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Han, L.; Camlibel, M.K.; Pang, J.-S.; Heemels, W.P.M.H.

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Convergence of Discrete-time Approximations of Constrained Linear-Quadratic Optimal Control Problems

L. Han M.K. Camlibel J.-S. Pang W.P.M.H. Heemels

Abstract— **Continuous-time linear constrained optimal control problems are in practice often solved using discretization techniques, e.g. in model predictive control (MPC). This requires the discretization of the (linear time-invariant) dynamics and the cost functional leading to discrete-time optimization problems. Although the question of convergence of the sequence of optimal controls, obtained by solving the discretized problems, to the true optimal continuous-time control signal when the discretization parameter (the sampling interval) approaches zero has been addressed in the literature, we provide some new results under less restrictive assumptions for a class of constrained continuous-time linear quadratic (LQ) problems with mixed state-control constraints by exploiting results from mathematical programming extensively. As a byproduct of our analysis, a regularity result regarding the costate trajectory is also presented.**

I. INTRODUCTION

Optimal control is a classical branch of applied mathematics with more than one hundred years of history and occupies a central place in many engineering applications. Among the optimal control problems, the unconstrained linear-quadratic (LQ) problem with a pure quadratic objective function and linear continuous-time dynamics is certainly the simplest and its literature is vast. Yet, this is not the case when there are hard algebraic constraints coupling states and controls, in fact, even the case with only input constraints is not extensively treated, although some recent studies of the latter problem exist, see [2], [10], [11], [18].

When it comes to practical computations by numerical methods, the constrained optimal control problem is much more challenging than the unconstrained case. Continuous-time LQ problems in practice are often solved using some sort of discretization. In particular, this requires discretization of the (linear time-invariant) dynamics and the cost functional leading to discrete-time optimization problems, which are closely connected to the optimization problems as used in MPC [19]. The question of whether refining the

discretization can lead to a better and better approximation and eventually converge in a certain sense to a solution of the original problem, or the consistency of the approximation, has also been considered in the literature. Among the existing research, some have studied direct approximations of the primal problem, see [3]–[7], [18], [22], while others focus on the approximations of the dual problem, see [13], [15], [21]. In [27], a primal-dual representation for approximation in optimal control is discussed. The convergence rate is studied in [8], [9]. The method presented in this paper can be regarded as a primal method. However, different to existing work on primal methods, we also investigate the convergence to the costate trajectory by considering the discretization of the costate trajectory as the multiplier of the constraint corresponding to the discretization of the differential equation. By doing this and exploiting extensive the results from mathematical programming, especially convex quadratic programming, we are able to avoid assumptions such as boundedness of the constraint set, convergence of the objective value, and so on. In addition, our approach is implementable (in terms of relaxing the constraints to ensure the feasibility of the discretizations) under less restrictive conditions. Moreover, this approach leads to a new regularity result regarding the Lipschitz Continuity of the costate trajectory for the class of LQ optimal control problems with mixed control and state constraint, see [14], [25] . A different convergence problem in MPC that did receive considerable attention is the relation between finite horizon control problems and the corresponding infinite horizon problems, see e.g. [12], [23].

II. THE LQ CONTROL PROBLEM

The main topic of this paper is the following continuous-time, finite-horizon, linear-quadratic (LQ) optimal control problem with mixed state and control constraints:

Problem 2.1: Find an absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ and an integrable function $u : [0, T] \to \mathbb{R}^m$, where $T > 0$ is a given time horizon, to

$$\underset{x,u}{\text{minimize }} V(x, u) \equiv \tfrac{1}{2} x(T)^T S x(T)$$
$$+ \int_0^T \left[\tfrac{1}{2} x(t)^T P x(t) + x(t)^T Q u(t) + \tfrac{1}{2} u(t)^T R u(t) \right] dt$$

subject to $x(0) = \xi$ and for almost all $t \in [0, T]$:

$$\dot{x}(t) = A x(t) + B u(t) \text{ and } C x(t) + D u(t) + f \geq 0, \tag{1}$$

where S, P, Q, R, A, B, C, D are constant matrices and f is a constant vector of appropriate dimensions.

Let $U(x) \triangleq \{ u \mid Cx + Du + f \geq 0 \}$ denote the (possibly unbounded) polyhedron of admissible controls given the state

x. We say that a pair of trajectories (x, u) is feasible to (1) if x is absolutely continuous and u is integrable and (x, u) satisfies the constraints as stated in (1). Part of the goal of the paper is to provide a constructive proof for the existence of such an optimal solution, under a set of assumptions to be stated next.

A. Model assumptions

We first introduce some notation used throughout the paper. We let $\| \bullet \|$ be the 2-norm of vectors and matrices and write $A_{J\bullet}$ for the submatrix consisting of the rows of A indexed by the set J and A_{JJ} for the principal submatrix of A indexed by J. Given a set Z and a vector z, the distance from z to Z is denoted by $\text{dist}(z, Z) \triangleq \min\{\|z - \bar{z}\| \mid \bar{z} \in Z\}$. Finally, we let $z^- \triangleq \max(0, -z)$ denote the non-positive part of the vector z.

We will use the following technical assumptions to analyze the numerical method for solving (1):

(A) the matrices S and $\Xi \triangleq \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}$ are symmetric positive semidefinite and R is positive definite;

(B) a continuously differentiable function $\widehat{x}_{\text{fs}}$ with $\widehat{x}_{\text{fs}}(0) = \xi$ and a continuous function $\widehat{u}_{\text{fs}}$ exist such that for all $t \in [0, T]$: $d\widehat{x}_{\text{fs}}(t)/dt = A\widehat{x}_{\text{fs}}(t) + B\widehat{u}_{\text{fs}}(t)$ and $\widehat{u}_{\text{fs}}(t) \in U(\widehat{x}_{\text{fs}}(t))$;

(C) $[D^T\mu = 0, \mu \geq 0]$ implies $(CA^iB)^T\mu = 0$ for all nonnegative integers i (a dual condition).

Condition (B) is clearly needed as it states feasibility of (1). In the case of pure control constraints ($C = 0$) when an admissible control exists it is obviously satisfied. In the existing literature of numerical methods, it is often assumed that the optimal control problem possesses an *optimal* solution with certain nice smoothness properties, see e.g. [8], [9], while we only assume in (B), the existence of a *feasible* solution with some desirable smoothness property rather than a "nice optimal solution". Condition (C) is trivially satisfied for pure control constraints. A condition that implies (C) is the existence of a constant $\delta > 0$ such that $\|C^T\mu\| \leq \delta\|D^T\mu\|$ for all $\mu \in \mathbb{R}^m_+$. It should be noted that condition (C) rules out the case where $D = 0$, i.e., in the pure state constrained problem. This case of pure constraints is even more involved than the control and mixed control/state constraints and is a topic for future research.

Under the set of assumptions (A–C), the main contribution of the paper is threefold: (a) to provide a numerical scheme for linear quadratic optimal control problem with convex (not necessarily strictly convex) cost integrand and mixed polyhedral (possibly unbounded) state and control constraint with provable convergence; (b) to show the existence of a Lipschitz continuous costate trajectory; and (c) to provide a relaxation method which can guarantee the feasibility of the discretizations under less restrictive assumptions.

Before introducing the discretized MPC problems related to (1), we first derive some properties of the optimal control problem in (1).

III. OPTIMALITY IN TERMS OF VARIATIONAL INEQUALITIES

We briefly review some fundamental results for finite-dimensional convex quadratic programs after which we present variational conditions that the optimal control functions for (1) satisfy. These conditions are directly related to Pontryagin's maximum principle.

A. Convex quadratic programs: A review

Given a polyhedral set $Z \subseteq \mathbb{R}^m$, the affine variational inequality (AVI) defined by a vector $e \in \mathbb{R}^m$ and a matrix $M \in \mathbb{R}^{m \times m}$, denoted by $\text{AVI}(Z, e, M)$, is to find a vector $z \in Z$ so that

$$(z' - z)^T(e + Mz) \geq 0, \quad \forall z' \in Z.$$

The set of solutions of the $\text{AVI}(Z, e, M)$ is denoted by $\text{SOL}(Z, e, M)$. If Z has the linear inequality representation: $Z \triangleq \{z \in \mathbb{R}^m \mid Ez \geq b\}$ for some matrix $E \in \mathbb{R}^{\ell \times m}$ and vector $b \in \mathbb{R}^\ell$, then a vector $z \in \text{SOL}(Z, e, M)$ if and only if there exists a multiplier vector $\mu \in \mathbb{R}^\ell$ such that the following Karush-Kuhn-Tucker (KKT) conditions hold:

$$\begin{aligned} 0 &= e + Mz - E^T\mu \\ 0 &\leq \mu \perp Ez - b \geq 0, \end{aligned} \quad (2)$$

where $v \perp w$ means that the two vectors v and w are perpendicular, i.e., $v^Tw = 0$.

In the definition of the AVI, the matrix M is not required to be symmetric. When M is symmetric positive semidefinite, the AVI is equivalent to the convex quadratic program, which we denote $\text{QP}(Z, e, M)$:

$$\underset{z \in Z}{\text{minimize}} \; e^T z + \tfrac{1}{2} z^T M z.$$

Just like the AVI formulation of a convex QP, the LQ optimal control problem (1) admits an equivalent differential affine variational inequality (DAVI) formulation derived from the Pontryagin Principle that starts with the Hamiltonian function

$$H(x, u, \lambda) \triangleq \tfrac{1}{2} x^T P x + \tfrac{1}{2} u^T R u + \lambda^T (Ax + Bu),$$

where λ is the costate (also called adjoint) variable of the ODE $\dot{x}(t) = Ax(t) + Bu(t)$, and the Lagrangian function:

$$L(x, u, \lambda, \mu) \triangleq H(x, u, \lambda) - \mu^T (Cx + Du + f),$$

where μ is the Lagrange multiplier of the algebraic constraint $Cx + Du + f \geq 0$. By the Pontryagin Principle [26, Section 6.2] and [17], [24], it follows that a necessary condition for the pair (x, u) to be an optimal solution of (1) is the existence of λ and μ such that the boundary conditions and the following differential-algebraic conditions hold for almost all $t \in (0, T)$:

$$\begin{pmatrix} \dot{\lambda}(t) \\ \dot{x}(t) \end{pmatrix} = \begin{bmatrix} -A^T & -P \\ 0 & A \end{bmatrix} \begin{pmatrix} \lambda(t) \\ x(t) \end{pmatrix} + \begin{bmatrix} -Q \\ B \end{bmatrix} u(t) + \\ + \begin{bmatrix} C^T \\ 0 \end{bmatrix} \mu(t) \quad (3a)$$

$$0 = q(t) + Q^T x(t) + Ru(t) + B^T\lambda(t) - D^T\mu(t) \quad (3b)$$

$$0 \leq \mu(t) \perp Cx(t) + Du(t) + f \geq 0 \quad (3c)$$

$$x(0) = \xi \quad \text{and} \quad \lambda(T) = Sx(T). \quad (3d)$$

Note that $u(t) \in \operatorname*{argmin}_{u \in U(x(t))} H(x(t), u, \lambda(t))$ is equivalent to (3b)-(3c). The conditions (3) are clearly a dynamical variant of the AVI introduced earlier, thereby explaining the term DAVI. It is known that the set of necessary conditions (3) is also sufficient for optimality. The sufficiency is due to the convexity of the objective function in (x, u) and the linearity of the dynamics and the algebraic constraint. Among the sources for a proof of sufficiency, we mention two. One is [1, Theorem 7.2.1] that pertains to an abstract control constrained Mayer problem under a convexity assumption, and the other is [24, Theorem 3.1] specifically for the mixed inequality constrained case that is directly applicable to the LQ problem (1). In the special case of nonnegative control constraints, a proof is also given in [18] using the Hamilton-Jacobi-Bellman equation and establishing a connection between the costate and the gradient of the value function.

To make the statement of the necessary conditions more formal, we introduce the following.

Definition 3.1: The tuple (x, u, λ, μ) is a weak solution of (3) if (i) (x, λ) is absolutely continuous and (u, μ) is integrable on $[0, T]$, (ii) the differential equation and the two algebraic conditions hold for almost all $t \in (0, T)$, and (iii) the initial and boundary conditions are satisfied.

While we have used the Pontryagin Principle to motivate this DAVI (3), the proof of Theorem 3.1 below does not make use of this principle. The proof is omitted for space reasons, but can be found in [16].

Theorem 3.1: Under conditions (A–C), the following statements hold.

(I) **[Solvability of the DAVI]** The DAVI (3) has a weak solution $(x^*, \lambda^*, u^*, \mu^*)$ with both x^* and λ^* being Lipschitz continuous on $[0, T]$.

(II) **[Sufficiency of Pontryagin]** If $(x^*, \lambda^*, u^*, \mu^*)$ is any weak solution of (3), then the pair (x^*, u^*) is an optimal solution of the problem (1).

(III) **[Necessity of Pontryagin]** Let $(x^*, \lambda^*, u^*, \mu^*)$ be the tuple obtained from part (I). A feasible tuple $(\widetilde{x}, \widetilde{u})$ of (1) is optimal if and only if $(\widetilde{x}, \lambda^*, \widetilde{u}, \mu^*)$ is a weak solution of (3).

(IV) **[Uniqueness]** Any two optimal solutions $(\widehat{x}, \widehat{u})$ and $(\widetilde{x}, \widetilde{u})$ of (1) satisfy $\widehat{x} = \widetilde{x}$ everywhere on $[0, T]$ and $\widehat{u} = \widetilde{u}$ almost everywhere on $[0, T]$. In this case (1) has a unique optimal solution $(\widehat{x}, \widehat{u})$ such that $\widehat{x}$ is continuously differentiable and $\widehat{u}$ is Lipschitz continuous on $[0, T]$, and for any optimal $\widehat{\lambda}$, $\widehat{u}(t) \in \operatorname*{argmin}_{u \in U(\widehat{x}(t))} H(\widehat{x}(t), u, \widehat{\lambda}(t))$ for all $t \in [0, T]$.

IV. DISCRETE-TIME APPROXIMATIONS

In this section we present a numerical discretization of (1). A general time-stepping method for solving the LQ problem (1) is proposed next. Let $h > 0$ be an arbitrary step size such that $N_h \triangleq \frac{T}{h} - 1$ is a positive integer (the latter integrality condition on h will not be mentioned from here on). We

partition the interval $[0, T]$ into $N_h + 1$ subintervals each of equal length h:

$$0 \triangleq t_{h,0} < t_{h,1} < t_{h,2} < \cdots < t_{h,N_h} < t_{h,N_h+1} \triangleq T.$$

Thus $t_{h,i+1} = t_{h,i} + h$ for all $i = 0, 1, \cdots, N_h$.

Selecting piecewise constant controls given by

$$u(t) = u^{h,i+1}, \ t \in [ih, (i+1)h),$$

results in the state trajectory

$$x(s + ih) = e^{As}x^{h,i} + \Gamma(s)u^{h,i} \text{ when } s \in [0, h] \quad (4)$$

and $i = 0, 1, \ldots, N_h$, where $\Gamma(s) \triangleq \int_0^s e^{A\tau}B d\tau$ for all $s \in \mathbb{R}$. Although various ways exist to discretize $V(x, u)$ for which similar convergence results as below can be derived, here we use a simple integration routine based on forward Euler to integrate the costs $V(x, u)$. This leads to the following quadratic program:

$$(\text{QP}^h) : \operatorname*{minimize}_{\{x^{h,i}, u^{h,i}\}_{i=1}^{N_h}} (x^{h,N_h+1})^T \left(\frac{1}{2}Sx^{h,N_h+1}\right) +$$
$$+ \frac{h}{2}\sum_{i=0}^{N_h} \left\{ (x^{h,i})^T \left[Px^{h,i} + Qu^{h,i+1} \right] + \right.$$
$$\left. + (u^{h,i+1})^T \left[Q^T x^{h,i} + Ru^{h,i+1} \right] \right\}$$

subject to $x^{h,0} = \xi$ and for $i = 0, \cdots, N_h$

$$x^{h,i+1} = \underbrace{e^{Ah}}_{=:A(h)} x^{h,i} + \underbrace{\int_0^h e^{As}B ds}_{=:B(h)} u^{h,i+1},$$

and $u^{h,i+1} \in U(x^{h,i+1})$

Note that we defined $A(h) := e^{Ah}$ and $B(h) := \int_0^h e^{As}B ds$ for all $h \in \mathbb{R}$. Due to the mixed state-control constraint, it is not easy to guarantee the feasibility of these subproblems. This drawback necessitates a relaxation of the algebraic inequality constraint in $U(x^{h,i+1})$ that leads to a relaxed unified scheme to be presented in Section V. Based on these relaxed schemes, which are guaranteed to be feasible and yield optimal sequences of controls and states, we can calculate $\mathbf{x}^h \triangleq \left\{x^{h,i}\right\}_{i=0}^{N_h+1}$ and $\mathbf{u}^h \triangleq \left\{u^{h,i}\right\}_{i=1}^{N_h+1}$ by solving N_h finite-dimensional convex quadratic subprograms. From these discrete-time iterates, continuous-time numerical trajectories are constructed by piecewise linear and piecewise constant interpolation, respectively. Specifically, define the functions $\widehat{x}^h$ and $\widehat{u}^h$ on the interval $[0, T]$ as follows. For all $i = 0, \cdots, N_h$ and all $t \in (t_{h,i}, t_{h,i+1}]$ define

$$\begin{aligned} \widehat{x}^h(t) &\triangleq x^{h,i} + \frac{t - t_{h,i}}{h}(x^{h,i+1} - x^{h,i}), \\ \widehat{u}^h(t) &\triangleq u^{h,i+1}. \end{aligned} \quad (5)$$

The convergence of these trajectories as the step size $h \downarrow 0$ to an optimal solution of the LQ control problem (1) is a main concern in the subsequent analysis. However, first we introduce the mentioned relaxation schemes, which are guaranteed to be always feasible, while (QP^h) is in general not.

V. THE RELAXED QUADRATIC PROGRAM

There is in general no guarantee that the (QP^h) is even feasible. The culprit is the state-dependent constraint $Cx^{h,i+1} + Du^{h,i+1} + f \geq 0$. Although the original continuous-time problem (1) is assumed to be feasible, the discretized problems (QP^h) might not inherit this property as the class of control signals for the discretization scheme is essentially restricted to piecewise constant controls with step size h. Clearly, due to the positive definiteness of R feasibility implies solvability. We provide two different methods to relax (QP^h) (in particular the constraints) in order to ensure feasibility without loosing the convergence properties that we aim to provide.

A. Minimal residual method

In order to obtain a feasible QP, we consider the minimum residual of the constraints in (QP^h) and relax them accordingly. Specifically, for an initial vector ξ and a scalar $h > 0$, define the optimum objective value of the linear program (LP):

$$
\rho_h(\xi) \triangleq \underset{\rho;\, \{x^{h,i}, u^{h,i}\}_{i=1}^{N_h}}{\text{minimum}} \quad \rho
$$

$$
\text{subject to} \quad x^{h,0} = \xi, \quad \rho \geq 0
$$

$$
\text{and} \quad \text{for } i = 0, 1, \cdots, N_h :
$$

$$
\begin{cases}
x^{h,i+1} = A(h)x^{h,i} + B(h)u^{h,i+1} \\
Cx^{h,i+1} + Du^{h,i+1} + f + \rho\mathbf{1} \geq 0
\end{cases}
$$

(6)

where $\mathbf{1}$ is the vector of all ones. It is not difficult to see that the above linear program must have a finite optimal solution; thus $\rho_h(\xi)$ is well defined.

For the convergence analysis of the relaxed, unified time-stepping method, we need to establish a limiting property of the minimum residual $\rho_h(\xi)$ as $h \downarrow 0$; this is accomplished by invoking the assumption (B) introduced in Section II.

Proposition 5.1: If assumption (B) holds, then $\lim_{h \downarrow 0} \rho_h(\xi) = 0$.

See [16] for the proof.

Employing the minimum residual $\rho_h(\xi)$, the relaxed, unified time-stepping method solves the following (feasible) convex quadratic program at time $t_{h,i+1}$:

Problem 5.1: $(\widehat{\textbf{QP}}^h)$**:**

$$
\underset{\{x^{h,i}, u^{h,i}\}_{i=1}^{N_h+1}}{\text{minimize}} \quad V_h(\mathbf{x}^h, \mathbf{u}^h) \triangleq \frac{1}{2}(x^{h,N_h+1})^T S x^{h,N_h+1} +
$$

$$
\frac{h}{2} \sum_{i=0}^{N_h} \left(\begin{array}{c} x^{h,i} \\ u^{h,i+1} \end{array} \right)^T \left[\begin{array}{cc} P & Q \\ Q^T & R \end{array} \right] \left(\begin{array}{c} x^{h,i} \\ u^{h,i+1} \end{array} \right) \Big\}
$$

subject to $\quad x^{h,0} = \xi$,

and $\quad$ for $i = 0, 1, \cdots, N_h$

$$
\begin{cases}
x^{h,i+1} = A(h)x^{h,i} + B(h)u^{h,i+1} \\
f + Cx^{h,i+1} + Du^{h,i+1} + \rho_h(\xi)\mathbf{1} \geq 0
\end{cases}
$$

An alternative relaxation utilizing a Lipschitz constant: In forming $(\widehat{\text{QP}}^h)$, one needs to calculate the minimum residual $\rho_h(\xi)$ by first solving the LP (6). If one knows in advance a feasible trajectory (x, u) of the original optimal control problem (1) with the u-trajectory being Lipschitz continuous with a known Lipschitz constant, say $L > 0$, then one can bypass the LP step and consider directly the following QP:

$$
\underset{\{x^{h,i}, u^{h,i}\}_{i=1}^{N_h+1}}{\text{minimize}} \quad \frac{1}{2}(x^{h,N_h+1})^T S x^{h,N_h+1} +
$$

$$
\frac{h}{2} \sum_{i=0}^{N_h} \left(\begin{array}{c} x^{h,i} \\ u^{h,i+1} \end{array} \right)^T \left[\begin{array}{cc} P & Q \\ Q^T & R \end{array} \right] \left(\begin{array}{c} x^{h,i} \\ u^{h,i+1} \end{array} \right) \Big\}
$$

s.t. $\quad x^{h,0} = \xi$,

and $\quad$ for $i = 0, 1, \cdots, N_h$:

$$
\begin{cases}
x^{h,i+1} = A(h)x^{h,i} + B(h)u^{h,i+1} \\
f + Cx^{h,i+1} + Du^{h,i+1} + hTL\mathbf{1} \geq 0
\end{cases},
$$

where the minimum residual $\rho_h(\xi)$ is replaced by the product hTL. One can show, under the definition of the constant L, that the above QP is feasible. In the rest of the paper, we will not consider this variant of the basic scheme because the explicit knowledge of the Lipschitz constant L could restrict the application of this scheme in practice.

VI. CONVERGENCE ANALYSIS

The technical challenge of the convergence analysis lies in the derivation of the bounds which is the main topic of the following subsection. The technical details are rather long and can be found in [16]. Here we summarize the main bounds that are needed in the main proof.

A. Key bounds for solutions of $(\widehat{QP}^h)$

Proposition 6.1: Let assumptions (A)–(C) hold. Positive scalars $\bar{h}$, η, Ψ_u and L exist such that for all $h \in (0, \bar{h}]$, KKT multipliers $\left(\boldsymbol{\lambda}^h, \boldsymbol{\mu}^h\right)$ exist such that for all optimal solutions $\left(\mathbf{x}^h, \mathbf{u}^h\right)$ of the $(\widehat{\text{QP}}^h)$,

$$
\max\left(\|x^{h,i+1}\|, \|u^{h,i+1}\|, \|\lambda^{h,i}\|, h^{-1}\|\mu^{h,i+1}\| \right)
$$
$$
\leq \eta\,(1 + \Psi_u), \quad \forall i = 0, \cdots, N_h, \quad (7)
$$

and for all $i = 0, \cdots, N_h - 1$,

$$
\max\left\{ \|u^{h,i+2} - u^{h,i+1}\|, \; h^{-1}\|D^T(\mu^{h,i+2} - \mu^{h,i+1})\| \right\}
$$

$$
\leq L \left[\|x^{h,i+2} - x^{h,i+1}\| + \right.
$$
$$
\left. + \|x^{h,i+1} - x^{h,i}\| + \|\lambda^{h,i+1} - \lambda^{h,i}\| \right].
$$

(8)

B. The main convergence theorems

We consider the convergence of the numerical trajectories considering two cases: piecewise constant and piecewise linear interpolation of the control sequences.

1) Piecewise constant control signals: For this purpose, we recall the trajectories $(\widehat{x}^h, \widehat{u}^h)$ introduced in the opening paragraph of Section IV; see (5). In addition, we define the λ-trajectory similarly to the x-trajectory; namely, for $i = 0, \cdots, N_h$,

$$\widehat{\lambda}^h(t) \triangleq \lambda^{h,i} + \frac{t - t_{h,i}}{h}(\lambda^{h,i+1} - \lambda^{h,i}), \quad \forall t \in [t_{h,i}, t_{h,i+1}],$$

with $\lambda^{h, N_h+1} \triangleq c + S x^{h, N_h+1}$, and the μ-trajectory similarly to the u-trajectory; namely, for $i = 0, \cdots, N_h$,

$$\widehat{\mu}^h(t) \triangleq h^{-1}\mu^{h,i+1}, \quad \forall t \in (t_{h,i}, t_{h,i+1}],$$

Besides the convergence, an immediate consequence of the theorem below is the existence of an optimal solution to the DAVI (3), and thus to the QP (1), under assumptions (A)–(C), where the optimal state and costate variables are Lipschitz continuous.

Theorem 6.1: Let assumptions (A)–(C) hold. Let $\widehat{x}^h(t)$ and $\widehat{u}^h(t)$ be as defined by (5) and $\widehat{\lambda}^h(t)$ and $\widehat{\mu}^h(t)$ as above. The following statements hold.

(a) There exists a sequence of step sizes $\{h_\nu\} \downarrow 0$ such that the two limits exist: $\left(\widehat{x}^{h_\nu}, \widehat{\lambda}^{h_\nu}\right) \to \left(\widehat{x}, \widehat{\lambda}\right)$ uniformly on $[0, T]$ and $\left(\widehat{u}^{h_\nu}, \widehat{\mu}^{h_\nu}\right) \to (\widehat{u}, \widehat{\mu})$ weakly in $L^2(0, T)$; moreover, $\widehat{x}$ and $\widehat{\lambda}$ are Lipschitz continuous.

(b) Any limit tuple $(\widehat{x}, \widehat{u}, \widehat{\lambda}, \widehat{\mu})$ from (a) is a weak solution of (3); thus $(\widehat{x}, \widehat{u})$ is an optimal solution of (1) due to Theorem 3.1.

Proof. For the convergence of the sequences, we first show that

$$\left\{ \frac{\| x^{h,i+1} - x^{h,i} \|}{h} \right\}_{i=0}^{N_h} \text{and} \left\{ \frac{\| \lambda^{h,i+1} - \lambda^{h,i} \|}{h} \right\}_{i=0}^{N_h} \quad (9)$$

are both bounded uniformly for all $h > 0$ sufficiently small. By (7), we have for all $h > 0$ sufficiently small and all $i = 1, \cdots, N_h$,

$$
\begin{aligned}
\| \lambda^{h,i-1} - \lambda^{h,i} \| &= \big\| \big[(A(h))^T - I \big] \lambda^{h,i} + \\
&\quad h + P x^{h,i} + Q u^{h,i+1} \big] + C^T \mu^{h,i} \big\| \\
&\leq h \big[2\|A\| \eta(1 + \Psi_u) + \psi_p + \|P\| \eta(1 + \Psi_u) + \\
&\quad + \|P\| \eta(1 + \Psi_u) + \|C\| \eta(1 + \Psi_u) \big] \\
&\triangleq h L_\lambda, \quad \text{for some constant } L_\lambda > 0,
\end{aligned}
$$

which implies

$$\frac{\| \lambda^{h,i-1} - \lambda^{h,i} \|}{h} \leq L_\lambda$$

for all $i = 1, \cdots, N_h$ and all $h > 0$ sufficiently small. The same holds for $i = N_h + 1$ also. Similarly, we can establish the same bound for the x-variable: for some constant $L_x > 0$,

$$\frac{\| x^{h,i+1} - x^{h,i} \|}{h} \leq L_x,$$

for $i = 0, \cdots, N_h$ and all $h > 0$ sufficiently small. By (8), this implies the existence of a scalar $L' > 0$ such that $\max \left\{ \big\| u^{h,i+2} - u^{h,i+1} \big\|, h^{-1} \big\| D^T(\mu^{h,i+2} - \mu^{h,i+2}) \big\| \right\} \leq hL'$, for all $i = 0, \cdots, N_h - 1$ and all $h > 0$ sufficiently

small. From the above uniform bounds, we may conclude that the families of functions $\{\widehat{x}^h\}$, $\{\widehat{\lambda}^h\}$, $\{\widehat{u}^h\}$, and $\{D^T \widehat{\mu}^h\}$ for all $h > 0$ sufficiently small are equicontinuous families of functions. By the Arzela-Ascoli theorem, there is a sequence $\{h_\nu\} \downarrow 0$ such that $\{\widehat{x}^{h_\nu}\}$ and $\{\widehat{\lambda}^{h_\nu}\}$ converge in the supremum norm to Lipschitz functions $\widehat{x}$ and $\widehat{\lambda}$, respectively, on $[0, T]$. Similar to [20, Theorem 7.1], by the uniform boundedness of $(u^{h,i+1}, h^{-1}\mu^{h,i+1})$ and by looking at a proper subsequence of $\{h_\nu\}$ if necessary, we may conclude that $\{(\widehat{u}^{h_\nu}, \widehat{\mu}^{h_\nu})\}$ converges weakly to a pair of functions $(\widehat{u}, \widehat{\mu})$ in $L^2(0, T)$ with $\{\widehat{u}^{h_\nu}\}$ and $\{D^T \widehat{\mu}^{h_\nu}\}$ converging to $\widehat{u}$ and $D^T \widehat{\mu}$ uniformly. This proves (a). To show that $(\widehat{x}, \widehat{\lambda}, \widehat{u}, \widehat{\mu})$ is a weak solution to (3), we first notice that

$$\widehat{x}(0) = \xi, \quad \text{and} \quad \widehat{\lambda}(T) = \lim_{\nu \to \infty} \big[S\widehat{x}^{h_\nu}(T) \big] = S\widehat{x}(T).$$

Therefore the boundary conditions are satisfied. The rest of the proof to show that any such limit tuple $(\widehat{x}, \widehat{u}, \widehat{\lambda}, \widehat{\mu})$ is a weak solution of (3) is similar to that of [20, Theorem 7.1] and is omitted. ∎

2) Piecewise linear control signals: We can establish the uniform convergence of the u-variable by redefining the discrete-time trajectory $\widehat{u}^h$ using piecewise linear interpolation instead of the piecewise constant interpolation in the semidefinite case. First notice that $u^{h,0}$ is not included in the $(\widehat{QP}^h)$. By letting $u^{h,0}$ be the unique solution of the $\text{QP}\big(U(\xi), q^{h,0} + h^{-1}B(h)^T \lambda^{h,0} + Q^T\xi, R\big)$, we redefine

$$\widehat{u}^h(t) \triangleq u^{h,i} + \frac{t - t_{h,i}}{h}(u^{h,i+1} - u^{h,i}) \quad (10)$$

for all $t \in [t_{h,i}, t_{h,i+1}]$. Theorem 6.2 sharpens the convergence conclusions of Theorem 6.1 in this case and also establishes the sequential convergence of the state and control trajectories $\{\widehat{x}^h\}$ and $\{\widehat{u}^h\}$ to the unique optimal solution $(\widehat{x}, \widehat{u})$ of the problem (1) with $\widehat{x}$ being continuously differentiable and $\widehat{u}$ Lipschitz continuous on $[0, T]$.

Theorem 6.2: Assume that the hypotheses of Theorem 6.1 hold. Let $\widehat{x}^h(t)$, $\widehat{\lambda}^h(t)$, and $\widehat{\mu}^h(t)$ be as before, and $\widehat{u}^h(t)$ be defined by (10). The sequence $\{(\widehat{x}^h, \widehat{u}^h)\}$ converges uniformly to the unique optimal solution pair (x^*, u^*) of (1) where x^* is continuously differentiable and u^* is Lipschitz continuous on $[0, T]$.

Proof. Since $u^{h,i+1}$ is the unique optimal solution of the quadratic program

$$\underset{u}{\text{minimize}} \quad u^T \left\{ Q^T \big[x^{h,i} \big] + h^{-1}B(h)^T \lambda^{h,i} \right\} + \tfrac{1}{2} u^T R u$$

$$\text{subject to} \quad C x^{h,i+1} + Du + f + \rho_h(\xi) \mathbf{1} \geq 0,$$

by the positive definiteness of R and the uniform boundedness of the vectors in (9), it follows that a constant $\eta_u > 0$ exists such that for $i = 0, \cdots, N_h$ and all $h > 0$ sufficiently small,

$$\| u^{h,i+1} - u^{h,i} \| \leq h \eta_u.$$

This bound is sufficient to establish the subsequential uniform convergence of the sequence $\{\widehat{u}^h\}$ to a Lipschitz

function $\widehat{u}$ on $[0, T]$. Since

$$\widehat{x}(t) = \xi + e^{At} \int_0^t e^{-A\tau} B\widehat{u}(\tau) \, d\tau$$

and $\widehat{u}(t)$ is Lipschitz continuous, it follows that $\widehat{x}(t)$ is continuously differentiable. Thus by part (IV) of Theorem 3.1, the limiting pair $(\widehat{x}, \widehat{u})$ is the unique optimal solution of (1) with $\widehat{x}$ being continuous differentiable and $\widehat{u}$ Lipschitz continuous. Hence, the entire sequence $\{(\widehat{x}^h, \widehat{u}^h)\}$ converges uniformly to this optimal pair as any equicontinuous family of Lipschitz functions in a Hilbert space with a unique accumulation function must converge to that function. ■

VII. CONCLUDING REMARKS

In this paper, we have established the convergence of a discretization method for approximating an optimal solution of the continuous-time constrained linear-quadratic (LQ) optimal problem with mixed linear state-control constraints under suitable assumptions. Although the convergence of such discretizations have been addressed extensively under general settings in the literature, we provide sharper results for the LQ case by exploiting the specially linear or affine structure of the problem as well as many results from mathematical programming. In the process of proving this results, we also showed that Pontryagin's maximum principle was both necessary and sufficient and that the resulting optimal continuous-time solution has both the state x and costate λ variables Lipschitz continuous. The latter property is largely due to the last condition (C). Whether weaker conditions could yield the same regularity property and ensure similar convergence of the time-stepping methods remains to be investigated. The case of pure state constraints failing condition (C) is another topic that requires further study. For such problems, the costate variable is very likely not even continuous [17]. These and other related open issues will be considered as we continue our research in this area.

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