinear systems, the author of [2] derived stability results for the class of stabilizable systems. Typically this approach results in quantized control laws which are particularly easy to implement ([3]). Both [9], [2] did not explicitly take into account the notion of solutions for quantized systems, which is a delicate issue since quantized systems are systems with a discontinuous vector field. Different kind of solutions for quantized systems were discussed in [1], where the analysis was carried out relying on stability theory for differential inclusions. The solutions studied in [1] included the hysteretic solution adopted in [5] for systems with logarithmic quantization.

All the results discussed above assume that the parameters of the quantizer at the coder and of the quantizer at the decoder are the same. The recent papers [7], [6] have posed the problem of studying the stability of quantized linear control systems when the ranges of the two quantizers are different. The two papers present slightly different points of view on the problem: While the former is interested in understanding under what conditions stability is retained despite of the mismatch, the latter redesign the coder/decoder

to cope with the mismatch. On the other hand, both the papers deal with either a sampled-data or a discrete-time model of the quantized system.

In this paper, we want to propose an approach to the problem which, although close in spirit to [7], is substantially different. In fact, we are interested to investigate quantized *continuous-time* systems in the presence of mismatch, without relying on a sampled-data model of the system, but dealing directly with the continuous-time system in the presence of a discontinuous map representing the quantizer. This is a major difference, because in this way the transmission of the information from the encoder to the decoder occurs whenever the state crosses certain thresholds, while the adoption of a sampled-data model for the system implicitly assumes that the transmission occurs at the sampling times. We adopt logarithmic quantizers with hysteresis ([5]) to bypass the problem of defining a notion of solution for a system which presents a nonlinear right-hand side with discontinuous (quantized) terms ([1]). The same class of quantizers with hysteresis has been adopted in [5] to study quantized adaptive control systems, in [3] to study robustness of quantized control systems with respect to parametric uncertainties, and in [4], where quantized control systems are designed to be robust with respect to pointwise delays. Dealing with hysteresis is simple when the quantizers are logarithmic, but other choices are possible (see e.g. [8] for the case of uniform quantizers with hysteresis). In fact, in principle, the methods we present can be applied with any kind of quantizers. Finally, unlike [7], [6], our focus is on nonlinear systems. The results for linear systems are given as a special case.

In Section II, we introduce some preliminaries, namely the class of systems under consideration, the quantizers, the notion of mismatch and the formulation of the problem. In Section III, the main results of the paper are discussed, and conclusions are drawn in Section IV.

Notation. $\mathbb{R}_>$, $\mathbb{R}_\geq$ denote respectively the set of positive and non-negative real numbers. Given a symmetric and positive definite matrix P, $\lambda_{\max}(P)$, $\lambda_{\min}(P)$ denote respectively the largest and the smallest eigenvalue of P. A class-$\mathcal{K}_\infty$ function $\alpha : \mathbb{R}_\geq \to \mathbb{R}_\geq$ is continuous, strictly increasing, zero at zero, and unbounded. With a slight abuse of terminology, we define a class $\mathcal{KL}$-function $\beta(r,s) : \mathbb{R}_\geq \times \mathbb{R}_\geq \to \mathbb{R}_\geq$ as a function such that $\beta(\cdot,s)$ is a class-$\mathcal{K}_\infty$ function for every fixed s, and $\beta(r,\cdot)$ is a decreasing function for which $\lim_{s \to +\infty} \beta(r,s) = 0$ for each fixed r. The system

$$
\begin{array}{rcll}
\dot{X}(t) & = & F(X(t)) & t \neq t_k \\
X(t^+) & = & G(X(t)) & t = t_k
\end{array}
$$

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represents an impulsive system, i.e. a system whose state undergoes the reset $X(t^+) = G(X(t))$ at the (switching) times $t_0, t_1, t_2, \ldots$, and flows continuously according to the equation $\dot{X}(t) = F(X(t))$ during the inter-switching time.

II. PRELIMINARIES

Process. We consider non-linear *continuous-time* systems of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and f, g continuously differentiable maps. There is no special reason for considering input-affine systems as (1) except that of giving the stability conditions in the simplest possible form.

We assume the origin $x = 0$ to be an unstable equilibrium point which can be stabilized by a locally Lipschitz control law $u = k(x)$. Namely there exist a continuously differentiable Lyapunov function $V(x)$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\frac{\partial V}{\partial x}[f(x) + g(x)k(x)] \leq -\alpha V(x) , \tag{2}$$

with α_1, α_2 class-$\mathcal{K}_\infty$ functions, and $\alpha > 0$ a real number. The existence of a Lyapunov function (2) is the standing assumption for the results below.

Quantizer. The quantizer we focus on is the logarithmic quantizer introduced in [10] for linear discrete-time systems and adopted for non-linear continuous-time system also in [5], [1]. Following [5], the quantizer includes a hysteretic switching mechanism to avoid the difficulties with the definition of a notion of solutions with quantized nonlinear systems ([1]) and to avoid the occurrence of chattering. Actually, one of the reasons for us to adopt a specific class of quantizers rather than general quantizers lies in the fact that the analysis of quantized systems with hysteresis is particularly simple when quantizers are logarithmic ([5], [1]), or uniform ([8]). But the same results can be given for other classes of quantizers as well.

By logarithmic quantizer we mean the multi-valued map (see Fig. 1):

$$\psi(s) =$$

$$\begin{cases} \psi_0 & \dfrac{\psi_0}{1+\delta} < s \\[2ex] \dfrac{\psi_0}{1+\delta} & \dfrac{\psi_0}{(1+\delta)^2} < s \leq \dfrac{\psi_0}{(1+\delta)(1-\delta)} , \\[2ex] \psi_i & \dfrac{\psi_i}{1+\delta} < s \leq \dfrac{\psi_i}{1-\delta} , \\ & \qquad\qquad 0 \leq i \leq j \\[2ex] \dfrac{\psi_i}{1+\delta} & \dfrac{\psi_i}{(1+\delta)^2} < s \leq \dfrac{\psi_i}{(1+\delta)(1-\delta)} , \\ & \qquad\qquad 0 \leq i \leq j \\[2ex] 0 & 0 \leq s \leq \dfrac{\psi_j}{1+\delta} \\[2ex] -\psi(-s) & s < 0 . \end{cases} \tag{3}$$

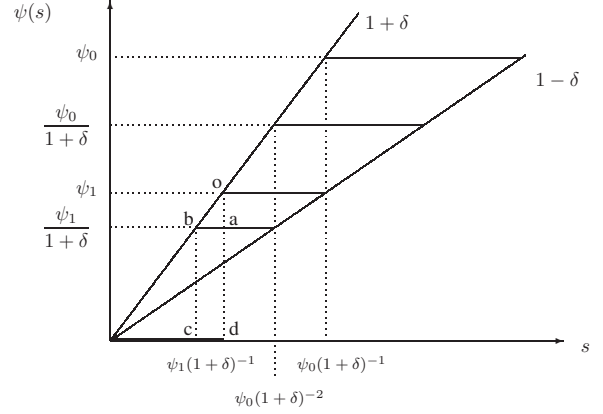Fig. 1. The multi-valued map $\psi(s)$ for $s > 0$, and with $j = 1$.

where $\psi_i = \rho^i \psi_0$, $i = 0, 1, \ldots, j$, $\rho = \frac{1-\delta}{1+\delta}$, $\delta \in (0, 1)$, j is a positive integer and ψ_0 is a positive real number. In what follows, we let ψ_0 be positive real number arbitrarily fixed and δ, j parameters to design. The number of quantization levels is equal to $4j + 1$. We refer to the set of points s such that $|s| \leq \psi_0(1 - \delta)^{-1}$ as the range of the quantizer, and we say that the quantizer undergoes overflow whenever the argument of the quantizer is outside the range of the quantizer. Moreover, the set of points such that $|s| \leq \psi_j(1 + \delta)^{-2}$ is the deadzone of the quantizer.

The law according to which $\psi(s)$ takes value as s evolves with time is described by the automaton in Fig. 2. The initial state at which the automaton lies is chosen according to the law below:

$$\psi(s(0)) =$$

$$\begin{cases} \psi_0 & \dfrac{1}{1+\delta}\psi_0 < s(0) \\[2ex] \psi_i & \dfrac{1}{1+\delta}\psi_i < s(0) \leq \dfrac{1}{1-\delta}\psi_i , \\ & \qquad\qquad 0 \leq i \leq j \\[2ex] 0 & 0 \leq s(0) \leq \dfrac{1}{1+\delta}\psi_j \\[2ex] -\psi_m(-s(0)) & s(0) < 0 . \end{cases}$$

This law is also used to determine the state of the automaton each time the argument of ψ is reset. This happens during the zooming-in phase (see Theorem 1 below). The value of $\psi(s(0))$ identifies a node of the graph. If the value of $s(0)$ fulfills one of the conditions of the edges leaving the node, then a transition is triggered and the quantizer takes the new value which is denoted by $\psi(s(0^+))$ given by the destination node. For $t > 0$, $\psi(s(t))$ remains constant until $s(t)$ triggers a transition of $\psi(s(t))$ to the new value, denoted by $\psi(s(t^+))$, again chosen according to the graph of Fig. 2.

It is straightforward to verify that, by definition, $\psi(s)$ satisfies the following inclusion (cf. [1]):
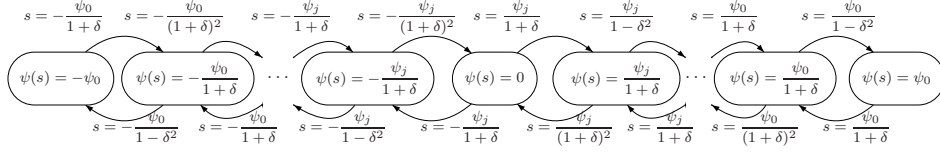
Fig. 2. The graph illustrates how the function $\psi(s)$ takes values depending on s. Each edge connects two nodes, and is labeled with the condition (*guard*) which triggers the transition from the starting node to the destination node.

$$\psi(s) \in \begin{cases} \{(1 + \lambda\delta)s, \lambda \in [-1, 1]\}, \\ \qquad \dfrac{\psi_j}{1+\delta} < |s| \leq \dfrac{\psi_0}{1-\delta} \\ \{\lambda(1 + \delta)s, \lambda \in [0, 1]\}, \\ \qquad \dfrac{\psi_j}{1+\delta} \geq |s|. \end{cases} \qquad (4)$$

Observe that in the former case $|\psi(s) - s| \leq \delta|s| \leq \delta\frac{\psi_0}{1-\delta}$, while in the latter case $|\psi(s) - s| \leq |s| \leq \frac{\psi_j}{1+\delta}$.

Quantized measurements. We assume that n sensors are available, each one measuring one and only one of the state components. Each component is quantized. The ranges of the quantizers are adjusted dynamically through the positive real number μ_c which changes over time. The vector of quantized measurements is then

$$\Psi_c(x) = \mu_c\Psi\left(\frac{x}{\mu_c}\right) := \mu_c\begin{pmatrix} \psi\left(\frac{x_1}{\mu_c}\right) \\ \vdots \\ \psi\left(\frac{x_n}{\mu_c}\right) \end{pmatrix}.$$

Encoder/decoder mismatch. The vector of quantized measurements $\Psi(\frac{x}{\mu_c})$ is received at the other end of the channel, where each decoder quantizer uses the range parameter μ_d. Hence, the decoder generates the signal

$$\Psi_d(x) = \mu_d\Psi\left(\frac{x}{\mu_c}\right).$$

Typically it is assumed that the ranges of the quantizers at the encoder and at the decoder are the same, but due to uncertainty in the parameters of the quantizers, this may not be always the case. Following [7], we consider here the case in which there is a mismatch between the ranges of the two quantizers, namely $\mu_d(t) = r^{-1}\mu_c(t)$, with $r \in (0, 1)$ an unknown parameter which measures the discrepancy between the two parameters.

III. RESULTS

The line of the arguments is the following. We first give a preparatory lemma in which we study the conditions for practical stability. Afterwards, we show how to iterate the argument to make the origin asymptotically stable. We consider the case in which both the encoder and the decoder know an upper bound on the size of the set of initial conditions of (1), so that they can choose their initial range

to avoid overflow. The case in which the bound is unknown poses no challenge and could be tackled similarly to [7], [9].

In the lemma, the ranges of the quantizers are kept constant and equal to their initial value. Hence, the closed-loop system obeys the equations (we regard μ_c and μ_d as state variables)

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))k\left(\mu_d(t)\Psi\left(\frac{x(t)}{\mu_c(t)}\right)\right) \\ \dot{\mu}_c(t) = 0 \\ \dot{\mu}_d(t) = 0. \end{cases} \qquad (5)$$

Because of the encoder/decoder mismatch, at the time $\bar{t}$, $\mu_d(\bar{t}) = r^{-1}\mu_c(\bar{t})$, and therefore $\mu_d(t) = r^{-1}\mu_c(t)$ for all $t \geq \bar{t}$ along the solutions of (5). Hence, we have

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))k\left(\frac{1}{r}\mu_c(t)\Psi\left(\frac{x(t)}{\mu_c(t)}\right)\right) \\ \dot{\mu}_c(t) = 0 \\ \dot{\mu}_d(t) = 0. \end{cases} \qquad (6)$$

We can rewrite the x-subsystem as [1] ([2])

$$\begin{aligned} \dot{x}(t) = \ & f(x(t)) + g(x(t))k(x(t)) + \\ & g(x(t))h\left(x(t), \frac{1}{r}\mu_c(t)\Psi(\frac{x(t)}{\mu_c(t)})\right) \cdot \frac{1}{r}\mu_c(t) \cdot \\ & \cdot \left(\Psi(\frac{x(t)}{\mu_c(t)}) - \frac{x(t)}{\mu_c(t)} + (1 - r)\frac{x(t)}{\mu_c(t)}\right), \end{aligned} \qquad (7)$$

where

$$h(x, w) = \int_0^1 \left[\frac{\partial k(y)}{\partial y}\right]_{y=(1-a)x+aw} da.$$

---

[1] First observe that:

$$\begin{aligned} f(x) + g(x)k\left(\mu_d\Psi\left(\frac{x}{\mu_c}\right)\right) = \ & f(x) + g(x)k(x) + g(x) \cdot \\ & \cdot \left(k\left(\mu_d\Psi\left(\frac{x}{\mu_c}\right)\right) - k(x)\right) \end{aligned}$$

Set $w = \mu_d\Psi\left(\frac{x}{\mu_c}\right)$ and $\tilde{k}(a) = k((1-a)x + aw)$, with $a \in [0, 1]$. Then $k\left(\mu_d\Psi\left(\frac{x}{\mu_c}\right)\right) - k(x) = \tilde{k}(1) - \tilde{k}(0)$ and

$$\tilde{k}(1) - \tilde{k}(0) = \int_0^1 \frac{d\tilde{k}(a)}{da}da = \int_0^1 \left[\frac{\partial k(y)}{\partial y}\right]_{y=(1-a)x+aw} (w - x)da.$$

Hence

$$k\left(\mu_d\Psi\left(\frac{x}{\mu_c}\right)\right) - k(x) = h\left(x, \mu_d\Psi\left(\frac{x}{\mu_c}\right)\right) \cdot \left(\mu_d\Psi\left(\frac{x}{\mu_c}\right) - x\right),$$

and also

$$\begin{aligned} f(x) + g(x)k\left(\mu_d\Psi\left(\frac{x}{\mu_c}\right)\right) = \ & f(x) + g(x)k(x) + g(x) \cdot \\ & \cdot h\left(x, \mu_d\Psi\left(\frac{x}{\mu_c}\right)\right) \cdot \left(\mu_d\Psi\left(\frac{x}{\mu_c}\right) - x\right). \end{aligned}$$

Recalling that $\mu_d = \frac{1}{r}\mu_c$, (7) follows.

Let υ, κ_r be continuous, non-decreasing (υ also zero at zero) functions defined as

$$\upsilon(s) = \tilde{\upsilon}(\alpha_1^{-1}(s)) \ , \ \tilde{\upsilon}(s) \geq \max_{|x| \leq s} \left| \frac{\partial V}{\partial x} g(x) \right| \ ,$$

$$\kappa_r(s) = \tilde{\kappa}_r(\alpha_1^{-1}(s)) \ , \ \tilde{\kappa}_r(s) \geq \max_{\substack{|x|_\infty \leq s \\ \frac{r}{2}|w|_\infty \leq s}} |h(x, w)| \ .$$

Without loss of generality assume that $\upsilon(s) > 0$ for $s > 0$. For the system (7), the following result states conditions on the quantization density δ, the number of quantization levels j, ad the mismatch parameter r under which any trajectory which starts from the (arbitrarily large) level set $\Omega_\theta = \{x \in \mathbb{R}^n : V(x) \leq \theta\}$ converges in finite time to the (arbitrarily small) inner level set $\Omega_{\gamma^2\theta}$:

Lemma 1: Let $M > 0$, $0 < \gamma < 1$, $\psi_0 \in \mathbb{R}_>$, and $\theta \in [0, M]$. Suppose:
(i) There exist $0 < \delta, r < 1$ and an integer j such that

$$2\upsilon(\theta)\kappa_r(\theta)\alpha_1^{-1}(\theta)\Delta \leq \alpha\gamma^2\theta \qquad (8)$$

with

$$\Delta = \frac{\sqrt{n}(\rho^j + \delta) + 1 - r}{r} \text{ and } \rho = \frac{1-\delta}{1+\delta} \ .$$

(ii) The initial condition $(x(\bar{t}), \mu_c(\bar{t}), \mu_d(\bar{t}))$ satisfies

$$V(x(\bar{t})) \leq \theta \ , \ \frac{\alpha_1^{-1}(\theta)}{\mu_c(\bar{t})} = \frac{\psi_0}{1+\delta} \ , \ \mu_d(\bar{t}) = r^{-1}\mu_c(\bar{t}) \ .$$

Then any solution of the system (5) satisfies

$$V(x(t)) \leq \max\left\{ e^{-\frac{\alpha}{2}(t-\bar{t})}\theta, \gamma^2\theta \right\} \ , \quad \forall t \geq \bar{t} \ ,$$

and there exists a finite T such that $V(x(t)) \leq \gamma^2\theta$ for all $t \geq \bar{t} + T$.

Proof: As far as $x(t) \in \Omega_\theta$, the Lyapunov function $V(x(t))$ computed along the solution of the system away from the switching times (the x-system is a switched system) satisfies

$$\dot{V}(x(t)) \leq -\alpha V(x(t)) + \upsilon(\theta)\kappa_r(\theta)\frac{\mu_c(t)}{r}\left| \psi\left(\frac{x(t)}{\mu_c(t)}\right) + \right.$$
$$\left. -\frac{x(t)}{\mu_c(t)} + (1-r)\frac{x(t)}{\mu_c(t)} \right|$$

where we have exploited the fact that $\alpha_1(|x(t)|) \leq \theta$ and [2]

$$\left| h\left(x(t), \frac{1}{r}\mu_c(t)\Psi(\frac{x(t)}{\mu_c(t)}) \right) \right| \leq \kappa_r(\theta) \ .$$

If we prove that $\dot{V}(x(t)) < 0$ for almost all t such that $x(t) \in \Omega_\theta \setminus \overset{\circ}{\Omega}_{\gamma^2\theta}$, then the thesis holds. In particular, observe

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[2] Indeed observe that, by (4),

$$\left| \Psi\left(\frac{x(t)}{\mu_c(t)} \right) \right|_\infty = \psi\left(\frac{|x(t)|_\infty}{\mu_c(t)} \right) \leq (1+\delta)\frac{|x(t)|_\infty}{\mu_c(t)} < 2\frac{|x(t)|_\infty}{\mu_c(t)}$$

and therefore $\left| \frac{1}{r}\mu_c(t)\Psi(\frac{x(t)}{\mu_c(t)}) \right|_\infty < \frac{1}{r}\mu_c(t) \cdot 2\frac{|x(t)|_\infty}{\mu_c(t)}$. Hence, if $\alpha_1(|x(t)|) \leq s$, for some $s \geq 0$, then $\left| \frac{1}{r}\mu_c(t)\Psi(\frac{x(t)}{\mu_c(t)}) \right|_\infty < \frac{2}{r}\alpha_1^{-1}(s)$.

---

that $x(t) \in \Omega_\theta$ and $\frac{\alpha_1^{-1}(\theta)}{\mu_c(t)} = \frac{\psi_0}{1+\delta}$ imply $\frac{|x(t)|}{\mu_c(t)} \leq \frac{\psi_0}{1+\delta}$, i.e. each quantizer is never in overflow.

Further observe that there exists an integer $0 \leq m \leq n$, whose value depends on $\frac{x(t)}{\mu_c(t)}$, such that $n - m$ components of $\frac{x(t)}{\mu_c(t)}$ are in absolute value larger than $\frac{\psi_j}{1+\delta}$. Hence

$$\left| \Psi\left(\frac{x(t)}{\mu_c(t)} \right) - \frac{x(t)}{\mu_c(t)} \right| \leq \sqrt{n}\frac{\psi_j}{1+\delta} + \sqrt{n}\delta\frac{|x(t)|}{\mu_c(t)}$$

Then, since $\mu_c(t) = \mu_c(\bar{t}) = \frac{1+\delta}{\psi_0}\alpha_1^{-1}(\theta)$, the bound on $\dot{V}(x(t))$ writes as

$$\dot{V}(x(t)) \leq -\alpha V(x(t)) + \upsilon(\theta)\kappa_r(\theta)\frac{\mu_c(t)}{r}\left[\sqrt{n}\frac{\psi_j}{1+\delta} + \right.$$
$$\left. + (\sqrt{n}\delta + 1 - r)\frac{|x(t)|}{\mu_c(t)} \right]$$
$$= -\alpha V(x(t)) + \upsilon(\theta)\kappa_r(\theta)\frac{1}{r}\sqrt{n}\rho^j\alpha_1^{-1}(\theta) + \qquad (9)$$
$$+ \upsilon(\theta)\kappa_r(\theta)\frac{\sqrt{n}\delta + 1 - r}{r}|x(t)|$$
$$\leq -\alpha V(x(t)) + \upsilon(\theta)\kappa_r(\theta)\alpha_1^{-1}(\theta)\Delta \ .$$

Because of the condition in (8), observe that $-\frac{1}{2}\alpha V(x(t)) + \upsilon(\theta)\kappa_r(\theta)\alpha_1^{-1}(\theta)\Delta \leq 0$ and $\dot{V}(x(t)) \leq -\frac{1}{2}\alpha V(x(t))$. As a consequence the thesis holds. In particular, since $\dot{V}(x(t)) \leq -\frac{\alpha V(x(t))}{2} \leq -\frac{\alpha}{2}\gamma^2\theta$, if $T \geq \frac{(1-\gamma^2)}{\frac{\alpha}{2}\gamma^2}$, then $V(x(t)) \leq \max\left\{ e^{-\frac{\alpha}{2}(t-\bar{t})}\theta, \gamma^2\theta \right\}$ for $t \in [\bar{t}, \bar{t} + T]$. ∎

Remark. The right-hand side of (8) models the perturbation due to the presence of the quantization and the encoder/decoder mismatch in the Lyapunov inequality (9). Condition (i) makes sure that such a perturbation does not destroy the stability property of the (unperturbed) system guaranteed by (2). Condition (ii), on the other hand, makes sure that the initial state of the system (1) lies within the quantization range of the encoder (and the decoder).

We illustrate the lemma above by a simple example:
Example. Consider the nonlinear system

$$\dot{x} = -x + x^2 + u \qquad (10)$$

and take as stabilizing feedback the law $u = k(x) = -x^2$. Hence, condition (2) is satisfied with $V(x) = x^2/2$, $\alpha_1(|x|) = \alpha_2(|x|) = |x|^2/2$, $a = 2$. It is easy to check that $h(x, w) = -(x + w)$, from which we derive that $\tilde{k}_r(s) = (1 + \frac{2}{r})s$, and $k_r(s) = (1 + \frac{2}{r})\alpha_1^{-1}(s) = (1 + \frac{2}{r})\sqrt{2s}$. Moreover, since $\frac{\partial V}{\partial x}g(x) = x$, we have $\tilde{\upsilon}(s) = s$ and therefore $\upsilon(s) = \alpha_1^{-1}(s) = \sqrt{2s}$. Condition (8) in this case rewrites as

$$2(1 + \frac{2}{r})\frac{\rho^j + \delta + 1 - r}{r}\sqrt{2\theta} \leq \gamma^2 \qquad (11)$$

Thus, Lemma 1 applies provided that the parameters r, j, δ satisfy the inequality above. This is indeed possible no matter what the values of $\theta > 0$ and $0 < \gamma < 1$ are. ◁

To obtain asymptotic stability the ranges of the encoder and the decoder quantizer must be updated. The rationale ([9]) is that, since after a finite time the state is closer to

the origin of the state space (see previous lemma), then the range of the quantizers can be decreased. Since the number of quantization levels is the same, reducing the range of the quantizers implies the quantization errors to be smaller, and this in turn yields that the state will approach even further the origin. The update of the ranges of the encoder and the decoder is done through the nonlinear map

$$\Omega_{in}(\mu) = \frac{1+\delta}{\psi_0}\alpha_1^{-1}\left(\gamma^2\alpha_1\left(\frac{\psi_0}{1+\delta}\mu\right)\right) . \qquad (12)$$

The closed-loop system then takes the form

$$
\begin{aligned}
\dot{x}(t) &= f(x(t)) + g(x(t))\kappa_r\left(\mu_d(t)\psi\left(\frac{x(t)}{\mu_c(t)}\right)\right) \\
\dot{\mu}_c(t) &= 0 \\
\dot{\mu}_d(t) &= 0 & t \neq t_k \quad (13) \\
x(t^+) &= x(t) \\
\mu_c(t^+) &= \Omega_{in}(\mu_c(t)) \\
\mu_d(t^+) &= \Omega_{in}(\mu_d(t)) & t = t_k
\end{aligned}
$$

Using the lemma above repeatedly, it is not hard to show asymptotic stability of the system under mismatch:

Theorem 1: Let $\psi_0 \in \mathbb{R}_+$, $\gamma \in (0,1)$ and $R > 0$. Suppose:
(i) There exist $0 < \delta, r < 1$ and a positive integer j such that for all $\theta \in [0, M]$, with $M = \alpha_2(R)$, (8) holds.
(ii) The initial conditions $(x(t_0), \mu_c(t_0), \mu_d(t_0))$ satisfy

$$|x(t_0)| \leq R, \quad \mu_c(t_0) = \frac{1+\delta}{\psi_0}\alpha_1^{-1}(\gamma^2 M) , \qquad (14)$$
$$\mu_d(t_0) = r^{-1}\mu_c(t_0) .$$

Define the sequence of times $t_k = t_0 + kT$, with $T \geq \frac{2}{\alpha}(\frac{1}{\gamma^2} - 1)$.
Then there exists a class-$\mathcal{KL}$ function β such that any solution of (13) satisfies $|X(t)| \leq \beta(R, t-t_0)$ for all $t \geq t_0$, with $X = (x^T \, \mu_c \, \mu_d)^T$.

Proof: As initial step, apply the previous lemma for $t \in [t_0, t_1]$, with $\theta = M$. This is possible because $|x(t_0)| \leq R$ implies $V(x(t_0)) \leq M$, and $\mu_c(t_0) = \frac{1+\delta}{\psi_0}\alpha_1^{-1}(\gamma^{-2}M)$ and the definition of $\Omega_{in}(\mu_c)$ imply $\frac{\alpha_1^{-1}(M)}{\mu_c(t_0^+)} = \frac{\psi_0}{1+\delta}$. As a result, $V(x(t)) \leq \max\left\{e^{-\frac{\alpha}{2}(t-t_0)}M, \gamma^2 M\right\}$ for $t \in [t_0, t_1]$ and $V(x(t_1)) \leq \gamma^2 M$. Now suppose that for some $k \geq 0$

$$V(x(t)) \leq \max\left\{e^{-\frac{\alpha}{2}(t-t_k)}\gamma^{2k}M, \gamma^{2(k+1)}M\right\}$$
$$\text{for } t \in [t_k, t_{k+1}] ,$$
$$V(x(t_{k+1})) \leq \gamma^{2(k+1)}M ,$$
$$\frac{\alpha_1^{-1}(\gamma^{2k}M)}{\mu_c(t_k^+)} \leq \frac{\psi_0}{1+\delta} .$$

We want to prove that the same inequalities hold with k replaced by $k+1$. We resort once again to the lemma above, this time applied with $\bar{t} = t_{k+1}$, $\theta = \gamma^{2(k+1)}M$. The first two properties immediately follow. Regarding the last one, namely

$$\frac{\alpha_1^{-1}(\gamma^{2(k+1)}M)}{\mu_c(t_{k+1}^+)} \leq \frac{\psi_0}{1+\delta} ,$$

recall that $\mu_c(t_{k+1}^+) = \Omega_{in}(\mu_c(t_k^+))$, and therefore the inequality above is true if and only if

$$\frac{\alpha_1^{-1}(\gamma^{2(k+1)}M)}{\frac{1+\delta}{\psi_0}\alpha_1^{-1}\left(\gamma^2\alpha_1\left(\frac{\psi_0}{1+\delta}\mu_c(t_k^+)\right)\right)} \leq \frac{\psi_0}{1+\delta} .$$

This is in turn equivalent to

$$\frac{\alpha_1^{-1}(\gamma^{2k}M)}{\mu_c(t_k^+)} = \frac{\psi_0}{1+\delta}$$

which actually holds true by hypothesis. By induction we conclude that for each $k \geq 0$

$$V(x(t)) \leq \max\left\{e^{-\frac{\alpha}{2}(t-t_k)}\gamma^{2k}M, \gamma^{2(k+1)}M\right\}$$

for $t \in [t_k, t_{k+1}]$ or, what is the same [3]

$$V(x(t)) \leq e^{-\tilde{\alpha}(t-t_0)}M ,$$

with $\tilde{\alpha} = \min\left\{\frac{\alpha}{2}, \frac{2|\ln\gamma|}{T}\right\}$. Hence,

$$|x(t)| \leq \alpha_1^{-1}\left(e^{-\tilde{\alpha}(t-t_0)}M\right) , \quad M = \alpha_2(R) .$$

On the other hand, the following holds:

$$
\begin{aligned}
\mu_c(t) &= \mu_c(t_k^+) = \frac{1+\delta}{\psi_0}\alpha_1^{-1}\left(\gamma^{2k}M\right) \\
&\leq \frac{1+\delta}{\psi_0}\alpha_1^{-1}\left(\gamma^{2k}e^{-\frac{2|\ln\gamma|}{T}(t-t_0)}M\right)
\end{aligned}
$$

It is easy now to find a class-$\mathcal{KL}$ function β such that $|X(t)| \leq \beta(R, t-t_0)$. In fact,

$$
\begin{aligned}
&\left|\left(\, x(t)^T \quad \mu_c(t) \quad \mu_d(t) \,\right)^T\right| \\
&\leq |x(t)| + \mu_c(t) + \mu_d(t) = |x(t)| + \left(1 + \tfrac{1}{r}\right)\mu_c(t) \\
&\leq \alpha_1^{-1}\left(e^{-\tilde{\alpha}(t-t_0)}M\right) + \\
&\quad + \left(1 + \tfrac{1}{r}\right)\frac{1+\delta}{\psi_0}\alpha_1^{-1}\left(\gamma^{2k}e^{-\frac{2|\ln\gamma|}{T}(t-t_0)}M\right) \\
&\leq \max\left\{1, \left(1 + \tfrac{1}{r}\right)\frac{1+\delta}{\psi_0}\right\}\left[\alpha_1^{-1}\left(e^{-\tilde{\alpha}(t-t_0)}M\right) + \right. \\
&\quad \left. +\alpha_1^{-1}\left(\gamma^{2k}e^{-\frac{2|\ln\gamma|}{T}(t-t_0)}M\right)\right] \\
&\leq 2\max\left\{1, \left(1 + \tfrac{1}{r}\right)\frac{1+\delta}{\psi_0}\right\}\alpha_1^{-1}\left(e^{-\tilde{\alpha}(t-t_0)}M + \right. \\
&\quad \left. +\gamma^{2k}e^{-\frac{2|\ln\gamma|}{T}(t-t_0)}M\right) \\
&\leq 2\max\left\{1, \left(1 + \tfrac{1}{r}\right)\frac{1+\delta}{\psi_0}\right\}\alpha_1^{-1}\left(2e^{-\tilde{\alpha}(t-t_0)}\alpha_2(R)\right) \\
&= \beta(R, t-t_0) .
\end{aligned}
$$

∎

Example. (Cont'd) Consider again the system (10). In this case, $M = R^2/2$ and and we need to verify that (11) holds

---

[3] The following relations are easily derived:

$$(i) \max\left\{e^{-\frac{\alpha}{2}(t-t_k)}\gamma^{2k}M, \gamma^{2(k+1)}M\right\} = $$
$$\max\left\{e^{-\frac{\alpha}{2}(t-t_k)}, \gamma^2\right\}\gamma^{2k}M$$
$$(ii) \gamma^2 = e^{2\ln\gamma} = e^{-2|\ln\gamma|} \leq e^{-2\frac{|\ln\gamma|}{T}(t-t_k)} \Rightarrow$$
$$\max\left\{e^{-\frac{\alpha}{2}(t-t_k)}, \gamma^2\right\} \leq e^{-\min\left\{\frac{\alpha}{4}, 2\frac{|\ln\gamma|}{T}\right\}(t-t_k)}$$
$$(iii) \gamma^{2k} = e^{-\frac{2|\ln\gamma|}{T}(t_k-t_0)} .$$

for all $\theta \in [0, R^2/2]$. This is guaranteed if (11) holds for $\theta = R^2/2$, i.e. if

$$2(1 + \frac{2}{r})\frac{\rho^j + \delta + 1 - r}{r}R \le \gamma^2 \ .$$

Then the theorem applies provided that the map which updates the ranges of the quantizers is chosen as in (12), which now becomes $\Omega_{in}(\mu) = \gamma\mu$. From the proof of the theorem, we also observe that the function β which describes the convergence of the state $X = (x^T \ \mu_c \ \mu_d)^T$ to the origin takes the form

$$\beta(R, t - t_0) = 2\sqrt{2}R \max\left\{1, \left(1 + \frac{1}{r}\right)\frac{1 + \delta}{\psi_0}\right\} e^{-\frac{\tilde{\alpha}}{2}(t - t_0)},$$

with $\widetilde{\alpha} = \min\left\{1, \frac{2\gamma^2|\ln\gamma|}{1 - \gamma^2}\right\}$.

Remark. The condition under which the theorem holds is (8). It captures through Δ the interplay of the design parameters δ, j, r to guarantee stability of the closed-loop system. Loosely speaking, a larger mismatch ($r \to 0$) can be counteracted by a denser quantization $\delta \to 0$, and a coarser quantization ($\delta \to 1$) is tolerant to a smaller mismatch. Conditions which guarantee robustness with respect to the encoder/decoder mismatch can be given forms different from (8). Other conditions, for instance, can be derived from the results of [9] and [2] taking into account the mismatch parameter as in the results above. $\triangleleft$

There is a special class of systems for which the condition takes a particularly simple form, and this is the class of linear systems

$$\dot{x}(t) = Ax(t) + Bu(t) \ , \tag{15}$$

with (A, B) a stabilizable pair, considered in [7], [6]. The standing assumption is the existence of a symmetric positive definite matrix P, and a matrix K such that $(A + BK)^T P + P(A + BK) = -I$. Then the Lyapunov function $V(x) = x^T P x$ satisfies (2), with $k(x) = Kx$. Theorem 1 for linear systems can be stated as follows:

Corollary 1: Let $\psi_0 \in \mathbb{R}_+$, $\gamma \in (0, 1)$, $R > 0$. Suppose:
(i) There exist $0 < \delta, r < 1$ and a positive integer j such that for all $\theta \in [0, M]$, with $M = \lambda_{\max}(P)R^2$,

$$\frac{\sqrt{n}(\rho^j + \delta) + 1}{r} \le \gamma^2 \frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}\frac{1}{2||PB||\,||K||} + 1 \ . \tag{16}$$

(ii) The initial conditions $(x(t_0), \mu_c(t_0), \mu_d(t_0))$ satisfy

$$|x(t_0)| \le R, \quad \mu_c(t_0) = \gamma\frac{1 + \delta}{\psi_0}\sqrt{\frac{M}{\lambda_{\min}(P)}} \ ,$$
$$\mu_d(t_0) = r^{-1}\mu_c(t_0) \ .$$

Let $t_k = t_0 + kT$, with $T = 2\lambda_{\max}(P)(\frac{1}{\gamma^2} - 1)$. Then any solution of

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + BK\mu_d(t)\Psi\left(\frac{x(t)}{\mu_c(t)}\right) \\
\dot{\mu}_c(t) &= 0 \\
\dot{\mu}_d(t) &= 0 & t \ne t_k \\[6pt]
x(t^+) &= x(t) \\
\mu_c(t^+) &= \gamma\mu_c(t) \\
\mu_d(t^+) &= \gamma\mu_d(t) & t = t_k
\end{aligned}
$$

satisfies $|X(t)| \le \tilde{\delta}e^{-\frac{\tilde{\alpha}}{2}(t - t_0)}R$, for all $t \ge t_0$, with $\tilde{\delta} = 2\max\{1, \left(1 + \frac{1}{r}\right)\frac{1 + \delta}{\psi_0}\}\sqrt{\frac{2\lambda_{\max}(P)}{\lambda_{\min}(P)}}$.

Proof: The statement is obtained by considering the form taken by the functions α_i, $i = 1, 2$, υ, κ_r in the case of linear systems. Clearly, $\alpha = \frac{1}{\lambda_{\max}(P)}$, $\upsilon(\theta) = 2||PB||\alpha_1^{-1}(\theta) = 2||PB||\sqrt{\frac{\theta}{\lambda_{\min}(P)}}$, $\kappa_r(\theta) = ||K||$. Therefore condition (8) becomes

$$2||PB||\frac{\theta}{\lambda_{\min}(P)}||K||\Delta \le \alpha\gamma^2\theta \ ,$$

that is (16). The exponential bound on $|X(t)|$ derives from the expression of the function β given in the proof of the previous theorem, again specialized to the case in which α_1 and α_2 are quadratic functions. Details are straightforward and therefore omitted. ∎

IV. Conclusion

We derived a simple condition which guarantees the stability of quantized continuous-time nonlinear systems in the presence of encoder/decoder mismatch and which describes the interplay between quantization density and mismatch. We consider quantizers with hysteresis and the resulting system is a switched system.

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