VARIATIONAL PRINCIPLE FOR FUZZY GIBBS MEASURES

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Dedicated to the memory of R. L. Dobrushin

ABSTRACT. In this paper we study a large class of renormalization transformations of measures on lattices. An image of a Gibbs measure under such transformation is called a fuzzy Gibbs measure. Transformations of this type and fuzzy Gibbs measures appear naturally in many fields. Examples include the hidden Markov processes (HMP), memoryless channels in information theory, continuous block factors of symbolic dynamical systems, and many renormalization transformations of statistical mechanics. The main result is the generalization of the classical variational principle of Dobrushin–Lanford–Ruelle for Gibbs measures to the class of fuzzy Gibbs measures.

1. INTRODUCTION

Let $d \geqslant 1$ and $\mathscr{X} = \mathscr{A}^{\mathbb{Z}^d}$, $\mathscr{Y} = \mathscr{B}^{\mathbb{Z}^d}$, where $\mathscr{A}$, $\mathscr{B}$ are finite alphabets. Suppose $|\mathscr{A}| > |\mathscr{B}|$ and $\pi \colon \mathscr{A} \to \mathscr{B}$ is onto. We refer to π as a fuzzy coding or factor map. We will use the same letter π to denote the componentwise extension of π to a mapping from $\mathscr{A}^V$ onto $\mathscr{B}^V$ for any subset $V \subseteq \mathbb{Z}^d$. Note also that if $T_k \colon \mathscr{X} \to \mathscr{X}$, $S_k \colon \mathscr{Y} \to \mathscr{Y}$, $k \in \mathbb{Z}^d$, denote the natural translations by k, then the diagram

$$
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\ T_k\ } & \mathscr{X} \\
\pi \downarrow & & \downarrow \pi \\
\mathscr{Y} & \xrightarrow{\ S_k\ } & \mathscr{Y}
\end{array}
$$

commutes. Therefore, if μ is a translation invariant measure on $\mathscr{X}$, then $\nu = \mu \circ \pi^{-1}$, the image of μ under π, is a translation invariant measure on $\mathscr{Y}$. The sets of translation invariant measures and Gibbs measures on $\mathscr{X}$ and $\mathscr{Y}$ are denoted by $\mathcal{M}_T(\mathscr{X})$, $\mathcal{M}_S(\mathscr{Y})$, $\mathcal{G}_{\mathscr{X}}$ and $\mathcal{G}_{\mathscr{Y}}$, respectively. In this paper we consider only

translation invariant Gibbs measures, therefore we assume $\mathcal{G}_{\mathscr{X}} \subset \mathcal{M}_T(\mathscr{X})$, $\mathcal{G}_{\mathscr{Y}} \subset \mathcal{M}_S(\mathscr{Y})$.

If μ is a Gibbs state ($\mu \in \mathcal{G}_{\mathscr{X}}$), and $\nu = \mu \circ \pi^{-1}$, we refer to ν as a *fuzzy Gibbs state*. The set of all fuzzy Gibbs measures on $\mathscr{Y}$ obtained from Gibbs measures on $\mathscr{X}$ under π, will be denoted by

$$\mathcal{FG}^{\pi}_{\mathscr{X},\mathscr{Y}} = \{\nu \in \mathcal{M}_S(\mathscr{Y}) \colon \nu = \mu \circ \pi^{-1},\, \mu \in \mathcal{G}_{\mathscr{X}}\}.$$

The study of transformed (renormalized) Gibbs states is an active area of research. Many of renormalization transformation considered in the literature can be represented as fuzzy factors. For example, hidden Markov models ($d = 1$) and functions of Markov chains ($d = 1$) [23]. Moreover, in higher dimensions ($d \geqslant 2$), fuzzy Potts measures [14], [7]; decimated Ising models [21], [12], "copy-with-noise" (Griffiths–Pearce transformation) [21], [9]. Finally, infinite temperature spin-flip dynamics [20] also leads to fuzzy transformations.

One has to mention that there exist also important transformations of Gibbs states studied in the literature which cannot be represented as fuzzy factors: namely, projections on lower-dimensional sublattices (e.g., $\mathbb{Z}^d \mapsto \mathbb{Z}^v$ for $v < d$), see [18], [13], [5].

Transformations of Gibbs states often produce measures which are strictly speaking *non-Gibbsian*. Examples mentioned above demonstrate that in many cases transformed measures are not that bad: namely, Gibbsian property fails on a set of configurations which is exceptional, typically of measure 0.

Dobrushin proposed a restoration program consisting of two important ingredients [19], [10]. Firstly, he asked for an extension of a classical definition of Gibbs states to incorporate the new "nearly" Gibbs examples. Dobrushin himself introduced the notion of a *weakly Gibbs* state [3]. Latter, other notions such as almost Gibbs and intuitively weak Gibbs were introduced.

As a second step, one would like to recover some the of the thermodynamic results valid for Gibbs states; most importantly, the variational principle. The "classical" variational principle provides a characterization of the simplex of Gibbs states for a given potential: if μ is a Gibbs state for some potential and λ is an arbitrary translation invariant state, then the functional $i(\lambda|\mu)$ known as the relative entropy density, vanishes if and only if λ is Gibbs for the same potential.

Variational principle was extended to some classes of generalized Gibbs states under additional assumptions, e.g., [4], [8], [22]. Without additional assumptions, variational principle might fail, as demonstrated by a weakly Gibbs example constructed in [8].

In the present paper we prove that the variational principle remains valid for fuzzy Gibbs states in complete generality. Specifically, suppose μ is Gibbs for potential Φ, and $\nu = \mu \circ \pi^{-1}$ is the fuzzy image of μ. Then $i(\rho|\nu)$ vanishes if and only if ρ is fuzzy Gibbs, $\rho = \lambda \circ \pi^{-1}$ for some λ which is Gibbs for the same potential Φ. Note that ν itself is not necessarily Gibbs, as the examples cited above demonstrate.

The key tools used in the proof are the notion of compensation function and the relativized variational principle, originating in dynamical systems.

Finally, the validity of the variational principle for fuzzy Gibbs states strongly suggests that the possible singularities are not very severe. It would be interesting to understand what sort of non-Gibbsianity might the fuzzy Gibbs states exhibit.

2. Large Deviations for Fuzzy Gibbs Measures: Motivation for Variational Principle

Let us discuss briefly the large deviation (LD) principle for fuzzy Gibbs measures. Suppose $f \colon \mathscr{Y} \to \mathbb{R}$ is some function. We say that (ν, f) satisfies the large deviations principle with a rate function $I_{\nu,f}(\cdot)$ if for all closed F and open G, $F, G \subset \mathbb{R}$, one has

$$\limsup_{n \to \infty} \frac{1}{|\Lambda_n|} \log \nu \Big(\Big\{ y \in \mathscr{Y} \colon \frac{1}{|\Lambda_n|} \sum_{k \in \Lambda_n} f(S_k y) \in F \Big\} \Big) \leqslant - \inf_{\alpha \in F} I_{\nu,f}(\alpha),$$

$$\liminf_{n \to \infty} \frac{1}{|\Lambda_n|} \log \nu \Big(\Big\{ y \in \mathscr{Y} \colon \frac{1}{|\Lambda_n|} \sum_{k \in \Lambda_n} f(S_k y) \in G \Big\} \Big) \geqslant - \inf_{\alpha \in G} I_{\nu,f}(\alpha),$$

where $\Lambda_n = [0, n]^d$.

Note that since $\nu = \mu \circ \pi^{-1}$ and $S_k \circ \pi = \pi \circ T_k$ for all k, one immediately obtains that for all $C \subset \mathbb{R}$

$$\nu \Big(\Big\{ y \in \mathscr{Y} \colon \frac{1}{|\Lambda|} \sum_{k \in \Lambda} f(S_k y) \in C \Big\} \Big) = \mu \Big(\Big\{ x \in \mathscr{X} \colon \frac{1}{|\Lambda|} \sum_{k \in \Lambda} (f \circ \pi)(T_k y) \in C \Big\} \Big).$$

Therefore, if $(\mu, f \circ \pi)$ satisfies a LD principle with a rate function $I_{\mu,f \circ \pi}$, then so is (ν, f), and

$$I_{\nu,f}(\alpha) = I_{\mu,f \circ \pi}(\alpha) \quad \text{for all } \alpha.$$

The opposite is of course also true.

Typically, for a Gibbs measure μ on $\mathscr{X}$ and $f \colon \mathscr{X} \to \mathbb{R}$ such that the LD principle holds for ergodic averages of g one has

$$I_{\mu,f}(\alpha) = \inf_{\lambda \in \mathcal{M}_T(\mathscr{X}) \colon \int f \, d\lambda = \alpha} i(\lambda|\mu),$$

where $i(\lambda|\mu)$ is a relative entropy density (see Definition 5 below), which is a well defined quantity for all translation invariant measures λ.

One can show [16], [4] that

$$I_{\nu,f}(\alpha) = \inf_{\rho \in \mathcal{M}_T(\mathscr{Y}) \colon \int f \, d\rho = \alpha} i(\rho|\nu),$$

thus extending the LD formalism to fuzzy Gibbs states. Moreover,

$$I_{\nu,f}(\alpha) = \inf_{\rho \in \mathcal{M}_T(\mathscr{Y}) \colon \int f \, d\rho = \alpha} i(\rho|\nu)$$

$$= I_{\mu,f \circ \pi}(\alpha) = \inf_{\lambda \in \mathcal{M}_T(\mathscr{X}) \colon \int f \circ \pi \, d\lambda = \alpha} i(\lambda|\mu)$$

$$= \inf_{\rho \in \mathcal{M}_T(\mathscr{Y}) \colon \int f \, d\rho = \alpha} \quad \inf_{\lambda \in \mathcal{M}_T(\mathscr{X}) \colon \lambda \circ \pi^{-1} = \rho} i(\lambda|\mu).$$

This suggests that

$$i(\rho|\nu) = \inf_{\lambda \in \mathcal{M}_T(\mathscr{X}) \colon \lambda \circ \pi^{-1} = \rho} i(\lambda|\mu).$$

In Section 4 we show that this is indeed the case, and this equality is in fact very useful, as it helps to extend the full classical variational principle for Gibbs measures to the fuzzy Gibbs case.

3. Thermodynamic Formalism and Gibbs Measures

In this section we introduce the pressure, relative entropy density, potentials, Gibbs measures and formulate the variational principles. We do it for the space $\mathscr{X} = \mathscr{A}^{\mathbb{Z}^d}$ with a finite alphabet $\mathscr{A}$. As above, $T_k \colon \mathscr{X} \to \mathscr{X}$ are the translations by $k \in \mathbb{Z}^d$. For the space $\mathscr{Y} = \mathscr{B}^{\mathbb{Z}^d}$ the same definitions clearly apply with an obvious substitution of T_k by S_k. All the definitions and results of this section are well-known and can be found in [17], [6], [21].

For a point $x \in \mathscr{X}$ and any finite Λ, let $x_\Lambda := x|_\Lambda \in \mathscr{A}^\Lambda$ and

$$[x_\Lambda] = \{\tilde{x} \in \mathscr{X} : \tilde{x}_\Lambda = x_\Lambda\}.$$

For any $n \in \mathbb{N}$, $\Lambda_n = [0, n]^d \subset \mathbb{Z}^d$. $C(\mathscr{X})$ denotes the set of real-valued continuos functions on $\mathscr{X}$.

3.1. Entropy

Definition 1. Suppose μ is a translation invariant measure on $\mathscr{X}$, the entropy of μ is

$$h(\mu) = \lim_{n \to \infty} -\frac{1}{|\Lambda_n|} \sum_{x_{\Lambda_n}} \mu([x_{\Lambda_n}]) \log \mu([x_{\Lambda_n}]).$$

It is well known that the limit exists, $h \colon \mathcal{M}_T(\mathscr{X}) \to [0, \log |\mathscr{A}|]$, and h is a convex functional on $\mathcal{M}_T(\mathscr{X})$.

3.2. Pressure

Definition 2. For any continuous function $f \colon \mathscr{X} \to \mathbb{R}$ define the *pressure of f* as follows

$$P(f) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \log \left[\sum_{x_{\Lambda_n}} \sup_{\tilde{x} \in [x_{\Lambda_n}]} \exp\left(\sum_{k \in \Lambda_n} f(T_k \tilde{x}) \right) \right]. \tag{1}$$

Note that in dynamical systems literature one usually defines the pressure as $\limsup$. However, for symbolic space $\mathscr{X} = \mathscr{A}^{\mathbb{Z}^d}$, $\mathscr{A}$ is finite, the limit exists. Pressure $P(\cdot) \colon C(\mathscr{X}) \to \mathbb{R}$ is a finite Lipschitz continuous convex functional.

Theorem 3 ("Weak" variational principle)**.** *For any continuous function f on $\mathscr{X}$*

$$P(f) = \sup_{\mu \in \mathcal{M}_T(\mathscr{X})} \left[h(\mu) + \int f \, d\mu \right].$$

Definition 4. A measure $\mu \in \mathcal{M}_T(\mathscr{X})$ is called an *equilibrium state for f* if $h(\mu) + \int f \, d\mu = P(f)$. The set of all equlibrium states for f is denoted by $\mathcal{ES}_T(X, f)$.

3.3. Pressure relative to a measure. Important quantity for the large deviations of ergodic averages and the variational principle, is the *pressure of f relative to μ* denoted by $p(f|\mu)$. For $f \in C(\mathscr{X})$ and $\mu \in \mathcal{M}_T(\mathscr{X})$, $p(f|\mu)$ is defined as (provided the limit exists)

$$p(f|\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \int_{\mathscr{X}} \exp\Big(\sum_{k\in\Lambda_n} f(T_k x) \Big) \, d\mu. \tag{2}$$

3.4. Relative entropy density

Definition 5. Let λ and μ be translation invariant probability measures on $\mathscr{X}$. The *relative entropy density* is defined as

$$i(\lambda|\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{\omega_{\Lambda_n}} \lambda(\omega_{\Lambda_n}) \log \frac{\lambda(\omega_{\Lambda_n})}{\mu(\omega_{\Lambda_n})},$$

provided the limit exists.

A natural question is under what conditions does the relative entropy density exist. There are examples when it is not the case, e.g., [21]. When establishing variational principles, one is usually interested in measures μ with the property that $i(\lambda|\mu)$ exists for all translation invariant measures λ. Gibbs measures are known to have this property, as well as the *asymptotically decoupled measures* introduced by Pfister [16], which form the most general class with such property discovered so far.

3.5. Potentials, Hamiltonians and Gibbs measures. A potential

$$\Phi = \big\{\Phi(\Lambda, \cdot)\big\}_{\Lambda\in\mathcal{S}(\mathbb{Z}^d)}$$

is a family of functions indexed by finite subsets of $\mathbb{Z}^d$ ($\Lambda \in \mathcal{S}(\mathbb{Z}^d)$) with the property that $\Phi(\Lambda, x)$ depends only on x_Λ. We say that a potential Φ is translation invariant if

$$\Phi(\Lambda + k, x) = \Phi(\Lambda, T_k x) \quad \text{for all } x \in \mathscr{X} \text{ and every finite } \Lambda \subset \mathbb{Z}^d.$$

Denote by $\mathfrak{B}_1(\mathscr{X})$ the set of all absolutely summable potentials, i.e., $\Phi \in \mathfrak{B}_1(\mathscr{X})$ if

$$\sum_{0\in\Lambda\in\mathcal{S}(\mathbb{Z}^d)} \sup_{x_\Lambda} |\Phi(\Lambda, x_\Lambda)| < \infty.$$

A Hamiltonian $H_\Lambda = H_\Lambda^\Phi$ is defined by

$$H_\Lambda^\Phi(x) = \sum_{\Lambda'\cap\Lambda\neq\varnothing} \Phi(\Lambda', x).$$

For a potential $\Phi \in \mathfrak{B}_1(\mathscr{X})$, the Hamiltonian H_Λ is convergent in all $x \in \mathscr{X}$, i.e., the sum on the right hand side is convergent in the *net convergent sense.*

For a potential $\Phi \in \mathfrak{B}_1(X)$, put

$$f_\Phi(x) = - \sum_{0\in\Lambda\in\mathcal{S}(\mathbb{Z}^d)} \frac{1}{|\Lambda|} \Phi(\Lambda, x).$$

It turns out that f_Φ is well defined for all x and is a continuous function of x. Introduce the following semi-norm on $\mathfrak{B}_1(\mathscr{X})$:

$$\|\Phi_1 - \Phi_2\|_{\mathscr{X}} = \|f_{\Phi_1} - f_{\Phi_2}\|_{C(\mathscr{X})_{/(\mathfrak{J}+\text{const})}},$$

where for a continuous function f the semi-norm is given by

$$\|f\|_{C(\mathscr{X})_{/(\mathfrak{J}+\text{const})}} = \inf_{c\in\mathbb{R}, g\in\mathfrak{J}} \|f - (g+c)\|_{C(\mathscr{X})},$$

here $\mathfrak{J}$ is set the of continuous functions g such that

$$\int_{\mathscr{X}} g\,d\mu = 0 \quad \text{for all } \mu \in \mathcal{M}_T(\mathscr{X}).$$

The Gibbs specification $\gamma^\Phi = \{\gamma_\Lambda^\Phi\}_{\Lambda\in\mathcal{S}(\mathbb{Z}^d)}$ is defined by

$$\gamma_\Lambda^\Phi(x_\Lambda | \tilde{x}_{\Lambda^c}) = \frac{\exp(-H_\Lambda^\Phi(x_\Lambda \tilde{x}_{\Lambda^c}))}{\sum\limits_{\bar{x}_\Lambda \in \mathscr{A}^\Lambda} \exp(-H_\Lambda^\Phi(\bar{x}_\Lambda \tilde{x}_{\Lambda^c}))}.$$

Definition 6. A measure $\mu \in \mathcal{M}_T(\mathscr{X})$ is called a Gibbs measure for potential $\Phi \in \mathfrak{B}_1(\mathscr{X})$, if for all $\Lambda \in \mathcal{S}(\mathbb{Z}^d)$, one has

$$\mu(x_\Lambda | x_{\Lambda^c}) = \gamma_\Lambda^\Phi(x_\Lambda | x_{\Lambda^c}) \quad (\mu\text{-a.s.}).$$

or, equivalently, for all $f \in C(\mathscr{X})$,

$$\int_{\mathscr{X}} f(x)\,d\mu(x) = \int_{\mathscr{X}} \sum_{\bar{x}_\Lambda \in \mathscr{A}^\Lambda} f(\bar{x}_\Lambda x_{\Lambda^c}) \gamma_\Lambda^\Phi(\bar{x}_\Lambda | x_{\Lambda^c})\,d\mu(x).$$

3.6. Variational principles for Gibbs measures

Theorem 7. *Suppose* $\Phi \in \mathfrak{B}_1(\mathscr{X})$. *Then* $\mu \in \mathcal{M}_T(\mathscr{X})$ *is a Gibbs measure for* Φ *if and only if* μ *is an equilibrium state for* f_Φ.

Theorem 8. *Suppose* $\mu \in \mathcal{M}_T(\mathscr{X})$ *is a Gibbs measure for a potential* $\Phi \in \mathfrak{B}_1(\mathscr{X})$. *Then for all* $\lambda \in \mathcal{M}_T(\mathscr{X})$, $f \in C(\mathscr{X})$, $i(\lambda|\mu)$ *and* $p(f|\mu)$ *exist and are given by*

$$i(\lambda|\mu) = P(f_\Phi) - h(\lambda) - \int_{\mathscr{X}} f_\Phi\,d\lambda,$$
$$p(f|\mu) = P(f + f_\Phi) - P(f_\Phi).$$

Moreover,

$$p(f|\mu) = \sup_{\lambda\in\mathcal{M}_T(\mathscr{X})} \left[\int_X f\,d\lambda - i(\lambda|\mu)\right],$$
$$i(\lambda|\mu) = \sup_{f\in C(\mathscr{X})} \left[\int f\,d\lambda - p(f|\mu)\right].$$

Theorem 9. *Suppose* $\mu \in \mathcal{M}_T(\mathscr{X})$ *is a Gibbs measure for a potential* $\Phi \in \mathscr{B}_1(\mathscr{X})$ *and* $\lambda \in \mathcal{M}_T(\mathscr{X})$. *Then* $i(\lambda|\mu) = 0$ *if and only if* λ *is a Gibbs measure for potential* Φ.

4. Thermodynamic Formalism for Fuzzy Factors

In this section, we discuss the relation between thermodynamic quantities defined on $\mathscr{X}$ and $\mathscr{Y}$ in case $\mathscr{Y} = \mathscr{B}^{\mathbb{Z}^d}$ is a fuzzy factor of $\mathscr{X} = \mathscr{A}^{\mathbb{Z}^d}$.

4.1. Entropy. If ν is a fuzzy image of μ, then the entropies of μ and ν are related by the following result of Walters and Ledrappier [11].

Theorem 10. *If $\mu \in M_T(\mathscr{X})$ and $\nu \in M_S(\mathscr{Y})$ are such that $\nu = \mu \circ \pi^{-1}$, then*

$$h(\mu) = h(\nu) + h_\pi(\mu),$$

where

$$h_\pi(\mu) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} H(A_n | \pi^{-1} B_n),$$

where A_n and B_n are partitions into cylinders

$$A_n = \{[x_{\Lambda_n}] \colon x_{\Lambda_n} \in \mathscr{A}^{\Lambda_n}\}, \quad B_n = \{[y_{\Lambda_n}] \colon y_{\Lambda_n} \in \mathscr{B}^{\Lambda_n}\},$$

In general, computation of $h(\nu)$ is not an easy task, even for Markov measures in dimension one ($d = 1$). In information theory one often has to deal with such questions, and special methods have been developed [15], [25].

4.2. Pressure and relative pressure. If $f \in C(\mathscr{Y})$, then $f \circ \pi \in C(\mathscr{X})$. It turns out that for fuzzy factors π, there exist functions $F \in C(\mathscr{X})$ such that

$$P(f) = P(F + f \circ \pi) \quad \text{for all } f \in C(\mathscr{Y}).$$

Such function F is called a *compensation function*. This notion was introduced by Boyle and Tuncel [2], and further developed by Walters [24] in $d = 1$. However, the results are readily generalized to higher dimensions.

For fuzzy factors π, at least one compensation function is easily identified: namely,

$$F(x) = -\log |\pi^{-1}(\pi x_0)|,$$

i. e., $F(x)$ is minus the logarithm of the number of symbols in $\mathscr{A}$ which are mapped into $\pi(x_0) \in \mathscr{B}$. Note that $F(x) = G \circ \pi(x)$, where $G(y) = -\log |\pi^{-1} y_0|$. Such compensation functions are often called *saturated*.

Definition 11. Let $f \in C(\mathscr{X})$, define the *relative pressure of f at point $y \in \mathscr{Y}$* as follows

$$P(f, \pi)(y) = \limsup_{n \to \infty} \frac{1}{|\Lambda_n|} \log \sum_{x_{\Lambda_n} : \pi x_{\Lambda_n} = y_{\Lambda_n}} \sup_{\tilde{x} \in [x_{\Lambda_n}]} \exp\Big(\sum_{k \in \Lambda_n} f(T_k x)\Big),$$

where $\Lambda_n = [0, n]^d$.

The relative pressure $P(f, \pi)$ is also can be seen as the pressure of f restricted to a fibre

$$\mathscr{X}_y = \pi^{-1} y = \{x \in \mathscr{X} : \pi(x) = y\}.$$

Note that $\mathscr{X}_y$ is a closed, but not necessarily translation invariant subset of $\mathscr{X}$.

Theorem 12 (Walters, [24])**.** *The relative pressure has the following properties*:

(a) $P(f, \pi)(y)$ *is a measurable function of y; moreover, $P(f, \pi)$ is translation invariant: for all $y \in \mathscr{Y}$ and $k \in \mathbb{Z}^d$*

$$P(f, \pi)(y) = P(f, \pi)(S_k y);$$

(b) $P(\cdot, \pi)(y) \colon C(\mathscr{X}) \to \mathbb{R}$ *is a convex, real-valued function on $C(\mathscr{X})$ for every $y \in \mathscr{Y}$;*

(c) *for $f, g \in C(\mathscr{X})$, $|P(f, \pi) - P(g, \pi)| \leqslant \|f - g\|_{C(\mathscr{X})}$;*

(d) *for $c \in \mathbb{R}$, $f, g \in C(\mathscr{X})$, one has*

$$P(f + g \circ T_k - g + c, \pi) = P(f, \pi) + c;$$

(e) *denote by $\mathscr{Y}_0$ the set of points y such that*

$$P(f, \pi)(y) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \log \sum_{x_{\Lambda_n} : \pi x_{\Lambda_n} = y_{\Lambda_n}} \sup_{\tilde{x} \in [x_{\Lambda_n}]} \exp\Big(\sum_{k \in \Lambda_n} f(T_k x) \Big),$$

then $\mathscr{Y}_0$ is a total probability set, *i. e., for every $\nu \in \mathbb{M}_S(\mathscr{Y})$*

$$\nu(\mathscr{Y}_0) = 1.$$

An important result is the *relativized variational principle* of Ledrappier and Walters, [11].

Theorem 13. *Let $\pi \colon \mathscr{X} \to \mathscr{Y}$ be a fuzzy factor. Then for an arbitrary $f \in C(\mathscr{X})$ and every translation invariant measure ν on $\mathscr{Y}$*

$$h(\nu) + \int_{\mathscr{Y}} P(f, \pi)(y)\nu(dy) = \sup_{\mathbb{M}_T(\mathscr{X}, \nu)} \left\{ h(\mu) + \int_{\mathscr{X}} f \, d\mu \right\}, \qquad (3)$$

where $\mathbb{M}_T(\mathscr{X}, \nu)$ is the fibre over ν

$$\mathbb{M}_T(\mathscr{X}, \nu) = \left\{ \mu \in \mathbb{M}_T(\mathscr{X}) \colon \nu = \mu \circ \pi^{-1} \right\}.$$

This result has been established in [11] for subshifts $\mathscr{X} \subset \mathscr{A}^{\mathbb{Z}}$ and $\mathscr{Y} \subset \mathscr{B}^{\mathbb{Z}}$ and continuous factors π commuting with shifts. Again, generalization to lattices $\mathbb{Z}^d$, $d \geqslant 1$, in case of full shifts is straightforward.

Definition 14. If μ attains maximum in right-hand side of (3), then μ is called a *relative equilibrium state for f over ν.*

The relative pressure $P(f, \pi)$ is not a continuous function on $\mathscr{Y}$. Nevertheless, we can define equilibrium states for $P(f, \pi)$.

Definition 15. A measure $\nu \in \mathbb{M}_S(\mathscr{Y})$ is an equilibrium state for $P(f, \pi)$, denoted $\nu \in \mathcal{ES}_S(\mathscr{Y}, P(f, \pi))$, if

$$h(\nu) + \int_Y P(f, \pi)(y)\nu(dy) = \sup_{\rho \in \mathbb{M}_S(\mathscr{Y})} \left\{ h(\rho) + \int_Y P(f, \pi)(y)\rho(dy) \right\}.$$

The following result seems to be new, however, it is quite similar in spirit to the result of Walters [24], for measures of relative maximal entropy.

Theorem 16. *A measure $\mu \in \mathbb{M}_T(\mathscr{X})$ is an equilibrium state for f, $f \in C(\mathscr{X})$, if and only if $\nu = \mu \circ \pi^{-1} \in \mathcal{ES}_S(\mathscr{Y}, P(f, \pi))$ and μ is a relative equilibrium state for f over ν.*

Proof. Let $\mu \in \mathcal{M}_T(\mathscr{X})$ be an equilibrium state for f, $\nu = \mu \circ \pi^{-1}$ and ρ be an arbitrary measure in $\mathcal{M}_S(\mathscr{Y})$. Then

$$h(\nu) + \int_{\mathscr{Y}} P(f, \pi)(y) \, d\nu \overset{(a)}{\geqslant} h(\mu) + \int_{\mathscr{X}} f \, d\mu$$

$$\overset{(b)}{=} P(f) \overset{(c)}{=} \sup_{\lambda \in \mathcal{M}_T(\mathscr{X})} \left[h(\lambda) + \int_{\mathscr{X}} f \, d\lambda \right]$$

$$\geqslant \sup_{\lambda \in \mathcal{M}_T(\mathscr{X}): \lambda \circ \pi^{-1} = \rho} \left[h(\lambda) + \int_X f d\lambda \right]$$

$$\overset{(d)}{=} h(\rho) + \int_Y P(f, \pi) \, d\rho,$$

where we have used (a) relativized variational principle (Theorem 13) applied to ν and f, (b) $\mu \in \mathcal{ES}_T(X, f)$, (c) variational principle (Theorem 3), (d) relativized variational principle for ρ and f. Therefore, $\nu \in \mathcal{ES}_S(\mathscr{Y}, P(f, \pi))$. But also, μ is a relative equilibrium state for f over ν. Indeed, on one hand ((a) and (b) above)

$$h(\nu) + \int_{\mathscr{Y}} P(f, \pi) \, d\nu \geqslant P(f),$$

and on the other, by the relativized and classical variational principles,

$$h(\nu) + \int_{\mathscr{Y}} P(f, \pi) \, d\nu \leqslant \sup_{\lambda \in \mathcal{M}_T(\mathscr{X})} \left[h(\lambda) + \int_X f \, d\lambda \right] = P(f).$$

Therefore μ attains a maximum in (3), and hence μ is a relative equilibrium state for f over ν.

To prove the opposite implication assume that $\nu \in \mathcal{ES}_S(\mathscr{Y}, P(f, \pi))$, and μ is a relative equilibrium state for f over ν.

Suppose $\lambda \in \mathcal{M}_T(\mathscr{X})$, and let $\rho = \lambda \circ \pi^{-1}$. Then

$$h(\mu) + \int_{\mathscr{X}} f \, d\mu \overset{(a)}{=} h(\nu) + \int_{\mathscr{Y}} P(f, \pi) \, d\nu \overset{(b)}{\geqslant} h(\rho) + \int_{\mathscr{Y}} P(f, \pi) \, d\rho \overset{(c)}{\geqslant} h(\lambda) + \int_{\mathscr{X}} f \, d\lambda,$$

where we used (a) μ is a relative equilibrium state for f over ν, (b) ν is an equilibrium state for $P(f, \pi)$, (c) relativized variational principle applied to ρ and f. Therefore

$$h(\mu) + \int_{\mathscr{X}} f \, d\mu \geqslant h(\lambda) + \int_{\mathscr{X}} f \, d\lambda$$

for any $\lambda \in \mathcal{M}_T(\mathscr{X})$, and hence μ is an equilibrium state for f. $\square$

4.3. Pressure relative to a measure. Again, since $f \circ \pi \in C(\mathscr{X})$ for $f \in C(\mathscr{Y})$, we are able to relate $p(f|\nu)$ and $p(f \circ \pi | \mu)$ for $\nu = \mu \circ \pi^{-1}$.

Lemma 17. *If $\nu = \mu \circ \pi^{-1}$, then $p(f|\nu)$ is defined if and only if $p(f \circ \pi | \mu)$ is defined, and they are equal.*

Proof. For any $g \in C(\mathscr{Y})$ one has

$$\int_{\mathscr{Y}} g \, d\nu = \int_{\mathscr{X}} g \circ \pi \, d\mu.$$

Take $g(y) = \exp(\sum_{k \in \Lambda} f(S_k y))$, then

$$
\begin{aligned}
g \circ \pi(x) &= \exp\Big(\sum_{k \in \Lambda} f(S_k \circ \pi(x)) \Big) \\
&= \exp\Big(\sum_{k \in \Lambda} f(\pi \circ T_k(x)) \Big) = \exp\Big(\sum_{k \in \Lambda} (f \circ \pi)(T_k x)) \Big),
\end{aligned}
$$

where we used that $\pi \circ T_k = S_k \circ \pi$ for all k. Therefore, the integrals (see Definition 2) are equal for any finite Λ, and the corresponding limits exists or do not exists simultaneously. □

Corollary 18. *If ν is fuzzy Gibbs, then $p(f|\nu)$ is defined for all $f \in C(\mathscr{Y})$.*

Proof. Follows from the previous lemma and the fact that for Gibbs μ, $p(g|\mu)$ is defined for all $g \in C(\mathscr{X})$ (Theorem 8). □

4.4. Relative entropy density

Theorem 19. *If $\mu \in \mathcal{M}_T(\mathscr{X})$ is a Gibbs measure for potential $\Phi \in \mathfrak{B}_1(\mathscr{X})$ and $\nu = \mu \circ \pi^{-1}$, then $i(\rho|\nu)$ exists for all $\rho \in \mathcal{M}_S(\mathscr{Y})$ and given by*

$$
i(\rho|\nu) = P(f_\Phi) - h(\rho) - \int_{\mathscr{Y}} P(f_\Phi, \pi)\, d\rho.
$$

Moreover,

$$
i(\rho|\nu) = \inf_{\lambda \in \mathcal{M}_T(\mathscr{X}):\, \lambda \circ \pi^{-1} = \rho} i(\lambda|\mu). \tag{4}
$$

Proof. The existence of relative entropy density $i(\rho|\nu)$ for fuzzy Gibbs ν and arbitrary ρ was first observed in [4]. In fact, it is easy to show that fuzzy Gibbs measures inherit the property of being asymptotically decoupled from their Gibbs "parents", and hence by Pfister's general result [16] for such measures the relative entropy density exists. On the other hand, direct treatment of relative entropy densities of fuzzy Gibbs measure leads to a new result (4), which gives better insight into the nature of fuzzy Gibbs states and leads to the variational principle.

For a Gibbs measure μ a with potential $\Phi \in \mathfrak{B}_1(\mathscr{X})$ we have the following estimate ([6, Theorem 15.23], [21, Proposition 2.46]):

$$
\sup_{x \in \mathscr{X}} \left| \log \mu([x_{\Lambda_n}]) - \sum_{k \in \Lambda_n} f_\Phi(T_k x) + |\Lambda_n| P(f_\Phi) \right| = o(|\Lambda_n|). \tag{5}
$$

Therefore, since $\nu([y_{\Lambda_n}]) = \sum_{x_{\Lambda_n} \in \pi^{-1} y_{\Lambda_n}} \mu([x_{\Lambda_n}])$, we conclude that

$$
\frac{1}{|\Lambda_n|} \sum_{[y_{\Lambda_n}]} \rho([y_{\Lambda_n}]) \log \frac{\rho([y_{\Lambda_n}])}{\nu([y_{\Lambda_n}])} = \frac{1}{|\Lambda_n|} \sum_{[y_{\Lambda_n}]} \rho([y_{\Lambda_n}]) \log \rho([y_{\Lambda_n}]) -
$$

$$
\frac{1}{|\Lambda_n|} \sum_{[y_{\Lambda_n}]} \rho([y_{\Lambda_n}]) \log \sum_{x_{\Lambda_n} \in \pi^{-1} y_{\Lambda_n}} \sup_{\tilde{x} \in [x_{\Lambda_n}]} \exp\Big(\sum_{k \in \Lambda_n} f_\Phi(T_k \tilde{x}) - |\Lambda_n| P(f_\Phi) \Big) + \frac{o(|\Lambda_n|)}{|\Lambda_n|}.
$$

The first term on the right hand side converges to $-h(\rho)$ and the last term converges to 0. By the Lebesgue theorem on bounded convergence and Theorem 12, the second term converges to

$$\int_{\mathscr{Y}} P(f_\Phi - P(f_\Phi), \, \pi) \, d\rho = \int_{\mathscr{Y}} P(f_\Phi, \, \pi) \, d\rho - P(f_\Phi).$$

As a corollary, we also obtain that if $\mu_1 \in \mathcal{G}_{\mathscr{X}}(\Phi_1)$ and $\mu_2 \in \mathcal{G}_{\mathscr{X}}(\Phi_2)$ are such that $\mu_1 \circ \pi^{-1} = \mu_2 \circ \pi^{-1}$, then

$$\int_{\mathscr{Y}} P(f_{\Phi_1}, \, \pi) \, d\rho - P(f_{\Phi_1}) = \int_{\mathscr{Y}} P(f_{\Phi_2}, \, \pi) \, d\rho - P(f_{\Phi_2})$$

for every $\rho \in \mathcal{M}_S(\mathscr{Y})$.

Let us now prove the variational formula (4). Indeed, if $\mu \in \mathcal{G}_{\mathscr{X}}(\Phi)$ is such that $\mu \circ \pi^{-1} = \nu$, then by relativized variational principle for f_Φ and ρ, one has

$$h(\rho) + \int_{\mathscr{Y}} P(f_\Phi, \, \pi) \, d\rho = \sup_{\lambda \in \mathcal{M}_T(\mathscr{X}): \lambda \circ \pi^{-1} = \rho} \left[h(\lambda) + \int_{\mathscr{X}} f_\Phi \, d\lambda \right].$$

Therefore,

$$i(\rho|\nu) = P(f_\Phi) - \left[h(\rho) + \int_{\mathscr{Y}} P(f_\Phi, \, \pi) \, d\rho \right]$$

$$= \inf_{\lambda \in \mathcal{M}_T(\mathscr{X}): \lambda \circ \pi^{-1} = \rho} \left[P(f_\Phi) - h(\lambda) - \int_{\mathscr{X}} f_\Phi \, d\lambda \right]$$

$$= \inf_{\lambda \in \mathcal{M}_T(\mathscr{X}): \lambda \circ \pi^{-1} = \rho} i(\lambda|\mu),$$

where for the last equality we used the variational principle for Gibbs measures (Theorem 8). □

4.5. Variational principles for fuzzy gibbs states

Theorem 20 (cf. Theorem 7). *A measure $\nu \in \mathcal{M}_S(\mathscr{Y})$ is a fuzzy image of an equilibrium state for f, $f \in C(\mathscr{X})$, if and only if ν is an equilibrium state for $P(f, \pi)$.*

Theorem 21 (cf. Theorem 8). *Suppose $\nu \in \mathcal{M}_S(\mathscr{Y})$ is a fuzzy Gibbs measure. Then for all $\rho \in \mathcal{M}_S(\mathscr{Y})$, $f \in C(\mathscr{Y})$, $i(\rho|\nu)$ and $p(f|\nu)$ exist and given by*

$$i(\rho|\nu) = P(f_\Phi) - h(\rho) - \int_{\mathscr{Y}} P(f_\Phi, \, \pi) \, d\rho = \inf_{\lambda \in \mathcal{M}_T(\mathscr{X}): \lambda \circ \pi^{-1} = \rho} i(\lambda|\mu),$$

$$p(f|\nu) = P(f \circ \pi + f_\Phi) - P(f_\Phi) = p(f \circ \pi|\mu),$$

where $\mu \in \mathcal{M}_T(\mathscr{X})$ is an arbitrary Gibbs measure such that $\nu = \mu \circ \pi^{-1}$ and $\Phi \in \mathfrak{B}_1(\mathscr{X})$ is the potential for μ. Moreover,

$$p(f|\nu) = \sup_{\rho \in \mathcal{M}_S(\mathscr{Y})} \left[\int_{\mathscr{Y}} f \, d\rho - i(\rho|\nu) \right], \tag{6}$$

$$i(\rho|\nu) = \sup_{f \in C(\mathscr{Y})} \left[\int_{\mathscr{Y}} f \, d\rho - p(f|\nu) \right]. \tag{7}$$

Theorem 22 (cf. Theorem 9). *Suppose $\mu \in \mathcal{G}_{\mathscr{X}}(\Phi)$ and $\nu = \mu \circ \pi^{-1}$ is a fuzzy image of μ. Then for any $\rho \in \mathcal{M}_S(\mathscr{Y})$*

$$i(\rho|\nu) = 0 \quad \text{if and only if} \quad \rho = \lambda \circ \pi^{-1} \text{ for some } \lambda \in \mathcal{G}_{\mathscr{X}}(\Phi).$$

Proof of Theorem 20. In Theorem 16 we already established that if $\mu \in \mathcal{M}_T(\mathscr{X})$ is an equilibrium state for f, then ν is an equilibrium state for $P(f, \pi)$. We now have to show that if ν is an equilibrium state for for $P(f, \pi)$, then there exists a measure $\mu \in \mathcal{ES}_T(\mathscr{X}, f)$ such that $\nu = \mu \circ \pi^{-1}$. By the relativized variational principle for ν and f

$$h(\nu) + \int_{\mathscr{Y}} P(f, \pi) \, d\nu = \sup_{\mu \in \mathcal{M}_T(\mathscr{X}) : \mu \circ \pi^{-1} = \nu} \left[h(\mu) + \int_{\mathscr{X}} f \, d\mu \right].$$

Moreover, using the upper semicontinuity of the entropy function, one can easily show that the supremum is achieved. Suppose μ is relative equilibrium state for f over ν. But then

$$h(\mu) + \int_{\mathscr{X}} f \, d\mu = h(\nu) + \int_{\mathscr{Y}} P(f, \pi) \, d\nu = \sup_{\rho \in \mathcal{M}_S(\mathscr{Y})} \left[h(\rho) + \int_{\mathscr{Y}} P(f, \pi) \, d\rho \right]$$

$$= \sup_{\rho \in \mathcal{M}_S(\mathscr{Y})} \sup_{\lambda \in \mathcal{M}_T(\mathscr{X}) : \lambda \circ \pi^{-1} = \rho} \left[h(\lambda) + \int_{\mathscr{X}} f \, d\lambda \right]$$

$$= \sup_{\lambda \in \mathcal{M}_T(\mathscr{X})} \left[h(\lambda) + \int_{\mathscr{X}} f \, d\lambda \right] = P(f).$$

Therefore μ is an equilibrium state for f. This finishes the proof. $\square$

Proof of Theorem 21. Existence of $p(f|\nu)$ and $i(\rho|\nu)$ has been established in Corollary 18 and Theorem 19. Equality $p(f \circ \pi|\mu) = P(f \circ \pi + f_\Phi) - P(f_\Phi)$ follows from the variational principle for Gibbs measures. Now let us show that $p(f|\nu)$ and $i(\rho|\nu)$ are convex conjugated.

In fact we only have to show that $p(\cdot|\nu)$ is a convex conjugate of $i(\cdot|\nu)$, i.e.,

$$p(f|\nu) = \sup_{\rho \in \mathcal{M}_S(\mathscr{Y})} \left[\int_{\mathscr{Y}} f \, d\rho - i(\rho|\nu) \right].$$

Since, $i(\cdot|\nu)$ is a convex lower semicontinuous functional on $\mathcal{M}_S(\mathscr{Y})$, the convex conjugate of $p(\cdot|f)$ is equal to $i(\cdot|\nu)$, see, e.g., [1, Theorem 4.4.2]. This gives us (7).

Suppose $\nu = \mu \circ \pi^{-1}$, where $\mu \in \mathcal{G}_X$. Consider an arbitrary function $f \in C(\mathscr{Y})$, a measure $\rho \in \mathcal{M}_S(\mathscr{Y})$ and a measure $\lambda \in \mathcal{M}_T(\mathscr{X})$ such that $\rho = \lambda \circ \pi^{-1}$. Then

$$\int_{\mathscr{Y}} f \, d\rho - p(f|\nu) = \int_{\mathscr{X}} (f \circ \pi) \, d\lambda - p(f \circ \pi|\mu). \tag{8}$$

Since μ is Gibbs, and for Gibbs measures we do have the variational principle, we obtain

$$p(f \circ \pi|\mu) = \sup_{\theta \in \mathcal{M}_T(\mathscr{X})} \left[\int_{\mathscr{X}} (f \circ \pi) \, d\theta - i(\theta|\mu) \right] \geqslant \int_{\mathscr{X}} (f \circ \pi) \, d\lambda - i(\lambda|\mu).$$

Using this inequality and (8), we conclude that for any $f \in C(\mathscr{Y})$ and any λ such that $\rho = \lambda \circ \pi^{-1}$, one has

$$\int_{\mathscr{Y}} f \, d\rho - i(\lambda|\mu) \leqslant p(f|\nu).$$

Taking first the supremum over all $\lambda \in \mathcal{M}_T(\mathscr{X})$, $\lambda \circ \pi^{-1} = \rho$, and then supremum over all $\rho \in \mathcal{M}_S(\mathscr{Y})$, we conclude that

$$\sup_{\rho \in \mathcal{M}_S(\mathscr{Y})} \left[\int_{\mathscr{Y}} f \, d\rho - i(\rho|\nu) \right] \leqslant p(f|\nu),$$

where we also used the variational characterization of $i(\rho|\nu)$ from Theorem 19.

Let us prove the opposite inequalities. Using the variational principle for the Gibbs measure μ, $\mu \circ \pi^{-1} = \nu$, we have

$$p(f|\nu) = p(f \circ \pi|\mu) = \sup_{\lambda \in \mathcal{M}_T(\mathscr{X})} \left[\int_{\mathscr{X}} (f \circ \pi) \, d\theta - i(\lambda|\mu) \right].$$

Therefore there exists a sequence of measures $\{\lambda_n\}_{n \geqslant 1} \subset \mathcal{M}_T(\mathscr{X})$ such that

$$\int_X (f \circ \pi) \, d\lambda_n - i(\lambda_n|\mu) \geqslant p(f \circ \pi|\mu) - \frac{1}{n} = p(f|\nu) - \frac{1}{n}.$$

Let $\rho_n = \lambda_n \circ \pi^{-1}$. Since $i(\lambda_n|\mu) \geqslant i(\rho_n|\nu)$, we obtain that

$$\int_{\mathscr{Y}} f \, d\rho_n - i(\rho_n|\nu) \geqslant p(f|\nu) - \frac{1}{n}.$$

Therefore

$$\sup_{\rho \in \mathcal{M}_S(\mathscr{Y})} \left[\int_{\mathscr{Y}} f \, d\rho - i(\rho|\nu) \right] \geqslant p(f|\nu).$$

This finishes the proof. $\square$

Proof of Theorem 22. If $\rho = \lambda \circ \pi^{-1}$ for some $\lambda \in \mathcal{G}_{\mathscr{X}}(\Phi)$, then obviously

$$i(\rho|\nu) \leqslant i(\lambda|\mu) = 0,$$

by Theorem 19 and the Gibbs variational principle (Theorem 9) for λ, μ. Since $i(\rho|\nu) \geqslant 0$, we conclude that $i(\rho|\nu) = 0$.

Suppose $\rho \in \mathcal{M}_S(\mathscr{Y})$ is such that $i(\rho|\nu) = 0$. Then

$$0 = i(\rho|\nu) \stackrel{(a)}{=} P(f_\Phi) - h(\rho) - \int_{\mathscr{Y}} P(f_\Phi, \pi) \, d\rho$$

$$\stackrel{(b)}{=} P(f_\Phi) - \sup_{\lambda \in \mathcal{M}_T(\mathscr{X}): \lambda \circ \pi^{-1} = \rho} \left[h(\lambda) + \int_{\mathscr{X}} f_\Phi \, d\lambda \right],$$

where we used (a) the result of Theorem 19 for fuzzy Gibbs measures, (b) the relativized variational principle for f_Φ and ρ. It is not very difficult to see[1] that the

---

[1]Follows from upper semi-continuity of an entropy function $\lambda \to h(\lambda)$. Indeed, let $\{\lambda_n\}$ be a sequence of measures such that $\lambda_n \circ \pi = \rho$ and $P(f_\Phi) = \lim_{n \to \infty} (h(\lambda_n) + \int_{\mathscr{X}} f_\Phi d\lambda_n)$. Let λ be any limit point of the sequence $\{\lambda_n\}$. Then clearly, $\lambda \circ \pi^{-1} = \rho$, and by upper-semicontinuity of the entropy $h(\lambda) + \int_{\mathscr{X}} f d\lambda \geqslant \limsup_{n \to \infty} (h(\lambda_n) + \int_{\mathscr{X}} f_\Phi d\lambda_n) = P(f_\Phi)$.

supremum is achieved for some measure λ with $\rho = \lambda \circ \pi^{-1}$. This means that

$$h(\lambda) + \int_{\mathscr{X}} f_\Phi \, d\lambda = P(f_\Phi),$$

and hence λ is an equilibrium state for f_Φ. By the Gibbs variational principle (Theorem 7), this implies that λ is a Gibbs measure for Φ. This finishes the proof. $\qquad\square$

5. Applications of Variational Principle for Fuzzy Gibbs Measures

5.1. Three fundamental theorems. In the seminal paper [21], which formalized the treatment of renormalization transformations of Gibbs measures, two fundamental theorems (Theorems 23 and 24) have been established. We formulate these results for fuzzy transformation and give short proofs using the variational principles established in the previous sections.

Theorem 23 (First fundamental theorem: single valuedness of the fuzzy transformations). *Let $\mu, \lambda \in \mathcal{M}_T(\mathscr{X})$ be Gibbs measures for potential Φ, $\Phi \in \mathfrak{B}_1(\mathscr{X})$. Then $\nu = \mu \circ \pi^{-1}$ and $\rho = \lambda \circ \pi^{-1}$ are either both Gibbs for the same potential $\Psi \in \mathfrak{B}_1(\mathscr{Y})$, or not Gibbs.*

Proof. By the previous results $i(\rho|\nu) = i(\nu|\rho) = 0$. If ν is Gibbs for some potential Ψ, then $i(\rho|\nu) = 0$ and by the Gibbs variational principle, ρ is Gibbs with the same potential Ψ. $\qquad\square$

Theorem 24 (Second fundamental theorem: continuity of the fuzzy transformations). *If $\mathcal{G}_{\mathscr{X}}(\Phi_i) \circ \pi^{-1} \subset \mathcal{G}_{\mathscr{Y}}(\Psi_i)$, $i = 1, 2$, then*

$$\|\Psi_1 - \Psi_2\|_{\mathscr{Y}} \leqslant K \|\Phi_1 - \Phi_2\|_{\mathscr{X}}, \tag{9}$$

for some constant K.

Proof. There are several ways to establish this fact. One could repeat the original proof of Theorem 3.6 in [21], and use the results above to strengthen the claim by showing that in fact K can be set to 1. Alternative derivation (again with $K = 1$) is based on part (a) of Theorem 26 below: namely, for every $i = 1, 2$ there exists a constant c_i such that

$$g_i = f_{\Phi_i} - f_{\Psi_i} \circ \pi + c_i$$

is a compensation function. Important fact is that by the result of Walters [24, Theorem 3.3],

$$\int_{\mathscr{Y}} P(g_i, \pi) \, dm = 0$$

for all $m \in \mathcal{M}_S(\mathscr{Y})$. Hence, $P(g_i, \pi)$ is the so-called co-boundary, and can be well approximated by elements of $\mathfrak{J}$. Note that $f_{\Psi_i}(y) = P(f_{\Phi_i}, \pi)(y) - P(g_i, \pi)(y) + c_i$, and using the result of Theorem 12 (c), one can compare both sides of (9) with

$$\|P(f_{\Phi_1}, \pi) - P(f_{\Phi_2}, \pi)\|_{C(\mathscr{Y})/\mathfrak{J}+\mathrm{const}}$$

and derive desired result. $\qquad\square$

Results below actually show that we have a *fuzzy isometry*:

$$\|\!|\Psi_1 - \Psi_2|\!\|_{\mathscr{Y}} = \inf_{\Phi_i : \Phi_i \overset{\pi}{\to} \Psi_i} \|\!|\Phi_1 - \Phi_2|\!\|_{\mathscr{X}}.$$

Hence, we obtain another way to derive previous result:

$$\|\!|\Psi_1 - \Psi_2|\!\|_{\mathscr{Y}} = \inf_{\Phi'_i : \Phi'_i \overset{\pi}{\to} \Psi_i} \|\!|\Phi'_1 - \Phi'_2|\!\|_{\mathscr{X}}$$
$$= \inf_{\Phi'_1 : \Phi'_1 \overset{\pi}{\to} \Psi_1} \|\!|\Phi'_1 - \Phi_2|\!\|_{\mathscr{X}} \leqslant \|\!|\Phi_1 - \Phi_2|\!\|_{\mathscr{X}}.$$

As we have seen above (the first fundamental theorem), if a Gibbs measure for potential Φ is transformed by π into a Gibbs measure for potential Ψ, then so is the whole simplex $\mathcal{G}_{\mathscr{X}}(\Phi)$. The following result states that this transformation is surjective on simplexes of Gibbs measures.

Theorem 25 (Third fundamental theorem: morphism of Gibbs simplexes)**.** *If $\mu \in \mathcal{G}_{\mathscr{X}}(\Phi)$, $\nu \in \mathcal{G}_{\mathscr{Y}}(\Psi)$, and $\nu = \mu \circ \pi^{-1}$. Then $\mathcal{G}_{\mathscr{X}}(\Phi) \ni \lambda \to \lambda \circ \pi^{-1} \in \mathcal{G}_{\mathscr{Y}}(\Psi)$ is onto.*

Proof. By the first fundamental theorem,

$$\pi_*(\mathcal{G}_{\mathscr{X}}(\Phi)) = \big\{ \rho \colon \rho = \lambda \circ \pi^{-1} =: \pi_* \lambda, \ \lambda \in \mathcal{G}_{\mathscr{X}}(\Phi) \big\} \subset \mathcal{G}_{\mathscr{Y}}(\Psi).$$

By the fuzzy Gibbs variational principle π_* is onto. $\qquad\square$

A plausible strengthening of the previous result would be that in case $\mu \in \mathcal{G}_{\mathscr{X}}(\Phi)$, $\nu \in \mathcal{G}_{\mathscr{Y}}(\Psi)$, and $\nu = \mu \circ \pi^{-1}$, the map $\lambda \to \lambda \circ \pi^{-1}$ is not only surjective as a map from $\mathcal{G}_{\mathscr{X}}(\Phi)$ to $\mathcal{G}_{\mathscr{Y}}(\Psi)$, but is injective as well.

Following [21], denote by $\mathfrak{R}_\pi$ the set of all pairs $(\Phi, \Psi) \in \mathfrak{B}_1(\mathscr{X}) \times \mathfrak{B}_1(\mathscr{Y})$ such that there exists a Gibbs measure μ with potential Φ and $\nu = \mu \in \pi^{-1}$ is Gibbs for Ψ:

$$\mathfrak{R}_\pi = \big\{ (\Phi, \Psi) \in \mathfrak{B}_1(\mathscr{X}) \times \mathfrak{B}_1(\mathscr{Y}) \colon$$
$$\exists \mu \in \mathcal{G}_{\mathscr{X}}(\Phi), \ \nu \in \mathcal{G}_{\mathscr{Y}}(\Psi) \text{ such that } \nu = \mu \circ \pi^{-1} \big\}.$$

Theorem 26. *For each pair $(\Phi_0, \Psi_0) \in \mathfrak{R}_\pi$, the following holds*:

(a) *There exists $c \in \mathbb{R}$ such that $f_{\Phi_0} - f_{\Psi_0} \circ \pi + c$ is a compensation function.*

(b) *For $\Psi \in \mathfrak{B}_1(\mathscr{Y})$, let $\Phi := \Phi_0 - \Psi_0 \circ \pi + \Psi \circ \pi \in \mathfrak{B}_1(\mathscr{X})$. Then $(\Phi, \Psi) \in \mathfrak{R}_\pi$. In other words, there exists $\mu \in \mathcal{G}_{\mathscr{X}}(\Phi)$ such that $\nu = \mu \circ \pi^{-1} \in \mathcal{G}_{\mathscr{Y}}(\Psi)$.*

(c) *Let*
$$\mathfrak{R}_\pi(\Psi) = \{ \Phi \in \mathfrak{B}_1(\mathscr{X}) \colon (\Phi, \Psi) \in \mathfrak{R}_\pi \},$$
then for all $\Psi, \Psi' \in \mathfrak{B}_1(\mathscr{Y})$, $\mathfrak{R}_\pi(\Psi)$ and $\mathfrak{R}_\pi(\Psi')$ are isomorphic.

Before we proceed with the proof of the Theorem 26, let us establish two relatively simple facts.

Lemma 27. *The following holds*:

(a) *For every $f \in C(\mathscr{X})$, $g \in C(\mathscr{Y})$ and any $\nu \in \mathcal{M}_S(\mathscr{Y})$*
$$\int_{\mathscr{Y}} P(f + g \circ \pi, \pi) \, d\nu = \int_{\mathscr{Y}} P(f, \pi) \, d\nu + \int_{\mathscr{Y}} g \, d\nu.$$

(b) *If* $f \in C(\mathscr{X})$ *is such that for every* $\nu \in \mathcal{M}_S(\mathscr{Y})$

$$\int_{\mathscr{Y}} P(f, \pi) \, d\nu = 0, \tag{10}$$

then f *is a compensation function.*

Proof. (a) Applying the relativized variational principle first for ν and $f + g \circ \pi$, and then for ν and f one has

$$h(\nu) + \int_{\mathscr{Y}} P(f + g \circ \pi, \pi) \, d\nu = \sup_{\mu \in \mathcal{M}_T(\mathscr{X}): \mu \circ \pi^{-1} = \nu} \left\{ h(\mu) + \int_{\mathscr{X}} f \, d\mu + \int_{\mathscr{X}} g \circ \pi \, d\mu \right\}$$

$$= \sup_{\mu \in \mathcal{M}_T(\mathscr{X}): \mu \circ \pi^{-1} = \nu} \left\{ h(\mu) + \int_{\mathscr{X}} f \, d\mu \right\} + \int_{\mathscr{Y}} g \, d\nu$$

$$= h(\nu) + \int_{\mathscr{Y}} P(f, \pi) \, d\nu + \int_{\mathscr{Y}} g \, d\nu,$$

and the result follows.

(b) Suppose $f \in C(\mathscr{X})$ is such that (10) holds for any $\nu \in \mathcal{M}_S(\mathscr{Y})$. Take an arbitrary $g \in C(\mathscr{Y})$. Then

$$P(f + g \circ \pi) = \sup_{\mu \in \mathcal{M}_T(\mathscr{X})} \left\{ h(\mu) + \int_{\mathscr{X}} (f + g \circ \pi) \, d\mu \right\}$$

$$= \sup_{\nu \in \mathcal{M}_S(\mathscr{Y})} \sup_{\mu \in \mathcal{M}_T(\mathscr{X}): \mu \circ \pi^{-1} = \nu} \left\{ h(\mu) + \int_{\mathscr{X}} f \, d\mu + \int_{\mathscr{Y}} g \, d\nu \right\}$$

$$= \sup_{\nu \in \mathcal{M}_S(\mathscr{Y})} \left\{ h(\nu) + \int_{\mathscr{Y}} P(f, \pi) \, d\nu + \int_{\mathscr{Y}} g \, d\nu \right\}$$

$$= \sup_{\nu \in \mathcal{M}_S(\mathscr{Y})} \left\{ h(\nu) + \int_{\mathscr{Y}} g \, d\nu \right\} = P(g).$$

Hence, f is a compensation function. $\qquad\square$

Proof of Theorem 26. (a) Since $(\Phi_0, \Psi_0) \in \mathfrak{R}_\pi$, one can find $\mu \in \mathcal{G}_{\mathscr{X}}(\Phi_0)$ and $\nu \in \mathcal{G}_{\mathscr{Y}}(\Psi_0)$ such that $\nu = \mu \circ \pi^{-1}$. Let ρ be an arbitrary translation invariant measure on $\mathscr{Y}$, and let us compute the relative entropy density $i(\rho|\nu)$. In fact we can do so in 2 ways. First of all, we can use the Gibbs variational principle for ν and Ψ_0 (Theorem 21), and secondly, since ν is a fuzzy image of μ, we can use the result of Theorem 19. Therefore

$$i(\rho|\nu) = P_{\mathscr{Y}}(f_{\Psi_0}) - h(\rho) - \int_{\mathscr{Y}} f_{\Psi_0} \, d\rho$$

$$= P_{\mathscr{X}}(f_{\Phi_0}) - h(\rho) - \int_{\mathscr{Y}} P(f_{\Phi_0}, \pi) \, d\rho.$$

Finally, the previous equality implies that for

$$F = f_{\Phi_0} - f_{\Psi_0} \circ \pi - P_{\mathscr{X}}(f_{\Phi_0}) + P_{\mathscr{Y}}(f_{\Psi_0}),$$

we have

$$\int_{\mathscr{Y}} P(F, \pi)\, d\rho = 0 \tag{11}$$

for any $\rho \in \mathcal{M}_S(\mathscr{Y})$, and hence F is a compensation function by previous lemma.

(b) The proof of this statement is completely analogous to the proofs of [2] and [24, Corollary 3.3]. Let $(\Phi_0, \Psi_0) \in \mathfrak{R}_\pi$. Then $F = f_{\Phi_0} - f_{\Psi_0} \circ \pi + c$ is a compensation function for some c. Suppose $\Psi \in \mathfrak{B}_1(\mathscr{Y})$. We have to show that $\Phi := \Phi_0 - \Psi_0 \circ \pi + \Psi \circ \pi$ is such that $(\Phi, \Psi) \in \mathfrak{R}_\pi$, i.e., for some (and hence all) $\mu \in \mathcal{G}_{\mathscr{X}}(\Phi)$, $\nu = \mu \circ \pi^{-1}$ is a Gibbs measure for Ψ. Note that for any $c \in \mathbb{R}$, Φ and $\Phi + c$ are physically equivalent, and hence define the same simplexes of Gibbs measures. Hence, $\mu \in \mathcal{G}_{\mathscr{X}}(\Phi + c)$. Take $c \in \mathbb{R}$ such that $F = f_{\Phi_0} - f_{\Psi_0} \circ \pi + c$ is a compensation function. Therefore

$$
\begin{aligned}
P_{\mathscr{Y}}(f_\Psi) &= P_{\mathscr{X}}(F + f_\Psi \circ \pi) = h(\mu) + \int_{\mathscr{X}} F\, d\mu + \int_{\mathscr{X}} f_\Psi \circ \pi\, d\mu \\
&= \sup_{\lambda \in \mathcal{M}_T(\mathscr{X}): \lambda \circ \pi^{-1} = \mu \circ \pi^{-1}} \left\{ h(\lambda) + \int_{\mathscr{X}} F\, d\lambda + \int_{\mathscr{X}} f_\Psi \circ \pi\, d\lambda \right\} \\
&= \sup_{\lambda \in \mathcal{M}_T(\mathscr{X}): \lambda \circ \pi^{-1} = \nu} \left\{ h(\lambda) + \int_{\mathscr{X}} F\, d\lambda \right\} + \int_{\mathscr{Y}} f_\Psi\, d\nu \\
&= h(\nu) + \int_{\mathscr{Y}} P(F, \pi)\, d\nu + \int_{\mathscr{Y}} f_\Psi\, d\nu = h(\nu) + \int_{\mathscr{Y}} f_\Psi\, d\nu,
\end{aligned}
$$

where we have used the variational principle for μ, the relative variational principle for ν and F, and finally, the equality (11). The previous equality shows that ν is an equilibrium state for f_Ψ, and hence by the variational principle, ν is also a Gibbs state for Ψ.

(c) The fibre isomorphism was first observed in [2] in $d = 1$ with the transformation given by

$$\mathfrak{R}_\pi(\Psi) \to \mathfrak{R}_\pi(\Psi'): \Phi \mapsto \Phi - \Psi \circ \pi + \Psi' \circ \pi.$$

Part (b) establishes that the same map works in any dimension. $\qquad\square$

6. Concluding Remarks and Open Problemsx

6.1. Identification of potentials of fuzzy Gibbs measures. Suppose μ is a Gibbs measure on $\mathscr{X}$ for a known potential $\Phi \in \mathfrak{B}_1(\mathscr{X})$, and let $\nu = \mu \circ \pi^{-1}$. Is ν a Gibbs measure? And if so, can one identify the potential Ψ for ν? Let us assume that Φ is such that $P(f_\Phi) = 0$, and we will be looking for a potential $\Psi \in \mathfrak{B}_1(\mathscr{Y})$ with $P(f_\Psi) = 0$ as well.

In the preceding sections we showed that ν is an equilibrium state for $P(f_\Phi, \pi)(y)$. Moreover, in the case, ν is Gibbs for some Ψ

$$\int_{\mathscr{Y}} P(f_\Phi, \pi)(y)\rho(dy) = \int_{\mathscr{Y}} f_\Psi(y)\rho(dy) \tag{12}$$

for any translation invariant measure ρ. Note that $P(f_\Phi, \pi) : \mathscr{Y} \to \mathbb{R}$ is "severely" discontinuous. On the other hand, (12) implies that for any $\rho \in \mathcal{M}_S(\mathscr{Y})$ one has

$$\int_{\mathscr{Y}} P(f_\Phi, \pi)(y)\rho(dy) = \int_{\mathscr{Y}} f_\Psi^*(y)\rho(dy), \qquad (13)$$

where

$$f_\Psi^*(y) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{k \in \Lambda_n} f_\Psi(S_k y),$$

if the limit exists, and say $+\infty$, if the limit does not exist. Note that by ergodic theorem, $f_\Psi^*(y)$ is well defined on a total probability set and is an invariant function. Hence, for Gibbs ν and for every $\rho \in \mathcal{M}_S(\mathscr{Y})$

$$P(f_\Phi, \pi)(y) = f_\Psi^*(y) \quad \text{for } \rho\text{-a.a. } y \in Y.$$

The relative pressure $P(f_\Phi, \pi)$ was defined as an upper limit. If we introduce

$$P^*(f_\Phi, \pi) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \log \sum_{x_{\Lambda_n} : \pi x_{\Lambda_n} = y_{\Lambda_n}} \sup_{\tilde{x} \in [x_{\Lambda_n}]} \exp\Big(\sum_{k \in \Lambda_n} f(T_k x) \Big)$$

provided the limit exists, and $P^*(f_\Phi, \pi) = +\infty$ otherwise, then a slightly stronger statement is true.

Theorem 28. *Measure $\nu = \mu \circ \pi^{-1}$, $\mu \in \mathcal{G}_\mathscr{X}(\Phi)$ with $P(f_\Phi) = 0$ is a Gibbs measure for potential $\Psi \in \mathfrak{B}_1(\mathscr{Y})$, $P(f_\Psi) = 0$, if and only if*

$$P^*(f_\Phi, \pi)(y) = f_\Psi^*(y) \quad \text{for all } y \in \mathscr{Y}.$$

Proof. Define a local entropy of ν at $y \in \mathscr{Y}$ as

$$h(\nu, y) = \lim_{n \to \infty} -\frac{1}{|\Lambda_n|} \log \nu([y_{\Lambda_n}]),$$

provided the limit exists. It is not very difficult to see that $h(\nu, y)$ exists if and only if $P(f_\Phi, \pi)(y)$ exists. Similarly, since ν is Gibbs for Ψ, $h(\nu, y)$ exists if and only if $f_\Psi^*(y)$ exists. Hence, the statement follows. $\square$

6.2. Open questions. The problem of finding necessary and sufficient conditions for $\nu = \mu \circ \pi^{-1}$ to be Gibbs in terms of potential and the fuzzy factor map π, remains open. Some indications how these conditions should look like is given in [21]. However, at the present moment this question is far from being resolved. One would also like to have a practical test for Gibbsianity of a transformed measure which covers most of the known examples.

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