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The decentralized implementability problem

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Abstract— **This paper deals with the problems of decentralized implementability and decentralized regular implementability in the context of finite-dimensional linear differential system behaviors. Given a plant behavior with a pre-specified partition of the system variable and a desired behavior, the problem of decentralized implementability is to find a controller which is decentralized with respect to the given partition and implements (regularly) the desired behavior with respect to the plant. In this paper we formulate these problems in the behavioral framework, with control as interconnection and we also provide necessary and sufficient conditions for the solvability of these problems.**

Keywords: behaviors, implementability, regular interconnection, decentralized controllers.

I. Introduction

For large scale systems like power networks, digital communication networks, economic systems and flexible manufacturing systems, decentralized control is one of the prominent strategies for control. Such systems are often characterized by geographical separation, large dimension, or consists of many interconnected subsystems. For such systems it is computationally efficient to formulate control laws that use only locally available control variables. As it is easy to implement, and less cost is involved in communication overhead, this approach is also economical. In fact, the decentralized structure is an essential design constraint on controllers in situations where it is prohibited to exchange information between the subsystems. The analysis and the design of decentralized control has been intensively considered for over three decades. For the vast body of literature on decentralized control in an input-output framework, we refer the reader to the survey papers [10], [11], books [5], [12] and journal articles [2], [3], [13], [16], and references therein.

In this paper we will discuss the problem of decentralized control in the behavioral framework. In contrast to [2], [3] and [16], we work in the generality where we view systems in a behavioral sense, that is, as families of trajectories, and control is viewed as restricting the plant behavior by intersecting it with a controller behavior. In particular we will discuss the problem of implementability by decentralized control. The implementability problem may be considered as a basic question in control: given a plant behavior, together with some 'desired' behavior, the latter is called *implementable* (sometimes called: achievable) if it can be achieved as controlled behavior by interconnecting the plant with a suitable controller. The implementability problem was studied extensively in [18] and [1], and necessary and sufficient conditions were established for the full as well as partial interconnection case, both for general as well as regular interconnections. In this paper we will formulate the problem of *decentralized* implementability. Here the problem is, for a given plant, to characterize all desired behaviors that can be achieved (implemented) by means of decentralized controllers. A decentralized controller is a controller that only gives 'local' constraints on the control variable. In particular, for a given partition of the control variable into 'local' variables, a controller is called decentralized if it only involves laws on these local variables. In this paper, we will restrict ourselves to the full interconnection version of this problem. We will derive conditions for implementability and regular implementability using such decentralized controllers.

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A. Notation and nomenclature

A few words about the notation and nomenclature used. We use standard symbols for the fields of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$. $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. $\mathbb{R}[\xi]$ denotes the ring of polynomials in the indeterminate ξ with real coefficients. We use $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$ to denote the space of matrices with components in $\mathbb{R}[\xi]$. Elements of $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$ are called *real polynomial matrices*.

For $n \geq 1$ we use the notation $\underline{n}$ to represent the set $\{1,2,\ldots,n\}$. Given matrices A_i, $i \in \underline{n}$, we use the notation $\mathrm{blockdiag}(A_1,A_2,\ldots,A_n)$ to represent the block diagonal matrix with diagonal blocks A_i. Finally, we use the notation $\mathrm{col}(w_1,w_2)$ to represent the column vector formed by stacking w_1 over w_2.

II. Linear differential systems and polynomial kernel representations

In the behavioral approach a dynamical system is given by a triple $\Sigma = (T,W,\mathfrak{B})$, where T is the time axis, W is the signal space, and the behavior $\mathfrak{B}$ is a subset of W^T, the set of all functions from T to W. A *linear differential system* is a dynamical system with time axis $T=\mathbb{R}$, and whose signal space W is a finite dimensional Euclidean space, say, $\mathbb{R}^{\mathtt{w}}$. Correspondingly, the manifest variable is then given as $w=\mathrm{col}(w_1,w_2,\ldots,w_{\mathtt{w}})$. The behavior $\mathfrak{B}$ is a linear subspace of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exists a positive integer $\mathtt{g}$ and a polynomial matrix $R\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ such that

$$\mathfrak{B}=\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{\mathrm{d}}{\mathrm{d}t})w=0\}.$$

The set of linear differential systems with manifest variable w taking its value in $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{L}^{\mathtt{w}}$.

Let $R\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ be a polynomial matrix. If the behavior $\mathfrak{B}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w=0$ then we call this a *kernel representation* of $\mathfrak{B}$. Further, a kernel representation is said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least $\mathtt{g}$ rows. A given kernel representation, $R(\frac{\mathrm{d}}{\mathrm{d}t})w=0$, is minimal if and only if the polynomial matrix R has full row rank (see [8], Theorem 3.6.4). The number of rows in any minimal polynomial kernel representation of $\mathfrak{B}$ is equal to the *output cardinality* of $\mathfrak{B}$, denoted by $\mathtt{p}(\mathfrak{B})$. This number corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$.

We speak of a system as the behavior $\mathfrak{B}$, one of whose representations is given by $R(\frac{\mathrm{d}}{\mathrm{d}t})w=0$ or just $\mathfrak{B}=\ker(R)$. The '$\frac{\mathrm{d}}{\mathrm{d}t}$' is often suppressed to enhance readability.

The *controllable part* of a behavior $\mathfrak{B}$ is defined as the largest controllable sub-behavior of $\mathfrak{B}$. This is denoted by $\mathfrak{B}_{\mathrm{cont}}$ (see [8]).

Definition 2.1: Let $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable w partitioned as $w=(w_1,w_2)$. We will call w_2 *free in* $\mathfrak{B}$ if, for any $w_2\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$, there exists w_1 such that $(w_1,w_2)\in\mathfrak{B}$. We call w_2 *maximally free* if it is free, and we can not enlarge this set with components from w_1 and still continue to have freeness for this enlarged set of variables.

The following result was shown in [8]:

Proposition 2.2: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1, w_2). Let a minimal kernel representation of $\mathfrak{B}$ be given by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. Then w_2 is free in $\mathfrak{B}$ if and only if the polynomial matrix R_1 has full row rank.

We now review some facts on elimination of variables. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable $w = (w_1, w_2)$. Let P_{w_1} denote the projection onto the w_1-component. Then the set $P_{w_1}\mathfrak{B}$ consisting of all w_1 for which there exists w_2 such that $(w_1, w_2) \in \mathfrak{B}$ is again a linear differential system. We denote $P_{w_1}\mathfrak{B}$ by $(\mathfrak{B})_{w_1}$, and call it the behavior obtained by eliminating w_2 from $\mathfrak{B}$.

If $\mathfrak{B} = \ker\begin{pmatrix} R_1 & R_2 \end{pmatrix}$, then a representation for $(\mathfrak{B})_{w_1}$ is obtained as follows: choose a unimodular matrix U such that $UR_2 = \begin{pmatrix} R_{12} \\ 0 \end{pmatrix}$, with R_{12} full row rank, and conformably partition $UR_1 = \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix}$. Then $(\mathfrak{B})_{w_1} = \ker(R_{21})$ (see [8], section 6.2.2).

Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$, the behavior consisting of the trajectories w_1 with the variable w_2 put equal to zero is denoted by $\mathcal{N}_{w_1}(\mathfrak{B})$, and is called the *hidden behavior*. It is defined as

$$\mathcal{N}_{w_1}(\mathfrak{B}) = \{w_1 \mid (w_1, 0) \in \mathfrak{B}\}.$$

III. Implementability

The problem of implementability deals with the question which controlled behaviors can be achieved by interconnecting a given plant with a controller. This problem may actually be considered as a basic question in engineering design: a behavior is prescribed, and the question is whether this "desired" behavior can be achieved by inserting a suitably designed subsystem into the over-all system. Details on the implementability problem can be found in [18].

In this section we will review the *full interconnection case*. In that case we have a plant behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$, and a controller for $\mathcal{P}$ is also a behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. The full interconnection of $\mathcal{P}$ and $\mathcal{C}$ is the system whose behavior is the intersection $\mathcal{P} \cap \mathcal{C}$. This controlled behavior is again a linear differential system. Indeed, if $\mathcal{P} = \ker(R)$ and $\mathcal{C} = \ker(C)$, then $\mathcal{P} \cap \mathcal{C} = \ker\begin{pmatrix} R \\ C \end{pmatrix} \in \mathfrak{L}^{\mathtt{w}}$.

Definition 3.1: Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be a given behavior, to be interpreted as a desired behavior. If $\mathcal{K}$ can be achieved as controlled behavior, i.e., if there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$, then we call $\mathcal{K}$ *implementable by full interconnection*(with respect to $\mathcal{P}$).

Obviously, a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable with respect to $\mathcal{P}$ by full interconnection if and only if $\mathcal{K} \subset \mathcal{P}$. Indeed, if $\mathcal{K} \subset \mathcal{P}$, then with 'controller' $\mathcal{C} = \mathcal{K}$ we have $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$. It is well known [8] that if $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$ and $\mathfrak{B}_1 = \ker(R_1)$, $\mathfrak{B}_2 = \ker(R_2)$ are kernel representations, then $\mathfrak{B}_1 \subset \mathfrak{B}_2$ if and only if there exists a polynomial matrix F such that $R_2 = FR_1$. For easy reference we therefore state:

Proposition 3.2: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{P} = \ker(R)$ and $\mathcal{K} = \ker(K)$ be kernel representations. Then the following are equivalent:

1) $\mathcal{K}$ is implementable with respect to $\mathcal{P}$ by full interconnection,
2) there exists a polynomial matrix F such that $R = FK$.

In the behavioral framework one often needs to require that the interconnection of plant and controller is a *regular interconnection*. Detailed material can be found in [1]. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ be a plant behavior, and let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ be a controller.

Definition 3.3: The interconnection of $\mathcal{P}$ and $\mathcal{C}$ is called *regular* if

$$\mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}) = \mathtt{p}(\mathcal{P} \cap \mathcal{C}),$$

in other words, if the output cardinalities of the plant and the controller add up to the output cardinality of the controlled behavior. In that case, we also call the controller $\mathcal{C}$ regular (with respect to $\mathcal{P}$).

In terms of kernel representations this condition can be expressed as follows. Let $\mathcal{P} = \ker(R)$ and $\mathcal{C} = \ker(C)$ be minimal kernel representations of plant and controller, respectively. Then $\mathcal{P} \cap \mathcal{C} = \ker\begin{pmatrix} R \\ C \end{pmatrix}$ is a kernel representation of the controlled behavior. Since the output cardinality of a behavior is equal to the rank of the polynomial matrix in any of its kernel representations, the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is regular if and only if $\begin{pmatrix} R \\ C \end{pmatrix}$ has full row rank, equivalently yields a minimal kernel representation of $\mathcal{P} \cap \mathcal{C}$.

Definition 3.4: Given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$, a given behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is called *regularly implementable by full interconnection* (with respect to $\mathcal{P}$) if there exists a regular controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ that implements $\mathcal{K}$ by full interconnection.

The following result from [1] gives a characterization of all regularly implementable behaviors.

Proposition 3.5: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{P}_{\mathrm{cont}}$ be its controllable part. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Then the following statements are equivalent:

1) $\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$,
2) $\mathcal{K} + \mathcal{P}_{\mathrm{cont}} = \mathcal{P}$.

The previous result does not use representations of the behaviors involved. The following result characterizes regular implementability in terms of kernel representations (see [9]):

Proposition 3.6: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{P} = \ker(R)$ and $\mathcal{K} = \ker(K)$ be minimal kernel representations of plant and desired behavior. Then the following are equivalent:

1) $\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$,
2) there exists a polynomial matrix F with $F(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$ such that $R = FK$.

Having obtained necessary and sufficient conditions for implementability and regular implementability of a given desired behavior $\mathcal{K}$, we now aim at establishing characterizations of *all* controllers $\mathcal{C}$ that (regularly) implement it. Parameterizations of these controllers have been established before in [9]. In the following two lemmas we formulate alternative characterizations:

Lemma 3.7: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Assume that $\mathcal{K}$ is implementable with respect to $\mathcal{P}$. Let $\mathcal{P} = \ker(R)$ and $\mathcal{K} = \ker(K)$ be minimal kernel representations and let F be a polynomial matrix such that $R = FK$. Then the following statements are equivalent:

1) $\mathcal{C} = \ker(C)$ implements $\mathcal{K}$ by full interconnection,
2) there exists a polynomial matrix L such that $C = LK$, where $\begin{pmatrix} F(\lambda) \\ L(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$.

Proof:

((1) ⇒ (2))

If $\ker(C)$ implements $\ker(K)$ by full interconnection, then $\ker\begin{pmatrix} R \\ C \end{pmatrix} = \ker(K)$. Since K has full row rank, we must have

$$\begin{pmatrix} R \\ C \end{pmatrix} = U \begin{pmatrix} K \\ 0 \end{pmatrix}$$

for some unimodular matrix $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ and some zero-matrix 0 with an appropriate number of rows. This implies $R = U_{11}K$ and $C = U_{21}K$. It follows that $U_{11} = F$. Define $L := U_{21}$. Then $\begin{pmatrix} F(\lambda) \\ L(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$.

((2) ⇒ (1))

Assume $C = LK$. We have

$$\begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} F \\ L \end{pmatrix} K.$$

Clearly, since $\begin{pmatrix} F(\lambda) \\ L(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$, we have $\ker \begin{pmatrix} R \\ C \end{pmatrix} = \ker(K)$. $\square$

Lemma 3.8: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Assume that $\mathcal{K}$ is regularly implementable with respect to $\mathcal{P}$. Let $\mathcal{P} = \ker(R)$ and $\mathcal{K} = \ker(K)$ be minimal kernel representations and let F be a polynomial matrix with $F(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$ such that $R = FK$. Then the following statements are equivalent:

1) $\mathcal{C} = \ker(C)$ regularly implements $\mathcal{K}$ by full interconnection and $\ker(C)$ is a minimal representation of $\mathcal{C}$,
2) there exists a polynomial matrix L such that $C = LK$, where $\begin{pmatrix} F \\ L \end{pmatrix}$ is unimodular.

Proof:
((1) ⇒ (2))
If $\ker(C)$ regularly implements $\ker(K)$ by full interconnection, then $\ker \begin{pmatrix} R \\ C \end{pmatrix} = \ker(K)$. Since the interconnection is regular, both kernel representations are minimal. Hence there exists a unimodular matrix $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ such that

$$\begin{pmatrix} R \\ C \end{pmatrix} = UK,$$

which implies that $R = U_1 K$ and $C = U_2 K$. It follows that $U_1 = F$. Define $L := U_2$. Then $\begin{pmatrix} F \\ L \end{pmatrix}$ is unimodular.
((2) ⇒ (1))
Assume $C = LK$. We have

$$\begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} F \\ L \end{pmatrix} K.$$

Clearly, since $\begin{pmatrix} F \\ L \end{pmatrix}$ is unimodular, we have $\ker \begin{pmatrix} R \\ C \end{pmatrix} = \ker(K)$, so $\ker(C)$ implements $\mathcal{K}$. Also, the interconnection is regular since $\begin{pmatrix} R \\ C \end{pmatrix}$ has full row rank. $\square$

IV. Decentralized implementability

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ be a given plant behavior, with system variable w. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be a desired behavior. In this section we will deal with the problem to find *decentralized controllers* that implement $\mathcal{K}$ by full interconnection. A decentralized controller is a controller that only gives 'local' constraints on the control variable w. In particular, for a given partition of the variable w into

$$w = (w_1, w_2, w_3, \ldots, w_n) \tag{1}$$

with w_i taking values in $\mathbb{R}^{\mathtt{w}_i}$ ($i \in \underline{n}$), a controller is called decentralized if it only involves laws on the local variables w_i. More precisely:

Definition 4.1: Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$, with system variable w, to be interpreted as a controller. Let w be partitioned as $w = (w_1, w_2, \ldots, w_n)$ with w_i of dimension $\mathtt{w}_i$, $\mathtt{w} = \Sigma_{i=1}^{n} \mathtt{w}_i$. Then $\mathcal{C}$ is called *decentralized* with respect to the partition of the system variable if for all $i \in \underline{n}$ there exists $\mathcal{C}_i \in \mathfrak{L}^{\mathtt{w}_i}$ with system variable w_i such that $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n$.

The following proposition characterizes for a given behavior the property of being decentralized:

Proposition 4.2: Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ with system variable w partitioned as $w = (w_1, w_2, \ldots, w_n)$ with w_i of dimension $\mathtt{w}_i$. Then the following statements are equivalent.

1) $\mathcal{C}$ is decentralized with respect to the given partition.
2) There exists polynomial matrices $C_i \in \mathbb{R}^{\bullet \times \mathtt{w}_i}[\xi]$ such that $\mathcal{C}$ admits a kernel representation $\mathcal{C} = \ker(C)$, where $C = \text{blockdiag}(C_1, C_2, \ldots, C_n)$.
3) $(\mathcal{C})_{w_i} = \mathcal{N}_{w_i}(\mathcal{C})$ for all $i \in \underline{n}$.

Proof: From Definition 4.1 the equivalence between statements 1) and 2) is straightforward by defining $\mathcal{C}_i := \ker(C_i)$. We now prove the equivalence of statements 2) and 3) of the Proposition.
((2) ⇒ (3))
If $\mathcal{C} = \ker(\text{blockdiag}(C_1, C_2, \ldots, C_n))$ then we have $(\mathcal{C})_{w_i} = \ker(C_i)$ and $\mathcal{N}_{w_i}(\mathcal{C}) = \ker(C_i)$. Therefore we have $(\mathcal{C})_{w_i} = \mathcal{N}_{w_i}(\mathcal{C})$.
((3) ⇒ (2))
We prove this implication for $n = 2$. For the case $n > 2$ the proof can be given by induction. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1, w_2). Let $\mathcal{C} = \ker \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ be a minimal kernel representation. Then there exists a unimodular matrix U_1 such that $U_1 \begin{pmatrix} C_1 & C_2 \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}$ such that C_{22} has full row rank. Then we have $(\mathcal{C})_{w_1} = \ker(C_{11})$, $\mathcal{N}_{w_1}(\mathcal{C}) = \ker \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}$ and $\mathcal{N}_{w_2}(\mathcal{C}) = \ker(C_{22})$. As $(\mathcal{C})_{w_1} = \mathcal{N}_{w_1}(\mathcal{C})$ and C_{11} has full row rank, there exists a unimodular matrix $\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$ such that $\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} = \begin{pmatrix} C_{11} \\ 0 \end{pmatrix}$. Therefore we have $\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & V_{12}C_{22} \\ 0 & V_{22}C_{22} \end{pmatrix}$. We have

$$\mathcal{C} = \ker \begin{pmatrix} C_{11} & V_{12}C_{22} \\ 0 & V_{22}C_{22} \end{pmatrix}. \tag{2}$$

As C_{11} has full row rank we have $(\mathcal{C})_{w_2} = \ker(V_{22}C_{22})$. As $\mathcal{N}_{w_2}(\mathcal{C}) = (\mathcal{C})_{w_2}$, we have $\ker(C_{22}) = \ker(V_{22}C_{22})$, which implies that V_{22} is a unimodular matrix. It is easy to verify that $\begin{pmatrix} I & -V_{12}V_{22}^{-1} \\ 0 & V_{22}^{-1} \end{pmatrix}$ is a unimodular matrix and

$$\begin{pmatrix} I & -V_{12}V_{22}^{-1} \\ 0 & V_{22}^{-1} \end{pmatrix} \begin{pmatrix} C_{11} & V_{12}C_{22} \\ 0 & V_{22}C_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix}. \tag{3}$$

Therefore from Equations (2) and (3) we have

$$\mathcal{C} = \ker \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix}. \tag{4}$$

$\square$

Given a plant $\mathcal{P}$ together with a partition (1) of its variable, and a given desired behavior $\mathcal{K}$ we now deal with the question whether $\mathcal{K}$ can be implemented by means of a decentralized controller. We give the following definitions:

Definition 4.3: Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Assume the system variable w is partitioned as in (1). We call $\mathcal{K}$ *decentralized implementable* with respect to $\mathcal{P}$ if there exists a decentralized controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$.

Definition 4.4: Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Assume the system variable w is partitioned as in (1). We call $\mathcal{K}$ *decentralized regularly implementable* with respect to $\mathcal{P}$ if there exists a decentralized regular controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$.
In the following we want to establish conditions for a given desired behavior $\mathcal{K}$ to be decentralized (regularly) implementable with respect to $\mathcal{P}$. For simplicity, we assume that the system variable w is partitioned into two parts, $w = (w_1, w_2)$. The following theorem gives necessary and sufficient conditions for a behavior $\mathcal{K}$ to be decentralized implementable with respect to $\mathcal{P}$:

Theorem 4.5: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ with variable w partitioned as $w = (w_1, w_2)$, and with minimal kernel representation $\mathcal{P} = \ker(R)$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ with minimal kernel representation $\mathcal{K} = \ker(K)$, $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$. Assume that $\mathcal{K}$ is implementable by full interconnection with respect to $\mathcal{P}$ and let F be a polynomial matrix such that $R = FK$. Then $\mathcal{K}$ is decentralized implementable with respect to $\mathcal{P}$ if and only if there exist polynomial matrices L_1, L_2 such that

$$\begin{pmatrix} F(\lambda) \\ L_1(\lambda) \\ L_2(\lambda) \end{pmatrix}$$

has full column rank for all $\lambda \in \mathbb{C}$ and $L_1K_2 = 0$, $L_2K_1 = 0$. In this case a decentralized controller is given by $\mathcal{C} = \ker\begin{pmatrix} L_1K_1 & 0 \\ 0 & L_2K_2 \end{pmatrix}$.

Proof: A proof follows immediately from Lemma 3.7 and Proposition 4.2. □

Along the same lines, decentralized regular implementability is dealt with in the next theorem:

Theorem 4.6: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ with variable w partitioned as $w = (w_1, w_2)$, and with minimal kernel representation $\mathcal{P} = \ker(R)$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ with minimal kernel representation $\mathcal{K} = \ker(K)$, $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$. Assume that $\mathcal{K}$ is regularly implementable with respect to $\mathcal{P}$ and let F be a polynomial matrix with $F(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$ such that $R = FK$. Then $\mathcal{K}$ is decentralized regularly implementable with respect to $\mathcal{P}$ if and only if there exist polynomial matrices L_1, L_2 such that

$$\begin{pmatrix} F \\ L_1 \\ L_2 \end{pmatrix}$$

is unimodular and $L_1K_2 = 0$, $L_2K_1 = 0$. In this case a decentralized regular controller is given by $\mathcal{C} = \ker\begin{pmatrix} L_1K_1 & 0 \\ 0 & L_2K_2 \end{pmatrix}$.

Proof: Again, a proof follows immediately from Lemma 3.8 and Proposition 4.2. □

The following corollaries are immediate consequences of the foregoing:

Corollary 4.7: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ with variable w partitioned as $w = (w_1, w_2)$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ with minimal kernel representation $\mathcal{K} = \ker(K)$, $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$. Denote $\mathtt{k} := \mathtt{p}(\mathcal{K})$. Then $\mathcal{K}$ is decentralized implementable with respect to $\mathcal{P}$ if and only if

1) $\mathcal{K} \subset \mathcal{P}$,
2) there exist behaviors $\mathcal{H}_1 \in \mathfrak{L}^{\mathtt{k}}$, $\mathcal{H}_2 \in \mathfrak{L}^{\mathtt{k}}$ such that $\mathrm{im}(K_1) \subset \mathcal{H}_2$, $\mathrm{im}(K_2) \subset \mathcal{H}_1$ and $K\mathcal{P} \cap \mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$.

Proof: Assume $\mathcal{K}$ is decentralized implementable with respect to $\mathcal{P}$. Then clearly 1) holds. Let $\ker(R)$ be a minimal kernel representation of $\mathcal{P}$ and let F be a polynomial matrix such that $R = FK$. It is easily seen that $\ker(F) = K\mathcal{P}$. There exist L_1 and L_2 such that

$$\begin{pmatrix} F(\lambda) \\ L_1(\lambda) \\ L_2(\lambda) \end{pmatrix} \tag{5}$$

has full column rank for all $\lambda \in \mathbb{C}$ and $L_1K_2 = 0$, $L_2K_1 = 0$. Define $\mathcal{H}_1 := \ker(L_1)$ and $\mathcal{H}_2 := \ker(L_2)$. Then $K\mathcal{P} \cap \mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ and $\mathrm{im}(K_2) \subset \mathcal{H}_1$, $\mathrm{im}(K_1) \subset \mathcal{H}_2$.

Conversely, let $\mathcal{P} = \ker(R)$ be a minimal kernel representation of $\mathcal{P}$. By 1) there exists F such that $R = FK$. Let L_1 and L_2 be such that $\ker(L_1) = \mathcal{H}_1$ and $\ker(L_2) = \mathcal{H}_2$. Then $L_1K_2 = 0$, $L_2K_1 = 0$ and (5) has full column rank for all $\lambda \in \mathbb{C}$. By Theorem 4.5 this yields that $\mathcal{K}$ is decentralized implementable with respect to $\mathcal{P}$. □

In a similar way we can characterize decentralized regular implementability.

Corollary 4.8: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ with variable w partitioned as $w = (w_1, w_2)$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ with minimal kernel representation $\mathcal{K} = \ker(K)$, $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$. Denote $\mathtt{k} := \mathtt{p}(\mathcal{K})$. Then $\mathcal{K}$ is decentralized regularly implementable with respect to $\mathcal{P}$ if and only if

1) $\mathcal{K} + \mathcal{P}_{\mathrm{cont}} = \mathcal{P}$
2) there exist behaviors $\mathcal{H}_1 \in \mathfrak{L}^{\mathtt{k}}$, $\mathcal{H}_2 \in \mathfrak{L}^{\mathtt{k}}$ such that $\mathrm{im}(K_1) \subset \mathcal{H}_2$, $\mathrm{im}(K_2) \subset \mathcal{H}_1$ and $K\mathcal{P} \cap \mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ and $\mathtt{p}(\mathcal{H}_1) + \mathtt{p}(\mathcal{H}_2) = \mathtt{k} - \mathtt{p}(\mathcal{P})$.

For the case that the system variable w is partitioned as $w = (w_1, w_2, \ldots, w_n)$ with $n > 2$, analogous results can be formulated. The latter two corollaries express decentralized (regular) implementability of a given $\mathcal{K} = \ker(K)$ with $K = (K_1 \;\; K_2)$ in terms of geometric properties of the behaviors $K\mathcal{P}$, $\mathrm{im}(K_1)$ and $\mathrm{im}(K_2)$. Currently, we investigate how to actually verify these properties computationally.

V. Decentralized control and partial interconnection

In addition to full interconnection, in [18] and [1] results have been established on implementability by *partial* interconnection (see also [6], [7], [4]). In this section, we will establish necessary conditions for decentralized implementability by full interconnection (as introduced in the previous section) in terms of concepts around partial interconnection.

We will first briefly review implementability by partial interconnection. In control by partial interconnection, only a pre-specified subset of the plant variables is available for interconnection. Let $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be a linear differential system, with system variable (w, c), where w takes its values in $\mathbb{R}^{\mathtt{w}}$ and c in $\mathbb{R}^{\mathtt{c}}$. Before the controller acts, there are two behaviors of the plant that are relevant: the behavior $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ (the full plant behavior) of the variables w and c combined, and the behavior $(\mathcal{P}_{\mathrm{full}})_w$ of the to-be-controlled variables w (with the interconnection variable c eliminated). Hence

$$(\mathcal{P}_{\mathrm{full}})_w = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \text{ such that } (w, c) \in \mathcal{P}_{\mathrm{full}}\}.$$

By the elimination theorem, $(\mathcal{P}_{\mathrm{full}})_w \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. The controller $\mathcal{C}$ restricts the interconnection variables c. The *full controlled behavior* $\mathcal{P}_{\mathrm{full}} \wedge_c \mathcal{C}$ is obtained by the interconnection of $\mathcal{P}_{\mathrm{full}}$ and $\mathcal{C}$ through the variable c and is defined as:

$$\mathcal{P}_{\mathrm{full}} \wedge_c \mathcal{C} = \{(w, c) \mid (w, c) \in \mathcal{P}_{\mathrm{full}} \text{ and } c \in \mathcal{C}\}.$$

Eliminating c from the full controlled behavior, we obtain its restriction $(\mathcal{P}_{\mathrm{full}} \wedge_c \mathcal{C})_w$ to the behavior of the to-be-controlled variable w, defined by

$$(\mathcal{P}_{\mathrm{full}} \wedge_c \mathcal{C})_w = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, c \in \mathcal{C} \text{ such that } (w, c) \in \mathcal{P}_{\mathrm{full}}\}.$$

Note that, again by the elimination theorem, $(\mathcal{P}_{\mathrm{full}} \wedge_c \mathcal{C})_w \in \mathfrak{L}^{\mathtt{w}}$.

Definition 5.1: Given $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$, we say that $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ *implements* $\mathcal{K}$ *through* c if $\mathcal{K} = (\mathcal{P}_{\mathrm{full}} \wedge_c \mathcal{C})_w$.

The (partial interconnection) implementability problem is to characterize, for given $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, all $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ for which there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}$ through c. This problem has a very simple and elegant solution: it depends only on the projected full plant behavior $(\mathcal{P}_{\mathrm{full}})_w$ and on the hidden behavior $\mathcal{N}_w(\mathcal{P}_{\mathrm{full}})$ given by

$$\mathcal{N}_w(\mathcal{P}_{\mathrm{full}}) = \{w \mid (w, 0) \in \mathcal{P}_{\mathrm{full}}\}.$$

Theorem 5.2: [18] Let $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be the full plant behavior. Then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable with respect to $\mathcal{P}_{\mathrm{full}}$ by a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ acting on the interconnection variable c if and only if $\mathcal{N}_w(\mathcal{P}_{\mathrm{full}}) \subset \mathcal{K} \subset (\mathcal{P}_{\mathrm{full}})_w$.

Theorem 5.2 shows that $\mathcal{K}$ can be *any* behavior of $\mathfrak{L}^{\mathtt{w}}$ that is wedged in between the given behaviors $\mathcal{N}_w(\mathcal{P}_{\mathrm{full}})$ and $(\mathcal{P}_{\mathrm{full}})_w$. The implementability problem was also studied in [6], [15] and [9]. In particular, the question when a particular controlled behavior can be implemented by a feedback processor remains a very important one, and was discussed e.g. in [17] and [14].

Next, we turn to *regular implementability* by partial interconnection.

Definition 5.3: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through c is called regular if

$$\mathtt{p}(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}) = \mathtt{p}(\mathcal{P}_{\text{full}}) + \mathtt{p}(\mathcal{C}),$$

i.e., the output cardinalities of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ add up to that of the full controlled behavior $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}$. In that case we also call the controller $\mathcal{C}$ regular.

Definition 5.4: A given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is called *regularly implementable* through c with respect to $\mathcal{P}_{\text{full}}$ if there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $\mathcal{K}$ is implemented by $\mathcal{C}$, and the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular.

Similar to implementability by full interconnection, an important question is under what conditions a given behavior $\mathcal{K}$ is regularly implementable through c with respect to $\mathcal{P}_{\text{full}}$. The following theorem from [1] provides a solution to this problem:

Theorem 5.5: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let $(\mathcal{P}_{\text{full}})_w$ and $\mathcal{N}_w(\mathcal{P}_{\text{full}})$ be the corresponding projected plant behavior and hidden behavior, respectively. Let $(\mathcal{P}_{\text{full}})_{w,\text{cont}}$ be the controllable part of $(\mathcal{P}_{\text{full}})_w$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Then $\mathcal{K}$ is regularly implementable with respect to $\mathcal{P}_{\text{full}}$ by interconnection through c if and only if the following two conditions are satisfied:

- $\mathcal{N}_w(\mathcal{P}_{\text{full}}) \subset \mathcal{K} \subset (\mathcal{P}_{\text{full}})_w$
- $\mathcal{K} + (\mathcal{P}_{\text{full}})_{w,\text{cont}} = (\mathcal{P}_{\text{full}})_w$

The above theorem has two conditions. The first one is exactly the condition for implementability through c. The second condition formalizes the notion that the autonomous part of $(\mathcal{P}_{\text{full}})_w$ is taken care of by $\mathcal{K}$. While the autonomous part of $(\mathcal{P}_{\text{full}})_w$ is not unique, $(\mathcal{P}_{\text{full}})_{w,\text{cont}}$ is. This makes verifying the regular implementability of a given $\mathcal{K}$ computable. As a consequence of this theorem, note that if $(\mathcal{P}_{\text{full}})_w$ is controllable, then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable with respect to $\mathcal{P}_{\text{full}}$ by interconnection through c if and only if it is implementable with respect to $\mathcal{P}_{\text{full}}$ by interconnection through c.

We now return to the decentralized regular implementability problem (by full interconnection). The following theorem gives necessary conditions:

Theorem 5.6: Let $\mathcal{P}, \mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$, with system variable w partitioned as $w = (w_1, w_2, \ldots, w_n)$ with w_i of dimension $\mathtt{w}_i$, $\sum_{i=1}^{\mathtt{n}} \mathtt{w}_i = \mathtt{w}$. Then $\mathcal{K}$ is decentralized regularly implementable with respect to $\mathcal{P}$ by full interconnection only if the following conditions hold:

1) $\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$,
2) w_i is not free in $\mathcal{K}$ for all $i \in \underline{n}$, and
3) for every $i \in \underline{n}$ there exists $\mathcal{S}_i \in \mathfrak{L}^{\mathtt{w}_i}$ such that
    a) $\mathcal{S}_i$ is regularly implementable with respect to $\mathcal{P}$ by interconnection through $(w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)$,
    b) $(\mathcal{K})_{w_i}$ regularly implementable by full interconnection with respect to $\mathcal{S}_i$.

Proof: Clearly $\mathcal{K}$ regularly implementable by full interconnection with respect to $\mathcal{P}$ is a necessary condition. From Proposition 3.5 we have $\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$.

Let $\mathcal{P} = \ker \begin{pmatrix} R_1 & R_2 & \ldots & R_n \end{pmatrix}$. Let $\mathcal{C}$ be a decentralized controller with respect to the partition of the system variable $w = (w_1, w_2, \ldots, w_n)$ regularly implementing $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$. Then from Proposition 4.2, $\mathcal{C}$ admits a minimal kernel representation $\mathcal{C} = \ker(C)$, where $C = \text{blockdiag}(C_1, C_2, \ldots, C_n)$. Therefore we have

$$\mathcal{K} = \mathcal{P} \cap \mathcal{C} = \ker \begin{pmatrix} R_1 & R_2 & \ldots & R_n \\ C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & C_n \end{pmatrix}. \qquad (6)$$

From (6) and using Proposition 2.2, for all $i \in \underline{n}$, w_i is not free in $\mathcal{K}$. For $i \in \underline{n}$ there exists unimodular matrices U_i such that

$$U_i \begin{pmatrix} R_1 & R_2 & \ldots & R_n \\ C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & C_n \end{pmatrix} = \begin{pmatrix} 0 & \ldots & 0 & C_i & 0 & \ldots & 0 \\ 0 & \ldots & 0 & S_i & 0 & \ldots & 0 \\ L_1 & \ldots & L_{i-1} & L_i & L_{i+1} & \ldots & L_n \end{pmatrix} \qquad (7)$$

and $\begin{pmatrix} L_1 & \ldots & L_{i-1} & L_{i+1} & \ldots & L_{\mathtt{n}} \end{pmatrix}$ has full row rank. Therefore we have

$$(\mathcal{K})_{w_i} = \ker \begin{pmatrix} C_i \\ S_i \end{pmatrix}. \qquad (8)$$

Define $\mathcal{S}_i := \ker(S_i)$. From (7) it is evident that for all $i \in \underline{n}$, $\mathcal{S}_i$ is regularly implementable with respect to $\mathcal{P}$ by interconnection through $(w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)$. From (8) it is clear $(\mathcal{K})_{w_i}$ is regularly implementable with respect to $\mathcal{S}_i$. ☐

VI. Conclusions

In this paper we have introduced the problems of decentralized implementability and decentralized regular implementability. Given a plant behavior and a desired behavior, the problem is to give conditions for the existence of a decentralized controller that (regularly) implements the desired behavior. In the first part of this paper we have established necessary and sufficient conditions in terms of geometric properties of the desired behavior. In the second part of the paper we have obtained a set of necessary conditions, expressed in terms of regular implementability by partial interconnection.

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