Interdependencies

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Essays on Cross-Shareholdings, Social Networks, and Sectoral Linkages

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To my parents

Ба волидони азизам
Хилватшо ва Ниёзбегим мебахшам

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Umed Temurshoev
January 22, 2010
Groningen, the Netherlands

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CHAPTER 1

Prologue

1.1 Introduction

This thesis was supposed to be only about cross ownership of firms, which was reflected in the title of my research proposal "*The consequences of cross-shareholding for ownership structures and economic behavior*". The following quotation from this proposal, written in May 2006, gives a brief description of my original research intentions: "... this project aims at theoretically and empirically (i) constructing and analyzing national and (inter-) regional frameworks of ownership structures, (ii) quantifying the complex network of direct and indirect property relations, and (iii) studying the implication of cross-shareholding for several topics in industrial organization". Of course, as is, in general, the case for the majority of PhD theses, the current final output addresses only parts of the main issues in my proposal. Other issues addressed in this thesis were not included in the proposal, and were raised and investigated "along the way".

It is obvious that interdependencies of any kind at very different levels (e.g., individuals, firms, industries, regions, countries) may have a crucial impact on and implications for the activity of agents involved in such networks of bilateral and multilateral interactions. Therefore, it is not surprising that economists devote considerable attention to the complex interrelations between economic agents. As mentioned above, the aim of this research was to analyze the consequences of cross ownership of firms on their behavior and ownership structure. To give a simple example that sketches the complex network of interdependent owners, suppose that individual A owns a share in company B, which has a share in company C. In its turn, firm C owns a share in B. A few readily observable implications of such shareholdings are the following. Although individual A has no direct interest in C, there is an indirect relation via B. If the operating surplus (profits from ordinary production) of C increases, A benefits through its shares in B. If the operating surplus of B increases, A benefits not only directly but also indirectly (for instance, via the gains in C that are beneficial to B again). It turns out that using ownership distributions of private stockholders and companies, one can derive an analytical framework that totally redistributes ownership from firms to the "real" equityholders (e.g., individuals, the state, municipalities), which provides a basis for evaluating the true ownership structure of an economy. As a result many interesting questions arise: What is the value of the property embedded in shares that real owners hold in companies? How to assess decision making power in the presence of complex ownership links between firms? What is the role of the state or any other owner? What are the implications of firms' cross ownership on control power of shareholders, and does it have any impact on tacit collusive arrangements of firms? What is the effect of cross-shareholding on prices, outputs, profits, and social welfare? What happens if the structure of cross-shareholding changes? And many more.

While studying these issues, I came across the paper by Ballester et al. (2006) on finding *a* key player in social networks, where the key player exerts the largest impact on the overall (equilibrium) activity of the network.[1] This important study raised some related questions to me, focusing on which ultimately resulted in two papers that constitute two chapters (5 and 6) of this thesis. By doing so I also crossed the borders of my original research plan, from topics mainly in Industrial Organization to issues in such fields as Network Economics, Interindustry Economics and Social Network Analysis. In what follows these issues will be discussed in more detail.

1.2 Industrial organization and finance

Often it is argued that Continental Europe and Japan have an enterprise oriented system of ownership structure, while the Anglo-American system is market oriented. One of the important factors in determining such orientation of the ownership structure of an economy is the presence or absence of complex webs of intercorporate holdings. These are believed to play a prominent role in Continental

[1] I would like to thank José Luis Moraga-González for bringing this paper to my attention, and the industrial organization reading group of the University of Groningen, led by Marco Haan, that made me to delve deeper into the topic.

Europe and Japan. The question is how this ownership complexity can be quantified (which was, in fact, the second point of my original research proposal). Chapter 2 focuses on this issue, where two types of owners are distinguished: primary owners that own intermediary institutions but cannot be owned themselves (e.g., individuals, the state, municipalities), and secondary owners that can own other intermediary institutions but are surely owned themselves by other owners (e.g., companies, banks, industrial corporations). In quantifying ownership interrelatedness both the size of direct and indirect shareholdings and the "average distance" between primary owners and secondary owners are taken into account. The latter is obtained from the average number of secondary owners via whom ownership links between primary owners and secondary owners run. Combining the linkage size and the distance allows us to visualize the cross-shareholding interlocks and the true ownership relations in an industry (economy). The methodology is applied to the banking sector in the Czech Republic, where the complexity of the network of relations between primary and secondary owners are quantified, and the relevant shareholding chains are graphed.

Chapter 2 further explores the link between the proposed measures of ownership network complexity and the degree of separation of dividend and control rights, widely studied in the finance literature. To give an idea of the issue at stake, suppose we have the following ownership chain: $A \rightarrow B \rightarrow C$. That is, firm A owns a share in firm B, which in its turn owns a share in company C. Hence, although the dividend rights of A in firm C are zero, there is an indirect ownership connection via B that makes it possible for firm A to have positive control rights in C (which may be very large depending of the size of these direct shareholdings). Thus, it is not surprising that there are ample studies in Finance that focus on the issue of separation of ownership and control rights due to pyramiding ownership structures and cross-holdings. It is obvious that in the presence of mutual cross-shareholdings the chains of ownership stakes are not at all easy to trace. Thus, quantifying the control power embedded in such complex ownership networks is also far from trivial. For example, using the well-known "weakest link" methodology that defines the minimum stake along the ownership chain as the corresponding control right is simply unpractical. This is because in the presence of cross-shareholdings there exists an *infinite* number of ownership paths of different lengths. On the other hand, our proposed measures of ownership complexity fully take into account such means of enhancing control as non-pyramidal cross ownership links, where also the sizes of shareholdings and distances between owners are explicitly accounted for. Hence, we consider these indicators as alternative measures of the separation of ownership and control rights. That is, the more complex the network of non-negligible relations is, the larger is the degree of control enhancement due to cross-shareholding links among firms. Therefore, also the difference between the control and the ownership stakes of primary owners in secondary owners is larger. The empirical results confirm this for the Czech banking sector, where the results are compared to the "weakest link" and "dominant shareholder" approaches of identifying control rights.

In reality, shareholdings are often silent (or partial) by their nature, meaning that they do not give control power for their owners. However, as partial cross ownership (PCO) results in commonality of interests of firms engaged in such shareholding interlocks, it is interesting to investigate what are the effects of PCO on the market performance and market power of the individual firms in an industry. This is the subject of study in Chapter 3. For this purpose we modify the well-known framework of the "structure-conduct-performance paradigm" for estimating firms' market power and the degree of tacit collusion inherent to the market by considering both direct and indirect PCO holdings among firms. It is shown that, unlike in the no-PCO case, the link between firms' price-cost margins and the degree of tacit collusion is nonlinear in the presence of PCO. Thus, if PCO is present, ignoring it will most likely lead to biased results due to model misspecification. The modified framework is applied to the Japanese banking sector in 2003. We find that Japanese banks compete in a modest collusive environment. If, however, PCO is neglected, the results indicate a Cournot oligopoly. Further, it is shown that banks with passive holdings in rivals exert a strictly larger market power than those without any PCO. In particular, city banks with many shareholdings are found to exercise a much larger market power than regional banks with none or few stockholdings. Hence, the hypothesis is confirmed that acquiring shares in rivals is one of the crucial means for a firm to enhance its market power.

Passive investments of firms in rivals were either granted a de facto exemption from antitrust liability or have gone unchallenged by antitrust agencies in recent cases. However, recently Gilo et al. (2006) showed that there are cases in which PCO arrangements can facilitate tacit collusion among rival firms, hence such a lenient approach towards passive investments in rivals may be misguided. However, Gilo et al. (2006) assumed that firms are symmetric and have the same marginal cost functions. Chapter 4 relaxes this assumption and examines the effect of PCO on the incentives of asymmetric firms to collude. Unlike Chapter 3, which studies somewhat similar issues in a one-period conjectural variations setting, Chapter 4 posits an infinitely repeated Bertrand oligopoly model. We first consider the case where only the most efficient firm in the industry invests in rivals. Since there are no other firms involved in partial ownership acquisition, we refer to this case as partial ownership (PO) case. (PCO, on the other hand, indicates that more than one firm are involved in stockholdings, thus could mutual shareholdings are possible.) It is shown that even unilateral partial ownership by this firm may facilitate a market-sharing scheme in which all firms charge the same collusive price and divide the market between them. Unlike the case where firms have the same marginal costs, here firms have different monopoly prices on which they wish to collude, and the collusive price is assumed to be a compromise between the monopoly prices of the different firms. We show that when the most efficient firm invests in rivals, the collusive price increases relative to the case where there are no PO arrangements.

Further, Chapter 4 shows that in the case of multilateral PCO arrangements an increase in the stake that firm r holds in firm s will never hinder collusion and it will strictly facilitate collusion if and only if (i) the industry maverick (the firm with the strongest incentive to deviate from a collusive agreement) has a direct or indirect stake in firm r, and (ii) firm s is not the industry maverick. When either (i) or (ii) fails to hold, the increase in firm r's stake in firm s does not affect tacit collusion. These results extend the earlier findings in Gilo et al. (2006) and show that the results for firms with symmetric cost functions generalize to the asymmetric costs case. Then Chapter 4 investigates the effect of a transfer of PCO between firms on tacit collusion, and shows that depending on the initial structure of shareholdings of firms directly involved in the ownership transfer, tacit collusion may be facilitated, be hindered, or remain unchanged.

1.3 The link to network economics and social network analysis

In the sociology literature, the problem of identifying the most important actors in social networks has been studied extensively, and still remains an essential topic of concern. In particular, within the field of Social Network Analysis a vast number of indicators, the so-called network centralities, have been proposed in order to identify *key actors* in networks. For example, the best-known and most often used measures are centralities of degree, closeness, betweenness, information, Katz status measure, and Bonacich centrality (see e.g., Wasserman and Faust, 1994, pp.

169-219).

A similar problem from an economic perspective was first analyzed by Ballester et al. (2006), who introduce a network game, where actors' payoffs depend on each other through network embeddedness. Players choose a level of activity in a game with negative global externalities (e.g., competition) and local positive externalities (e.g., learning, collaboration) that come through the network. Obviously, such system has feedback effects, which are taken into account in the Nash equilibrium activity levels that are dependent on the underlying network topology. The authors show that individual equilibrium levels of agents are proportional to their Katz-Bonacich centrality measures. Hence, they provide a behavioral foundation to the status measure of Katz (1953) and the network centrality measure of Bonacich (1987). However, these measures are not sufficient to identify a *key player* – the player with the largest impact on the overall equilibrium outcome. Hence, Ballester et al. (2006) propose a new measure of network centrality, named the *intercentrality measure*, that is derived from the planner's optimization concern. Since it internalizes all the network payoff externalities of agents, the intercentrality measure identifies the key player.

Chapter 5 considers a more general setting of finding a *key group* in such network games, and also takes explicitly players' ex ante heterogeneity into account. Similar to the key player definition, the key group is a group of players that exert the maximum possible impact on the overall equilibrium activity level of the network. It should be noted that the assumption of ex ante identical players in the search of a key player used in Ballester et al. (2006) is quite restrictive from a practical point of view, because in that case all observable differences between individuals are ignored. These heterogeneity factors include, for example, a player's age, education, occupation, race, gender, parents education, or family size. We show that once this exogenous heterogeneity is accounted for, the results of the key player/group problem may change dramatically. In searching for the key group we make use of weighted and unweighted Katz-Bonachich (KB) centralities and *group intercentrality measures*, where the weights are the observable differences of the players.

Chapter 5 also endogenizes the size of the key group. The need for such endogenization arises because in reality targeting a certain set of players also incurs costs, next to benefits. In a majority of cases, these benefits and costs are directly related to the group size. As an example, suppose that a planner wants to maximally disrupt the functioning of a network of criminals in some location. It is obvious that the larger the size of the key group of criminals is, the larger is the benefit in terms of reducing criminal activity in this society. However, there are costs involved in the "elimination" of criminals, such as costs related to gathering information, time, hiring people, and other costs for planning and implementation of such an annihilating aim. All these costs are generally higher for a larger key group. We show that within the class of network games studied in Ballester et al. (2006) the optimal size of the key group is determined by the minimal *key group loss* measure that depends on players' weighted and unweighted KB centralities and key group intercentralities, and the costs of group targeting.

1.4 Interindustry economics and game theory

The key group problem within the network games discussed in the previous section has a close relation (at least, technically) to the problem of finding *key sectors* in the framework of input-output (IO) linkage analysis. Key sectors are the industries with the largest potential of spreading growth impulses throughout the economy. There are several methods for identifying key sectors in Interindustry Economics, but for our purposes we focus on the *hypothetical extraction method* (HEM) developed in the 1960s, which is extensively used in the IO literature. The HEM in identifying key sectors measures the importance of industries in terms of their contribution to the overall gross output of an economy by extracting them from the production structure. We show that this approach is similar to that of finding the key player in a social network in Network Economics and Social Network Analysis, where players are eliminated from the network of local interactions, which enables one to quantify these players' marginal contribution to the overall activity level and/or network functioning.

The main contribution of Chapter 6 to the literature on key sectors identification from the HEM perspective is that it distinguishes between and *explicitly* formulates the optimization problems of finding a *key sector* and a *key group of sectors*, and derives analytical solutions for these problems in terms of simple measures called industries' *factor worths*. The term "factor" refers to any indicator that is of interest in identifying the most important industries. This might be any social, environmental, and/or economic factor (e.g., employment, water use, GDP, etc.), or any combination of these factors. Our formal formulation of the HEM problems has several important implications, one of which is that the key group of $k > 1$ sectors is, in general, different from the set of top k sectors with the largest individual contributions to the overall factor production/consumption. This is confirmed in the empirical application of the key sector and the key group problems to the Australian economy in case of water use and CO_2 emissions. This (expected) finding is important, since up to date, to the best of our knowledge, the linkage literature (implicitly) accepted the top k sectors (selected on the basis of the key sector problem) as the key group. Technically speaking, this incongruence is due to the fact that while the key sector problem looks for the effect of the (hypothetical) extraction of one sector, the key group problem considers the effect of a *simultaneous* extraction of $k \geq 2$ sectors that takes differently into account the cross-contributions of the extracted industries to total factor arising within and outside the group. Its economic interpretation has to do with what sociologists call the *redundancy principle* (see e.g., Burt, 1992). In the IO framework, this means that sectors might be redundant with respect to each other if they have similar patterns of production linkages with other industries, and similar structures of final demand and factor generation capabilities. Hence, the optimal target should consist of rather *non*redundant sectors that have different patterns of (significant) interindustry linkages and factor generation ability. Therefore, which sectors will be part of the key group is largely dependent on the (dis)similarity of the production linkage patterns of sectors to each other and of their final demand and factor generation structures. At this point we have to mention that the redundancy principle also plays an important role in identifying the key group of players within the network games that are discussed in detail in Chapter 5.

Revealing the connection of the HEM to the well-known *fields of influence* approach in the IO literature (Sonis and Hewings 1989, 1992) gives an alternative economic interpretation of the HEM problems in terms of the overall impact on aggregate factor generation due to an incremental change in sectors' input self-dependencies. Further, we explore the related issues of finding the key *region* and the key group of regions in an interregional IO setting, and discuss the effect of netting out (nullifying) the intrasectoral transactions on industries' (or regions') factor worths. Also discussed in Chapter 6 is the link of the (generalized) HEM approach of finding the key sector to the coalitional game literature on fair allocation of gains from cooperation. In particular, the properties (axioms) of the well-known Shapley value are given, and it is elaborated whether these properties also hold for the industry's factor worth. Hence, there is also a connection to Game Theory, and to measuring the power of players, in particular.

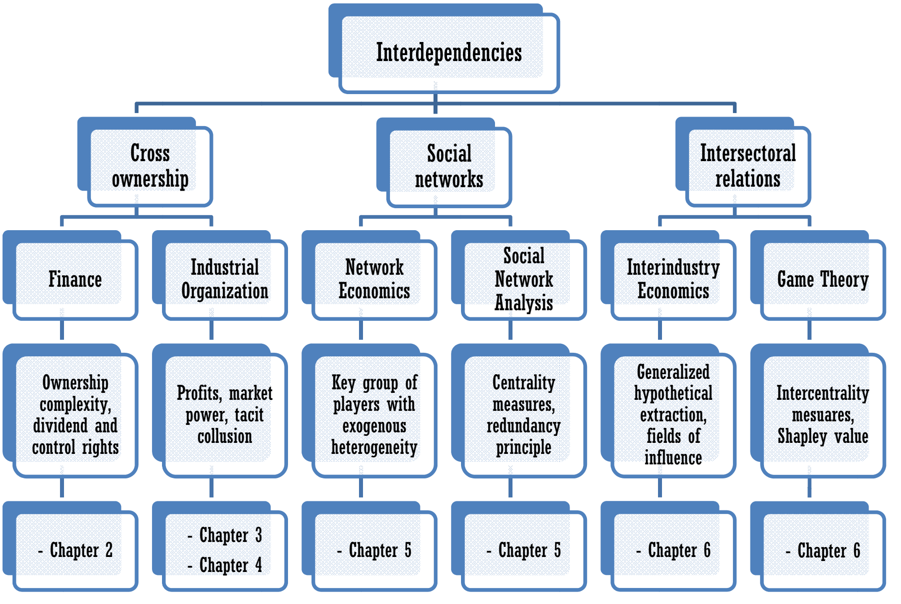
1.5 Outline of the study

A rough sketch of the present study is presented in Figure 1.1. As mentioned in the previous sections, the interdependencies that we are interested in are of three types, namely, cross ownership (or shareholding interlocks) of firms, social networks, and intersectoral relations. Obviously, since these interrelationships are different in their own nature, the analytical frameworks that are used in their analysis are also different. Hence, there are several well-established and seemingly independent fields of economics and sociology that are the focus of this study. As discussed in some detail in the previous sections these are Finance, Industrial Organization, Network Economics, Social Network Analysis, Interindustry Economics, and Game Theory.

It is worth noting, however, that all the issues considered in this work are closely related, because in the end the analysis boils down to focusing on all kinds of impacts due to the presence of the complex networks of linkages between firms, individuals, and/or sectors of an economy. Moreover, the mathematical techniques developed for one type of analysis (say, in Interindustry Economics) can be readily used to address related issues in the other fields (e.g., Industrial Organization, Network Economics). For example, the well-known open Leontief model in Input-Output Analysis, which is capable of quantifying both direct and indirect sectoral relations in an economy, is quite useful in modeling and analyzing cross ownership links of firms and easily allows to distinguish between the direct and indirect shareholdings. As will be discussed in the text this setting has important theoretical and practical implications. Similarly, our Lemmas 5.1 and 6.2 that are in fact mathematically equivalent, are the building blocks of the studies in Chapter 5 and Chapter 6. Thus, they directly connect the analysis of key players search in network games and key sectors identification in an input-output setting. Therefore, this thesis in fact shows that the above mentioned fields are not totally independent of each other, but are closely related, at least, when the focus is the analysis of interdependencies.

Some main issues of each chapter are also given Figure 1.1. For example, one of the main aims of Chapter 5 is the study of the problem of identifying key group of players in social networks, where the observable differences (or exogenous heterogeneity) of individuals are taken into account. This links our study to the Network Economics' topics on network games. But since in the course of this analysis such important sociology notions as centrality measures and redundancy principle play a crucial role, there is also a close relation to the Social Network Analysis of

Figure 1.1: A rough sketch of the thesis



finding the most important actors in networks. Similarly, Chapter 6 extends the traditional hypothetical extraction method (HEM) in Interindustry Economics in finding a key sector with the maximum potential of spreading total output growth impulses throughout the economy to the problem of identifying a key group of sectors with the highest economy-wide impact on factor generation/consumption. This generalized HEM is then linked to another widely used approach in the same field, namely, the fields of influence method, which will be also discussed in detail in the chapter. The solutions of the HEM problems have direct connection to the intercentrality measures (discussed in Chapter 5) and the so-called Shapley value in the coalitional game literature. The Shapley value identifies the worth (or importance) of each participant of the coalition to its functioning. In this way, Chapter 6 also discusses briefly this link to Game Theory.

1.6 Some general notations

Due to the nature of our study, matrix algebra will be extensively used throughout the book. Therefore, it makes sense to introduce some important notations at this point.

Vectors and matrices. Adopting usual convention, matrices are given in bold, capital letters (e.g., $\mathbf{X}$); vectors in bold, lower case letters (e.g., $\mathbf{x}$); and scalars in italicized, lower case letters (e.g., x). Vectors are columns by definition, thus row vectors are obtained by transposition, indicated by a prime (e.g., $\mathbf{x}'$). $\hat{\mathbf{x}}$ denotes the $n \times n$ diagonal matrix with the elements of the vector $\mathbf{x}$ on its main diagonal and zeros elsewhere. The zero matrix and the zero vector are, respectively, denoted by $\mathbf{O}$ and $\mathbf{0}$. The summation vector $\boldsymbol{\imath}$ consists of ones, i.e., $\boldsymbol{\imath}' = (1\,1\ \cdots\ 1)$.

Matrix (and vector) inequalities. The following notation for inequalities between matrices (and vectors) is adopted.

$\mathbf{X} \leq \mathbf{Y}$ means $x_{ij} \leq y_{ij}$ for all i and all j;
$\mathbf{X} < \mathbf{Y}$ means $\mathbf{X} \leq \mathbf{Y}$, but $\mathbf{X} \neq \mathbf{Y}$, i.e., $x_{ij} \leq y_{ij}$ for all i, j, with at least one strict inequality;
$\mathbf{X} \ll \mathbf{Y}$ implies $x_{ij} < y_{ij}$ for all i and all j.

CHAPTER 2

Ownership relations in the presence of cross-shareholding*

2.1 Introduction

The ownership structure of an economy is nowadays often characterized by a complex network of interdependent owners. For example, individual A owns a share in company B, which has a share in company C. In its turn, C owns a share in B. Although A has no direct interest in C, there is an indirect relation via B. If the operating surplus of C increases, A benefits through its shares in B. If the operating surplus of B increases, A benefits not only directly but also indirectly (for instance, via the gains in C that are beneficial to B again). This is just a very simple case, but it suffices to sketch the setting. Using pure accounting identities, Bolle and Güth (1992) constructed a general model of such interdependent property structures and arrived at the Leontief input-output scheme (see also Turnovec, 1999, 2005). In particular, they showed that eliminating all indirect ownership relations results in the final or true distribution of property over the individual owners.

There is a huge body of literature on ownership structures, but only few papers deal with the indirect effects arising from the so called "cross-shareholding" of companies. Due to cross-shareholding, companies have indirect interests in each other. In the literature this structure of ownership and control is also called an "insider system", which is an integral feature of Japanese, German and Swedish business groups in particular (see e.g., Kester, 1992). Franks and Mayer (1995) distinguish

* A shorter version of this chapter is published in the *Journal of Economics*, vol. 95, no. 3, pp. 189-212, 2008 (joint work with Erik Dietzenbacher).

two types of ownership structures: insider and outsider systems. An insider (or enterprise-oriented) system has a small number of listed companies, an illiquid capital market with infrequent trade of ownership and control, and complex systems of intercorporate holdings. In contrast, an outsider (or market oriented) system is characterized by the existence of a large number of listed companies, a liquid capital market with frequent trade of ownership and control rights, and few intercorporate holdings. In general, it is believed that Continental Europe and Japan have an insider system of ownership structure, while the Anglo-American system is market oriented.

Although indirect interests (such as the one sketched above) have been recognized in the literature (see e.g., Bresnahan and Salop, 1986; Reynolds and Snapp, 1986; Flath, 1989, 1991), only few papers take them into full account by implementing such interests in the models that are used. For example, Ellerman (1991, 1995) studies the cross ownership relations between corporations and uses the input-output framework to develop the so called primal and dual theories of ownership and control. His model is particularly relevant for control questions and the proportional representation scheme in voting systems. In a series of papers, Flath (1992a, 1992b, 1993) measures indirect shareholding for six major *keiretsu* groups in Japan. He shows that indirect shareholding in these groups is large, and should not be neglected because there are gains from indirect shareholding (which might explain the existence of keiretsu groups). Such gains were quantified by Dietzenbacher et al. (2000) in an empirical study for the Dutch financial sector. The effects of cross-shareholding for collusion were studied by Reitman (1994); Alley (1997); Gilo et al. (2006) and in Chapter 3 of this thesis.

The cross-shareholding of companies may result in a complex network of interdependent relations between economic agents.[1] Analyzing complexity and relatedness of national production structures has induced a considerable amount of input-output research (going back to Yan and Ames, 1965). In this study we want to quantify ownership interrelatedness (and ownership network complexity) in an economy between primary owners (e.g., individuals, the state, municipalities, individuals' non-profit associations) and secondary owners (e.g., companies, banks, industrial corporations) that is the consequence of cross-shareholding links. In doing so, we take into account not only the size of direct and indirect shareholdings, but

[1] It should be noted that the term "cross-shareholding" as used in the industrial organization literature includes all kinds of ownership relations that are distinguished in the finance literature. These include the pyramiding structure, one-sided shareholdings, and mutual (reciprocal) shareholdings (which also includes "ring-form" links).

also the "average distance" between primary owners and secondary owners. The latter is obtained from the average number of secondary owners via whom such shareholding links between primary owners and secondary owners run. Combining the linkage size and the distance, allows us to visualize the cross-shareholding interlocks and the true ownership relations.

Taking indirect ownership relations into full account has important theoretical and empirical implications. First, a primary owner may indirectly own a substantial part of some secondary owner, although there may be no direct interest at all. Using the observed property distribution for various purposes (e.g., valuing the property embedded in shares that primary owners hold in secondary owners, assessing decision making power, identifying the role of the state or any other primary owner, finding the distribution of national property or profits) may be quite misleading. In the presence of cross-shareholding, the observed direct ownership distribution may be very different from the true property distribution that incorporates indirect linkages as well.

Second, quantifying indirect ownership relations allows for comparing different sectors in an economy and/or different economies. In some cases, however, qualitative judgments are immediately clear because the shareholding matrices exhibit certain characteristics (such as reducibility).

Third, it would be of interest to link measures of indirect property relations to financial performance indicators. For example, the empirical evidence of the effect of cross-shareholding on corporate performance is ambiguous (see e.g., Prowse, 1990; Flath, 1993; Lichtenberg and Pushner, 1994; Weinstein and Yafeh, 1995, 1998; Morck et al., 2000; Yafeh and Yosha, 2003). The stable shareholding in Japan, which persisted for almost three decades, began to unwind dramatically in the 1990s.[2] This raised many questions about causes, effects and implications of the changes in the Japanese ownership structure. Quantifying ownership relations may shed a new light on the link between ownership structure and corporate performance.

Finally, there is a clear link between our measures of ownership network complexity and the degree of separation of dividend and control rights, widely studied in the finance literature (see e.g., La Porta et al., 1999, 2002; Bebchuk et al., 2000; Claessens et al., 2000; Faccio et al., 2001; Faccio and Lang, 2002; Attig and Gadhoum, 2003; Gadhoum et al., 2005; Dorofeenko et al., 2008). Cross-shareholding is

[2] According to Nippon Life Insurance Research Institute the stable shareholder ratio, defined as the ratio of shares owned by commercial banks, insurance companies, and other non-financial firms (business partners and the parent company) to the total of issued shares of listed firms (calculated on a value basis), was 45% in the early 1990s, but decreased to 27% in 2002.

one of the most important control enhancement devices and our proposed measures of distance take this into full account. We show that, as a consequence, our distance concept can be used as an alternative measure of separation of ownership and control. Ownership relations that are more complex and non-negligible in size may be expected to exhibit a larger gap between dividend and control rights.

The rest of this chapter is organized as follows. Section 4.2 describes the Leontief-type model of property structure. New measures of ownership network complexity are developed in Section 4.3. The method has been applied to the banking sector of the Czech Republic, the results of which are discussed in Section 4.4. Also in that section we explore the link to the finance literature, studying the separation of ownership and control. The summary and conclusions are presented in Section 4.5.

2.2 Basics of the Leontief-type model of ownership structure

The main point of departure is the model of property structure, developed by Bolle and Güth (1992) to study a complex network of interdependent owners (see also Turnovec, 1999, 2005). Essentially, there are two types of economic agents: principal or primary owners (e.g., individuals, the state, municipalities) and intermediary or secondary owners (e.g., companies, banks, industrial corporations). Principal owners can own intermediary institutions, but cannot be owned themselves. Intermediary institutions can own other intermediary institutions, but are surely owned themselves (by primary and other secondary owners). Due to this cross-shareholding of intermediary owners, principal owners may have no (or little) direct interest in some intermediary owner, but a huge indirect interest (via other secondary institutions).

Suppose there are m primary owners and n secondary owners. The $n \times m$ matrix $\mathbf{P}$ gives the direct primary property distribution. Element p_{ik} indicates the share in company i $(= 1, \ldots, n)$ that is held by primary owner k $(= 1, \ldots, m)$. The $n \times n$ matrix $\mathbf{S}$ denotes the secondary property distribution. That is, element s_{ij} gives the share in company i that is held by company j $(= 1, \ldots, n)$.[3] It is assumed that the shares are all non-negative and that their sum equals one. That is, $\sum_{j=1}^{n} s_{ij} + \sum_{k=1}^{m} p_{ik} = 1$ holds for all i. In matrix notation we thus have $\mathbf{S}\boldsymbol{\imath}_n + \mathbf{P}\boldsymbol{\imath}_m = \boldsymbol{\imath}_n$,

[3] Usually, it is assumed that no secondary owner holds shares in itself, so that the main diagonal of $\mathbf{S}$ is zero. However, as noted in previous chapters, this assumption is not always true because due to the tax advantage of capital gains, the share repurchases have recently become the dominant payout policy for corporations. From a mathematical point of view, it is no problem to allow for $s_{ii} > 0$.

where $\boldsymbol{\imath}_n$, for example, indicates the n-dimensional summation vector consisting of ones. This assumption simply states that any secondary owner i is totally owned by principal and other intermediary owners.

P and **S** give the direct property distributions that are actually observed. Eliminating indirect ownership relations results in a total property distribution, which may be significantly different from the observed ownership scheme. The first step in eliminating indirect ownership relations follows from the observation that primary owner k directly holds a share p_{ik} in company i, but it also holds a share p_{hk} in company h which holds a share s_{ih} in i itself. This holds for all h, so that primary owner k holds an indirect share in i that amounts to $\sum_{h=1}^{n} s_{ih} p_{hk}$ and which runs via one intermediate owner. The link $k \to h \to i$ thus involves two steps, which indicates the "distance" between k and i. The "two-step" indirect property distribution is given by the matrix **SP**. In the same way, primary owner k also holds an indirect share in i via two intermediate owners and thus involving three steps (i.e., $k \to h \to l \to i$). This yields the "three-step" indirect property distribution $\sum_{l=1}^{n} \sum_{h=1}^{n} s_{il} s_{lh} p_{hk}$, which is element (i, k) of matrix $\mathbf{S}^2\mathbf{P}$. And so forth.

Taking all such indirect property distributions into consideration (next to the direct initial property structure), gives the total property distribution. It is given by $(\mathbf{I} + \mathbf{S} + \mathbf{S}^2 + \mathbf{S}^3 + \cdots)\mathbf{P}$, where **I** denotes the identity matrix. It is well known that the power series expansion of a non-negative matrix **S** equals $(\mathbf{I} - \mathbf{S})^{-1}$, under certain conditions. In the present context, it suffices to assume that for each secondary owner, there is a primary owner that holds a positive share, i.e., matrix **P** has some positive element in each row (see Takayama, 1985 for a concise overview of all mathematical details). This implies that the total or "true" property distribution is given by the matrix

$$\mathbf{T} = (\mathbf{I} - \mathbf{S})^{-1}\mathbf{P}. \tag{2.1}$$

Because $\mathbf{S}\boldsymbol{\imath}_n + \mathbf{P}\boldsymbol{\imath}_m = \boldsymbol{\imath}_n$ implies $(\mathbf{I} - \mathbf{S})\boldsymbol{\imath}_n = \mathbf{P}\boldsymbol{\imath}_m$, we have $\boldsymbol{\imath}_n = (\mathbf{I} - \mathbf{S})^{-1}\mathbf{P}\boldsymbol{\imath}_m = \mathbf{T}\boldsymbol{\imath}_m$. This means that **T** satisfies the properties of a distribution. Note that in the end, all property is owned - directly or indirectly - only by principal owners, and all the secondary owners are left with nothing.

The primary property distribution matrix **P** with direct shareholding gives the direct ownership relations that are also observed in practice. The indirect relations run via one or more secondary owners and are given by the matrix

$$\mathbf{Y} = (\mathbf{S} + \mathbf{S}^2 + \mathbf{S}^3 + \cdots)\mathbf{P} = [(\mathbf{I} - \mathbf{S})^{-1} - \mathbf{I}]\mathbf{P} = \mathbf{T} - \mathbf{P}. \tag{2.2}$$

An alternative formulation yields $\mathbf{Y} = \mathbf{S}(\mathbf{I} + \mathbf{S} + \mathbf{S}^2 + \cdots)\mathbf{P} = \mathbf{S}(\mathbf{I} - \mathbf{S})^{-1}\mathbf{P} = \mathbf{ST}$.

Consider a simple example of a hypothetical ownership structure with two principal owners ($PO1$ and $PO2$), and three secondary owners ($SO1$, $SO2$, $SO3$). The observed primary and secondary property distributions **P** and **S** are

$$\mathbf{P} = \begin{pmatrix} 0.4 & 0 \\ 0.3 & 0.3 \\ 0 & 0.3 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} 0 & 0.5 & 0.1 \\ 0.4 & 0 & 0 \\ 0.7 & 0 & 0 \end{pmatrix}.$$

The matrix **T** with the true property distribution and the matrix **Y** with indirect property distribution become

$$\mathbf{T} = \begin{pmatrix} 0.753 & 0.247 \\ 0.601 & 0.399 \\ 0.527 & 0.473 \end{pmatrix} \text{ and } \mathbf{Y} = \begin{pmatrix} 0.353 & 0.247 \\ 0.301 & 0.099 \\ 0.527 & 0.173 \end{pmatrix}.$$

This example exhibits several features that are characteristic for cross-shareholding. First, note that each primary owner has an indirect interest in every secondary owner. Second, while $PO1$ holds no direct shares in $SO3$, it turns out that indirectly it owns no less than 53% of the property of $SO3$, which is more than the property held by $PO2$, who has a direct share of 30% in $SO3$. The same applies (albeit to a lesser extent) for the ownership of $SO1$ by $PO2$. Third, both primary owners hold a direct share of 30% in $SO2$, whereas the true property distribution shows that $PO1$ owns 50% more than $PO2$ does. The example clearly shows that focusing only on the observed property distribution in **P** may be quite misleading.

The element t_{ik} of matrix **T** gives the direct and indirect property of secondary owner i that is held by primary owner k.[4] Denote the elements of $\mathbf{L} \equiv (\mathbf{I} - \mathbf{S})^{-1}$ by l_{ij}. Then (2.1) implies that $t_{ik} = \sum_{h=1}^{n} l_{ih} p_{hk}$, where $l_{ih} p_{hk}$ indicates the part of the property t_{ik} that is embedded in the share that primary owner k holds in secondary owner h. This implies that for determining what is embedded in a certain share, the elements l_{ij} are crucial. For our example we find

$$\mathbf{L} = (\mathbf{I} - \mathbf{S})^{-1} = \begin{pmatrix} 1.370 & 0.685 & 0.137 \\ 0.548 & 1.274 & 0.055 \\ 0.959 & 0.480 & 1.096 \end{pmatrix}.$$

For instance, the elements in the first column show that a 10% direct share in $SO1$

[4] Dorofeenko et al. (2008) calls the element t_{ik} the *imputed ownership share* of investor k in firm i.

held by some primary owner, embeds 13.7% of the property of $SO1$ (as follows from l_{11}), 5.5% of the property of $SO2$ (from l_{12}), and no less than 9.6% of the property of $SO3$ (from l_{13}). In contrast, holding a 10% direct share in $SO3$ embeds only 1.4% of the property of $SO1$, 0.6% of $SO2$, and 11.0% of $SO3$. Similar direct shares (e.g., of 10%) in secondary owners may thus embed true property holdings that are very different. This suggests that the matrix **L** may also play a role for the value of the shares.

If the value of the secondary owners is known, we can determine the value of the property that is embedded in, for example, a 1% share in secondary owner j. Let $\mathbf{v}'$ denote the row vector of the values for the firms (i.e., secondary owners). Then the jth element of the row vector $0.01 \times \mathbf{v}'\mathbf{L}$ gives the value embedded in a 1% share in secondary owner j. In the example above, suppose that the values of the firms are equal to each other (say $100v$). It then turns out that a 1% share in secondary owner 1 is worth $2.877v$, $2.439v$ in case of owner 2, and only $1.288v$ for a 1% share in secondary owner 3. So a share in secondary owner 3 is worth much less than a share in the other secondary owners. It is now also possible to evaluate the properties of the primary owners. The kth element of the row vector $\mathbf{v}'\mathbf{LP} = \mathbf{v}'\mathbf{T}$ gives the property value of primary owner k. If we suppose again that the values of the firms are the same, it follows that primary owner 1 has a property of 63% of the total property (i.e., $\mathbf{v}'\boldsymbol{\imath} = v_1 + v_2 + v_3$) and primary owner 2 only 37%. This is in sharp contrast to the finding - which follows from the observed data - that the primary owners hold similar amounts in secondary owners.[5] Note that the entire property of the secondary owners is distributed over the primary owners, as follows from $\mathbf{v}'\mathbf{T}\boldsymbol{\imath}_m = \mathbf{v}'\boldsymbol{\imath}_n$.

2.3 A measure of ownership network complexity

Given the importance of the indirect relations (or linkages), we will study the complexity of their underlying network. It turns out in the empirical analysis that this allows us to get some insight into the hidden property structures. As will be clear from the next section, the network complexity measure is a useful indicator of separation of control and ownership rights, since the more complex is the system of non-negligible ownership links, the larger is the control power on firms exerted by primary owners through firms cross-holdings. Secondly, network complexity

[5] That is, $PO1$ holds 40% of $SO1$ and 30% of $SO2$, while $PO2$ holds 30% of both $SO2$ and $SO3$. These holdings are fairly similar, because in this example all secondary owners were assumed to have the same value.

measures can be used for comparative analysis. For instance, one economy can be compared with an other in terms of the overall degree of complexity of indirect relations, which identifies the market– or enterprise–orientedness of their ownership structures.

The complexity of the indirect ownership relations between principal and intermediary owners is quantified by their weighted average distance. Distance is defined as the number of intermediary owners via whom the relation runs, plus one. For example if the link between a primary owner k and a secondary owner i runs through secondary owner h (i.e., $k \rightarrow h \rightarrow i$), the distance is 2. It indicates the number of steps that are required to get from k to i. The weighted average distance is defined as one plus the average number of participating intermediary owners. In determining the latter average, we use a technique originally developed in the context of input-output models by Harthoorn (1988) and later extended by Dietzenbacher et al. (2005).

Consider the matrix of indirect property relations

$$\mathbf{Y} = (\mathbf{S} + \mathbf{S}^2 + \mathbf{S}^3 + \cdots)\mathbf{P} = \mathbf{SP} + \mathbf{S}^2\mathbf{P} + \mathbf{S}^3\mathbf{P} + \cdots . \tag{2.3}$$

Denote element (i,k) of matrix $\mathbf{S}^r\mathbf{P}$ as $(\mathbf{S}^r\mathbf{P})_{ik}$. In building up the total indirect relation y_{ik}, a share $(\mathbf{SP})_{ik}/y_{ik}$ reflects all relations with distance 2 (i.e., running through exactly one secondary owner). Note that element (i,k) of matrix $\mathbf{SP}$ yields $\sum_{j=1}^{n} s_{ij}p_{jk}$, where the relationship $s_{ij}p_{jk}$ between primary owner k and secondary owner i runs through secondary owner j. In the same way, the share $(\mathbf{S}^2\mathbf{P})_{ik}/y_{ik}$ gives the connections between k and i with distance 3 that run via two secondary owners, because $\sum_{j=1}^{n}\sum_{h=1}^{n} s_{ih}s_{hj}p_{jk}$. In general, the share $(\mathbf{S}^r\mathbf{P})_{ik}/y_{ik}$ gives all indirect relationships with distance $r+1$ that require r secondary owners.

The weighted average distance between primary owner k and intermediary owner i is given by the weighted average of the distances $r+1$ with corresponding weights $(\mathbf{S}^r\mathbf{P})_{ik}/y_{ik}$, where $r = 1, 2, 3, \ldots$. That is,

$$\begin{aligned} &\left[2(\mathbf{SP})_{ik} + 3(\mathbf{S}^2\mathbf{P})_{ik} + 4(\mathbf{S}^3\mathbf{P})_{ik} + \cdots + (r+1)(\mathbf{S}^r\mathbf{P})_{ik} + \ldots\right] / y_{ik} \\ &\quad = \frac{1(\mathbf{SP})_{ik} + 2(\mathbf{S}^2\mathbf{P})_{ik} + 3(\mathbf{S}^3\mathbf{P})_{ik} + \cdots + r(\mathbf{S}^r\mathbf{P})_{ik} + \cdots}{y_{ik}} + 1. \end{aligned} \tag{2.4}$$

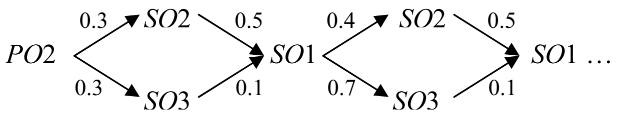
The second line of (2.4) is due to the fact that the shares $(\mathbf{S}^r\mathbf{P})_{ik}/y_{ik}$ are non-negative and sum to one (i.e., $\sum_{r=1}^{\infty}(\mathbf{S}^r\mathbf{P})_{ik}/y_{ik} = 1$). This shows that the weighted average distance equals one plus the weighted average number of intermediary owners involved. The numerator on the right hand side of expression (2.4) yields q_{ij}, with $\mathbf{Q} = \sum_{r=1}^{\infty} r\mathbf{S}^r\mathbf{P}$. Premultiplication by $(\mathbf{I} - \mathbf{S})$ and using (2.3) gives

$$(\mathbf{I}-\mathbf{S})\left(\sum_{r=1}^{\infty} r\mathbf{S}^r\mathbf{P}\right) = \sum_{r=1}^{\infty} r\mathbf{S}^r\mathbf{P} - \mathbf{S}\sum_{r=1}^{\infty} r\mathbf{S}^r\mathbf{P} = \sum_{r=1}^{\infty} r\mathbf{S}^r\mathbf{P} - \sum_{r=1}^{\infty} r\mathbf{S}^{r+1}\mathbf{P} = \sum_{r=1}^{\infty} \mathbf{S}^r\mathbf{P} = \mathbf{Y}.$$

Hence, $(\mathbf{I} - \mathbf{S})\mathbf{Q} = \mathbf{Y}$ and thus $\mathbf{Q} = (\mathbf{I} - \mathbf{S})^{-1}\mathbf{Y}$. This yields a simple expression for the weighted average number of secondary owners involved as defined in (2.4). The weighted average distance of the indirect linkages yields $WADIL_{ik} = (q_{ik}/y_{ik}) + 1$. The corresponding values in our hypothetical example are given in the matrix

$$\begin{pmatrix} 3.158 & 2.740 \\ 3.013 & 3.740 \\ 3.013 & 3.740 \end{pmatrix}.$$

Analyzing for instance the smallest and the largest elements $WADIL_{ik}$ is relatively easy, because the underlying example is fairly simple. The smallest weighted average distance is found for the link between $PO2$ and $SO1$, i.e., corresponding to $WADIL_{12} = 2.740$. Using the matrices $\mathbf{P}$ and $\mathbf{S}$ of this example given in Section 4.2 the connections between $PO2$ and $SO1$ can be graphed as follows.



It shows that the "shortest" indirect connection between $PO2$ and $SO1$ involves two steps and runs through one secondary owner, i.e., either via $SO2$ or via $SO3$. The property in $SO1$ that is attributed to $PO2$ in this link is 0.180 (i.e., the $(1,2)$-th element of $\mathbf{SP}$), which is 73% of the total property of $SO1$ (i.e., $y_{12} = 0.247$) that is redistributed to $PO2$ through indirect relations. The next connection between $PO2$ and $SO1$ involves three secondary owners (and thus four steps) and note that there are four different "paths" (via $SO2 \rightarrow SO1 \rightarrow SO2$; $SO3 \rightarrow SO1 \rightarrow SO2$; $SO2 \rightarrow SO1 \rightarrow SO3$; and $SO3 \rightarrow SO1 \rightarrow SO3$). The property attributed in this way to $PO2$ amounts to $(\mathbf{S}^3\mathbf{P})_{12} = 0.049$, which is 20% of the total. The next connection involves eight different paths, each via five secondary owners (i.e., six steps) and attributes to the primary owner 5% (i.e., 0.013) of the total property of $SO1$ that is redistributed to $PO2$ through indirect relations. And so forth. The weighted average distance (or number of steps involved) then equals $2 \times 0.73 + 4 \times 0.20 + 6 \times 0.05 + \cdots = 2.740$.

The largest value $WADIL_{ik}$ for an indirect connection between a primary and secondary owner is found between $PO2$ and either $SO2$ or $SO3$ (because $WADIL_{22} = WADIL_{23}$). Also in this case, the above graph illustrates the connections. If we focus on the connections between $PO2$ and $SO2$, we see that the "shortest" connection is a direct connection. The shortest indirect connection between $PO2$ and $SO2$ involves three steps and runs through two intermediary owners (i.e., via $SO2 \rightarrow SO1$ and via $SO3 \rightarrow SO1$). The next connection has five steps and runs through four intermediary owners and involves four different paths, etcetera.

In real world cases, the number of primary and secondary owners may become substantial implying numerous indirect relations. The distance becomes larger and the number of paths with the same distance grows rapidly when the number of primary and secondary owners increases. The complexity of this network of indirect relations between primary owner k and secondary owner i is summarized by the corresponding weighted average distance of the indirect linkages $WADIL_{ik} = (q_{ik}/y_{ik}) + 1$. A larger distance indicates a more complex network involving a larger number of different paths and is indicated by a larger value of $WADIL_{ik}$.

Several remarks seem to be in place. First, it may happen that $WADIL_{ik}$ cannot be determined, because $y_{ik} = 0$. This occurs for example if the matrix $\mathbf{S}$ is reducible. In that case, the secondary owners can be reclassified into two clusters (I and II) and no owner in cluster II holds a share in any of the owners in cluster I. If primary owner k holds only shares in the secondary owners of cluster II, we have that there is no indirect relation between k and secondary owners in cluster I. That is, $y_{ik} = 0$ for all i in cluster I. Using partitioned matrices in (2.2), and denoting the direct property distribution of primary owner k by the vector $\mathbf{p}$ and the indirect property distribution by the vector $\mathbf{y}$, we have

$$\left[\mathbf{I} - \begin{pmatrix} \mathbf{S}_{\text{I,I}} & \mathbf{O} \\ \mathbf{S}_{\text{II,I}} & \mathbf{S}_{\text{II,II}} \end{pmatrix}\right]^{-1} \begin{pmatrix} \mathbf{0} \\ \mathbf{p}_{\text{II}} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ \mathbf{p}_{\text{II}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{y}_{\text{II}} \end{pmatrix},$$

where, for example, $\mathbf{S}_{\text{II,I}}$ indicates the shares in a secondary owner in cluster II that are held by a secondary owner in cluster I, $\mathbf{p}_{\text{II}}$ is the shares in secondary owners in cluster II that are held by primary owner k, $\mathbf{y}_{\text{II}}$ gives the indirect property of secondary owners in cluster II as attributed to primary owner k, and $\mathbf{O}$ and $\mathbf{0}$ are, respectively, the null matrix and the null vector. As a matter of fact, we have $\mathbf{y}_{\text{II}} = (\mathbf{I} - \mathbf{S}_{\text{II,II}})^{-1}\mathbf{p}_{\text{II}} - \mathbf{p}_{\text{II}}$.

Whenever $y_{ik} = 0$ we define $WADIL_{ik} = 0$, so that the formal definition of the WADIL becomes

$$WADIL_{ik} = \begin{cases} (q_{ik}/y_{ik}) + 1 & \text{if} \quad y_{ik} > 0, \\ 0 & \text{if} \quad y_{ik} = 0. \end{cases} \tag{2.5}$$

with $\mathbf{Q} = (\mathbf{I} - \mathbf{S})^{-1}\mathbf{Y}$. Note that whenever an indirect relation exists (i.e., $y_{ik} > 0$), the number of secondary owners involved in this link cannot be smaller than one. Therefore, also the average number of secondary owners (i.e., q_{ik}/y_{ik}) cannot be smaller than one and $WADIL_{ik}$ is thus at least two.

Second, if we are interested in the total linkages (i.e., direct plus all indirect linkages) between a primary and a secondary owner, matrix $\mathbf{T}$ can be analyzed in the same way. We have $\mathbf{T} = \mathbf{LP} = (\mathbf{I} + \mathbf{S} + \mathbf{S}^2 + \cdots + \mathbf{S}^r + \cdots)\mathbf{P}$ and let r denote the number of secondary owners that act as an intermediate in the total link between primary owner k and secondary owner i. Note that this implies that a direct relation between k and i has zero intermediary owners and involves one step. Then, the weighted average distance or number of steps involved is, similar to (2.4), given by

$$\begin{aligned} &\left[p_{ik} + 2(\mathbf{SP})_{ik} + 3(\mathbf{S}^2\mathbf{P})_{ik} + 4(\mathbf{S}^3\mathbf{P})_{ik} + \cdots + (r+1)(\mathbf{S}^r\mathbf{P})_{ik} + \ldots\right] / t_{ik} \\ &\quad = \frac{0p_{ik} + 1(\mathbf{SP})_{ik} + 2(\mathbf{S}^2\mathbf{P})_{ik} + 3(\mathbf{S}^3\mathbf{P})_{ik} + \cdots + r(\mathbf{S}^r\mathbf{P})_{ik} + \cdots}{t_{ik}} + 1 \\ &\quad = (q_{ik}/t_{ik}) + 1. \end{aligned}$$

In line with (2.5), we define the weighted average distance of total linkages (WADTL) as

$$WADTL_{ik} = \begin{cases} (q_{ik}/t_{ik}) + 1 & \text{if} \quad t_{ik} > 0, \\ 0 & \text{if} \quad t_{ik} = 0. \end{cases} \tag{2.6}$$

Note that if there are neither direct nor indirect linkages we have $t_{ik} = 0$ and it makes no sense to examine the average distance. Hence, $WADTL_{ik} = 0$ by definition. Also observe that in the case when there is a direct linkage but no indirect linkages, we have that $p_{ik} > 0$ and $y_{ik} = 0$ imply $q_{ik} = 0$ and $t_{ik} > 0$, which yields $WADTL_{ik} = 1$.

In general, however, there are indirect linkages between k and i, implying that $q_{ik}/t_{ik} > 0$. Values close to one indicate that the link is essentially of a direct nature, while larger values express that the link is brought about by a complex network of relations. The reason is that (if $p_{ik} > 0$) generally a large part of the total link is of a direct nature and thus involves only one step (i.e., no intermediary owner). Note that because $\mathbf{Y} = \mathbf{T} - \mathbf{P}$, we have that $WADTL_{ik} = (q_{ik}/t_{ik}) + 1 \leq (q_{ik}/y_{ik}) + 1 = WADIL_{ik}$. That is, the WADTL between k and i is smaller than the WADIL (unless, of course, there are no direct linkages).[6]

2.4 An empirical application to the banking sector in the Czech Republic

For our empirical analysis, we have used the data in Turnovec (1999) for the banking sector in the Czech Republic at the end of 1997.[7] There are 13 primary owners and 12 secondary owners (see Appendix 2.A for a list, see Turnovec 1999 for further details). The primary property distribution **P** and the secondary property distribution **S** are given in Appendix 2.B.

2.4.1 Analyzing the ownership structure

For three secondary owners ($SO5$, $SO8$ and $SO12$) we observe that their shares are held only by primary owners. Since the corresponding rows in **S** contain only zeros, it is not possible to own a part of these secondary owners indirectly (i.e., via one or more secondary owners). This implies that the relation between a primary owner and these three secondary owners can only be direct. The matrix **Y** with indirect linkages will thus show only zeros in the corresponding rows. In other words, for these three secondary owners, the primary property distribution in matrix **P** tells the whole story. Since there are no indirect linkages, also the matrix with the average distances of the indirect linkages ($WADIL_{ik}$) shows rows with only zeros for these secondary owners.

The matrices **T** and **Y** with the sizes of total and indirect shares in secondary owners held by primary owners are given in Table 2.1. Because all the shares that are held by secondary owners are now accrued to primary owners, the matrix **Y** has more positive elements than the matrix **P**. In analyzing this matrix **Y**, let us focus first on the zero elements. Next to the rows for $SO5$, $SO8$ and $SO12$ (which contain only zeros, as has been explained above), we observe that also the row for $SO4$ contains primarily zeros. Note that $SO4$ is owned by four primary owners

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[6] Absent of direct linkages, all total linkages are indirect. That is, if $p_{ik} = 0$ we have $t_{ik} = y_{ik}$ and thus $WADTL_{ik} = WADIL_{ik}$.

[7] See for example the study by Kenway and Klvacova (1996) on Czech financial institutions, who argue that "... cross-ownership is not only a web but also a mask, hiding the extent to which the state remains an owner" (p. 800).

Table 2.1: Matrices **T** and **Y** for the banking sector in the Czech Republic, end of 1997

	*PO*1	*PO*2	*PO*3	*PO*4	*PO*5	*PO*6	*PO*7	*PO*8	*PO*9	*PO*10	*PO*11	*PO*12	*PO*13
	Matrix **T** with total linkages (in %)												
*SO*1	59.23	0.38	0.28	15.04	0.32	0.06	2.15	0.37	0.09	2.33	0.25	0.37	19.11
*SO*2	38.40	3.71	2.74	0	3.13	0.58	20.86	0	0.86	0	0	3.61	26.11
*SO*3	50.90	0.03	0.02	0.23	0.02	0.00	0.14	13.01	0.01	0.04	0.00	0.02	35.58
*SO*4	31.49	0	0	0	18.20	3.36	0	0	5.02	0	0	0	41.93
*SO*5	19.59	26.51	19.59	0	0	0	0	0	0	0	0	25.78	8.53
*SO*6	14.84	0.10	0.07	3.77	0.08	0.01	0.54	0.09	0.02	30.58	0.06	0.09	49.74
*SO*7	14.80	0.10	0.07	3.76	0.08	0.01	0.54	0.09	0.02	30.58	10.06	0.09	39.79
*SO*8	0	0	0	0	0	0	100.00	0	0	0	0	0	0
*SO*9	5.30	0.51	0.38	0	0.43	0.08	2.88	0	0.12	0	0	0.50	89.80
*SO*10	24.25	0.37	0.28	0	0.32	0.06	2.11	0	0.09	0	0	0.36	72.17
*SO*11	14.77	0.01	0.01	0.07	0.01	0.00	0.04	3.77	0.00	0.01	0.00	0.01	81.31
*SO*12	0	0	0	0	41.10	42.70	0	0	0	0	0	0	16.20
	Matrix **Y** with indirect linkages (in %)												
*SO*1	6.43	0.38	0.28	0.29	0.32	0.06	2.15	0.37	0.09	2.33	0.25	0.37	7.16
*SO*2	8.15	3.71	2.74	0	3.13	0.58	20.86	0	0.86	0	0	3.61	8.40
*SO*3	2.16	0.03	0.02	0.23	0.02	0.00	0.14	0.09	0.01	0.04	0.00	0.02	5.75
*SO*4	0	0	0	0	3.23	3.36	0	0	0	0	0	0	1.27
*SO*5	0	0	0	0	0	0	0	0	0	0	0	0	0
*SO*6	14.84	0.10	0.07	3.77	0.08	0.01	0.54	0.09	0.02	0.58	0.06	0.09	4.79
*SO*7	14.80	0.10	0.07	3.76	0.08	0.01	0.54	0.09	0.02	0.58	0.06	0.09	4.78
*SO*8	0	0	0	0	0	0	0	0	0	0	0	0	0
*SO*9	5.30	0.51	0.38	0	0.43	0.08	2.88	0	0.12	0	0	0.50	3.60
*SO*10	3.88	0.37	0.28	0	0.32	0.06	2.11	0	0.09	0	0	0.36	2.64
*SO*11	14.77	0.01	0.01	0.07	0.01	0.00	0.04	3.77	0.00	0.01	0.00	0.01	10.32
*SO*12	0	0	0	0	0	0	0	0	0	0	0	0	0

Note: Very small numbers close to zero are denoted by 0.00.

($PO1$, $PO5$, $PO9$, $PO13$) and one secondary owner ($SO12$). In its turn, however, all shares in this secondary owner are held by primary owners $PO5$, $PO6$, and $PO13$. Therefore, $PO5$, $PO6$, and $PO13$ are the only primary owners that have an indirect link to $SO4$, so that $y_{4k} = WADIL_{4k} = 0$ for $k \neq 5, 6, 13$. Note that the indirect link between, for example, $PO5$ and $SO4$ runs only through $SO12$. We have $y_{45} = s_{4,12} \times p_{12,5} = 0.0786 \times 0.4110 = 0.0323$. $SO12$ being the only intermediary owner also explains why $WADIL_{45} = WADIL_{46} = WADIL_{4,13} = 2$ in Table 2.2 (which gives the matrices with the WADILs and WADTLs). $PO1$ and $PO9$ have a direct link to $SO4$ (i.e., $p_{41}, p_{49} > 0$), but not an indirect link.

Additional zeros in the matrix $\mathbf{Y}$ and WADILs are found for the linkages between primary owners $PO4$, $PO8$, $PO10$ and $PO11$, and secondary owners $SO2$, $SO9$ and $SO10$. It turns out that each of these primary owners, only holds shares in (and thus has a direct link to) one or more secondary owners in the cluster $SO1$, $SO3$, $SO6$, $SO7$, $SO11$. In addition, each of these secondary owners only holds shares in one or more other members of the cluster. So, the indirect linkages only involve members of the cluster and it is thus impossible to achieve a link between one of the four primary owners ($PO4$, $PO8$, $PO10$ and $PO11$) and a secondary owner other than $SO1$, $SO3$, $SO6$, $SO7$, $SO11$. This implies that all remaining entries (i.e., in rows 2, 4, 5, 8, 9, 10, 12) are zero in the columns 4, 8, 10 and 11.

Another interesting issue is the case where some primary owner k has no direct link to a certain secondary owner i (i.e., $p_{ik} = 0$), but substantial indirect linkages (i.e., $y_{ik} > 0$). In this case, the information in the actually observed matrix $\mathbf{P}$ does not at all reflect the true ownership structure. For example, $PO7$ has no direct share in $SO2$, but indirectly it owns more than 20% of the property of $SO2$. Similarly, about 15% of the property of $SO6$, $SO7$ and $SO11$ is indirectly held by $PO1$ although there is no direct interest in them. The same applies to the majority of primary and secondary owners relations, but to a much lesser extent. Tables 2.1 and 2.2 also show that the presence of a direct interest does not necessarily mean that there is an indirect link as well. For instance, $y_{41} = WADIL_{41} = 0$, although $p_{41} > 0$.

Note that in Table 2.2 we have used two types of numbers. Integer numbers (i.e., without any decimals) are exact. For example, $WADIL_{ik} = 3$ indicates that all indirect connections between primary owner k and secondary owner i involve always exactly three steps (i.e., two intermediary owners). Using the initial property distribution in $\mathbf{P}$, we see that, say, $WADIL_{93} = WADIL_{10,3} = 3$ are both brought about through the intermediation of $SO5$ and $SO2$. An exceptional case is underlying $WADIL_{21} = 2$. All indirect linkages between $PO1$ and $SO2$ involve exactly

Table 2.2: WADILs and WADTLs for the banking sector in the Czech Republic, end of 1997

	*PO*1	*PO*2	*PO*3	*PO*4	*PO*5	*PO*6	*PO*7	*PO*8	*PO*9	*PO*10	*PO*11	*PO*12	*PO*13
	Weighted average distances of indirect linkages ($WADIL_{ik}$)												
*SO*1	2.341	3.043	3.043	3.040	3.220	4.043	3.043	2.053	3.043	2.040	2.040	3.043	2.220
*SO*2	2	2	2	0	2.177	3	2	0	2	0	0	2	2.026
*SO*3	2.334	4.023	4.023	2.053	4.200	5.023	4.023	3.016	4.023	3.053	3.053	4.023	2.102
*SO*4	0	0	0	0	2	2	0	0	0	0	0	0	2
*SO*5	0	0	0	0	0	0	0	0	0	0	0	0	0
*SO*6	2.146	4.043	4.043	2.040	4.220	5.043	4.043	3.053	4.043	3.040	3.040	4.043	2.457
*SO*7	2.146	4.043	4.043	2.040	4.220	5.043	4.043	3.053	4.043	3.040	3.040	4.043	2.457
*SO*8	0	0	0	0	0	0	0	0	0	0	0	0	0
*SO*9	2.212	3	3	0	3.177	4	3	0	3	0	0	3	2.330
*SO*10	2.212	3	3	0	3.177	4	3	0	3	0	0	3	2.330
*SO*11	2.057	5.023	5.023	3.053	5.200	6.023	5.023	2.014	5.023	4.053	4.053	5.023	2.178
*SO*12	0	0	0	0	0	0	0	0	0	0	0	0	0
	Weighted average distances of total linkages ($WADTL_{ik}$)												
*SO*1	1.146	3.043	3.043	1.040	3.220	4.043	3.043	2.053	3.043	2.040	2.040	3.043	1.457
*SO*2	1.212	2	2	0	2.177	3	2	0	2	0	0	2	1.330
*SO*3	1.057	4.023	4.023	2.053	4.200	5.023	4.023	1.014	4.023	3.053	3.053	4.023	1.178
*SO*4	1	0	0	0	1.177	2	0	0	1	0	0	0	1.030
*SO*5	1	1	1	0	0	0	0	0	0	0	0	1	1
*SO*6	2.146	4.043	4.043	2.040	4.220	5.043	4.043	3.053	4.043	1.039	3.040	4.043	1.140
*SO*7	2.146	4.043	4.043	2.040	4.220	5.043	4.043	3.053	4.043	1.039	1.013	4.043	1.175
*SO*8	0	0	0	0	0	0	1	0	0	0	0	0	0
*SO*9	2.212	3	3	0	3.177	4	3	0	3	0	0	3	1.053
*SO*10	1.194	3	3	0	3.177	4	3	0	3	0	0	3	1.049
*SO*11	2.057	5.023	5.023	3.053	5.200	6.023	5.023	2.014	5.023	4.053	4.053	5.023	1.150
*SO*12	0	0	0	0	1	1	0	0	0	0	0	0	1

two steps, but there are two of such paths. One runs via $SO4$ and the other via $SO5$. Outcomes that are not given as an integer reflect that there are at least multiple paths of different lengths.

Closer inspection of the numbers in $\mathbf{Y}$ and WADILs suggests that there is an inverse relationship between $WADIL_{ik}$ and the size y_{ik} of the indirect shareholdings. This should not be too much of a surprise as follows from (2.4). A "large" value of $WADIL_{ik}$ (say, 5 or so) indicates that the weight $(\mathbf{S}^r\mathbf{P})_{ik}/y_{ik}$ must be reasonably large for values $r = 4$, 5, and 6, for example. Thus $(\mathbf{S}^4\mathbf{P})_{ik}$, $(\mathbf{S}^5\mathbf{P})_{ik}$, and $(\mathbf{S}^6\mathbf{P})_{ik}$ have a considerable contribution to y_{ik}. In general, however, $(\mathbf{S}^r\mathbf{P})_{ik}$ declines rapidly when r increases. This explains why in many cases "large" values of $WADIL_{ik}$ are found for values y_{ik} close to zero. The correlation coefficient between size and weighted average distance is 0.534 for the indirect linkages (based on the 95 cases with $y_{ik} > 0$). Hence, smaller average distances are associated, to some extent, with larger indirect shares.

Figure 2.1: Indirect connection between $PO6$ and $SO11$ for the banking sector in the Czech Republic

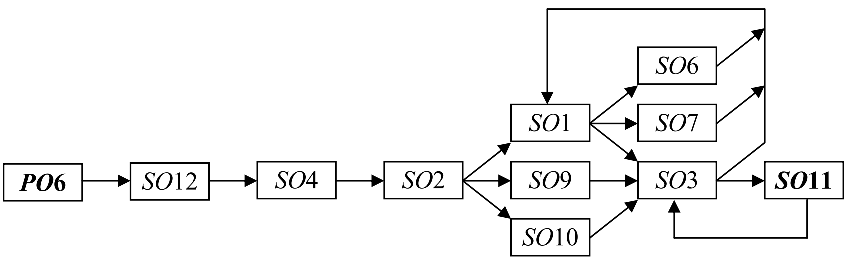


Table 2.2 shows that $WADIL_{11,6} = 6.023$ is the largest, i.e., $PO6$ owns shares of $SO11$ through 5.023 secondary owners on average. Using the primary and secondary property distributions in Appendix 2.B, this connection is graphed in Figure 2.1. The graph shows that the "shortest" connection between $PO6$ and $SO11$ can be established through three paths, each involving six steps (via five secondary owners: $SO12 \rightarrow SO4 \rightarrow SO2 \rightarrow SO1 \rightarrow SO3$; $SO12 \rightarrow SO4 \rightarrow SO2 \rightarrow SO9 \rightarrow SO3$; and $SO12 \rightarrow SO4 \rightarrow SO2 \rightarrow SO10 \rightarrow SO3$). Because of mutual shareholdings (between the cluster $SO6$, $SO7$, $SO3$ on the one hand and $SO1$ on the other hand, and between $SO3$ and $SO11$) there is in fact an infinite number of paths through which $PO6$ indirectly owns property of $SO11$. It should be stressed, however, that the property attributed to $PO6$ through paths involving eight or more steps is practically zero. Figure 2.1 also graphs the connections of $PO6$ with any other secondary owner. For

example, $PO6$ owns 0.58% of $SO2$ via exactly two intermediary owners (i.e., three steps), thus we have $y_{26} = 0.58$ and $WADIL_{26} = 3$.

So far we have discussed the WADIL, which is of particular interest to detect the interests that cannot be seen straightforwardly from the observed data. One might be more interested in the total linkages between a principal and a secondary owner, no matter whether they are direct or indirect. The total linkages are obtained from the matrix $\mathbf{T} = \mathbf{P} + \mathbf{Y}$. The number of intermediary owners involved in any specific link is given by $WADTL_{ik} = (q_{ik}/t_{ik}) + 1$, expressing the weighted average distance of total linkages between primary owner k and secondary owner i. The results are given in the bottom part of Table 2.2. Note that $WADTL_{ik} = 0$ if $t_{ik} = 0$ which indicates that there is no link, neither directly nor indirectly. The cases where $t_{ik} \neq 0$ and $q_{ik} = 0$ yield $WADTL_{ik} = 1$ and reflect that there is a direct link but no indirect link. This implies that any cell for which $WADIL_{ik} = 0$ in the upper part of Table 2.2, shows a 0 or a 1 for corresponding WADTLs. No indirect linkages (i.e., $WADIL_{ik} = 0$) means that there are only direct linkages (i.e., $WADTL_{ik} = 1$) or no linkages at all (i.e., $WADTL_{ik} = 0$).

Comparing the non-zero elements of WADILs and WADTLs in Table 2.2 shows that $0 < WADTL_{ik} < WADIL_{ik}$ if and only if there is a direct link between k and i (i.e., $p_{ik} > 0$).[8] In most cases, the average distance falls substantially once direct links are taken into account, because the direct linkage is a large part of the total linkage. Hence, values of $WADTL_{ik}$ that are close to one indicate that the link is mainly direct. As was the case with $WADIL_{ik}$, large values hint at the existence of a complex network of relations that underlie a certain link.

2.4.2 Visualizing the ownership structure

It is important to note that focusing entirely on either $\mathbf{T}$ or WADTL (or similarly on either $\mathbf{Y}$ or WADIL) would be misleading. This holds in particular if we are interested in obtaining a rough picture of the structure of relations between primary and secondary owners. The drawback of only considering the matrix $\mathbf{T}$ and/or $\mathbf{Y}$ is that it does not reflect the complexity of a certain relation. The elements of $\mathbf{T}$ and/or $\mathbf{Y}$ do not allow to distinguish how many intermediary owners are involved in a certain link. The limitation of focusing entirely on WADTL and/or WADIL is that the size of the total and/or indirect linkages is ignored. The only issue that matters is the "distance" between a primary and a secondary owner. For example, for

[8] $WADIL_{ik} > 0$ implies $y_{ik} > 0$ and thus $q_{ik} > 0$ (because $\mathbf{Q} \geq \mathbf{Y}$). Then $t_{ik} > y_{ik}$ if and only if $p_{ik} > 0$, and using the definitions in (2.5) and (2.6) straightforwardly proves the equivalence.

the case with the largest WADIL (which was graphed in Figure 2.1), we have that $t_{11,6} = y_{11,6} = 0.00$, which is a very negligible share. The issue thus arises whether it makes sense to consider this specific relation if one is interested in an overview of the ownership structure of a sector (or in making a graphical representation of its main characteristics).

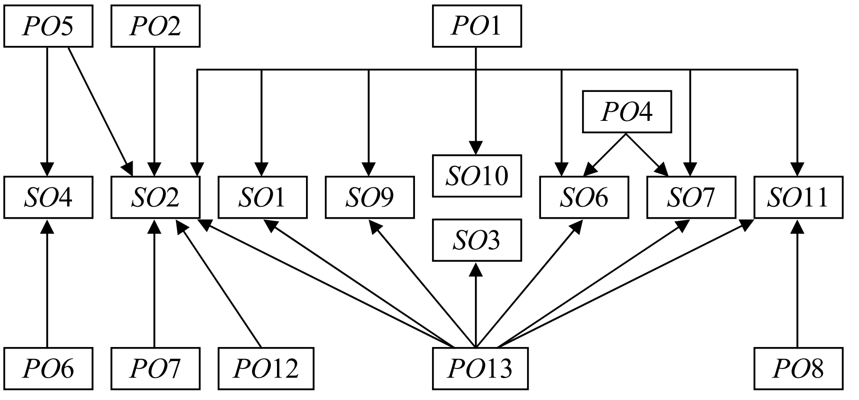
To solve this problem both types of indicator are combined. That is, we take the average distance into account only if the size of the linkage is sufficiently large, using a threshold value a. In our application we will focus on analyzing the indirect linkages between primary and secondary owners. The reason is that this reflects a part of the ownership structure that is "hidden" in the sense that it cannot be directly observed from the data, while it represents a substantive part. For example, the data in Appendix 2.B show that the shares of three secondary owners ($SO5$, $SO8$, and $SO12$) are held just by primary owners. For the other nine secondary owners, on average 21% (with a maximum of 52% for $SO2$) of their shares are held by secondary owners and need to be redistributed and attributed to primary owners.

In our first exercise we take only the indirect linkages into account that are at least 3% (i.e., $y_{ik} \geq 0.03$). This results in 23 combinations of a primary owner k and a secondary owner i that exhibit non-negligible indirect linkages. It should be stressed that the choice of the threshold is arbitrary. Lower thresholds imply a larger number of admissible combinations which renders complex graphs, causing that one cannot see the wood for the trees anymore. Setting the threshold too high implies that one admits only few combinations, which causes the corresponding graph to be overly simplistic. For each of the 23 combinations, the $WADIL_{ik}$ is rounded to the nearest integer. It turns out that in all cases the rounded average distance is 2 (indicating that on average there is approximately one intermediary owner involved in the connection).

The corresponding graph for the 23 combinations is given in Figure 2.2. We would like to make three observations. First, $PO1$ and $PO13$ are the most important primary owners of indirect shares in secondary owners. Both have an indirect interest in no less than seven secondary owners. Second, taking the opposite viewpoint, we see that $SO2$ is the secondary owner in which the largest number (i.e., six) of primary owners has an indirect interest. Third, four primary owners ($PO3$, $PO9$, $PO10$, and $PO11$) show no indirect interests in any of the secondary owners. Next we will analyze each of these observations in more detail.

With respect to the first observation, note that $PO1$ (Fund of National Property)

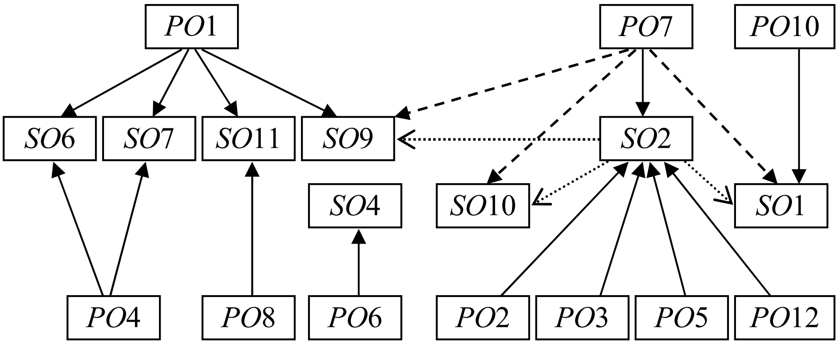
Figure 2.2: Indirect linkages for the banking sector in the Czech Republic



and $PO13$ (all minority investors taken together) hold large amounts of shares directly and their portfolios are very diversified in the sense that they have a direct interest in many secondary owners. Recall that the size of the two-step indirect linkages between primary owner k and secondary owner i is given by $y_{ik}^{(1)} = \sum_j s_{ij} p_{jk}$. For $k = 1$ and $k = 13$, we see that p_{jk} is fairly large for many secondary owners j. So $y_{ik}^{(1)}$ easily surpasses the threshold level 0.03 for quite a number of secondary owners i. With respect to the second observation, note that 47.96% of the shares in $SO2$ are directly held by primary owners and 52.04% by secondary owners. These 52.04% have to be redistributed and attributed to primary owners by means of indirect linkages. This is a very large part of the property of $SO2$. When we compare this with the other secondary owners we see that the direct share held by secondary owners is 29% for $SO11$; ranges between 20 and 25% for $SO1$, $SO6$ and $SO7$; and is less than 15% for the others. With respect to the third observation, consider the columns 3, 9, 10 and 11 of the matrix $\mathbf{Y}$ and note that the largest element y_{23} reflects an indirect share of 2.74%, which is lower than the 3% threshold. This indirect linkage runs exclusively through $SO5$. That is, $PO3$ has a 19.59% direct interest in $SO5$, which in its turn holds a 14.00% share in $SO2$. The indirect interest of $PO3$ in $SO2$ thus amounts to $0.1959 \times 0.1400 = 0.0274$.

Figure 2.2 provided a quick overview of the most important indirect linkages from the perspective of the entire banking sector. However, it might also be insightful to adopt the perspective of the individual primary owner. Returning to our third observation above, note that the indirect property (2.74%) that $PO3$ holds of $SO2$ is quite substantial in comparison to the total linkages of $PO3$. The question

Figure 2.3: Indirect linkages for the Czech banking sector, based on the matrix **T** and WADTLs



is whether this indirect linkage should have been neglected. As an alternative, one might thus want to focus on indirect shareholding that is important for a primary owner when compared to its direct and indirect shareholding (i.e., total linkages).

To this end, we have applied the same analysis to the WADTLs. The threshold was chosen such that finally we arrived at a comparable number of combinations to include in the graph. This led to the criteria that $t_{ik} \geq 0.02$ and for all combinations that satisfied this criteria the corresponding $WADTL_{ik}$ was rounded to the nearest number. Next, only the combinations with a rounded average distance of at least 2 were taken into account. That is, pure direct linkages and indirect linkages that are dominated by direct linkages were left out. This resulted in 17 combinations (i, k) that satisfy $t_{ik} \geq 0.02$ and $WADTL_{ik} \geq 1.500$. Their graphical representation is given in Figure 2.3.

We see that 14 combinations between a primary and a secondary owner have a rounded average distance 2, and three combinations have a distance 3. They are graphed by solid ($\longrightarrow$) and dashed ($\dashrightarrow$) arrows respectively. Cross-shareholdings between secondary owners, represented by dotted arrows ($\cdots>$), are included for the three cases with distance 3. For example, the distance 3 link between $PO7$ and $SO10$ is built up from a distance 2 link between $PO7$ and $SO2$ and a direct link between $SO2$ and $SO10$.

Observe that $PO13$ (which was one of the most important primary owners in Figure 2.2) has completely vanished. Although its total linkages are huge (as reflected by column 13 of the matrix **T**) and many indirect linkages are very large in size, the indirect linkages turn out to be relatively minor in comparison to the total linkages. As a consequence, the total linkages are dominated by the direct linkages, which explains why the average distance between $PO13$ and any secondary owner is always less than 1.500. To a lesser extent, the same applies to $PO1$. Of the four primary owners that were absent in Figure 2.2, two ($PO3$ and $PO10$) are now included in Figure 2.3. For example, $PO10$ has a 30% direct share in $SO6$ and $SO7$, who in their turn have a 5.10% and 2.50% share in $SO1$, respectively. As was the case in Figure 2.2, the primary owners $PO9$ and $PO11$ are also absent in Figure 2.3.

In contrast to Figure 2.2, $PO7$ turns up in Figure 2.3 as an important primary owner in terms of indirect linkages with secondary owners. Just like $PO1$, it shows four connections in the last figure, three of which have a distance 3 and run through $SO2$. From the initial property distribution in Appendix 2.B, it follows that $PO7$ holds a 100% share in $SO8$. In its turn, $SO8$ has a 20.86% shares in $SO2$ so that indirectly $PO7$ owns 20.86% of the property of $SO2$. Next, $SO2$ holds shares in $SO1$ (10.10%), $SO9$ (13.80%) and $SO10$ (10.10%), which explains the indirect distance 3 connections. As was the case in Figure 2.2, $SO2$ has the largest number of shareholders also in Figure 2.3. Five primary owners indirectly own a relevant part of $SO2$'s property.

Another application of the proposed indirect measures is that they can be used for comparative analyses. For example, the banking sector could be compared with another sector, or the banking sector could be compared for different years, or the banking sector in one country could be compared with that in another country. The overall degree of complexity of the network of indirect relations is reflected by the overall average of the weighted average distances (based on indirect or total linkages). In our empirical application, this average of the weighted average distances is 1.949 for the indirect linkages and 1.861 for the total linkages. The first indicator is the simple average of all the WADILs and the second of the WADTLs. Recall that $WADIL_{ik} = 0$ if there is no indirect linkage between k and i, and $WADIL_{ik} \geq 2$ otherwise. Similarly, we have that $WADTL_{ik} = 0$ if there is no linkage between k and i, $WADTL_{ik} = 1$ if there is only a direct link, and $WADTL_{ik} > 1$ if there is an indirect link.[9] Thus, the results for the overall average distance indicate that shareholding linkages in the Czech banking sector are brought about by a complex network of

[9] It should be emphasized that the WADILs (WADTLs) that are zero are included in calculating the overall average indirect (total) linkages. Neglecting the zero-elements would give us the average of the weighted average distances for the cases in which an indirect linkage exists. As an overall measure for comparative purposes, however, this would have a clear drawback that is sketched by the following example. Consider a sector with many secondary owners but no cross-shareholding except for a mutual interest between, say, $SO1$ and $SO2$. The overall average distance could well be substantial if the zero-elements were not taken into consideration, whereas the network of indirect relations is extremely simple. Taking all WADILs into account would in this example yield a very low overall average, in line with the simplicity of the structure. So, in determining the overall complexity of a network, it is important to take all possible connections into consideration.

relations, and that they cannot be associated with linkages of a direct nature only.

2.4.3 Ownership network complexity and separation of dividend and control rights

An important issue in the finance literature is the separation of control and dividend rights due to the pyramiding structure and cross-holdings (see e.g., La Porta et al., 1999, 2002; Bebchuk et al., 2000; Claessens et al., 2000; Faccio et al., 2001; Faccio and Lang, 2002; Attig and Gadhoum, 2003; Gadhoum et al., 2005; Dorofeenko et al., 2008). One of the main findings in these studies is that control (or voting) rights in the presence of cross-holdings, pyramiding structures, and dual class shares usually exceed dividend (or cash-flow) rights. Control rights are obtained from the so called "weakest link" approach used in all studies cited above, except Dorofeenko et al. (2008) who propose the "dominant shareholder" methodology for this purpose. In this subsection we will argue that our measure of property network complexity may be used as an alternative.

As a very brief introduction to cash-flow and control rights, consider the following simple hypothetical cases of a pyramidal structure:

Family a: 50% in firm $\alpha \rightarrow$ 11% in $\beta \rightarrow$ 11% in $\gamma \rightarrow$ 10% in $\delta \rightarrow$ 11% in ε
Family b: 10% in ε

The percentages represent the shares held in the "next" firm. For example, family a holds 50% of the shares in firm α, which holds 11% of the shares in firm β, etcetera. The question is what the cash flow (O) and control (C) rights of each family are in the last firm ε.

Clearly, family b owns 10% of both O and C rights in ε, if we assume that there are no dual-class shares (i.e., under the one-share-one-vote rule). For family a, the ownership stake is equal to the product of all cash-flow rights along the property chain. Hence, family a owns only $0.50 \times 0.11 \times 0.11 \times 0.10 \times 0.11 \approx 0.007\%$ of the O stake in firm ε. According to the "weakest link" methodology, the C stake of family a is equal to the minimum of the control stakes in the ownership chain. Hence, the family has 10% C rights in ε, provided that the threshold level is not larger than 10%.[10] For family a, the separation of ownership and control thus is huge (O$\approx$ 0.007% and C = 10%). According to the "weakest link" approach, both

[10] It should be mentioned that usually only control stakes that exceed certain threshold levels (typically 10 and 20%) are considered.

families have the same control power in ε, although there is a large difference in their O stakes.

One might argue that it is more reasonable that the direct control of 10% for family b is much larger in terms of real power than the indirect control of 10% via four intermediate firms in case of family a. It thus seems that the "weakest link" approach misses such incongruencies between O and C rights, which occur in the presence of cross-ownership relations among firms. Precisely these relations are fully captured by our notion of distance.

Following the above line of reasoning, one might expect that countries with the largest (smallest) separation of ownership and control have a more (less) complex network of ownership relations. Consequently, one would expect the proposed distance measures to be large (small). Claessens et al. (2000); Faccio and Lang (2002); Attig and Gadhoum (2003) and Gadhoum et al. (2005) report the following O/C ratios (in ascending order). Japan: 0.602; Switzerland: 0.740; Italy: 0.743; Indonesia: 0.784; Singapore: 0.794; Germany: 0.842: Canada: 0.850; Philippines: 0.908; Portugal: 0.924; France: 0.930; USA: 0.940; Thailand: 0.941; and Spain: 0.941. Some average O/C ratios are Canada-USA: 0.895; Western Europe: 0.868; and East Asia: 0.746. These results are in line with the reciprocal relationship between complexity and the O/C ratio. In East Asia and Japan, in particular, firms are historically interlinked through strong shareholding interlocks, yielding complex ownership structures. This suggests that our notion of distance can be considered as an alternative measure of separation of ownership and control, fully taking into account means of enhancing control such as non-pyramidal cross-ownerships (both one-sided and reciprocal).[11]

Also Dorofeenko et al. (2008) observe that "... for more complicated cross-ownership the product of shares and the minimum share along the chain are insufficient statistics for ownership and control, respectively" (p. 77). This implies that the "weakest link" approach is suitable only for pyramidal cross-ownership relations, where the chains of ownership stakes are easily tractable. Hence, for non-pyramidal cross-ownership relations more general measures for O and C rights are required. They suggest to trace a controlling primary owner for each firm, and propose the methodology that rests on the construction of control assignments on the

[11] It should be noted that dual-class shares are not covered by our approach. At the same time, multiple class shares are not the most common equity structure. According to Bebchuk et al. (2000), the reason is that "... the corporate law of some jurisdictions restricts both the voting ratio between high- and low-ratio shares and the numerical ratio between high- and low-vote shares that a firm is permitted to issue" (p. 297). The studies mentioned above, report that only 19.91, 16.10, and 8.19% of firms in, respectively, Europe, Canada, and the US issue multiple class equity as a mean to enhance control.

Table 2.3: Control rights according to the "weakest link" approach for the Czech banking sector (in %)

	*PO*1	*PO*2	*PO*3	*PO*4	*PO*5	*PO*6	*PO*7	*PO*8	*PO*9	*PO*10	*PO*11	*PO*12	*PO*13
*SO*1	65.70	12.90	12.90	14.75	12.90	10.66	12.90	2.80	7.82	7.60	2.50	12.90	24.85
*SO*2	61.43	14.00	14.00	0	17.18	7.86	20.86	0	5.02	0	0	14.00	43.42
*SO*3	55.04	6.30	6.30	1.53	6.30	6.30	6.30	12.92	6.30	1.53	1.53	6.30	36.13
*SO*4	31.49	0	0	0	22.81	7.86	0	0	5.02	0	0	0	48.52
*SO*5	19.59	26.51	19.59	0	0	0	0	0	0	0	0	25.78	8.53
*SO*6	25.05	12.90	12.90	14.75	12.90	10.66	12.90	2.80	7.82	32.50	2.50	12.90	69.80
*SO*7	24.99	12.90	12.90	14.75	12.90	10.66	12.90	2.80	7.82	35.10	10.00	12.90	59.86
*SO*8	0	0	0	0	0	0	100.00	0	0	0	0	0	0
*SO*9	13.80	13.80	13.80	0	13.80	7.86	13.80	0	5.02	0	0	13.80	100.00
*SO*10	30.47	10.10	10.10	0	10.10	7.86	10.10	0	5.02	0	0	10.10	79.63
*SO*11	29.01	6.30	6.30	1.53	6.30	6.30	6.30	12.92	6.30	1.53	1.53	6.30	100.00
*SO*12	0	0	0	0	41.10	42.70	0	0	0	0	0	0	16.20

Note: Control rights percentages are given without any imposed threshold level, hence the row sums are bigger than 100% except for *SO*5, *SO*8, and *SO*12.

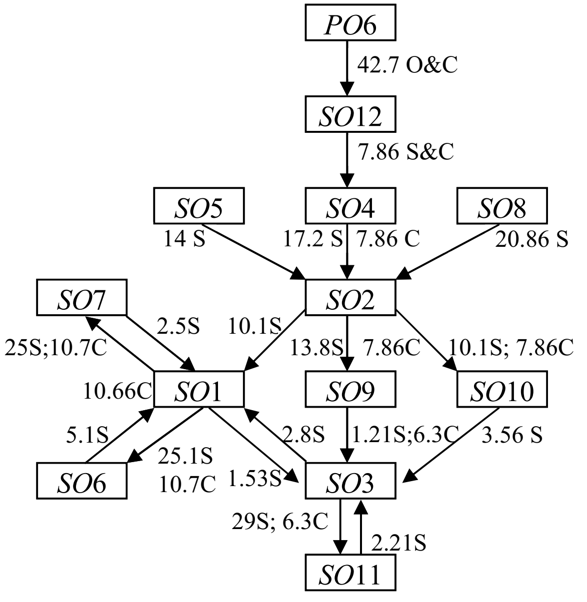
base of the "dominant shareholder" theorem that identifies controllers according to *relative* majority of (both direct and indirect) voting shares. In short, their approach is as follows. The $n \times m$ matrix $\mathbf{C}$ gives the control coefficients. Its typical element c_{ik} is an indicator function that takes a positive value if company i $(= 1, \ldots, n)$ is controlled by primary owner k $(= 1, \ldots, m)$, and zero otherwise. Then the share of votes in company i by some primary owner k is given by $p_{ik} + \sum_{j=1}^{n} s_{ij}c_{jk}$, which in matrix form yields $\mathbf{P} + \mathbf{SC}$. After this reassignment of shares, the remaining "uncontrolled" voting shares reduce to $\mathbf{S}(\mathbf{I} - \widehat{\mathbf{C}\boldsymbol{\imath}_m})$, where $\widehat{\mathbf{C}\boldsymbol{\imath}_m}$ is the $n \times n$ diagonal matrix with the row sums of $\mathbf{C}$ along its diagonal. The authors show that *relative* majority, unlike *absolute* majority as the criterion relevant for control, ensures that *every* firm is controlled *only* by primary owner(s), because a largest shareholder always exists. Thus the last matrix of "uncontrolled" shares is a zero matrix implying that $\mathbf{C}\boldsymbol{\imath}_m = \boldsymbol{\imath}_n$.[12] So the control rights are given by the matrix $\mathbf{P} + \mathbf{SC}$, which also adds to one for each firm i, i.e., $(\mathbf{P} + \mathbf{SC})\boldsymbol{\imath}_m = \boldsymbol{\imath}_n$.

In the remainder of this subsection, we compare our WADTL and WADIL measures with the "weakest link" and "dominant shareholder" approaches, applied to the Czech banking sector. Control rights of primary owners according to the "weakest link" methodology are given in Table 2.3, where for the moment we do not impose any threshold level on their sizes (hence, some row sums are bigger than 100%). To illustrate, we examine the derivation of control rights for the case of *PO*6, which is also graphically illustrated in Figure 2.4.

All possible shareholding linkages among secondary owners are illustrated in Figure 2.4. Assuming the one-share-one-vote rule, first, it is easy to see that *PO*6 has 42.70% control and cash flow stakes in *SO*12. The minimum stake in the ownership chain until *SO*4 is 7.86%, which is thus the control stake of *PO*6 in *SO*4. By the same logic, *PO*6 owns 7.86% of control rights in *SO*2, *SO*9, and *SO*10. To find the control stake in *SO*3, consider the three ownership chains that pass through *SO*1, *SO*9 and *SO*10, respectively. Then the corresponding control share is equal to the sum of the minimum stakes along these three chains, that is 6.30% (= 1.53% + 1.21% + 3.56%) gives the control rights of *PO*6 in *SO*3. Similarly, the control stake of *PO*6 in *SO*1 equals 10.66%, the sum of the minimum stakes in the links via *SO*2 and *SO*3 (= 7.86% + 2.80%). Notice that the mutual cross-holdings of *SO*1 with *SO*6 and of *SO*1 with *SO*7 have been disregarded. The reason is that we would consider such

[12] The control coefficients satisfy the following two conditions. (1) If $c_{ik} > 0$, then $p_{ik} + \sum_j s_{ij}c_{jk} \geq p_{il} + \sum_j s_{ij}c_{jl}$ for all $l = 1, \ldots, m$, for all $k = 1, \ldots, m$, and all $i, j = 1, \ldots, n$; and (2) $\sum_{k=1}^{m} c_{ik} = 1$ for all $i = 1, \ldots, n$. The first condition says that only relative majority shareholders can control a firm, while the second property states that every firm is controlled by some primary owner. See Dorofeenko et al. (2008) for further details.

Figure 2.4: Identification of control rights of *PO6* according to the "weakest link" methodology (without threshold)



Note: S stands for the size of shareholding between secondary owners, and C for the control rights of *PO6* in corresponding company.

mutual links only if *SO*1 owned at least 50% of *SO*6 and *SO*7, which in turn own *SO*1.[13] Since the control stake of 10.66% in *SO*1 is smaller than the ownership stakes of the *SO*1 in *SO*6 and *SO*7, we have that *PO*6 owns 10.66% of the control rights in *SO*6 and *SO*7 as well. Finally, the control stake of *PO*6 in *SO*11 is 6.30%, equal to that in *SO*3 and smaller than the direct stake of *SO*3 in *SO*11 (i.e., 29.01%).

Because our dataset is fairly small, the matrix of control coefficients on the basis of relative majority of votes is easily found. The underlying intuition is thoroughly explained in Dorofeenko et al. (2008). For a small dataset the control coefficients can be derived iteratively as follows. First, take the $n \times m$ matrix $\mathbf{C}^{(0)}$ that has unity in all cells that correspond to positive cells in the matrix **P**, and zeroes elsewhere. Next, from the secondary property distribution matrix **S**, the firms are found that are not owned by any other secondary owner. In our case, these are *SO*5, *SO*8, and *SO*12. Because these firms cannot be owned indirectly, we search in the corresponding rows of **P** for the shareholder(s) with the largest stake. These are assigned

[13] If this would have been the case, we would have added 5.1% and 2.5% to the 10.66% of *PO*6 control rights in *SO*1.

Table 2.4: Matrices **C** and **P** + **SC** for the banking sector in the Czech Republic, end of 1997

	*PO*1	*PO*2	*PO*3	*PO*4	*PO*5	*PO*6	*PO*7	*PO*8	*PO*9	*PO*10	*PO*11	*PO*12	*PO*13
	The control assignment matrix, **C**												
*SO*1	1	0	0	0	0	0	0	0	0	0	0	0	0
*SO*2	0	0	0	0	0	0	0	0	0	0	0	0	1
*SO*3	1	0	0	0	0	0	0	0	0	0	0	0	0
*SO*4	0	0	0	0	0	0	0	0	0	0	0	0	1
*SO*5	0	1	0	0	0	0	0	0	0	0	0	0	0
*SO*6	0	0	0	0	0	0	0	0	0	0	0	0	1
*SO*7	0	0	0	0	0	0	0	0	0	0	0	0	1
*SO*8	0	0	0	0	0	0	1	0	0	0	0	0	0
*SO*9	0	0	0	0	0	0	0	0	0	0	0	0	1
*SO*10	0	0	0	0	0	0	0	0	0	0	0	0	1
*SO*11	0	0	0	0	0	0	0	0	0	0	0	0	1
*SO*12	0	0	0	0	0	1	0	0	0	0	0	0	0
	Control rights according to the "dominant shareholder" approach, **P** + **SC** (in %)												
*SO*1	55.60	0	0	14.75	0	0	0	0	0	0	0	0	29.65
*SO*2	30.25	14.00	0	0	0	0	20.86	0	0	0	0	0	34.89
*SO*3	50.27	0	0	0	0	0	0	12.92	0	0	0	0	36.81
*SO*4	31.49	0	0	0	14.97	7.86	0	0	5.02	0	0	0	40.66
*SO*5	19.59	26.51	19.59	0	0	0	0	0	0	0	0	25.78	8.53
*SO*6	25.05	0	0	0	0	0	0	0	0	30.00	0	0	44.95
*SO*7	24.99	0	0	0	0	0	0	0	0	30.00	10.00	0	35.01
*SO*8	0	0	0	0	0	0	100.00	0	0	0	0	0	0
*SO*9	0	0	0	0	0	0	0	0	0	0	0	0	100.00
*SO*10	20.37	0	0	0	0	0	0	0	0	0	0	0	79.63
*SO*11	29.01	0	0	0	0	0	0	0	0	0	0	0	70.99
*SO*12	0	0	0	0	41.10	42.70	0	0	0	0	0	0	16.20

a positive value in the corresponding cell(s) of $\mathbf{C}^{(0)}$, and the remaining elements become zero. If there is more than one shareholder with the same (largest) stake, they receive equal control coefficients, such that the sum of coefficients equals one. From $\mathbf{P}$ in Appendix 2.B, it follows that $c^{(0)}_{5,2} = 1$, $c^{(0)}_{8,7} = 1$, $c^{(0)}_{12,6} = 1$, and $c^{(0)}_{ik} = 0$ for $i =$ 5, 8, and 12, and all other k's in these rows. Now, in the second stage we compute $\mathbf{P} + \mathbf{SC}^{(0)}$, find from the matrix $\mathbf{S}$ firms that are owned only by one other secondary owner, and for these corresponding rows again search the largest shareholder(s) in $\mathbf{P} + \mathbf{SC}^{(0)}$, and assign a positive value of control coefficients in $\mathbf{C}^{(0)}$, which after this adjustment is denoted by $\mathbf{C}^{(1)}$. Then compute $\mathbf{P} + \mathbf{SC}^{(1)}$, and the same procedure is applied until every firm is assigned to some primary owner(s). Table 2.4 gives both the final control assignment matrix $\mathbf{C}^{\text{final}}$ and the matrix of control rights $\mathbf{P} + \mathbf{SC}^{\text{final}}$ according to the "dominant shareholder" methodology. Note that the conditions in footnote 13 are satisfied.

In order to examine the relation between network complexity measures and the degrees of separation of C and O rights, in Table 2.5 we give the simple correlations between the various indicators. Note that ownership (O) in all cases is represented by the matrix $\mathbf{T}$. Like in Section 4.4.2, we will mainly focus on non-negligible linkages by combining both the distance and size of each ownership link, i.e., we take the average distance into account only if the size of the linkage is sufficiently large, using a threshold level. First, the relation between control and ownership differences (C-O), with C measured by the "weakest link" (WL) and "dominant shareholder" (DS) approaches, is given in the bottom row of Table 2.5. The full sample takes C-O into account for every pair of primary and secondary owner (thus, $n \times m = 12 \times 13 = 156$ observations). No significant correlation is found between the WL and the DS indicator of separation of O and C. However, if the average is taken over all secondary owners (for each primary owner, which yields $m = 13$ observations), positive correlations are found. Moreover, the correlations are much higher with a threshold level of 20% for the "weakest link" control rights. Hence, for the Czech banking sector, the two "standard" approaches result in approximately the same outcomes for the separation of O and C when the average control rights of the primary owner are compared, while the results differ significantly when all specific control rights are considered.

Next, we consider the results for the case where the control rights are obtained from the distance measures WADIL and WADTL. When size is not taken into account (i.e., distance indicators are considered without any threshold), there is no clear link between the WL and DS measures of separation of C and O, and the cor-

Table 2.5: Simple correlation between ownership network complexity measures and the degrees of separation of control and dividend rights

	$(C-O)_{ik}$ 156 observations			Average	$(\overline{C}-\overline{O})_k$ 13 observations		
	WL10	WL20	DS		WL10	WL20	DS
WADTL	0.318	−0.026	−0.014	WADTL	0.253	−0.284	−0.145
WADTL2	0.315	0.063	0.059	WADTL2	0.583	0.728	0.470
WADTL5	0.181	0.282	0.245	WADTL5	0.548	0.913	0.746
WADTL10	0.212	0.358	0.310	WADTL10	0.575	0.920	0.739
WADIL	0.377	0.085	−0.023	WADIL	0.332	−0.126	−0.025
WADIL2	0.479	0.385	0.050	WADIL2	0.630	0.843	0.600
WADIL5	0.301	0.451	0.238	WADIL5	0.524	0.860	0.731
WADIL10	0.213	0.357	0.306	WADIL10	0.460	0.681	0.566
DS (C-O)	−0.070	0.089	1	DS (C-O)	0.416	0.817	1

Note: $C-O$ is the difference between control and ownership rights. $(C-O)_{ik}$ does so for the interests of primary owner k $(=1,\ldots,13)$ in each secondary owner i $(=1,\ldots,12)$, $(\overline{C}-\overline{O})_k$ takes the average over the secondary owners. WL and DS stand for, respectively, the "weakest link" and "dominant shareholder" approaches of identifying control rights. WL10 means that the threshold level for control rights is 10% (otherwise the corresponding cell in Table 2.3 is set to zero). WADTL5 (WADIL5) takes the positive values of WADTL (WADIL) if the corresponding total (indirect) ownership is at least 5%.

responding distances. However, once ownership size is taken into account (with threshold levels of 2%, 5% or 10%) the WADILs and WADTLs show a positive correlation with the WL and DS measures for C and O differences, i.e., the wedge between C and O rights is greater (i.e., C-O is larger) when there is more complex network of ownership links (i.e., when WADTL and WADIL are larger).

Focusing on the cases where a threshold level is applied to the distance measures yields the following conclusions. First, the correlation is larger for average indicators (in the right panel of Table 2.5) than for the individual indicators (in the left panel). Second, in the full sample the correlations of C-O measures from the "weakest link" approach are on average larger when WADILs are used than when WADTLs are used, while the opposite holds for C-O measures computed by the DS approach. In the sample with C-O measures averaged over secondary owners, the correlations are stronger for WADIL than for WADTL at the 2% threshold level, whereas the opposite holds for the 5% and 10% threshold levels. Thus, for higher threshold levels (imposed on total and indirect linkages), WADTL measures are preferred in indicating C and O gap. This is because the WADTLs take direct and indirect shareholding linkages into account and both matter in determining the control power of a primary owner.

The empirical results clearly suggest that the distance measures WADTL and WADIL can be considered as alternative measures of the degree of separation of C and O due to pyramiding structures and cross-holding. When compared to the WL and DS methodologies, using WADIL and WADTL as indicators of separation of C and O has several clear advantages. First, its computation is extremely simple. Second, in contrast to the WL approach and similar to the DS methodology, the notion of distance takes all possible webs of property relations due to cross-ownership into full account. Furthermore, the distances are weighted by their corresponding contributions to total and indirect links, which make them preferable to, say, the "minimum distance" approach used in the sociology literature. Third, unlike the DS approach, there is no such notion as the multiplicity of control assignments (hence multiple control rights values).[14] Consequently, every initial primary and secondary property distributions have *unique* WADTL and WADIL matrices. Fourth, like the DS approach, WADIL and WADTL also consider the notion of "management control", when a firm is (partially) controlled by an owner without ownership in dividend rights at all.[15] On the other hand, similar to the DS methodology, the distance concept has a disadvantage that it focuses on the effects of cross-shareholding and does not consider other control arrangements, like dual class shares and voting caps. But given our observations mentioned in the beginning of this subsection, we expect the bias from the one-share-one-vote assumption to be small.[16]

2.5 Conclusions

In this chapter, we have studied ownership relations between primary owners (such as individuals and the state) and secondary owners (such as companies and banks). In the presence of cross-shareholding among secondary owners, the property structure may become quite complex. Cross-shareholding is widely observed in modern economies and is an important characteristic of Japanese, German and Swedish business groups in particular. The observed property distribution reflects

[14] To deal with this issue, Dorofeenko et al. (2008) introduce the notion of control "tightness" that gives the maximal stable control assignment.

[15] In our empirical application this applies to significant indirect ownership of *PO*7 in *SO*2, or *PO*1 in *SO*6, *SO*7, and *SO*11 with zero direct dividend rights.

[16] Dorofeenko et al. (2008), in supporting their assumption of the absence of other control arrangements, argue that "... these other control devices will most likely reinforce the control assignment emerging from the pure one-share-one vote arrangement, as they are presumably designed by controlling shareholders" (p. 80).

only direct shareholding and may be highly misleading because it hides the true property distribution. This true property distribution can only be obtained by taking also all indirect shareholding into full account. As a consequence, all property that is held by secondary owners accrues to the primary owners. The true property distribution allows for the calculation of the total property that is embedded in a 1% share in some corporation and the total property that is held by some primary owner.

For analyzing the ownership relations or linkages, two aspects are important. These are the size of the indirect or total (i.e., direct and indirect) linkages between a primary owner and a secondary owner, and the average distance of the linkages between the two. The last is obtained from the average number of secondary owners via whom the relation runs. The average distance indicates the complexity of the indirect linkages between a primary and a secondary owner and is taken into account only for the important linkages (i.e., those that are larger than a pre-specified threshold).

The methodology has been applied to the banking sector in the Czech Republic, which allowed us to get some insight into the "hidden property structures" of this sector. The complexity of the network of relations between primary and secondary owners was quantified, and the relevant shareholding chains were graphed. There is ample evidence that indirect ownership relations play a crucial role in the Czech banking sector.

Further, we found a clear link between ownership complexity measures proposed in this study and the degree of separation of dividend and control rights, largely investigated in the finance literature. The idea is that the more complex the network of non-negligible relations is, the larger the degree of control enhancement due to cross-shareholding links among firms. Hence, the larger the difference is between the control and the ownership stakes of primary owners in secondary owners. The empirical results confirm this for the Czech banking sector.

As a final remark, it should be noted that the empirical analysis of the Czech banking sector was carried out as if it were a closed, domestic system. However, some of the primary owners are in fact secondary owners in other countries. It may thus be the case that, say, 20% of the property of one of the Czech investment funds accrues to a German bank, for example. This points at foreign holding of Czech property. In its turn, it is in principle possible that, for instance, the Czech National Bank holds (directly and indirectly) 50% of this German bank. This would imply then that only 10% of the property of this Czech investment fund flows abroad, while 10% accrues to the Czech National Bank.

It is clear that the first part (i.e., foreign ownership of Czech property) of the example above is included in our analysis. The second part (i.e., Czech ownership of foreign property), however, is not. To do so, would at least require detailed information on shareholding in Germany. In general, if international cross-shareholding occurs, insight into the property structure and the international ownership relations would require a full interregional input-output framework.[17] That is, the necessary information would be given by expanded initial property distribution matrices **P** and **S**. Element p_{ik}^{RU} would give the share in secondary owner i in country R, that is held by primary owner k in country U. Similarly, s_{ij}^{RU} would indicate the share in secondary owner i in country R, that is held by secondary owner j in country U. Given the ongoing internationalization of shareholding, constructing and analyzing a full-fledged interregional database will be a major challenge for the future.

[17] See e.g., Miller and Blair (2009) for an excellent introduction, and Dietzenbacher and Romero (2007) for an application of distance to interregional production chains.

2.A The list of primary and secondary owners of the banking sector in the Czech Republic

*PO*1 - Fond národního majetku (Fund of National Property), state agency
*PO*2 - Česká národní banka (Czech National Bank), central bank
*PO*3 - Ministerstvo financí (Ministry of Finance), state agency
*PO*4 - Sdružení měst (Association of Municipalities)
*PO*5 - Bank Holding, non-state
*PO*6 - J. Ring stock company, non-state
*PO*7 - First Privatization Holding, non-state
*PO*8 - The Bank of New York
*PO*9 - Nomura Group
*PO*10 - The Midland Bank
*PO*11 - The Bankers Trust Investment
*PO*12 - Slovak Republic
*PO*13 - minority investors
*SO*1 - Česká spořitelna (Czech Saving Bank)
*SO*2 - Česká pojišťovna (Czech Insurance)
*SO*3 - Komerční banka (Commercial Bank)
*SO*4 - Invesičcní a poštovní banka (Investment and Post bank)
*SO*5 - Československá obchodní banka (Czecho-Slovak Trade Bank)
*SO*6 - Spořitelní privatizační fond Český (investment fund)
*SO*7 - Spořitelní privatizační fond výnosový (investment fund)
*SO*8 - První privatizační fond (investment fund)
*SO*9 - První investiční fond (investment fund)
*SO*10 - Restituční investiční fond (investment fund)
*SO*11 - Investiční privatizační fond Komerční banky (investment fund)
*SO*12 - Vojenskě stavby (stock company)

2.B The initial property distribution of the banking sector in the Czech Republic

	Primary property distribution - **P**												
	*PO*1	*PO*2	*PO*3	*PO*4	*PO*5	*PO*6	*PO*7	*PO*8	*PO*9	*PO*10	*PO*11	*PO*12	*PO*13
*SO*1	52.80			14.75									11.95
*SO*2	30.25												17.71
*SO*3	48.74							12.92					29.83
*SO*4	31.49				14.97				5.02				40.66
*SO*5	19.59	26.51	19.59									25.78	8.53
*SO*6										30.00			44.95
*SO*7										30.00	10.00		35.01
*SO*8							100.00						
*SO*9													86.20
*SO*10	20.37												69.53
*SO*11													70.99
*SO*12					41.10	42.70							16.20
	Secondary property distribution - **S**												
	*SO*1	*SO*2	*SO*3	*SO*4	*SO*5	*SO*6	*SO*7	*SO*8	*SO*9	*SO*10	*SO*11	*SO*12	
*SO*1		10.10	2.80			5.10	2.50						
*SO*2				17.18	14.00			20.86					
*SO*3	1.53								1.21	3.56	2.21		
*SO*4												7.86	
*SO*5													
*SO*6	25.05												
*SO*7	24.99												
*SO*8													
*SO*9		13.80											
*SO*10		10.10											
*SO*11			29.01										
*SO*12													

Source: Turnovec (1999).

CHAPTER 3

Cross-shareholding in the Japanese banking sector*

3.1 Introduction

There is ample evidence that nowadays firms often acquire shares in their rivals, and mostly these shareholdings do not give control power. For example, Hansen and Lott (1996, Table 1) give evidence for substantial cross-ownership relations in the American computer and automobile industries for 1994-1995, and state that "slightly over 77 percent of Intel and 71 percent of Compaq are owned by institutions that have holdings in at least one of the other five computer industry companies listed [Apple, Compaq, IBM, Intel, Microsoft, Motorola]. Fully 56 percent of Chrysler is held by institutions that simultaneously hold shares in Ford and/or General Motors" (p. 49). In 2002, the leader of the wireless communications businesses in Korea – SK Telekom – acquired 11.3% of Korea Telecom, the leader in the wireline communications business, which in its turn already owned 9.3% of equity of the first company (see Choi et al., 2003, p.498). Firms' acquisitions of stocks largely cross the national borders as well. For instance, in 2001, General Motors increased its equity holding in Suzuki Motor from 10.0% to 20.0%, and acquired also a 21.1% stake in Fuji Heavy Industries.[1] Since shareholding interlocks of firms is a widespread phenomenon,[2] it is essential to analyze the implication of the presence

* Section 3.2 is partly based on a paper published in the *Journal of Economic Studies*, vol. 36, no. 3, pp. 296-306, 2009a, while the rest of this chapter is based on joint work with Stanislav Stakhovych.

[1] See *Industrial Groupings in Japan. The Changing Face of Keiretsu*, 14th Edition, Brown & Company Ltd., Tokyo, 2001.

[2] See Gilo (2000) for more cases of equity acquisitions in various industries.

of ownership links on the behavior of firms.

Cross-shareholding is, in particular, an important characteristic of Japanese, German and Swedish business groups (see e.g., Kester, 1992). However, due to antitrust concerns most cross ownership is *silent* (or partial) by its nature. Financial interests are silent when firms do not control the policies (e.g., outputs, prices) of their competitors.[3] That is, firms take the choices of these competitors as given, although in the presence of cross ownership decisions of one firm affect also the profits of its rivals. It has been shown that partial cross ownership (PCO) of firms, when compared to the case without PCO, leads to higher prices,[4] lower industry outputs, and thus lower welfare (see e.g., Reynolds and Snapp, 1986; Flath, 1992a; Reitman, 1994; Dietzenbacher et al., 2000). Nonetheless, Farrell and Shapiro (1990) show that welfare may still rise even if prices increase, which occurs when a small firm acquires shares in a rival in which it previously had no financial interest.

Given the fact that passive investments in rivals were largely neglected by antitrust agencies (see e.g., Gilo, 2000), much attention in the literature was given to the study on explicit links between PCO and tacit collusion. Reitman (1994) shows that for any number of firms an individually rational PCO equilibrium exists if the market is more rivalrous than Cournot oligopoly and is close to price competition. Malueg (1992) concludes that passive investments have an ambiguous effect on the likelihood of collusion. In a repeated Cournot game, he shows that the effect of an increase in cross ownership on tacit collusion depends critically on the form of the market demand. However, Gilo et al. (2006) find that in a Bertrand supergame an increase in PCO never hinders tacit collusion and surely facilitates it under certain conditions. They show that an increase of firm r's stake in firm s strictly facilitates collusion if (i) firm s is not an industry maverick (a firm with the strongest incentive to deviate from a collusive agreement), and (ii) each industry maverick has a direct and/or an indirect stake in firm r (firm i has an *indirect* stake in firm r if it has a share in a firm that has a stake in firm r, or has a stake in a firm that has a stake in a firm that holds a stake in firm r, and so on).[5]

The results of empirical research on the effect of PCO on market structure mostly support the collusion hypothesis, which states that a complex web of PCO is an

[3] The term "silent financial interests" was introduced by Bresnahan and Salop (1986). Equivalently, such equity interests in the literature are also termed passive investments, partial ownership arrangements, and partial cross ownership links. We will also use all these terms interchangeably throughout this chapter.

[4] Interestingly, Weinstein and Yafeh (1995) find that keiretsu firms had price-cost margins *lower* by as much as 2.5 percentage points than those of non-keiretsu firms.

[5] An extension of Gilo et al. (2006) to the case where firms have asymmetric costs will be presented in Chapter 4.

important factor for the existence of collusive prices. The focus of such studies are specific industries, such as the US mobile telephone industry (Parker and Röller, 1997), the Dutch financial sector (Dietzenbacher et al., 2000), and the Norwegian-Swedish electricity market (Amundsen and Bergman, 2002). Alley (1997) finds that tacit collusion does occur in both the Japanese and the US domestic automobile industries, but its degree is lower in Japan.

In this chapter we take into *full* account both direct and indirect interests of firms in each other due to PCO, which is ignored, to the best of our knowledge, in all empirical estimations of the level of tacit collusion.[6] As mentioned above, for example, if firm i owns a share in firm k that has a share in j then firm i is said to have an indirect share in firm j (via firm k). In general, the number of intermediate firms in the indirect links can be infinity when there are cycles present in the ownership paths (for instance, when firm i holds shares in firm j and, vice versa, j has a stake in i). PCO is incorporated in the analysis of Alley (1997), but he considers only direct shareholdings. It has been shown that indirect interests might be significant in size, thus should not be neglected in the analysis of industries (economies) with the presence of PCO (see e.g., Flath, 1992b; Dietzenbacher and Temurshoev, 2008).

We first discuss different profit formulations of firms with cross-shareholdings that have been used in the literature, where the differences are due to the distinct ways of considering direct and/or indirect PCO links. Then using the conjectural variation model we find that (unlike in the case without PCO) the link between firms' price-cost margins and the degree of collusion is *nonlinear* in the presence of PCO. Hence, if shareholding links among firms are present, ignoring PCO would most likely give biased parameters' estimates due to model misspecification. It is shown that given market shares, number of firms, price elasticity of demand, and collusion degree, firms with shareholdings exert strictly higher market power than those without PCO, provided that the market conduct is consistent with Cournot or a more collusive environment. This is because shareholding interlocks among firms cause commonality of interests of firms, implying greater monopoly power for firms with PCO holdings.

The model is applied to the Japanese banking sector for the fiscal year 2003. The results of our estimations show that Japanese banks are competing in a modest collusive environment. However, disregarding banks' PCO gives biased result,

[6] Dietzenbacher et al. (2000) fully consider PCO links in a Cournot and a Bertrand setting, and find that such links reduce Dutch banks' price-cost margins, hence reduce competition. We, however, focus directly on the indicator of market performance that ranges from perfect competition to monopoly (perfect cartel).

indicating a Cournot oligopoly. It is further shown that banks with passive investments in rivals exert a strictly larger market power than those without any PCO, which confirms the hypothesis that acquiring shares in rivals is one of the crucial means for a firm to enhance its market power. In particular, city banks with many shareholdings are found to exercise a much higher market power than regional banks with none or few stockholdings.

The model presented here belongs to the conjectural variations (CV) literature. CV models are often used in empirical research in order to infer the degree of market power from real data (see e.g., Brander and Zhang, 1990; Haskel and Martin, 1994; Richards et al., 2001; Fischer and Kamerschen, 2003; Brissimis et al., 2008). It is well known that these models are subject to some criticism from a theoretical point of view because they describe the dynamics of firms' interaction using a static setting (see e.g., Tirole, 1988, pp. 244-45).[7] However, Cabral (1995) shows that CV models can be interpreted as a reduced form of the equilibrium in a quantity-setting supergame with linear demand and marginal cost functions, justifying their use in estimating the competition level among oligopolists. In the same fashion, for his infinite horizon adjustment cost model, Dockner (1992) shows that any steady state closed-loop (subgame-perfect) equilibrium coincides with the CV equilibrium. In addition, Pfaffermayr (1999) proves that CV models represent the joint profit maximizing reduced form of a price-setting supergame with product differentiation, which "...provides a comprehensive theoretical foundation of the widely criticized static CV models" (p. 323).

The rest of this chapter is organized as follows. Section 3.2 discusses different profit specifications of firms in the presence of PCO used in the literature. Section 3.3 describes the CV model with cross-shareholdings and examines the effect of PCO linkages on firms' market power. Section 3.4 focuses on the empirical estimation of the degree of tacit collusion in the Japanese commercial banking sector for 2003, and diagnoses market power of the banks. Section 3.5 concludes. All proofs are relegated to the Appendix at the end of the chapter.

[7] Some authors therefore believe that CV parameters have nothing to do with real conjectures or expectations of firms. To avoid this confusion Krouse (1998, p. 688), for example, refers to them as "equilibrium solution parameters".

3.2 Profits of horizontally interrelated firms

In this section we briefly present profit formulations of firms in the presence of partial cross ownership (PCO) that have been used in the literature. The differences in these profit specifications are the result of the different ways of taking account of a complex web of interfirm ownership links. Consider an industry with n firms that are interdependent through PCO ties. Reynolds and Snapp (1982) was one of the first studies that brought attention to the analysis of firms' PCO holdings and formulated the profit of firm i as follows

$$\pi_i = z_i + \sum_{k \neq i} w_{ik} z_k, \tag{3.1}$$

where π_i and z_i denote, respectively, the profits and the operating earnings of firm i, and w_{ik} $(i, k = 1, \ldots, n)$ represents the share in firm k that is held by firm i.[8] That is, equation (3.1) states that firm i's profits consists of its own operating earnings (profits from ordinary production) plus its *direct* shareholdings in operating earnings of all other firms. This formulation is also used in Bresnahan and Salop (1986), who study a competitive joint venture, in which parent firms own non-controlling ownership rights.

Reynolds and Snapp (1986) consider the case of *joint ventures*, whose profits are divided according to each partner's share of equity, and they define profits of firm i as[9]

$$\pi_i = \left(1 - \sum_{k \neq i} w_{ki}\right) z_i + \sum_{k \neq i} w_{ik} z_k, \tag{3.2}$$

which defers from (3.1) in that firm i also considers competitors' financial interests in its operating earnings. This specification of the firms' objective was used in Alley (1997) in analyzing the effect of non-controlling (partial) shareholdings on the degree of competition in the US and Japanese automobile industries.

The above specifications totally disregard *indirect* financial interests, when, for example, firm i has an indirect stake in firm j via intermediate firms. In many

[8] First and second subscripts in w_{ik} denote, respectively, the owner and the owned firm. Throughout this chapter it is assumed that a firm cannot own equity interest in itself, i.e., $w_{ii} = 0$ for all i. However, one can also allow for $w_{ii} > 0$, which would reflect, for example, the share repurchases by firms due to the tax advantage of capital gains. Note that while in Chapter 2 the cross-shareholding matrix was denoted by the matrix **S**, in this chapter its transpose is denoted by **W**.

[9] For other profit specifications depending on the kind of behavior imputed on the joint ventures see e.g., Bresnahan and Salop (1986) and Martin (2002, Chapter 12.10).

cases indirect shareholdings are significant in size and thus call for a proper consideration. Hence, equations (3.1) and (3.2) are not adequate when an industry is characterized by extensive shareholding interlocks. These shareholding links are fully taken into account in Flath (1991), who defines firm i's profit as the sum of its operating earnings and the revenue from shareholding in rivals' profits:

$$\pi_i = z_i + \sum_{k \neq i} w_{ik} \pi_k. \tag{3.3}$$

Equivalently, in matrix form, (3.3) can be rewritten as $\boldsymbol{\pi} = \mathbf{z} + \mathbf{W}\boldsymbol{\pi}$, where $\mathbf{W}$ is the n-square PCO matrix with its typical element w_{ij}, and $\boldsymbol{\pi}$ and $\mathbf{z}$ are, respectively, the column vectors of profits and operating earnings. Solving the last equation with respect to profits gives

$$\boldsymbol{\pi} = (\mathbf{I} - \mathbf{W})^{-1}\mathbf{z}, \tag{3.4}$$

where $\mathbf{I}$ is the n-square identity matrix.

Assuming that each firm has external shareholders (i.e., private owners and firms outside the industry) implies that the column sum of the matrix $\mathbf{W}$ is smaller than one, which guarantees non-singularity of the matrix $(\mathbf{I} - \mathbf{W})$ (see e.g., Solow, 1952).[10] Define $\mathbf{L} \equiv (\mathbf{I} - \mathbf{W})^{-1}$ that, similar to the Leontief inverse in input-output economics, can be written as the matrix power series expansion $\mathbf{L} = \mathbf{I} + \mathbf{W} + \mathbf{W}^2 + \ldots$ (see e.g., Miller and Blair, 2009). The last expression together with (3.4) allow us to separate direct and indirect effects of PCO. Namely, profits of firm i consist of three components (Dietzenbacher et al., 2000, p. 1226). First, its *own* operating earnings reflected by the i-th element of the vector $\mathbf{z}$. Second, firm i's *direct* shareholdings in rivals, reflected by the i-th element of the vector $\mathbf{Wz}$. Finally, the third term gives the *indirect* equity returns of firm i in other firms and is equal to the i-th element of the vector $(\mathbf{W}^2 + \mathbf{W}^3 + \ldots)\mathbf{z}$. So even if $w_{ij} = 0$, the entry (i, j) of the matrix $\mathbf{W}^3$ is positive if firm i partially owns firm k that has a share in firm h that in its turn holds a stake in firm j.

The profit specification in (3.4) is widely used in the literature (see e.g., Flath,

[10] Although, the existence of external shareholders perfectly corresponds with the real life observations, it is - mathematically speaking - not necessary that all column sums of $\mathbf{W}$ are smaller than one. For the existence of $(\mathbf{I} - \mathbf{W})^{-1}$ it suffices that no column sum of $\mathbf{W}$ is larger than one and, at least, one column sum is strictly less than one, provided that $\mathbf{W}$ is an indecomposable matrix. (A square matrix $\mathbf{A}$ is called *decomposable* if there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{O} & \mathbf{A}_{22} \end{pmatrix}$, where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are square submatrices, and $\mathbf{O}$ is a null matrix of appropriate dimension. If this is impossible, $\mathbf{A}$ is called *indecomposable*.) Hence, (if $\mathbf{W}$ is an indecomposable matrix) for all but one firm it may even be the case that no external shareholders exist.

1992a, 1992b; Dietzenbacher et al., 2000; Gilo et al., 2006; Dorofeenko et al., 2008). These profits "overestimate" industry-wide operating earnings. To see this, let $\boldsymbol{\imath}$ be the summation vector of ones. Then $\boldsymbol{\imath}'\boldsymbol{\pi} = \boldsymbol{\imath}'(\mathbf{I}-\mathbf{W})^{-1}\mathbf{z} = \boldsymbol{\imath}'(\mathbf{I}+\mathbf{W}+\mathbf{W}^2+\ldots)\mathbf{z} > \boldsymbol{\imath}'\mathbf{z}$ in the presence of PCO. However, this "overestimation" does not cause any problem since these profits indicate the *value* of the firms, and should increase when firms become interlinked. Say, in a two firms setting, PCO creates a multiplier effect in the sense that firm A gets a share in firm B's profit, which includes firm B's share in firm A's profit, which includes firm A's share in firm B's profit, and so on. However, what should concern us is whether there is a problem of overestimation of profits accruing to "real" (i.e., external) shareholders. The last is equal to $\boldsymbol{\imath}'(\mathbf{I}-\mathbf{W})\boldsymbol{\pi} = \boldsymbol{\imath}'(\mathbf{I}-\mathbf{W})(\mathbf{I}-\mathbf{W})^{-1}\mathbf{z} = \boldsymbol{\imath}'\mathbf{z}$, hence although the aggregate profits "...overstate the firms' cash flows ...the aggregate payoffs of 'real' equityholders are not overstated and do sum up to [industry operating earnings]" (Gilo et al., 2006, p. 86). This approach is very similar to the input-output technique, where multiplication of, say, the direct employment coefficients vector by the Leontief inverse gives total (direct and indirect) labor requirements per unit of final demand (see e.g., Miller and Blair, 2009). Here, similarly, multiplication of external shareholders' direct shares in firms, $\boldsymbol{\imath}'(\mathbf{I}-\mathbf{W})$, by the "Leontief inverse" of the form $(\mathbf{I}-\mathbf{W})^{-1}$ results in the total (direct and indirect) equity interests of owners in firms per unit of operating earnings, or, equivalently, in Gilo et al. (2006) terminology, in the *total effective stake* of the "real" equityholders in firms' profits.

The issue of profits overestimation in Flath's approach is considered in Merlone (2007). In terms of our notations, his proposed new formulation of *net* profits is $\boldsymbol{\pi}^{net} = (\mathbf{I}-\widehat{\boldsymbol{\imath}'\mathbf{W}})(\mathbf{I}-\mathbf{W})^{-1}\mathbf{z}$, where $\widehat{\boldsymbol{\imath}'\mathbf{W}}$ is the diagonal matrix with the column sums of $\mathbf{W}$ on its main diagonal and zero elsewhere. The last, unlike the profits in (3.4), sum up to the overall operating earnings, i.e., $\boldsymbol{\imath}'\boldsymbol{\pi}^{net} = \boldsymbol{\imath}'\mathbf{z}$ since $\boldsymbol{\imath}'(\mathbf{I}-\widehat{\boldsymbol{\imath}'\mathbf{W}}) = \boldsymbol{\imath}'(\mathbf{I}-\mathbf{W})$. However, as we just showed above, $\boldsymbol{\pi}^{net}$ is nothing else than the profits accruing to "real" equityholders of firms.[11]

A few studies focused only on the real cash flows due to firms' PCO links, hence effectively neglected the notion of a firm *value* considered in (3.4). Futatsugi (1978, 1986, 1987) writes firm i's profits as

[11] We should note that Merlone's (2007) view that his profit specification results in different cartelizing effects of shareholding interlocks than those based on equation (3.4) is entirely wrong. In fact, the Lerner indices for homogeneous and product-differentiated oligopolies proposed by Merlone (2007) are nothing else than the corresponding indicators in Merlone (2001). This is because Merlone's profit specification is a netted version of firms' objective in (3.4). Thus both profit formulations have exactly identical optimality conditions (from which Lerner indices are derived), since in the maximization process the structure of PCO is taken as given.

$$\pi_i = z_i + \sum_{k \neq i} w_{ik} r_k \pi_k, \tag{3.5}$$

where $r_k \in (0,1)$ is the payout ratio (dividend propensity) of firm k. Note that if $r_k = 1$ for all k, then (3.5) boils down to (3.3). Hence, unlike (3.3), the last equation considers only dividend returns of firms due to PCO. Its netted version, where dividend outflows due to PCO are also taken into account, is given in Temurshoev (2009a) as follows

$$\pi_i^{net} = (1 - r_i)\left(z_i + \sum_{k \neq i} w_{ik} r_k \frac{\pi_k^{net}}{1 - r_k}\right), \tag{3.6}$$

where π_i^{net} denotes firm i's profits after dividend payments, hence $\pi_i^{net}/(1 - r_i) = \pi_i$ is the gross profit including dividend payments.[12] Equations (3.5) and (3.6) in matrix form can be rewritten, respectively, as $\boldsymbol{\pi} = (\mathbf{I} - \mathbf{W}\hat{\mathbf{r}})^{-1}\mathbf{z}$ and $\boldsymbol{\pi}^{net} = (\mathbf{I} - \hat{\mathbf{r}})(\mathbf{I} - \mathbf{W}\hat{\mathbf{r}})^{-1}\mathbf{z}$, where $\hat{\mathbf{r}}$ is the diagonal matrix with payout ratios on its main diagonal and zero otherwise. Since in the analysis $\hat{\mathbf{r}}$ and $\mathbf{W}$ are given, the first-order conditions for profit maximization are exactly the same for (3.5) and (3.6).

However, equations (3.5) and (3.6) are *not* suitable for the economic analysis of cross-shareholdings. The main focus in economic analysis is the value of the firm, and not its total cash flows due to PCO. For instance, if no firm announces dividend payments (i.e., $r_i = 0$ for all i), then both (3.5) and (3.6) reduce to $\pi_i = \pi_i^{net} = z_i$. Although from a pure accounting view this is the correct amount of (current) earnings, it is a wrong representation of the PCO presence as far as economic analysis is concerned. This is because – in that case – (3.5) and (3.6) do not reflect the PCO links which give firms shares in the profits of rival firms (which in this case are held as retained earnings). Essentially, an investor's income from equity consists of dividends and retained earnings. The difference between the two is only the timing at which they are received: dividends are received whenever the firm distributes them, whereas retained earnings are realized either when the equityholder sells his shares or when the firm is liquidated. Equations (3.5) and (3.6) represent a one period model, where there should not be any difference between equity sales and firm liquidation, because the firm is effectively liquidated at the end of the period (after its profits are realized), and its profits are fully distributed. Therefore, divi-

[12] To see this, let $r_i = d_i/\pi_i$, where d_i denotes the dividend obligations of firm i. By definition $\pi_i^{net} = \pi_i - d_i$, which implies $\pi_i^{net}/(1 - r_i) = \pi_i$.

dends do not matter in a static one period model.[13] Hence, the only correct profit specification for economic analysis of PCO is Flath's formulation given in (3.3) or (3.4).

3.3 Theoretical framework

In order to diagnose market power of firms and analyze market performance in the presence of cross ownership links, we modify the well-known conjectural variation model of Clarke and Davies (1982) by taking into account both direct and indirect PCO linkages among firms. Assume there are n firms in an industry that are interdependent through PCO ties. The profit of firm i consists of its operating earnings plus the revenue from shareholding in other firms and is given in equation (3.3) in the previous section.

Consider a homogeneous product industry. Firm i's total cost $c_i(x_i)$ is a function of its own output level x_i. Further, the inverse demand function is $p(X)$, where $X = \sum_{i=1}^{n} x_i$. Let l_{ij} be the generic element of the matrix $\mathbf{L} = (\mathbf{I} - \mathbf{W})^{-1}$. Since the operating earnings of firm i is $z_i = p(X)x_i - c_i(x_i)$, using (3.4) firm i's profit can be written as

$$\pi_i = \sum_{j=1}^{n} l_{ij} \left[p(X)x_j - c_j(x_j) \right] .$$

We consider only passive financial interests of firms, thus in maximizing profits firms take the choices of their rivals as given. Following Clarke and Davies (1982) we further assume that in choosing its output, firm i forms a conjectural variation about the output response of all other firms to a unit change in its own output level. Denote the constant conjectural elasticity parameter of firm i by α, which is defined as

$$\frac{\partial x_j}{\partial x_i} = \alpha \frac{x_j}{x_i} \quad \text{for all } j \neq i. \tag{3.7}$$

The conjectural elasticity α is interpreted simply as the percentage change in firm

[13] In fact, Miller and Modigliani (1961) show that for a given investment policy, a firm's dividend policy is irrelevant to its current market valuation. In particular, they state: "[L]ike many other propositions in economics, the irrelevance of dividend policy, given investment policy, is 'obvious, once you think of it.' It is, after all, merely one more instance of the general principle that there are no 'financial illusions' in a rational and perfect economic environment. Values there are determined solely by 'real' considerations—in this case the earning power of the firm's assets and its investment policy—and not by how the fruits of the earning power are 'packaged' for distribution" (p. 414).

j's output that firm i expects in response to a one percent change in its own output. Note that this parameter is assumed to be the same for all firms and measures the degree of (tacit) collusion inherent in an industry. Positive values of α indicate the presence of collusion, and its degree is larger if α is larger. This is more obvious if we rewrite (3.7) as $\partial x_j/x_j = \alpha(\partial x_i/x_i)$. If $0 < \alpha < 1$, lower values of α imply that firm i's rivals will react with a smaller (percentage) change to the change in output i, so that firm i believes that there is some scope for improving its market share.[14] Let c_i' be the marginal cost of firm i, then the first-order condition (FOC) $\partial\pi_i/\partial x_i = 0$ is $\sum_j l_{ij}\left[(p - c_j')\partial x_j/\partial x_i + x_j \sum_k (dp/dX)(\partial x_k/\partial x_i)\right] = 0$.

Define firm i's price-cost margin by $m_i \equiv (p - c_i')/p$, its market share by $s_i \equiv x_i/X$, and the price elasticity of demand by $\varepsilon \equiv -(p/X)(\partial X/\partial p)$. Using $\partial x_j/\partial x_i = \alpha(s_j/s_i)$ as an equivalent expression for (3.7), firm i's FOC after some rearrangements yields [15]

$$m_i = \frac{1}{\varepsilon}\left[1 + \frac{\sum_{j\neq i} l_{ij}s_j}{l_{ii}s_i}\right][\alpha + (1-\alpha)s_i] - \alpha\frac{\sum_{j\neq i} l_{ij}s_j m_j}{l_{ii}s_i}. \tag{3.8}$$

To represent (3.8) succinctly in matrix form, let $\widehat{\mathbf{L}}$ be the diagonal matrix with l_{ii} along its main diagonal and zero otherwise, $\mathbf{m}$ and $\mathbf{s}$, respectively, be the vectors of firms' markups and market shares. Then (3.8) can be rewritten as[16] (see Appendix 3.A)

$$\mathbf{m} = \alpha\mathbf{Q}\mathbf{m} + \frac{\alpha}{\varepsilon}\mathbf{x}_1 + \frac{1-\alpha}{\varepsilon}\mathbf{x}_2, \tag{3.9}$$

where $\mathbf{Q} \equiv \hat{\mathbf{s}}^{-1}(\mathbf{I} - \widehat{\mathbf{L}}^{-1}\mathbf{L})\hat{\mathbf{s}}$, $\mathbf{x}_1 \equiv \hat{\mathbf{s}}^{-1}\widehat{\mathbf{L}}^{-1}\mathbf{L}\mathbf{s}$, and $\mathbf{x}_2 \equiv \widehat{\mathbf{L}}^{-1}\mathbf{L}\mathbf{s}$.

In empirical work equation (3.9) can be used for the estimation of the effect of PCO on the degree of market power of firms, and on the overall level of tacit collusion in an industry. For the first task it is obvious that a firm exercises market power if its markup is positive. In the context of this model, firm i exercises market power if m_i in (3.9) is significantly (in a statistical sense) positive. Without PCO,

[14] Throughout the paper the notions of market conduct, degree of tacit collusion, market performance, and market competitive intensity are used interchangeably for α.

[15] Equation (3) in Alley (1997) is $m_i = \frac{1}{\varepsilon}\left[1 + \frac{\sum_{j\neq i} w_{ij}s_j}{(1-\sum_{j\neq i} w_{ji})s_i}\right][\alpha + (1-\alpha)s_i] - \alpha\frac{\sum_{j\neq i} w_{ij}s_j m_j}{(1-\sum_{j\neq i} w_{ji})s_i}$. He disregards indirect shareholdings and since in the PCO presence $l_{ii} \geq 1$ and $l_{ij} \geq w_{ij}$ $(i \neq j)$, in general, these two equations will give different estimates of α and ε.

[16] Theoretically, we can allow for different conjectural elasticities, in which case the scalar α in (3.9) is replaced by the diagonal matrix $\hat{\boldsymbol{\alpha}}$ with α_i on its ii-th entry and zeros elsewhere. However, for empirical estimation we need to make an identical conjectural elasticity assumption, hence α instead of α_i or $\hat{\boldsymbol{\alpha}}$ is entered in all equations. Alley's model can be also written in the form of (3.9) with the redefinition of $\widehat{\mathbf{L}}^{-1}\mathbf{L} = \mathbf{I} + (\mathbf{I} - \widehat{\iota'\mathbf{W}})^{-1}\mathbf{W}$.

$\mathbf{L} = \mathbf{I}$, and the market power diagnosis of firm i reduces to the condition $m_i = [\alpha + (1-\alpha)s_i]/\varepsilon > 0$ (see Martin, 1988). In order to identify the market competitiveness, one needs to estimate the value of α empirically.[17]

Without PCO, $\mathbf{L} = \mathbf{I}$, hence (recalling that $\boldsymbol{\imath}$ is the summation vector of ones) (3.9) boils down to (see e.g., Martin, 2002)

$$\mathbf{m} = \frac{\alpha}{\varepsilon}\boldsymbol{\imath} + \frac{1-\alpha}{\varepsilon}\mathbf{s}. \tag{3.10}$$

The important difference between (3.9) and (3.10) is that without PCO price-cost margins are linearly related to the conjectural elasticity, while with PCO this relation is *nonlinear*. This is because the solution of (3.9) is $\mathbf{m} = (\mathbf{I} - \alpha\mathbf{Q})^{-1}\big(\frac{\alpha}{\varepsilon}\mathbf{x}_1 + \frac{1-\alpha}{\varepsilon}\mathbf{x}_2\big)$ and $(\mathbf{I} - \alpha\mathbf{Q})^{-1}$ is nonlinear in α. Hence, it follows that the failure of taking firms' direct and indirect cross-shareholdings in the presence of PCO is likely to give biased parameter estimates due to model misspecification.[18]

Using (3.9) the range of the market competitive intensity α consistent with the economic interpretations is given in the following result, which helps to infer the industry market performance.

Theorem 3.1. *Irrespective of whether PCO is present or absent, the reasonable range of the market competitive intensity is* $\alpha \in [-1/(n-1);\, 1]$.

In Cournot competition we have $\partial x_j/\partial x_i = 0$ for all $j \neq i$, which corresponds to zero conjectural elasticity, i.e., $\alpha = 0$. In this case markups in (3.9) become $\mathbf{m} = (1/\varepsilon)\hat{\mathbf{L}}^{-1}\mathbf{L}\mathbf{s}$ (Merlone, 2001, p. 335). The value of α equal to the lower bound of $-1/(n-1)$ characterizes the perfect competition outcome, because then price-cost margins equal zero. The case $\alpha = 1$ reflects the perfect cartel since then markups equal the inverse of the price elasticity of demand.[19]

Given the expressions for price-cost margins with and without PCO, respectively, in (3.9) and (3.10), the obvious question is how the two are interrelated. Clearly, it is impossible to compare two different real-world environments with and without PCO as all the endogenous variables (i.e., price-cost margins and mar-

[17] It is not possible to directly run an OLS regression of (3.9), since the inverse matrix $(\mathbf{I} - \alpha\mathbf{Q})^{-1}$ (which would solve (3.9) for the vector of markups) contains the unknown market conduct parameter α. This problem is similar to the so-called spatial autoregressive models in Spatial Econometrics, where $\mathbf{Q}$ and α can be reinterpreted as a spatial weight matrix, and a spatial autoregressive parameter, respectively (see Anselin, 1988). The only difference is that α is also included in the regression coefficient vector.

[18] Similarly, one may get biased estimates if only direct PCO holdings are taken into account, which in the model is equivalent to the case when $\mathbf{L} = \mathbf{I} + \mathbf{W}$ and $\widehat{\mathbf{L}} = \mathbf{I}$.

[19] Note also that if α is close to its lower bound, we say that the market competitive intensity is high, and, similarly, an increase in α is referred to as the decrease in the market competitive intensity. For the conjecture's range without PCO see e.g., Kwoka and Ravenscraft (1986).

ket shares) are different within the two frameworks. Hence, let us focus on the difference between the markups assuming that α, ε, n, and $\mathbf{s}$ are identical in both the PCO and the no PCO case.[20]

Theorem 3.2. *Let $m_i < 1/\varepsilon$ for all $i = 1, \ldots, n$. For given α, ε, n, and $\mathbf{s}$, price-cost margins of firms with PCO are higher than those of firms without PCO provided that $\alpha \in [0, 1)$.*

The intuition behind Theorem 3.2 is simple. In this setting, shareholding interlocks among firms cause a common interest of firms that in turn leads to greater monopoly power of firms with PCO holdings. Recall that the requirement $m_i < 1/\varepsilon$ means that firm i is not a monopolist (hence the above result excludes the perfect cartel case).

3.4 Empirical estimation and results

In practice, simple direct use of accounting price-cost margins is insufficient as marginal costs defined by economists are unobservable, i.e., firms' costs should also include opportunity costs. One way to deal with this problem in the literature is assuming constant returns to scale (CRS), which means that marginal costs equal average costs. Average costs of firm i, ac_i, besides costs of variable inputs, include also the normal rate of return on investments, i.e., $ac_i = (\mathbf{v}'\mathbf{l}_i + \mu K_i)/x_i$, where $\mathbf{l}_i$ and $\mathbf{v}$ are, respectively, the vectors of variable inputs of firm i and input prices, μ and K_i are, respectively, the rental cost of capital services and the value of capital assets of firm i. Plugging the last expression in the definition of the price-average cost margin, one gets firm i's *economic* earnings per unit of sales, or, equivalently, price-cost margins under the CRS assumption as (see e.g., Martin, 2002, p. 137)

$$m_i = \frac{px_i - \mathbf{v}'\mathbf{l}_i - \mu K_i}{px_i} = \frac{px_i - \mathbf{v}'\mathbf{l}_i}{px_i} - \mu\frac{K_i}{px_i} = PCM_i - \mu\frac{K_i}{px_i}, \tag{3.11}$$

which is equal to *accounting* price-cost margins (PCM_i) minus the normal rate of return on investments. Solving (3.9) for the vector of markups and combining it with (3.11) yields the final model for empirical estimation as[21]

[20] Note that the assumption $s_i = s_i^0$ for each i, where the superscript '0' refers to the no PCO case, does not necessarily imply that all firms have equal market shares of $1/n$.

[21] Evidently (3.12) is a nonlinear function of the unknown parameters α and ε. Therefore, we numerically estimate parameters in (3.12) using a nonlinear least-squares approach. In MATLAB this is implemented by the function *lsqnonlin*, which finds the minimum of the objective function on the basis of the Levenberg-Marquardt method.

$$PCM_i = \frac{\alpha}{\varepsilon}\left[(\mathbf{I}-\alpha\mathbf{Q})^{-1}\mathbf{x}_1\right]_i + \frac{1-\alpha}{\varepsilon}\left[(\mathbf{I}-\alpha\mathbf{Q})^{-1}\mathbf{x}_2\right]_i + \mu KS_i + \nu_i, \tag{3.12}$$

where $\left[(\mathbf{I}-\alpha\mathbf{Q})^{-1}\mathbf{x}_1\right]_i$ is the i-th element of the vector $(\mathbf{I}-\alpha\mathbf{Q})^{-1}\mathbf{x}_1$, $KS_i = K_i/(px_i)$ is firm i's capital-sales ratio, and ν_i is a random error term. Without PCO, $\mathbf{L} = \mathbf{I}$, thus $\mathbf{Q}$ is a null matrix, $\mathbf{x}_1 = \boldsymbol{\imath}$ and $\mathbf{x}_2 = \mathbf{s}$, and as a consequence (3.12) reduces to (Martin, 2002, eq. (6.11))

$$PCM_i = a_0 + a_1 s_i + \mu KS_i + \nu_i, \tag{3.13}$$

where $a_0 = \alpha/\varepsilon$ and $a_1 = (1-\alpha)/\varepsilon$. Hence, estimates for a_0 and a_1 provide the estimates of α and ε.

3.4.1 Data

As an empirical application, we study the banking sector in Japan. Conventional wisdom is that the Japanese economy is collusive due to the existence of *keiretsu* groups that are historically interlinked through strong shareholding interlocks. We select city and regional banks from the *Bankscope* database published by *Bureau van Dijk Electronic Publishing*. Trust banks, long-term credit banks, security firms, and other smaller cooperative institutions (such as Shinkin banks) are excluded from the analysis because the sample should be consistent with the homogeneity assumption of the theoretical model described in Section 3.3 in the sense that all banks face the same inverse market demand function. Trust banks (next to having banking business) are also engaged in trust business (i.e., asset management services). Security firms apparently have different lines of business than commercial banks, hence do not compete with each other in the same market either. Similarly, long-term credit banks are mainly specialized in the provision of long-term loans and debentures. Hence, the city banks and the regional banks constitute the "ordinary banks". Legally, the two are not distinguished from each other and it is basically the size and area of business that distinguishes them. Regional banks are much smaller and operate in restricted areas, whereas city banks have nation-wide branch networks and operation. Uchida and Tsutsui (2005) reports that in 1996 the shares of city and regional banks in the Japanese loan market were, respectively, 49.6% and 33.1%, thus by analyzing these two groups one is able to cover 82.7% of the total outstanding loans in Japan. (The total outstanding loan in Japan is de-

Table 3.1: Descriptive statistics

	Mean	St. deviation	Minimum	Maximum
	Overall sample (63 obs.)			
Markups (*PCM*)	0.2274	0.1274	0.0288	0.8574
Market share (*s*)	0.0159	0.0341	0.0008	0.2058
Capital-sales ratio (*KS*)	2.6753	0.8051	1.0616	4.6928
Growth rate (*GR*)	0.3808	1.9184	−0.3210	15.3398
	City banks (4 obs.)			
Markups (*PCM*)	0.4763	0.3181	0.1983	0.8574
Market share (*s*)	0.1377	0.0496	0.0878	0.2058
Capital-sales ratio (*KS*)	2.3848	0.8442	1.6474	3.1439
Growth rate (*GR*)	3.8812	7.6394	−0.0100	15.3398
	Regional banks (59 obs.)			
Markups (*PCM*)	0.2105	0.0869	0.0288	0.5069
Market share (*s*)	0.0076	0.0049	0.0008	0.0265
Capital-sales ratio (*KS*)	2.6950	0.8061	1.0616	4.6928
Growth rate (*GR*)	0.1435	0.1146	−0.3210	0.6172

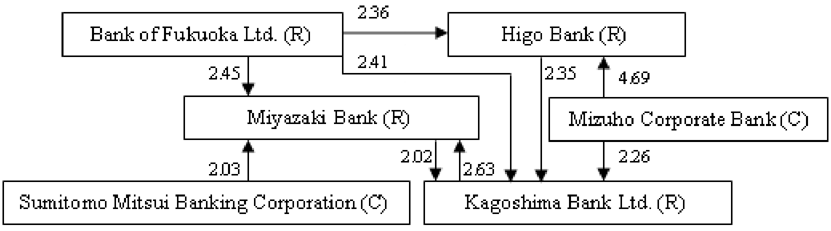
Note: Computations are based on the data given in thousands of US dollars. *GR* is the growth rate of banks' total revenue in 2003 relative to 2002.

fined as the sum of loans of city banks, long-term credit banks, trust banks, regional banks, Shinkin banks, and credit cooperations.)

After deleting unprofitable banks and those without necessary data information, we end up with a sample of 63 commercial banks for the fiscal year 2003, which includes 4 city banks and 59 regional banks. Data on accounting price-cost margins (PCM_i) and market shares (s_i) are derived from the banks' unconsolidated statements. The accounting price-cost margin is defined as the ratio of profit before tax over total revenue, where total revenue is the sum of net interest revenue and other operating income, and profit before tax is equal to the total revenue minus overheads, loan loss provisions, and other net expenses. The capital-sales ratio (KS_i) is proxied by the ratio of total equity over total revenue. Descriptive statistics are reported in Table 3.1. It shows that city banks have both economically and statistically significant larger means of accounting price-cost margins and market shares than regional banks. In particular, on average, city banks have higher accounting markups and market shares by factors of 2.3 and 18.1, respectively. The averages of the capital-sales ratios of these banks are roughly identical (i.e., with an insignificant difference).

Data on ownership are available only for the last year of the bank's reports, which varies from 2002 to 2005. Thus in constructing the cross-shareholding matrix for Japan, we assume that these direct shareholdings were also valid for 2003.

Figure 3.1: Partial ownership relations among the Japanese banks



Note: The figures are direct shareholdings in percentages. The arrows are directed from the shareholder to the bank(s) it owns. C and R stand for city banks and regional banks, respectively. Source: BankScope, Bureau van Dijk Electronic Publishing.

However, we should note that the ownership data, though crucial for this analysis, represent an incomplete picture of the shareholding ties due to its partial (and in some cases total) unavailability in the Bankscope dataset. In general, the city banks are the most intensive shareholders in the Japanese commercial banking sector.[22] For illustration purposes, a few banks from the sample are chosen and their partial ownership links are graphed in Figure 3.1. For the sake of simplicity, we disregard outside shareholding links of these banks, which do exist. As an example, Figure 3.1 shows that Bank of Fukuoka owns 2.36% of the shares in Higo Bank. Two remarks are in place. First, there are cases of *mutual* shareholding ties in the Japanese banking sector. In the figure this is the case for Kagoshima Bank and Miyazaki Bank. Second, given this mutual relationship, one *might* expect that *indirect* shareholdings could matter for the Japanese banks. It is easily seen that Mizuho Corporate Bank has an indirect share in Miyazaki Bank via, for example, Kagoshima Bank. However, in fact, because of the mutual shareholding described above there is an *infinite* number of paths of different length through which Mizuho Corporate Bank indirectly owns Miyazaki Bank (see Dietzenbacher and Temurshoev, 2008).

3.4.2 Estimation results

The results of the numerical nonlinear least-squares estimation are reported in Table 3.2. Since in the presence of local optima, finding the global optimal point de-

[22] In the entire financial system of Japan, besides city banks, also long-term credit banks and trust banks comprise the heavy shareholders of other financial and nonfinancial institutions due to their nature of operations. For example, trust banks are likely to hold shares in commercial banks as trustees of mutual funds. Thus it is expected that the effect of PCO would be much stronger if these banks had also been taken into account, but this would have required a different theoretical model for an industry with differentiated products. This is, however, beyond the scope of the current chapter.

Table 3.2: Empirical results (year 2003, obs.= 63)

	Full PCO	Direct PCO	Alley	No PCO	Full PCO	No PCO
$\hat{\alpha}$	0.0435 (0.0404)	0.0435 (0.0404)	0.0435 (0.0404)	0.0390 (0.0401)	0.0281* (0.0163)	0.0255 (0.0160)
$\hat{\varepsilon}$	0.6189* (0.3631)	0.6189* (0.3631)	0.6188* (0.3630)	0.6167* (0.3674)	0.3329** (0.1466)	0.3241** (0.1501)
$\hat{\mu}$	0.0495*** (0.0139)	0.0495*** (0.0139)	0.0495*** (0.0139)	0.0521*** (0.0147)	0.0410*** (0.0146)	0.0430*** (0.0152)
$\hat{\mu}_{GR}$					-0.0365** (0.0177)	-0.0371* (0.0190)
SSR	0.7382	0.7383	0.7383	0.7542	0.5850	0.6008

Note: The superscripts (*), (**), and (***) denote statistical significance of the coefficients at 10%, 5%, and 1% levels, respectively. SSR denotes the sum of squired residuals. The robust standard errors are given in parentheses.

pends on the initial parameters' values, in the estimation we first constructed grids for all parameters (i.e., we created a grid structure for α, ε and μ), and used *all* possible combinations of these grids as starting points. Then the minimum value of the sum of squared residuals (SSR) is chosen, and its corresponding estimates are given in Table 3.2. Column 2 gives the estimates of the parameters in (3.12) when both direct and indirect (full) PCO links are taken into account (i.e., when $\mathbf{L} = (\mathbf{I} - \mathbf{W})^{-1}$). Positive values of $\hat{\alpha}$ are indicative of cooperative behavior of banks. The full PCO model gives the market conduct estimate of $\hat{\alpha} = 0.0435$, which is not statistically different from zero.[23] Hence, at this point one may conclude that commercial banks in Japan in 2003 behaved as Cournot competitors. Table 3.2 also shows that the Japanese banking sector is characterized by inelastic demand (i.e., $\hat{\varepsilon} = 0.6189$ which is significant at 10% level). So, theoretically, banks in 2003 would have increased their revenues if they had raised the price. Finally, the sign of the capital-sales ratio coefficient is positive as expected, and is statistically significant for all estimated models. This is an estimate of the marginal rental cost of capital to the firm. One can also interpret the capital-sales ratio as a barrier to entry, and from this point of view its coefficient should also be positive, meaning that the higher the capital-sales ratio, the more difficult it is for a new firm to enter the industry.

The third column of Table 3.2 gives the results of the *direct* PCO model (i.e.,

[23] Standard errors are heteroscedastic-consistent. The error vector is $\boldsymbol{\nu}(\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a vector of k parameters. Denote the $n \times k$ Jacobian matrix by $\mathbf{J}(\boldsymbol{\theta})$ with $(\mathbf{J}(\boldsymbol{\theta}))_{ij} = \partial \nu_i(\boldsymbol{\theta})/\partial \theta_j$. Then the heteroscedastic-consistent estimate of the variance-covariance matrix of the estimate $\tilde{\boldsymbol{\theta}}$ is $\tilde{\boldsymbol{\Phi}} \equiv \frac{n}{n-k}[\tilde{\mathbf{J}}'\tilde{\mathbf{J}}]^{-1}\tilde{\mathbf{J}}'\boldsymbol{\Omega}\tilde{\mathbf{J}}[\tilde{\mathbf{J}}'\tilde{\mathbf{J}}]^{-1}$, where $\tilde{\mathbf{J}} = \partial \boldsymbol{\nu}/\partial \boldsymbol{\theta}'|_{\tilde{\theta}}$ and $\boldsymbol{\Omega}$ is a diagonal matrix with $\tilde{\nu}_i^2$ on its main diagonal (see e.g., Cameron and Trivedi, 2005, Chapter 5.8).

when $\mathbf{L} = \mathbf{I} + \mathbf{W}$). The estimates of the parameters are identical to those of the full PCO model, implying that for our sample *indirect* ownership links are insignificant and do not have any impact on the results. The fourth column of Table 3.2 gives the results for Alley's model (i.e., when $\widehat{\mathbf{L}}^{-1}\mathbf{L} = \mathbf{I} + (\mathbf{I} - \widehat{\imath'\mathbf{W}})^{-1}\mathbf{W}$, see footnotes 15 and 16), which also gives estimates of the parameters that are very close to those of the full PCO and direct PCO specifications. Alley's model is based on the profit specification given by (3.2), $\pi_i = (1 - \sum_{k \neq i} w_{ki})z_i + \sum_{k \neq i} w_{ik} z_k$, which is different from (3.3) for the full PCO model. However, the closeness of the outcomes of all these three models is due to the fact that there is a small number of PCO links in our sample (to be discussed later) and direct shareholdings are small in size (on average 3.2%), both of which imply that indirect PCO links are negligible.

Column 5 in Table 3.2 reports the estimates of the parameters in (3.12) when all the elements of the PCO matrix are set to zero (i.e, $\mathbf{L} = \mathbf{I}$, hence effectively (3.13) is estimated), which gives an estimate of the tacit collusion degree of $\hat{\alpha} = 0.0390$ that is not statistically different from zero either. Hence, without considering any other additional explanatory variable(s) in (3.12), neglecting PCO links does not give an economic bias in the results. That is, both the full PCO and the no PCO models predict that Japanese commercial banks compete in a Cournot oligopoly (although note that the point estimates are different).

Following Alley (1993, 1997) we re-estimate the full PCO and the no PCO models in Table 3.2 by adding the growth variable GR_i - the growth rate of a bank's operating income relative to the year 2002, which allows for changes in demand and thus in accounting price-cost margins to be taken into account.[24] Theoretically, the sign of the effect of the growth rate variable can be either positive, or negative. On the one hand, an increase in market demand may raise demand on inputs, thereby increasing their factor prices, hence may lead to lower accounting markups. On the other hand, the growth rate of demand may increase accounting price-cost margins by increasing output prices and/or expanding production volume. The results are given in the last two columns of Table 3.2, where the estimate $\hat{\mu}_{GR}$ for the growth variable is negative, and statistically significant.

Note that including GR_i in the full PCO model gives a market conduct estimate of $\hat{\alpha} = 0.0281$ that is statistically significant (at 10% level), while in the no PCO case $\hat{\alpha} = 0.0255$, which does not differ statistically significantly from zero. Hence, neglecting cross-shareholding links in this case yields different economic outcomes:

[24] We also estimated the models with other bank-specific factors, such as net loans and total fixed assets to account for risk and capacity differences. However, these coefficients were insignificant and did not change the results, hence are not reported.

the no PCO case predicts Cournot oligopoly in the 2003 Japanese banking sector, while the full PCO model predicts *modest* collusive environment in the industry. Although $\hat{\alpha}$ in the full PCO case is not highly statistically different from zero, this result suggests that ownership links should be taken into account in empirical studies of the Japanese banking sector. In addition, we think that the main reasons for the almost identical results of all the four models given in columns 2-5 of Table 3.2 are the following. First, as we already noted, the ownership data are incomplete, and it is quite difficult to obtain the true picture of these linkages. This yields underestimation of the PCO effects.[25] Second, some banks with partial ownership data were excluded from the sample for their unprofitability and/or unavailability of other required data. Third, in our 63×63 PCO matrix there are 67 cases of shareholding links, which comprises only 1.7% of the total number of possible ownership ties of $n(n-1) = 3906$. Fourth, in general, in the Japanese financial system city banks, long-term credit banks and trust banks are the main shareholders of other financial (and nonfinancial) institutions (see footnote 22). Hence, we expect that studies that concentrate also on the last two types of banks should consider PCO links, otherwise the (economic) bias of the results might be significant. In this chapter, however, we do not consider trust banks and long-term credit banks, which would require using a different model of a differentiated-product nature.

3.4.3 Comparison with related studies

There are few studies that estimate the degree of competition in the Japanese banking sector. Before comparing our results with these studies, we first briefly discuss different approaches in estimating the competition level (see for details e.g., Bresnahan, 1989). CV models are frequently used for this purpose, starting with the early important paper of Iwata (1974). The Clarke and Davies (1982) model, adopted in this chapter, also belongs to this strand of literature. Since CV models provide theoretical foundations for firms' structure-conduct-performance reduced-form relationships (which explains the term "structure-conduct-performance paradigm"), they are widely used to infer the degree of market competition. The disadvantage of using such models is that cost data are required, which in many cases are difficult to obtain. The attempt of avoiding cost data resulted in the so-called "new empirical industrial organization" (NEIO) literature pioneered by Bresnahan (1982) and

[25] Dietzenbacher et al. (2000) analyzed the sensitivity in their analysis of the Dutch banking sector, because banks were only required to report if shares were larger than 5%. They showed that direct interests below 5% are relevant and have a substantial effect on the estimates of banks' price-cost margins.

Lau (1982). Its econometric approach is structural because both demand and supply sides are explicitly considered. However, modeling the competition level does not differ from the CV literature, which is stated by Bresnahan (1989, p. 1027) as follows: "As a matter of fact, there is absolutely no difference between [CV and NEIO approaches to modeling collusion and] ... the two specifications can nest the same models".

Another widely used approach is that of Panzar and Rosse (1987). The Panzar-Rosse *H* statistic is the sum of the elasticities of the reduced-form revenues with respect to all factor prices.[26] Its advantage is that few data are required on endogenous variables (revenue is always observable even when price and quantity are not), though it will require information on all the variables that shift demand or cost. However, using *H* statistics in empirical work relies on the assumption that markets are in the long-run equilibrium in each point of time. In general, speaking about above methods and others including time-series data analysis, event studies, studies of the determinants of the price, and fully dynamic models, Martin (2002, p. 225) concludes: "No one of these are immune from criticism. Broadly speaking, these diverse methodologies yield consistent results, tending to support the hypotheses advanced by the structure-conduct-performance school".

The paper closest to our work in terms of the methodology used is Alley (1993), who uses exactly the same theoretical model, but without considering PCO linkages. The author finds that the degree of competitive intensity for 1986-87 Japanese regional and Sogo banks is $\hat{\alpha} = 0.6013$, indicating a high degree of collusion. This estimate is much larger than our estimate of $\hat{\alpha} = 0.0281$ for the Japanese commercial banking sector (column 6 in Table 3.2). Two remarks are in place in this regard. First, it might very well happen that the estimate of α is biased (upward), given the fact that back in the 1980s-1990s shareholding interlocks were quite extensive in the Japanese banking system compared to the current situation (see e.g., Miyajima and Kuroki, 2007). Second, if the result would not change with PCO consideration, then comparison of the two would suggest that competition has significantly improved between 1986-87 and 2003.

Molyneux et al. (1996) employing Panzar-Rosse *H* statistics, conclude that Japanese commercial (city and regional) banks behaved as if under monopoly in 1986, but the market conduct improved in 1988 whereby it becomes consistent with mo-

[26] If H is negative then firms' policies are consistent with the monopoly conduct, $0 < H < 1$ represents monopolistic competition and $H = 1$ under perfect competition. These interpretations can be deduced from the effect of an upward shift in firms' marginal, average and total cost curves on the firms' *equilibrium* revenues.

nopolistic competition. Using the NEIO approach and long-term panel data from 1974 to 2000, Uchida and Tsutsui (2005) conclude that market competition largely improved from the 1970s to the 1980s, but deteriorated after 1997. They also find that the degree of competition is higher for city banks than for regional banks. Finally, Lee and Nagano (2008) compare the pre-merger period of 1986-1997 to the post-merger wave period of 1998-2005 in a set of Japanese regional banks that is divided into seven regions. Essentially their results in terms of H statistics suggest that in six regions the monopolistic competition environment holds for both periods, while in only one region there is a tendency towards a more competitive environment.[27] In relation to this chapter, we think that similar to the first point made with regard to Alley's (1993) study, there is a possibility of bias in the estimates of the market conduct or H statistics due to ignorance of PCO linkages, which is again much more probable for the results on earlier periods in these studies. The market performance indicator for Japanese regional banks in 2000 in Uchida and Tsutsui (2005) shows a collusive environment, which is consistent with our result for 2003. However, their study does not reject Cournot competition for city banks in 2000. All in all, we think that taking into account PCO links in all these studies is crucial, which might even change the results, especially, for the period before the mid 1990s when cross-shareholding was believed to be one of the main distinguishing features of the Japanese business groups.

3.4.4 Market power test

In this section the market power test of each individual bank is carried out. Having the estimates of the market competitive intensity and the price elasticity of demand, one can estimate firms' markups using equation (3.9). Then in the context of our model, firm i exercises market power if its estimated price-cost margin is in a statistical sense significantly positive. As mentioned in Section 3.3, in an industry without PCO, the market power diagnosis of firm i reduces to the condition $[\hat{\alpha} + (1 - \hat{\alpha})s_i]/\hat{\varepsilon} > 0$ (see Martin, 1988).

The *delta method* is used in order to compute t-statistics of the markups in (3.9).[28] The estimated markups and their t-statistics based on the estimates of the full PCO

[27] We should note that the authors' own conclusion is, however, different. Lee and Nagano (2008) state that "... the banking sector in Japan's metropolitan area is very competitive, becoming more competitive than that of 1986-1997" (p. 614). This conclusion is *not* consistent with the values of the H statistics with their appropriate 95% confidence intervals given in their Table 1 on pp. 612-613.

[28] Let price-cost margins depend on k parameters given by the vector $\boldsymbol{\theta}$ and let $\mathbf{C}(\boldsymbol{\theta}) = \partial \mathbf{m}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}'$. Then according to the delta method, the estimated (asymptotic) variance-covariance matrix of the markups is given by $\tilde{\mathbf{C}}\tilde{\boldsymbol{\Phi}}\tilde{\mathbf{C}}'$, where $\tilde{\boldsymbol{\Phi}}$ is defined in footnote 23 (see e.g., Greene, 2003).

model from Table 3.2 (i.e., column 6) are reported in Table 3.3. Note that estimating markups for the *actual* no PCO case does not make sense, since we do not know anything about the real environment without cross-shareholdings between banks. That is, markups and market shares would be different in that case, implying that using our data for this purpose would be totally misleading.

The t-statistics of all these markups are computed on the null hypothesis that the true value of the statistics are zero, which is a market power test for each bank. Several conclusions can be drawn from Table 3.3. First, given that the smallest t-statistic in the entire sample is 2.183, we conclude that each bank exercises some degree of market power (at a 5% significance level). Second, on average, banks that hold shares in other banks have *higher* markups than banks without any stock-holdings in rivals (i.e., 0.271 vs. 0.106). This difference is statistically significant (the one-sided two-sample t significance test of means gives $p = 0.0184$ with 9 degrees of freedom), implying that PCO increases the market power of banks owning shares in rivals. In our sample there are in total 67 cases of shareholdings (the sum of the column "Sub" in Table 3.3, which denotes the number of subsidiaries, or, equivalently, the sum of the column "Share" for the number of shareholders) that are made by the 10 banks that hold shares in other banks, consisting of all four city banks and six regional banks. Note also that the regional banks, and not the city banks, are owned by others. Moreover, the correlation coefficient between the estimated markups and the number of banks' subsidiaries for the entire sample is 0.68, while that between the estimated price-cost margins and the number of banks' shareholders is equal to -0.26. All in all, this confirms the conjecture that owning shares in rivals increases (resp. decreases) market power of firms-owners (resp. owned firms). Third, city banks, on average, have significantly higher price-cost margins than regional banks (i.e., 0.501 vs. 0.107, and the difference is highly statistically significant with $p = 0.0054$). One of the explanations for this (in light of the second point made above) is that city banks own many more banks with larger shareholding size than regional banks. Table 3.3 shows that the four city banks, on average, own 14.5 banks with an average direct stake of 4.98%, while they are not owned themselves. On the other hand, on average, a regional bank owns only 0.2 banks with 0.21% as the average share, but 3.10% of its shares are owned by 1.1 banks. The six regional banks with shareholdings, on average, hold 2.01% shares in 1.5 banks, whereas 2.75% of their shares are owned by 2.5 banks (not shown in Table 3.3). Hence, among other factors, owning larger shares in many regional banks allows city banks to exercise a larger market power.

Table 3.3: Market power test of the Japanese commercial banks in 2003

No	Bank Name	Type	$\hat{m}_i$	t-stat.	Sub.	%%	Share.	%%
1	77 Bank	Reg.	0.117	2.634	0	-	2	2.47
2	Akita Bank Ltd	Reg.	0.098	2.391	0	-	1	1.65
3	Aomori Bank Ltd.	Reg.	0.098	2.392	0	-	2	3.25
4	Awa Bank	Reg.	0.102	2.443	0	-	2	2.97
5	Bank of Fukuoka Ltd.	Reg.	0.136	2.754	3	2.41	0	-
6	Bank of Ikeda	Reg.	0.100	2.415	0	-	1	3.04
7	Bank of Iwate, Ltd.	Reg.	0.100	2.418	0	-	1	3.71
8	Bank of Kyoto	Reg.	0.111	2.567	0	-	1	3.16
9	Bank of Okinawa	Reg.	0.095	2.342	0	-	2	1.48
10	Bank of the Ryukyus Ltd.	Reg.	0.098	2.381	0	-	1	1.89
11	Bank of Tokyo - Mitsubishi Ltd	City	0.467	2.571	24	3.45	0	-
12	Bank of Yokohama, Ltd.	Reg.	0.162	2.850	0	-	0	-
13	Chiba Bank Ltd.	Reg.	0.140	2.795	0	-	1	4.59
14	Chiba Kogyo Bank	Reg.	0.099	2.399	0	-	2	9.44
15	Chikuho Bank	Reg.	0.089	2.223	0	-	0	-
16	Chugoku Bank, Ltd.	Reg.	0.115	2.615	0	-	0	-
17	Daishi Bank Ltd.	Reg.	0.111	2.569	0	-	2	2.01
18	Eighteenth Bank	Reg.	0.101	2.439	0	-	1	4.85
19	Gunma Bank Ltd.	Reg.	0.124	2.686	1	1.20	3	2.60
20	Hachijuni Bank	Reg.	0.121	2.676	0	-	1	4.76
21	Higo Bank	Reg.	0.110	2.510	2	2.40	2	3.53
22	Hiroshima Bank Ltd.	Reg.	0.126	2.710	0	-	2	3.25
23	Hokkaido Bank	Reg.	0.111	2.575	0	-	2	2.86
24	Hokkoku Bank Ltd.	Reg.	0.105	2.493	0	-	0	-
25	Hokuetsu Bank Ltd.	Reg.	0.097	2.376	0	-	1	5.41
26	Hokuriku Bank Ltd.	Reg.	0.133	2.757	0	-	0	-
27	Hokuto Bank	Reg.	0.093	2.308	0	-	3	1.66
28	Hyakugo Bank Ltd.	Reg.	0.108	2.535	0	-	2	3.19
29	Hyakujushi Bank Ltd.	Reg.	0.106	2.503	0	-	1	2.69
30	Iyo Bank Ltd	Reg.	0.113	2.591	0	-	1	5.60
31	Joyo Bank Ltd.	Reg.	0.131	2.744	1	1.69	2	2.97
32	Juroku Bank Ltd.	Reg.	0.113	2.593	0	-	0	-
33	Kagoshima Bank Ltd.	Reg.	0.106	2.488	1	2.63	4	2.26
34	Kanto Tsukuba Bank Ltd.	Reg.	0.095	2.343	0	-	0	-
35	Kiyo Bank	Reg.	0.107	2.522	0	-	1	1.54
36	Michinoku Bank, Ltd.	Reg.	0.094	2.320	0	-	0	-
37	MIE Bank Ltd	Reg.	0.093	2.300	0	-	1	6.57
38	Miyazaki Bank	Reg.	0.100	2.380	1	2.02	4	2.39
39	Mizuho Bank	City	0.483	2.544	5	3.38	0	-
40	Mizuho Corporate Bank	City	0.362	2.662	23	3.32	0	-
41	Musashino Bank	Reg.	0.104	2.475	0	-	1	3.59
42	Nanto Bank Ltd.	Reg.	0.112	2.581	0	-	1	4.56
43	Nishi-Nippon City Bank Ltd.	Reg.	0.117	2.635	0	-	1	3.08
44	Ogaki Kyoritsu Bank	Reg.	0.106	2.508	0	-	1	0.40
45	Oita Bank Ltd.	Reg.	0.100	2.426	0	-	1	2.60
46	San-In Godo Bank, Ltd	Reg.	0.109	2.547	0	-	0	-
47	Senshu Bank Ltd.	Reg.	0.097	2.368	0	-	1	2.40
48	Shiga Bank, Ltd.	Reg.	0.109	2.549	0	-	2	2.68
49	Shikoku Bank Ltd.	Reg.	0.101	2.438	0	-	1	4.99
50	Shimizu Bank Ltd.	Reg.	0.095	2.340	0	-	1	5.25
51	Shizuoka Bank	Reg.	0.131	2.749	0	-	2	2.88
52	Shonai Bank	Reg.	0.091	2.255	0	-	1	42.18
53	Sumitomo Mitsui Banking Corporation	City	0.692	2.448	6	9.78	0	-
54	Suruga Bank, Ltd.	Reg.	0.110	2.561	0	-	0	-
55	Tajima Bank Ltd.	Reg.	0.089	2.231	0	-	0	-
56	Toho Bank Ltd.	Reg.	0.104	2.484	0	-	0	-
57	Tohoku Bank	Reg.	0.089	2.230	0	-	0	-
58	Tokyo Tomin Bank, Ltd.	Reg.	0.101	2.434	0	-	1	4.97
59	Tottori Bank	Reg.	0.090	2.248	0	-	0	-
60	Toyama Bank, Ltd.	Reg.	0.087	2.183	0	-	0	-
61	Yamagata Bank Ltd.	Reg.	0.096	2.361	0	-	1	4.80
62	Yamaguchi Bank	Reg.	0.115	2.615	0	-	0	-
63	Yamanashi Chuo Bank Ltd.	Reg.	0.099	2.406	0	-	3	2.92
	Overall sample average		0.132		1.1	0.51	1.1	2.91
	City banks average		0.501		14.5	4.98	0.0	-
	Regional banks average		0.107		0.2	0.21	1.1	3.10
	All shareholders average		0.271		6.7	3.23	1.5	1.37
	All non-shareholders average		0.106		0.0	-	1.0	3.19

3.5 Concluding remarks

Nowadays there is ample evidence of the presence of partial cross ownership (PCO) links among firms. This study examines empirically the influence of PCO on the degree of competitive intensity of an industry and on firms' market power. The model of Clarke and Davies (1982) is adopted and modified by taking into *full* account both direct and indirect interests of firms in each other via PCO ties. To the best of our knowledge, in all empirical estimations of the degree of tacit collusion, PCO is totally neglected, except for Alley (1997), who, however, disregards indirect shareholdings.

It has been shown that, unlike in the no PCO case, with cross-shareholding the link between firms' price-cost margins and the market competitive intensity is *non-linear*. Hence, in the presence of extensive shareholding links among firms, ignoring PCO leads to biased parameter estimates due to model misspecification. It has been shown that when market shares, number of firms, price elasticity of demand, and collusion degree are given, firms with shareholdings exert a strictly larger market power than those without PCO, provided that the market conduct is consistent with Cournot or a more collusive environment. This is because shareholding interlocks among firms cause a common interest of firms, implying greater monopoly power for firms with PCO holdings.

As an empirical application we have studied the Japanese banking sector in 2003. We found that the Japanese banks are competing in a modest collusive environment, while neglecting PCO yields a different economic outcome that indicates a Cournot oligopoly. (By modest we mean that the degree of collusion is relatively small being closer to the Cournot outcome rather than a monopoly.) Secondly, banks with passive investments in rivals exert a strictly larger market power than those without any PCO, which confirms the hypothesis that acquiring shares in rivals for a firm is one of the crucial means of enhancing its market power. Also, city banks with many shareholdings were found to exercise a much larger market power than regional banks with none or few stockholdings.

A few simplifying assumptions have been made throughout the chapter and need some clarification. First, we did not consider product differentiation, and focused only on homogeneous market environment, which, in general, does not hold in the real world. Analyzing a differentiated-product industry is rather complex, since one has to compute all the own- and cross-price elasticities, for instance.[29]

[29] See, for example, the "menu approach" for identifying market conduct proposed by Nevo (1998).

Nonetheless, the empirical results of the homogeneous model used here are still useful in discovering the collusion degree within an industry, as the estimates of the market competitive intensity indicate “the similarity of margins between firms of different size” (Clarke et al., 1984, p. 447). As a matter of fact this has been confirmed in our study, as the low degree of collusion implies rather different levels of firms’ market power. Second, the PCO structure has been assumed to be exogenous, which might not reflect the optimal decisions of firms. However, similar to the Gilo et al. (2006) study, our analysis was done from the perspective of antitrust agencies facing a given pattern of PCO. Third, in the empirical part we have disregarded the PCO of banks with other financial and non-financial institutions. This allowed us to focus on the commercial banking sector only, while neglecting the potential effect of banks’ shareholding interlocks with firms in other industries.[30] However, for that one needs to use a different theoretical model for an industry with differentiated products, which is beyond the scope of the current study.

[30] Ideally, one would like to consider all possible shareholding links, but this would be unfeasible or, at least, a complicated task in light of unavailability of (access to all) ownership data of firms in the all involved industries.

3.A Proofs

Derivation of equation (3.9). Equation (3.8) can be expressed in matrix form as

$$\mathbf{m} = \frac{\alpha}{\varepsilon}[\mathbf{I} + \widehat{\mathbf{L}}^{-1}\hat{\mathbf{s}}^{-1}(\mathbf{L} - \widehat{\mathbf{L}})\hat{\mathbf{s}}]\boldsymbol{\imath} + \frac{1-\alpha}{\varepsilon}[\mathbf{I} + \widehat{\mathbf{L}}^{-1}(\mathbf{L} - \widehat{\mathbf{L}})]\mathbf{s} - \alpha[\widehat{\mathbf{L}}^{-1}\hat{\mathbf{s}}^{-1}(\mathbf{L} - \widehat{\mathbf{L}})\hat{\mathbf{s}}]\mathbf{m}, \tag{3.A.1}$$

where $\boldsymbol{\imath}$ is the summation vector of ones.

All the three terms in square brackets can be further simplified as

$$\begin{aligned} &\mathbf{I} + \widehat{\mathbf{L}}^{-1}\hat{\mathbf{s}}^{-1}(\mathbf{L} - \widehat{\mathbf{L}})\hat{\mathbf{s}} = \hat{\mathbf{s}}^{-1}\widehat{\mathbf{L}}^{-1}\mathbf{L}\hat{\mathbf{s}}, \qquad \mathbf{I} + \widehat{\mathbf{L}}^{-1}(\mathbf{L} - \widehat{\mathbf{L}}) = \widehat{\mathbf{L}}^{-1}\mathbf{L}, \\ &\widehat{\mathbf{L}}^{-1}\hat{\mathbf{s}}^{-1}(\mathbf{L} - \widehat{\mathbf{L}})\hat{\mathbf{s}} = \hat{\mathbf{s}}^{-1}(\widehat{\mathbf{L}}^{-1}\mathbf{L} - \mathbf{I})\hat{\mathbf{s}}. \end{aligned} \tag{3.A.2}$$

Plugging results from (3.A.2) in (3.A.1) we obtain

$$\mathbf{m} = (1/\varepsilon)\left[\alpha\hat{\mathbf{s}}^{-1}\widehat{\mathbf{L}}^{-1}\mathbf{L}\mathbf{s} + (1-\alpha)\hat{\mathbf{L}}^{-1}\mathbf{L}\mathbf{s}\right] - \alpha\hat{\mathbf{s}}^{-1}(\hat{\mathbf{L}}^{-1}\mathbf{L} - \mathbf{I})\hat{\mathbf{s}}\mathbf{m}. \tag{3.A.3}$$

With definitions $\mathbf{Q} \equiv \hat{\mathbf{s}}^{-1}(\mathbf{I} - \widehat{\mathbf{L}}^{-1}\mathbf{L})\hat{\mathbf{s}}$, $\mathbf{x}_1 \equiv \hat{\mathbf{s}}^{-1}\widehat{\mathbf{L}}^{-1}\mathbf{L}\mathbf{s}$, and $\mathbf{x}_2 \equiv \widehat{\mathbf{L}}^{-1}\mathbf{L}\mathbf{s}$ the equation (3.A.3) yields (3.9). ■

Proof of Theorem 3.1. From economic point of view, the lower limit of α corresponds to zero price-cost margins for all firms $i = 1, \ldots, n$. Using $\mathbf{x}_2 = \hat{\mathbf{s}}\mathbf{x}_1$, (3.9) can be rewritten as $\mathbf{m} = (1/\varepsilon)(\mathbf{I} - \alpha\mathbf{Q})^{-1}[\alpha\mathbf{I} + (1-\alpha)\hat{\mathbf{s}}]\mathbf{x}_1$, which together with (3.10) imply that markups are zero both with and without PCO when $\alpha\mathbf{I} = -(1-\alpha)\hat{\mathbf{s}}$, or, equivalently, when $\alpha = -(1-\alpha)s_i$. This implies that market shares should be equal, thus plugging $s_i = s = 1/n$ in the last condition gives $\alpha = -1/(n-1)$. The highest possible price-cost margins are those of the monopolist (perfect cartel) that equal the inverse of the price elasticity of demand, which is the case when $\alpha = 1$ in (3.9), since then $\mathbf{m} = (1/\varepsilon)(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{x}_1 = (1/\varepsilon)(\mathbf{I} - \mathbf{I} + \hat{\mathbf{s}}^{-1}\widehat{\mathbf{L}}^{-1}\mathbf{L}\hat{\mathbf{s}})^{-1}\mathbf{x}_1 = (1/\varepsilon)\boldsymbol{\imath}$. The same holds for the no PCO case in (3.10), hence the (economic) upper bound of the conjectural elasticity both with and without PCO is $\alpha = 1$. ■

Proof of Theorem 3.2. For simplicity denote $\mathbf{A} \equiv \widehat{\mathbf{L}}^{-1}\mathbf{L}$ and $\mathbf{B} \equiv \mathbf{A} - \mathbf{I}$. Premultiplication of (3.9) by $\varepsilon\hat{\mathbf{s}}$ yields $\varepsilon\hat{\mathbf{s}}\mathbf{m} = \alpha\mathbf{A}\mathbf{s} + (1-\alpha)\hat{\mathbf{s}}\mathbf{A}\mathbf{s} - \varepsilon\alpha\mathbf{B}\hat{\mathbf{s}}\mathbf{m}$. Add to and subtract from the right-hand side (rhs) of the last equation $\alpha\mathbf{s} + (1-\alpha)\hat{\mathbf{s}}\mathbf{s}$, which in turn is equal to $\varepsilon\hat{\mathbf{s}}\mathbf{m}^0$ as follows from (3.10), where $\mathbf{m}^0$ is the vector of markups in the no PCO case provided that $\alpha^0 = \alpha$, $\varepsilon^0 = \varepsilon$, $n^0 = n$, and $\mathbf{s}^0 = \mathbf{s}$. This yields

$\varepsilon\hat{\mathbf{s}}\mathbf{m} = \alpha\mathbf{B}\mathbf{s} + (1-\alpha)\hat{\mathbf{s}}\mathbf{B}\mathbf{s} - \varepsilon\alpha\mathbf{B}\hat{\mathbf{s}}\mathbf{m} + \varepsilon\hat{\mathbf{s}}\mathbf{m}^0$. Hence,

$$\varepsilon\hat{\mathbf{s}}(\mathbf{m}-\mathbf{m}^0) = \alpha\mathbf{B}\mathbf{s} + (1-\alpha)\hat{\mathbf{s}}\mathbf{B}\mathbf{s} - \alpha\varepsilon\mathbf{B}\hat{\mathbf{s}}\mathbf{m} = (1-\alpha)\hat{\mathbf{s}}\mathbf{B}\mathbf{s} + \alpha\mathbf{B}\hat{\mathbf{s}}(\boldsymbol{\imath} - \varepsilon\mathbf{m}). \quad (3.A.4)$$

For $\alpha = 0$, the rhs in (3.A.4) is $\hat{\mathbf{s}}\mathbf{B}\mathbf{s}$ and its i-th element is strictly positive if and only if firm i owns shares in rival(s). Hence, with $\alpha = 0$ we have $m_i > m_i^0$ for all i with PCO holdings, otherwise $m_i = m_i^0$. Equally, for $\alpha \in (0,1)$ and $m_i < 1/\varepsilon$, the i-th element in the rhs of (3.A.4) is positive if i has shareholdings, implying again that $m_i > m_i^0$ for all firms i with PCO holdings. Note that if $\alpha = 1$ (or equivalently $\boldsymbol{\imath} = \varepsilon\mathbf{m}$), the rhs in (3.A.4) is a zero vector, hence we get an expectable outcome of $\mathbf{m} = \mathbf{m}^0$. ■

CHAPTER 4

Partial cross ownership and tacit collusion under cost asymmetries*

4.1 Introduction

There are many cases in which firms acquire their rivals' stock as passive investments that give them a share in the rivals' profits but not in the rivals' decision making. These investments are often multilateral; examples of industries that feature complex webs of partial cross ownerships are the Japanese and the US automobile industries (Alley, 1997), the global airline industry (Airline Business, 1998), the Dutch financial sector (Dietzenbacher et al., 2000), the Nordic power market (Amundsen and Bergman, 2002), and the global steel industry (Gilo et al., 2006). While horizontal mergers are subject to substantial antitrust scrutiny and are often opposed by antitrust authorities, passive investments in rivals were either granted a de facto exemption from antitrust liability or have gone unchallenged by antitrust agencies in recent cases (Gilo, 2000).[1] This lenient approach towards passive in-

* This chapter is based on joint work with David Gilo (The Buchman Faculty of Law, Tel-Aviv University) and Yossi Spiegel (Recanati Graduate School of Business Adminstration, Tel Aviv University).

[1] For example, to the best of our knowledge, Microsoft's investments in the nonvoting stocks of Apple and Inprise/Borland Corp. were not challenged by antitrust agencies while Gillette's 22.9% stake in Wilkinson Sword was approved by the US Department of Justice (DOJ) after the DOJ was assured that this stake would be passive (see *United States v. Gillette* Co. 55 Federal Register at 28312). The US Federal Trade Commission (FTC) approved Tele-Communications Inc.'s (TCI's) 9.0% stake in Time Warner which at the time was TCI's main rival in the cable TV industry and even allowed TCI to raise its stake in Time Warner to 14.99% in the future, after being assured that TCI's stake would be completely

vestments in rivals stems from the courts' interpretation of the exemption for stock acquisitions "solely for investment" included in Section 7 of the Clayton Act.

In an earlier work Gilo et al. (2006) began to investigate the merits of this lenient approach of courts and antitrust agencies towards passive investments in rivals. They showed that partial cross ownership (PCO) arrangements can facilitate tacit collusion among rival firms though cases exist in which such investments have no effect on the incentive of firms to collude. In particular it was shown that when firm r increases its stake in a rival firm s, then collusion is never hindered, and that it will be surely facilitated if and only if (i) each firm in the industry holds a stake in at least one rival, (ii) the *maverick firm* in the industry (the firm with the strongest incentive to deviate from a collusive agreement)[2] has a direct or an indirect stake in firm r,[3] and (iii) firm s is not the industry maverick. These results were established, however, under the assumption that firms are symmetric and have the same marginal cost functions. In the current study, we relax this assumption and examine the effect of PCO on the incentives of asymmetric firms to collude. This is obviously an important question since most industries feature cost asymmetries among firms.

To address this question we posit an infinitely repeated Bertrand oligopoly model in which firms have asymmetric marginal costs and they acquire some of their rivals' (nonvoting) shares. This simple setting allows us to deal with the complexity generated by multilateral PCO. This complexity arises since, in general, multilateral PCO arrangements create multiplier effects so the profit of each firm, both under collusion as well as under deviation from collusion, depends on the whole set of PCO in the industry and not only on the firm's own stake in rivals. Another advantage of this model is that PCO does not affect the equilibrium in the one shot case and therefore does not have any unilateral competitive effects. This allows us to focus on the effect of PCO on the ability of firms to engage in tacit collusion. We say that PCO arrangements facilitate tacit collusion if they expand the range of discount factors for which tacit collusion can be sustained.

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passive (see *Re Time Warner Inc.*, 61 Federal Register 50301, 1996). The FTC also agreed to a consent decree approving Medtronic Inc.'s almost 10% passive stake in SurVivaLink, one of the only two rivals of Medtronic's subsidiary in the automated External Defibriallators market (see *Re Medtronic, Inc.*, FTC File No. 981-0324, 1998).

[2] The Horizontal Merger Guidelines of the US Department of Justice and FTC define maverick firms as "firms that have a greater economic incentive to deviate from the terms of coordination than do most of their rivals," see http://www.usdoj.gov/atr/public/guidelines/hmg.htm. For an excellent discussion of the role that the concept of maverick firms plays in the analysis of coordinated competitive effects, see Baker (2002).

[3] Firm i has an indirect stake in firm r if it either has a stake in a firm that has a stake in firm r, or if it has a stake in a firm that has a stake in a firm that has a stake in firm r, and so on.

In the first part of this study we consider the case where only the most efficient firm in the industry invests in rivals. We show that even unilateral PCO by this firm may facilitate a market-sharing scheme in which all firms charge the same collusive price and divide the market between them. Unlike the case where firms have the same marginal costs, here firms have different monopoly prices on which they wish to collude. We assume that the collusive price is a compromise between the monopoly prices of the different firms. We show that when the most efficient firm invests in rivals, the collusive price would increase relative to the case where there are no PCO arrangements. Moreover, we show that the most efficient firm in the industry prefers to first invest in its most efficient rival both because this is the most effective way to promote tacit collusion and because such investment leads to a collusive price that is closer to the most efficient firm's monopoly price. Only if investment in the most efficient rival is insufficient to sustain a market-sharing scheme, then the most efficient firm begins to invest in less efficient rivals.

In the second part of this chapter, we turn to multilateral PCO arrangements. In that case, cost asymmetries raise the complexity of the analysis considerably because the most efficient firm earns a positive profit even after the collusive agreement breaks down. Consequently, an increase in a firm i's direct or indirect stake in the most efficient firm has conflicting effects on firm i's incentive to collude. On the one hand, a larger (direct or indirect) stake in the most efficient firm makes firm i less eager to deviate from collusion, because firm i obtains a larger share in the collusive profit of the most efficient firm. But on the other hand, the increased stake of firm i in the most efficient firm also gives it a larger share in the profit of the most efficient firm once the collusive agreement breaks down. This second effect weakens the incentive of firm i to collude.

Despite these complications, we are able to show that an increase in the stake of firm r in firm s never hinders collusion and it will strictly facilitate collusion if and only if (i) the industry maverick has a direct or indirect stake in firm r, and (ii) firm s is not the industry maverick. When either (i) or (ii) fails to hold, the increase in firm r's stake in firm s does not affect tacit collusion. These results extend the earlier findings in Gilo et al. (2006) and show that the results when firms have symmetric cost functions generalize to the asymmetric costs case.

Apart from Gilo et al. (2006), we are aware of only one other paper, Malueg (1992), that studies the coordinated effects of PCO. His paper differs from ours in several ways as he considers a repeated symmetric Cournot game in which firms hold identical stakes in one another, and moreover, in his paper, it is effectively the controllers rather than the firms that hold stakes in rivals. This difference is important because investments by controllers do not feature the complex chain-effect interaction between the profits of rival firms which is a main focus of our study. Other papers that look at the competitive effects of PCO include Reynolds and Snapp (1986), Bolle and Güth (1992), Flath (1991, 1992a), Reitman (1994), and Dietzenbacher et al. (2000). These papers, however, examine the unilateral effects of PCO arrangements in the context of static oligopoly models.[4]

The rest of the chapter is organized as follows. Section 4.2 examines the ability of firms to achieve the fully collusive outcome in the context of an infinitely repeated Bertrand model with asymmetric firms without PCO. Section 4.3 examines the case where only the most efficient firm in the industry invests in rivals. Section 4.4 examines multilateral PCO arrangements. Conclusions and final remarks are given in Section 4.5. All technical proofs are given in Appendix 4.A.

4.2 Tacit collusion absent PCO

We examine the coordinated competitive effects of PCO in the context of an infinitely repeated Bertrand oligopoly model with $n \geq 2$ firms. We assume that the n firms produce a homogenous product using a constant returns to scale technology and face a downward sloping demand function $Q(p)$. In every period, the n firms simultaneously choose prices and the lowest price firm captures the entire market. In case of a tie, the set of lowest price firms get equal shares of the total sales. The firms, however, have different marginal costs: let c_i be the (constant) marginal cost of firm i and assume $c_1 < c_2 < \ldots < c_n$. That is, higher indices represent higher cost firms. The profit of firm i when it serves the entire market at a price p is given by

$$y_i(p) = Q(p)(p - c_i).$$

We shall make the following assumptions on $y_i(p)$.

Assumption 1: $y_i(p)$ has a unique global maximizer, p_i^m.

[4] See also Bresnahan and Salop (1986) and Kwoka (1992) for a related analysis of static models of horizontal joint ventures. Alley (1997) and Parker and Röller (1997) provide empirical evidence on the effect of PCO on collusion. Alley (1997) finds that failure to account for PCO leads to misleading estimates of the degree of tacit collusion in the Japanese and US automobile industries (see also Chapter 3 for similar conclusion in the case of the Japanese banking sector). Parker and Röller (1997) find that cellular telephone companies in the US tend to collude more in one market if they have a joint venture in another market.

Assumption 2: $p_1^m > c_n$ and $y_1(c_2) > y_1(c_j)/(j-1)$ for all $j = 3, \ldots, n$.

Assumption 1 is standard and holds whenever the demand function is either concave or not too convex. Since $c_1 < c_2 < \ldots < c_n$, then $p_1^m < p_2^m < \ldots < p_n^m$, where $p_i^m \equiv \arg\max_p y_i(p)$ is the monopoly price from firm i's point of view.[5] That is, higher cost firms prefer higher monopoly prices. The first part of Assumption 2 ensures that all firms are effective competitors because it states that the monopoly price of the most efficient firm exceeds the marginal cost of the least efficient firm. The second part of Assumption 2 implies that in a static Bertrand game, firm 1 will prefer to set a price slightly below c_2 and capture the entire market than share the market with firm 2 at a price slightly below c_3, or share the market with firms 2 and 3 at a price slightly below c_4, and so on. Given this assumption, it is clear that absent collusion, firm 1 will prefer to monopolize the market by charging a price slightly below c_2.[6]

When the stage game is infinitely repeated, firms may be able to engage in tacit collusion. The fact that different firms have different monopoly prices raises the obvious question of which price would they coordinate on in a collusive equilibrium? If side payments were possible, firms would clearly let firm 1, which is the most efficient firm, serve the entire market at a price p_1^m (e.g., firms 2, ..., n would all set prices above p_1^m and would make no sales). The firms will then use side payments to share the monopoly profit

$$y_1^m \equiv Q(p_1^m)(p_1^m - c_1). \tag{4.1}$$

We rule out this possibility by assuming that side payments are not feasible, say due to the fear of antitrust prosecution.

Instead, we consider a collusive scheme led by firm 1. According to this scheme, firm 1 sets a price $\widehat{p}$, which is some compromise between the monopoly prices of the various firms, i.e., $p_1^m \leq \widehat{p} \leq p_n^m$. All firms adopt $\widehat{p}$ and consumers randomize between them.[7] Consequently, each firm i serves $1/n$ of the market and its profit in

[5] By revealed preferences, the fact that $y_i(\cdot)$ has a unique maximizer implies that $Q(p_i^m)(p_i^m - c_i) > Q(p_j^m)(p_j^m - c_i)$, and $Q(p_j^m)(p_j^m - c_j) > Q(p_i^m)(p_i^m - c_j)$. Summing up the two inequalities and simplifying, yields $Q(p_i^m)(c_j - c_i) > Q(p_j^m)(c_j - c_i)$. Assuming without loss of generality that $j > i$, and noting that $Q'(\cdot) < 0$, it follows that $p_j^m > p_i^m$.

[6] For example, the case of a linear demand function $Q(p) = a - bp$ and all c_j's at equal distance (i.e., $c_{j+1} - c_j = \vartheta > 0$ for all $j = 1, \ldots, n-1$) satisfies Assumption 2, since then $y_1(c_2) = Q(c_2)\vartheta > Q(c_j)\vartheta = y_1(c_j)/(j-1)$ for all $j = 3, \ldots, n$.

[7] That is, we study "pure" price fixing. A more elaborate collusive scheme might also involve market division in which case the market shares need not be equal. Such a scheme, however, will be in general much harder to enforce and easier for antitrust authorities to detect.

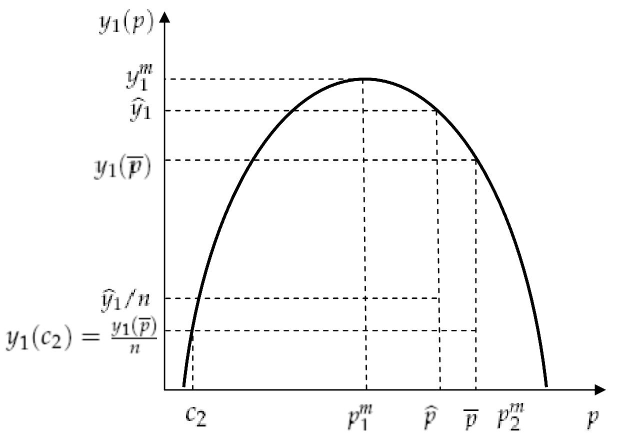
every period is $\widehat{y}_i/n$, where

$$\widehat{y}_i \equiv Q(\widehat{p})(\widehat{p} - c_i), \qquad i = 1, \ldots, n. \tag{4.2}$$

Although $\widehat{p}$ can exceed firm 1's monopoly price, p_1^m, it cannot exceed it by too much. To see why, note that firm 1 can always ensure itself a profit of $y_1(c_2)$ by setting a price slightly below c_2 and capturing the entire market. Hence, to ensure that firm 1 has an incentive to collude at $\widehat{p}$, it must be the case that $\widehat{y}_1/n \geq y_1(c_2)$. Since by Assumption 2, $c_2 < p_1^m \leq \widehat{p}$, it follows that $\widehat{p}$ is bounded from above by $\overline{p}$, where $\overline{p}$ is implicitly defined by $y_1(\overline{p})/n = y_1(c_2)$ (see Figure 4.1). If this is not the case, i.e., if $\widehat{p} > \overline{p}$, then firm 1 would be better off deviating to c_2 and capturing the entire market than colluding at $\widehat{p}$. Before proceeding, we add the following assumption which is illustrated in Figure 4.1:

Assumption 3: $\overline{p} < p_2^m$, where $\overline{p}$ is implicitly defined by $y_1(\overline{p})/n = y_1(c_2)$.

Figure 4.1: Illustrating Assumption 3



Recalling that $p_1^m < p_2^m < \ldots < p_n^m$, Assumption 3 implies that $\overline{p} < p_i^m$ for all $i = 2, \ldots, n$. Since $\widehat{p} \leq \overline{p}$, it follows that $\widehat{p} < p_i^m$ for all $i = 2, \ldots, n$: the collusive price is below the monopoly prices of all firms but 1. This implies in turn that the optimal deviation for firm $i = 2, \ldots, n$ is to set a price slightly below $\widehat{p}$, while the optimal deviation for firm 1 is to set a price p_1^m. Following any deviation from the collusive scheme (including a deviation by firm 1), firm 1 charges a price slightly below c_2 forever after and captures the entire market.

Recall that we have assumed that firm 1 prefers to set a price of c_j and share the market with firms $i = 2, \ldots, j-1$ than set a price of c_{j+1} and share the market with firms $i = 2, \ldots, j$ (second part of Assumption 2). Recalling that on the equilibrium path, it must be the case that $\widehat{y}_1/n \geq y_1(c_2)$ (otherwise firm 1 does not wish to collude), this implies that firm 1 prefers to collude with all $n-1$ rivals at $\widehat{p}$ than collude with only j firms by setting a price just below c_{j+1}.

We assume that the pricing decisions of each firm are effectively made by its controller (i.e., a controlling shareholder) whose ownership stake is γ_{ii}. We are now interested in finding conditions that will ensure that in a subgame perfect equilibrium of the infinitely repeated game, every controller will set $\widehat{p}$ in every period.

Using δ to denote the intertemporal discount factor, the condition that ensures that the controller of firm $i = 2, \ldots, n$ does not wish to deviate from the collusive scheme is given by

$$\gamma_{ii}\frac{\widehat{y}_i}{n(1-\delta)} \geq \gamma_{ii}\widehat{y}_i, \qquad i = 2, \ldots, n. \tag{4.3}$$

The left-hand side of (4.3) is the infinite discounted payoff of firm i's controller which consists of his share in firm i's collusive profit. The right-hand side of (4.3) is the controller's share in the one-time profit that firm i earns in the period in which it undercuts its rivals slightly and captures the entire market. Condition (4.3) can be rewritten as

$$\delta \geq \widehat{\delta} \equiv 1 - \frac{1}{n}.$$

That is, the controllers of firms $2, \ldots, n$ have an incentive to participate in the collusive scheme provided that they are sufficiently patient. This condition is identical to the well-known condition for tacit collusion in the context of an infinitely repeated Bertrand model with n identical firms (see e.g., Tirole, 1988, Ch. 6.3.2.1).[8]

As for firm 1, then its controller does not wish to deviate from the collusive scheme provided that

[8] To see how realistic this condition is, one can use the identity $\delta = 1/(1+r)$, where r is an interest rate. Then $\delta > 1 - 1/n$ is equivalent to $r < 1/(n-1)$. Thus, for $n = 10$ collusive scheme is sustainable if $r < 11.1\%$.

$$\gamma_{11}\frac{\widehat{y}_1}{n(1-\delta)} \geq \gamma_{11}\left(y_1^m + \frac{\delta y_1(c_2)}{1-\delta}\right), \tag{4.4}$$

where y_1^m is the one-time profit of firm 1 in the period in which it deviates and captures the entire market while charging p_1^m, and $y_1(c_2)$ is the per-period profit of firm 1 in all subsequent periods. Condition (4.4) can be rewritten as

$$\delta \geq \widehat{\delta}_1(\widehat{p}) \equiv \frac{y_1^m - \widehat{y}_1/n}{y_1^m - y_1(c_2)}. \tag{4.5}$$

Note that

$$\widehat{\delta}_1(\widehat{p}) > \frac{y_1^m - \widehat{y}_1/n}{y_1^m} \geq 1 - \frac{1}{n} \equiv \widehat{\delta},$$

where the weak inequality follows because $y_1^m \geq \widehat{y}_1$. Since $\widehat{\delta}_1(\widehat{p}) > \widehat{\delta}$, it is clear that if firm 1 wishes to collude then all other firms surely wish to collude. That is, firm 1 is the *maverick firm* in the industry, i.e., the firm with the strongest incentive to deviate from a collusive agreement. Hence, (4.5) is a necessary and sufficient condition for the collusive scheme led by firm 1 to be sustained as a subgame perfect equilibrium of the infinitely repeated game. Moreover, since $\widehat{p} \geq p_1^m$, it follows that $\widehat{y}_1$ increases as $\widehat{p}$ is lowered towards p_1^m. As a result, firm 1's controller would prefer to set $\widehat{p} = p_1^m$ and thereby maximize his infinite discounted stream of collusive profits while relaxing constraint (4.5). Hence,

Theorem 4.1. *Absent PCO by firms, firm* 1 *is the industry maverick and its controller would like to set the collusive price equal to p_1^m. Collusion at p_1^m can be sustained as a subgame perfect equilibrium of the infinitely repeated game provided that $\delta \geq \widehat{\delta}_1(p_1^m)$.*

4.3 Tacit collusion with unilateral partial ownership by firm 1

In this section we will only examine the competitive effects of unilateral partial ownership (PO) investments by firm 1 in rival firms. The competitive effects of multilateral PCO arrangements are considered in Section 4.4. We will now use $\widehat{\delta}_1(p_1^m)$ (the critical discount factor above which the collusive scheme characterized in the previous section can be sustained) as our measure of the ease of collusion;

accordingly, we will say that PCO facilitates tacit collusion if it lowers $\widehat{\delta}_1(p_1^m)$, and will say that PCO hinders tacit collusion if it raises $\widehat{\delta}_1(p_1^m)$.[9]

Specifically, assume that firm 1 invests in rivals and let $w_{12}, \ldots, w_{1n}$ be its ownership stakes in firms $2, \ldots, n$.[10] Since the collusive profit of each firm i is $\widehat{y}_i/n$, it follows that firm 1's infinite discounted stream of profits under collusion is

$$\frac{\widehat{y}_1 + \sum_{i\neq 1} w_{1i}\widehat{y}_i}{n(1-\delta)}.$$

If firm 1's controller deviates from the collusive scheme, all rivals make zero profits, so firm 1's payoff is

$$y_1^m + \frac{\delta y_1(c_2)}{1-\delta},$$

exactly as in the absence of PO. Consequently, the condition that ensures that firm 1's controller does not wish to deviate from the collusive scheme is now given by

$$\gamma_{11}\left(\frac{\widehat{y}_1 + \sum_{i\neq 1} w_{1i}\widehat{y}_i}{n(1-\delta)}\right) \geq \gamma_{11}\left(y_1^m + \frac{\delta y_1(c_2)}{1-\delta}\right), \tag{4.6}$$

or

$$\delta \geq \widehat{\delta}_1^{po}(\widehat{p}) \equiv \frac{y_1^m - \left(\widehat{y}_1 + \sum_{i\neq 1} w_{1i}\widehat{y}_i\right)/n}{y_1^m - y_1(c_2)}. \tag{4.7}$$

Notice that $\widehat{\delta}_1^{po}(\widehat{p})$ is decreasing with each w_{1i}: the larger the stakes of firm 1 in rival firms, the stronger is firm 1's incentive to collude. The reason for this is that the collusive payoff of firm 1 increases when it invests in rivals, while its payoff under deviation is unaffected because rival firms make a profit of 0 in the period in which firm 1 deviates as well as in all future periods. Clearly, firm 1 does not have an incentive to invest in rivals up to the point where $\widehat{\delta}_1^{po}(\widehat{p})$ drops below $\widehat{\delta}$ since then it might very well happen that firm 1 is no longer the industry maverick (and thus firm 1's stakes in rivals no longer facilitate tacit collusion). Hence, we shall assume in the rest of this section that firm 1 remains an industry maverick even when it holds PO stakes in rivals. A sufficient condition for that to be the case is that firm 1's profit when the collusive agreement breaks down, $y_1(c_2)$, is at

[9] Of course, the infinitely repeated game admits multiple subgame perfect equilibria. We restrict attention to the most collusive equilibrium and focus on $\widehat{\delta}_1(p_1^m)$ because this is a standard way to capture the notion of "ease of collusion".

[10] Hence, for PCO holdings we adopt the same notation as in Chapter 3.

least as large as firm 1's average stake in the profits of rival firms under collusion, $\left(\sum_{i\neq 1} w_{1i}\widehat{y}_i\right)/(n-1)$, because then,

$$\widehat{\delta}_1^{po}(\widehat{p}) \geq \frac{y_1^m - \left(y_1^m + \sum_{i\neq 1} w_{1i}\widehat{y}_i\right)/n}{y_1^m - y_1(c_2)} \geq \frac{y_1^m - \left(y_1^m + (n-1)y_1(c_2)\right)/n}{y_1^m - y_1(c_2)} = 1 - \frac{1}{n} \equiv \widehat{\delta},$$

where the first weak inequality follows because $y_1^m \geq \widehat{y}_1$.

Assuming then that firm 1 is the industry maverick, firm 1's controller selects $\widehat{p}$ to maximize the infinite discounted sum of firm 1's collusive profits given by the left-hand side of (4.6) subject to (4.7). The following result follows (see Appendix 4.A).

Theorem 4.2. *Suppose that firm* 1 *invests in rivals but still remains the industry maverick. Using* $\widehat{p}^*$ *to denote the optimal collusive price from firm* 1*'s perspective, the following holds:*

(i) $\widehat{p}^*$ *is increasing with each* w_{1i} *and is above firm* 1*'s monopoly price:* $\widehat{p}^* > p_1^m$.

(ii) $\widehat{\delta}_1^{po}(\widehat{p}^*)$ *is decreasing with each* w_{1i} *and is below* $\widehat{\delta}_1(p_1^m)$ *– the critical discount factor above which collusion can be sustained absent PO.*

(iii) PO in an efficient rival raises $\widehat{p}^*$ *by less and lowers* $\widehat{\delta}_1^{po}(\widehat{p}^*)$ *by more than a similar PO in a less efficient rival.*

Theorem 4.2 implies that investments by firm 1 in rivals do not only facilitate tacit collusion by lowering the critical discount factor above which tacit collusion can be sustained, but also lead to a higher collusive price. The latter result arises because, due to its investment in rivals, firm 1 is interested in maximizing a weighted average of its own profit and the profits of the firms it invests in. The higher firm 1's investments in rivals, the higher the weight that firm 1's assigns to the rivals' profits in its objective function. Maximizing the rivals' profits requires a higher monopoly price than the monopoly price from firm 1's own perspective.

The theorem suggests that to the extent that firm 1 invests in rivals, it always prefers to invest in its most efficient rival first, since this leads to a collusive price that is closer to firm 1's monopoly price and also expands the range of discount factors above which collusion can be sustained. This also implies that firm 1 will have an incentive to minimize its investments in rivals subject to being able to facilitate tacit collusion. If investment in the most efficient rival is not sufficient to sustain collusion, then firm 1 invests in the next efficient rival.

4.4 Tacit collusion with multilateral PCO

In this section we turn to the case where all firms potentially invest in rivals. To this end, let w_{ij} be firm i's partial cross ownership stake in firm j and define the following $n \times n$ PCO matrix:

$$\mathbf{W} = \begin{pmatrix} 0 & w_{12} & \cdots & w_{1n} \\ w_{21} & 0 & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & 0 \end{pmatrix}.$$

Row i in the matrix $\mathbf{W}$ specifies the stakes that firm i has in all rival firms, while column j in the matrix $\mathbf{W}$ specifies the stakes that rival firms hold in firm j. Since apart from rival firms each firm is also held by its controller and possibly by outside stakeholders, the sum of each column of $\mathbf{W}$ is strictly less than 1. It is also assumed that a firm cannot own shares in itself, i.e., all diagonal terms in the matrix $\mathbf{W}$ are equal to 0.

4.4.1 The accounting profits under PCO

When firms hold stakes in each other, the profit of each firm potentially depends on the profits of *all* other firms in the industry. For instance, firm 1 may get a share w_{12} of firm 2's profit while at the same time firm 2 owns a share w_{23} in the profit of firm 3, which in turn holds a share w_{31} in the profit of firm 1. Hence, we potentially have a multiplier effect that drives a wedge between the direct profit of each firm and its overall profit that also includes the firm's share in the profits of rival firms. Therefore, before characterizing the conditions that ensure that a collusive scheme can be supported as a subgame perfect equilibrium of an infinitely repeated game, we first need to express the profit of each firm under collusion and in the case of a deviation from collusion.

Under collusion, all firms charge the same price, $\widehat{p}$. Since the products are homogeneous, consumers choose which firm to buy from at random, so the market share of each firm is $1/n$. Hence, the direct profit of each firm i (excluding its share in the profits of rivals) is $\widehat{y}_i/n$, where $\widehat{y}_i$ is given by equation (4.2). Since by assumption, $c_1 < c_2 < \ldots < c_n$, we have $\widehat{y}_1 > \widehat{y}_2 > \ldots > \widehat{y}_n$: the direct profit of firm 1 exceeds that of firm 2, which in turn exceeds that of firm 3, and so on. In addition to its direct profit, each firm i also gets a share in its rivals' profits due to its cross ownership stake in these firms. Hence, the (column) vector of collusive profits, $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)'$, is given by the solution to the following system of n equations:

$$\boldsymbol{\pi} = (1/n)\mathbf{y} + \mathbf{W}\boldsymbol{\pi}, \tag{4.8}$$

where $\mathbf{y} \equiv (\widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_n)'$.

Next, we consider what happens when the controller of firm i deviates from the collusive scheme. If $i \neq 1$, then firm i will slightly undercut $\widehat{p}$, so the direct profit of all firms but i will be 0, while the direct profit of firm i will be arbitrarily close to $\widehat{y}_i$. Consequently, the vector of current profits, $\boldsymbol{\pi}^{d_i} = (\pi_1^{d_i}, \pi_2^{d_i}, \ldots, \pi_n^{d_i})'$, is defined by the solution to the following system:

$$\boldsymbol{\pi}^{d_i} = \mathbf{y}^{d_i} + \mathbf{W}\boldsymbol{\pi}^{d_i}, \qquad \text{for all} \quad i = 2, ..., n, \tag{4.9}$$

where $\mathbf{y}^{d_i} \equiv (0, \ldots, 0, \widehat{y}_i, 0, \ldots, 0)'$ is an n-dimensional vector with $\widehat{y}_i$ in the i-th entry and 0′s elsewhere.

If the deviant is firm 1 ($i = 1$), then it charges p_1^m and its profit in the current period will be y_1^m (see equation (4.1)). The current direct profits of all other firms will be 0. Hence, the vector of profits in period in which firm 1′s controller deviates, $\boldsymbol{\pi}^{d_1} = (\pi_1^{d_1}, \pi_2^{d_1}, \ldots, \pi_n^{d_1})'$, is defined by the solution to the system

$$\boldsymbol{\pi}^{d_1} = \mathbf{y}^{d_1} + \mathbf{W}\boldsymbol{\pi}^{d_1}, \tag{4.10}$$

where $\mathbf{y}^{d_1} \equiv (y_1^m, 0, \ldots, 0)'$ is an n-dimensional vector with y_1^m in the first entry and 0′s elsewhere.

Once the collusive agreement breaks down, firm 1 will charge a price slightly below c_2 in every period and will capture the entire market. Hence the vector of profits following a breakdown of the collusive agreement, $\boldsymbol{\pi}^f = (\pi_1^f, \pi_2^f, \ldots, \pi_n^f)'$, is defined by the solution to the following system:

$$\boldsymbol{\pi}^f = \mathbf{y}^f + \mathbf{W}\boldsymbol{\pi}^f, \tag{4.11}$$

where $\mathbf{y}^f \equiv (y_1(c_2), 0, \ldots, 0)'$ is an n-dimensional vector with $y_1(c_2)$ in the first entry and 0′s elsewhere.

To solve systems (4.8)-(4.11), note that since the PCO matrix, $\mathbf{W}$, is nonnegative and the sum of each of its columns is strictly less than 1, systems (4.8)-(4.11) are Leontief systems and have unique nonnegative solutions (see Sydsæter et al., 2005, Chapter 22; see also discussions after equation (3.4) in Chapter 3) defined by

$$\begin{aligned}
&\boldsymbol{\pi}(\widehat{p};\mathbf{W}) = (1/n)\mathbf{L}\mathbf{y},\\
&\boldsymbol{\pi}^{d_i}(\widehat{p};\mathbf{W}) = \mathbf{L}\mathbf{y}^{d_i}, \qquad i = 1,\ldots,n,\\
&\boldsymbol{\pi}^{f}(c_2;\mathbf{W}) = \mathbf{L}\mathbf{y}^{f},
\end{aligned} \tag{4.12}$$

where $\mathbf{L} \equiv (\mathbf{I}-\mathbf{W})^{-1}$ is the inverse Leontief matrix that specifies the aggregate imputed shares of "real" equityholders (i.e., outside equityholders that are not part of the n firms) in the accounting profits of the n firms.[11] That is, the ij-th entry in the matrix $\mathbf{L}$, denoted l_{ij}, is the aggregate imputed share that the real equityholders of firm i have in the accounting profit of firm j. Equation (4.12) implies that the accounting collusive profit of firm $i \neq 1$ is $\pi_i(\widehat{p};\mathbf{W}) = \left(\sum_{j=1}^{n} l_{ij}\widehat{y}_j\right)/n$, its one-time profit in the period in which it deviates from the collusive scheme is $\pi_i^{d_i}(\widehat{p};\mathbf{W}) = l_{ii}\widehat{y}_i$, and its profit in any subsequent period is $\pi_i^{f}(c_2;\mathbf{W}) = l_{i1}y_1(c_2)$. The corresponding accounting profits of firm 1 are $\pi_1(\widehat{p};\mathbf{W}) = \left(\sum_{j=1}^{n} l_{1j}\widehat{y}_j\right)/n$, $\pi_1^{d_1}(\widehat{p};\mathbf{W}) = l_{11}y_1^{m}$, and $\pi_1^{f}(c_2;\mathbf{W}) = l_{11}y_1(c_2)$.

Given the important role that the aggregate imputed shares matrix, $\mathbf{L}$, plays in our analysis, we state the following result whose proof appears in Gilo et al. (2006).

Lemma 4.1. *The aggregate imputed shares matrix* $\mathbf{L}$ *has the following properties:*

(i) $l_{ii} \geq 1$ *for all i, and* $0 \leq l_{ij} < l_{ii}$ *for all i and all* $j \neq i$.

(ii) Let i and j be two distinct firms. Then, $l_{ij} = 0$ *if and only if firm i does not have a direct and an indirect stake in firm j.*[12]

(iii) $l_{ii} > 1$ *if and only if firm i has a direct or an indirect stake in some firm j which in turn has a direct or an indirect stake in firm i (i.e.,* $l_{ij} > 0$ *and* $l_{ji} > 0$*).*

(iv) $\widehat{l}_i \equiv \sum_{j=1}^{n}\left(1-\sum_{k\neq j} w_{kj}\right)l_{ji} = 1$ *for all i.*

To interpret Lemma 4.1, recall that l_{ij} is the aggregate imputed share that the real equityholders of firm i have in the accounting profit of firm $j \neq i$ through the direct or indirect cross ownership of firm i in firm j and l_{ii} is the aggregate imputed share that the real equityholders of firm i have in the accounting profit of their own firm. Part (i) of Lemma 4.1 says that a 1% stake in firm i may give the real equityholders of firm i more than a 1% imputed share in the firm's profit (i.e., $l_{ii} \geq 1$), and the real equityholders always have larger imputed shares in their own firm's profit than in the profits of rival firms (i.e., $l_{ij} < l_{ii}$ for all i and all $j \neq i$). Part (ii) of the

[11] The terminology "imputed shares" is due to Dorofeenko et al. (2008).

[12] We will say that firm i has no indirect stake in firm j, if it has no stake in a firm that has a stake in firm j, and has no stake in a firm that has a stake in a firm that has a stake in firm j and so on.

lemma says that the real equityholders of firm i will get a share in the profit of a rival firm j if and only if firm i has a direct and/or indirect stake in firm j. Part (iii) of the lemma says that if the real equityholders of firm i have a direct or an indirect stake in some rival firm j and this firm's real equityholders in turn have a direct or an indirect stake in firm i, then the aggregate imputed share that a real equityholder of firm i will have in firm i will exceed 1. In other words, a 1% stake in firm i will give a "real" equityholder of firm i more than a 1% share in the firm's profit. The reason for this surprising property is that multilateral cross ownership arrangements create a multiplier effect that results in an overstatement of the firms' cash flows.[13] Part (iv) of the lemma ensures, however, that the aggregate *effective* shares of "real" equityholders in each firm i sum up to 1. Hence, while the accounting profits of firms will overstate the total cash flows, the aggregate payoff of all real equityholders will sum up exactly to the total cash flows.

4.4.2 Collusion with multilateral PCO

Given the accounting profits of the n firms under collusion and following a deviation from the fully collusive scheme, the condition that ensures that the collusive outcome can be sustained as a subgame perfect equilibrium is

$$\frac{\gamma_{ii}\pi_i(\widehat{p};\mathbf{W})}{1-\delta} \geq \gamma_{ii}\left(\pi_i^{d_i}(\widehat{p};\mathbf{W}) + \frac{\delta\pi_i^f(c_2;\mathbf{W})}{1-\delta}\right), \qquad i = 1,\ldots,n. \tag{4.13}$$

The left-hand side of (4.13) is the infinite discounted payoff of firm i's controller under collusion, consisting of the controller's share in firm i's collusive profit. The right-hand side of (4.13) is the controller's share in the profit that firm i earns when it undercuts its rivals slightly (the one-time profit $\pi_i^{d_i}(\widehat{p};\mathbf{W})$ in the period in which firm i deviates and $\pi_i^f(c_2;\mathbf{W})$ in all subsequent periods). If (4.13) holds, no controller wishes to unilaterally deviate from the fully collusive scheme.

Recalling that $\pi_i(\widehat{p};\mathbf{W}) = \left(\sum_{j=1}^n l_{ij}\widehat{y}_j\right)/n$ and $\pi_i^f(c_2;\mathbf{W}) = l_{i1}y_1(c_2)$ for all i, $\pi_1^{d_1}(\widehat{p};\mathbf{W}) = l_{11}y_1^m$, and $\pi_i^{d_i}(\widehat{p};\mathbf{W}) = l_{ii}\widehat{y}_i$ for all $i \neq 1$, and using $z_{ij} \equiv l_{ij}/l_{ii}$ to denote the *relative* imputed share that the equityholders of firm i have in firm j (relative to their imputed share in their "own" firm i), the necessary condition (4.13) for collusion can be rewritten as

[13] See Dietzenbacher et al. (2000), Dorofeenko et al. (2008), and Chapter 3 of this thesis for additional discussion of this effect of PCO.

$$\delta\left(y_1^m - y_1(c_2)\right) \geq y_1^m - \frac{1}{n}\sum_{j=1}^{n} z_{1j}\widehat{y}_j, \quad \text{and} \tag{4.14}$$

$$\delta\left(\widehat{y}_i - z_{i1}y_1(c_2)\right) \geq \widehat{y}_i - \frac{1}{n}\sum_{j=1}^{n} z_{ij}\widehat{y}_j, \qquad i = 2, \ldots, n. \tag{4.15}$$

Notice that by definition, $y_1^m > \widehat{y}_1/n \geq y_1(c_2)$ (see Figure 4.1) and recall that $\widehat{y}_1 > \widehat{y}_2 > \ldots > \widehat{y}_n$. Since, part (i) of Lemma 4.1 implies that $z_{ii} = 1$ for all i and $z_{ij} < 1$ for all i and all $j \neq i$, it follows that both sides of (4.14) are positive. Moreover, $\widehat{y}_1/n \geq y_1(c_2)$ implies that $\left(\sum_{j=1}^{n} z_{ij}\widehat{y}_j\right)/n = z_{i1}\widehat{y}_1/n + \left(\sum_{j\neq 1}^{n} z_{ij}\widehat{y}_j\right)/n \geq z_{i1}y_1(c_2)$, with strict inequality when $z_{ij} > 0$ for some j. Hence, $\widehat{y}_i - z_{i1}y_1(c_2) \geq \widehat{y}_i - \left(\sum_{j=1}^{n} z_{ij}\widehat{y}_j\right)/n$. Before proceeding we impose the following assumption on $\widehat{y}_i$:

Assumption 4: $\widehat{y}_i > \left(\sum_{j=1}^{n} z_{ij}\widehat{y}_j\right)/n$ for all $i \neq 1$.

Assumption 4 implies that each firm $i \neq 1$ earns more money when it unilaterally deviates from a collusive scheme than it earns under collusion. This assumption ensures that both sides of (4.15) are positive. Notice that in the presence of PCO this need not be the case because under collusion, firm i gets a share in the profits of its rivals, while under deviation it does not. If firm i is relatively inefficient, then its profit under collusion may exceed its profit when it deviates even though in the latter case the firm serves the entire market while under collusion it serves only $1/n$ of the market (but it gets a share in the profits of its rivals).

With Assumption 4 in place, (4.14) and (4.15) imply the following result.

Lemma 4.2. *Let $z_{ij} \equiv l_{ij}/l_{ii}$ be the relative imputed share that the equityholders of firm i have in firm j (relative to their imputed share in their "own" firm i). Then, the fully collusive outcome can be sustained as a subgame perfect equilibrium of the infinitely repeated game provided that*

$$\delta \geq \widehat{\delta}^{po}(\mathbf{W}) \equiv \max\left\{\widehat{\delta}_1(\mathbf{W}), \ldots, \widehat{\delta}_n(\mathbf{W})\right\},$$

where

$$\widehat{\delta}_1(\mathbf{W}) \equiv \frac{y_1^m - \frac{1}{n}\sum_{j=1}^{n} z_{1j}\widehat{y}_j}{y_1^m - y_1(c_2)}, \tag{4.16}$$

and

$$\widehat{\delta}_i(\mathbf{W}) \equiv \frac{\widehat{y}_i - \frac{1}{n}\sum_{j=1}^{n} z_{ij}\widehat{y}_j}{\widehat{y}_i - z_{i1}y_1(c_2)}, \qquad i = 2, ..., n. \tag{4.17}$$

It is easy to see that the incentives of firms to collude depend on cross ownership only through the matrix $\mathbf{Z}$ whose characteristic element is z_{ij}. In what follows we shall therefore examine how changes in cross ownership affect the matrix $\mathbf{Z}$ and consequently the critical discount factors above which firms wish to collude.

4.4.3 A firm increases its stake in a rival firm by buying shares from an outsider or from the rival's controller

Now, suppose that firm r increases its stake in firm s, w_{rs} by $\omega > 0$. The resulting new PCO matrix is $\mathbf{W}^{\omega}$; it differs from the original PCO matrix only in that its rs-th entry is $w_{rs} + \omega$ rather than w_{rs}. Our main question is whether $\widehat{\delta}_i(\mathbf{W}^{\omega})$ is higher or lower than $\widehat{\delta}_i(\mathbf{W})$.

To address this question, note from equation (4.16) that $\partial\widehat{\delta}_1(\mathbf{W})/\partial z_{1j} < 0$ for all j, and note from equation (4.17) that $\partial\widehat{\delta}_i(\mathbf{W})/\partial z_{ij} < 0$ for all $i \neq 1$ and all $j \neq 1$. Moreover, from equation (4.17) it follows that

$$\begin{aligned}
\frac{\partial\widehat{\delta}_i(\mathbf{W})}{\partial z_{i1}} &= \frac{-\frac{\widehat{y}_1}{n}\left(\widehat{y}_i - z_{i1}y_1(c_2)\right) + y_1(c_2)\left(\widehat{y}_i - \frac{1}{n}\sum_{j=1}^{n} z_{ij}\widehat{y}_j\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)^2} \\
&= \frac{-\widehat{y}_i\left(\frac{\widehat{y}_1}{n} - y_1(c_2)\right) + \frac{y_1(c_2)}{n}\left(z_{i1}\widehat{y}_1 - \sum_{j=1}^{n} z_{ij}\widehat{y}_j\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)^2} \\
&= -\frac{\widehat{y}_i\left(\frac{\widehat{y}_1}{n} - y_1(c_2)\right) + \frac{y_1(c_2)}{n}\sum_{j\neq 1}^{n} z_{ij}\widehat{y}_j}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)^2} < 0,
\end{aligned}$$

where the inequality follows because by assumption $\widehat{y}_1/n \geq y_1(c_2)$ (otherwise firm 1 has no incentive to collude) and $\sum_{j\neq 1}^{n} z_{ij}\widehat{y}_j = z_{ii}\widehat{y}_i + \sum_{j\neq 1,i}^{n} z_{ij}\widehat{y}_j \geq \widehat{y}_i > 0$ (recall that $z_{ii} = 1$). Hence,

Lemma 4.3. $\partial\widehat{\delta}_i(\mathbf{W})/\partial z_{ij} < 0$ *for all i and all j: the critical discount factor above which firm i wishes to collude is a strictly decreasing function of each of firm i's relative imputed shares in rival firms.*

Lemma 4.3 implies that in order to determine the effect of the increase in firm r's stake in firm s by ω on firm i's incentive to collude, we only need to know how it affects the i'th row in matrix $\mathbf{Z}$, which specifies the relative imputed shares of firm

i in all rival firms. To this end, note from Lemma A1 in Gilo et al. (2006) that

$$z_{ij}^{\omega} \equiv \frac{l_{ij}^{\omega}}{l_{ii}^{\omega}} = \frac{l_{ij} + \varepsilon_i l_{sj}}{l_{ii} + \varepsilon_i l_{si}}, \qquad \varepsilon_i = \frac{\omega l_{ir}}{1 - \omega l_{sr}} \geq 0, \tag{4.18}$$

where l_{ij}^{ω} is the typical element of the new matrix of aggregate imputed shares $\mathbf{L}^{\omega} = (\mathbf{I} - \mathbf{W}^{\omega})^{-1}$.

Straightforward differentiation yields

$$\frac{\partial z_{ij}^{\omega}}{\partial \omega} = \frac{l_{ii} l_{sj} - l_{si} l_{ij}}{(l_{ii} + \varepsilon_i l_{si})^2} \times \frac{l_{ir}}{(1 - \omega l_{sr})^2}. \tag{4.19}$$

Using this equation we are able to prove the following result, which generalizes Theorem 1 in Gilo et al. (2006) to the case of asymmetric firms (see Appendix 4.A).

Theorem 4.3. *Starting with a PCO matrix* $\mathbf{W}$, *suppose that firm* r *increases its stake in firm* s *by some* $\omega > 0$, *so that the new PCO matrix* $\mathbf{W}^{\omega}$ *differs from* $\mathbf{W}$ *only with respect to the rs-th entry which is increased by* ω. *Then*

(i) $\widehat{\delta}_s(\mathbf{W}^{\omega}) = \widehat{\delta}_s(\mathbf{W})$,

(ii) $\widehat{\delta}_i(\mathbf{W}^{\omega}) = \widehat{\delta}_i(\mathbf{W})$ *if* $l_{ir} = 0$ *(firm i has no direct and indirect stake in the acquiring firm r), and*

(iii) $\widehat{\delta}_i(\mathbf{W}^{\omega}) < \widehat{\delta}_i(\mathbf{W})$ *otherwise, i.e., for all* $i \neq s$ *and* $l_{ir} > 0$.

Theorem 4.3 shows that an increase in the stake of firm r in firm s never hinders collusion. In fact, the theorem shows that there are only two special cases in which collusion is not strictly facilitated: one case arises when the maverick firm is the target firm (firm s). Collusion is not facilitated in this case because the incentive of the target firm to collude is not affected by the fact that firm r has increased its stake in firm s. Intuitively, when firm r increases its stake in firm s, the relative imputed shares of firm s do not change because the imputed shares of firm s in all j, l_{sj}, must change by the same constant proportion.[14] This is because l_{sj} will change in this case if and only if the target firm s has a direct or an indirect stake in the acquiring firm r (i.e., when $l_{sr} > 0$), thus a change in l_{sj} for any j is only due to the link of firm s to firm r. The second special case arises when the maverick firm has no direct or indirect stake in the acquiring firm (firm r). Then, the increase in w_{rs} does not affect the relative imputed shares of the maverick firm in any way and hence its incentive to collude are not affected either. In all other cases collusion is strictly facilitated. We summarize this conclusions in the next corollary.

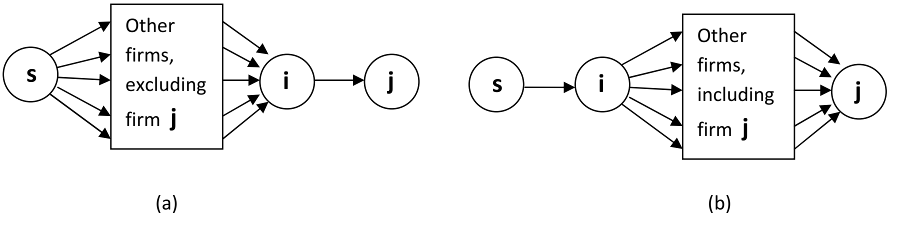
[14] In fact, from Lemma A1 in Gilo et al. (2006) it follows that $l_{sj}^{\omega} = [1/(1 - \omega l_{sr})] l_{sj}$ for all j.

Corollary 4.1. *An increase in firm r's cross ownership stake in firm s never hinders tacit collusion and surely facilitates it if and only if (i) each industry maverick has a direct or an indirect stake in firm r, and (ii) firm s is not an industry maverick.*

The proof of Theorem 4.3 provides a simpler proof for Theorem 1 in Gilo et al. (2006). To see why, note that in the special case where all firms have the same marginal cost (the case considered in Gilo et al., 2006), $\widehat{y}_1 = \ldots = \widehat{y}_n = y_1^m$ and $y_1(c_2) = 0$. Hence, equations (4.16) and (4.17) imply that $\partial\widehat{\delta}_i(\mathbf{W})/\partial z_{ij} < 0$ for all i and all j. Then, the proof of Theorem 4.3 implies immediately that $\widehat{\delta}_i(\mathbf{W}) \geq \widehat{\delta}_i(\mathbf{W}^{\omega})$ with strict equality if and only if $i = s$ or $l_{ir} = 0$.

To get some intuition for Theorem 4.3, note that the key step in the proof is the observation that $l_{ii}l_{sj} - l_{si}l_{ij} \geq 0$ for all j with strict inequality for $j = s$. In order to find an interpretation for $l_{ii}l_{sj} - l_{si}l_{ij} \geq 0$, consider the following two cases of firms' direct shareholdings as depicted in Figure 4.2.

Figure 4.2: PCO structure resulting in $l_{sj} = l_{si}l_{ij}/l_{ii}$



Note: The arrows are directed from a firm-holder of direct stakes to owned firm(s). These links can be mutual.

Case (a) applies when only firm i has a direct stake in firm j, and case (b) applies when only firm s holds a direct stake in firm i that has a direct or an indirect stake in firm j. In both cases the *absence* of firm i would immediately imply that firm s has no shares in firm j. Hence, in such cases we say that firm s has a share (direct and/or indirect) in firm j only *due to* firm i.[15]

To quantify such "dependence" of firm s ownership in firm j on the presence of an intermediate firm i, let $\mathbf{W}^{-i}$ be the modified PCO matrix derived from $\mathbf{W}$ by setting its i-th row and i-th column to zero, and let $\mathbf{L}^{-i} = (\mathbf{I} - \mathbf{W}^{-i})^{-1}$ be the associated matrix of imputed shares. Theorem 1 in Zeng (2001) implies that the

[15] For the sake of simplicity, in Figure 4.2 we did not draw double-sided arrows as we are primary interested in the ownership paths that stem from firm s and end at firm j. Notice that the number of such paths are infinite if there are cycles present, and potentially all firms can participate in such indirect links infinite number of times.

sj-th element of $\mathbf{L}^{-i}$, l_{sj}^{-i}, is equal to

$$l_{sj}^{-i} = \frac{l_{ii}l_{sj} - l_{si}l_{ij}}{l_{ii}}, \qquad \text{for all } \ i \neq s. \tag{4.20}$$

Therefore, the part of firm s's imputed share in firm j which is due to (the presence of) firm i ($\neq s$) is equal to

$$l_{sj} - l_{sj}^{-i} = \frac{l_{si}l_{ij}}{l_{ii}}.$$

This also implies that if firm s has a stake in firm j *only* due to firm i (i.e., in the absence of firm i there is no direct and/or indirect share of firm s in firm j, $l_{sj}^{-i} = 0$) then it must be true that $l_{sj} = l_{si}l_{ij}/l_{ii}$, which holds for any $i \neq s$. This exactly corresponds to the cases depicted in Figure 4.2.

With respect to the intuition behind the inequality $l_{ii}l_{sj} - l_{si}l_{ij} \geq 0$, note that it is equivalent to $l_{sj} \geq l_{si}l_{ij}/l_{ii}$. Here, l_{sj} is the imputed share of firm s in firm j and $l_{si}l_{ij}/l_{ii}$ is the part of firm s's imputed share in firm j which is due to (the presence of) firm i. It is then intuitively clear that $l_{sj} \geq l_{si}l_{ij}/l_{ii}$, because $l_{si}l_{ij}/l_{ii}$ takes into account only part of the imputed share of firm s in firm j.

Example: To illustrate the case of $l_{sj} = l_{si}l_{ij}/l_{ii}$ and Theorem 4.3, we will now examine the following example. Consider an industry with 3 firms where the PCO matrix is

$$\mathbf{W} = \begin{pmatrix} 0 & \alpha & 0 \\ \beta & 0 & \beta \\ \eta & 0 & 0 \end{pmatrix}.$$

The associated matrix of imputed shares is given by

$$\mathbf{L} = (\mathbf{I} - \mathbf{W})^{-1} = \frac{1}{1 - \alpha\beta(1+\eta)} \begin{pmatrix} 1 & \alpha & \alpha\beta \\ \beta(1+\eta) & 1 & \beta \\ \eta & \alpha\eta & 1 - \alpha\beta \end{pmatrix}.$$

Note that without firm 2, firm 1 has no stake in firm 3, so $l_{13}^{-2} = 0$. Since the total stake of firm 1 in firm 3 due to (the presence of) firm 2 is given by $l_{13} - l_{13}^{-2} = l_{12}l_{23}/l_{22}$, it follows that $l_{13} = l_{12}l_{23}/l_{22}$, which can be verified to hold in the example. Likewise, firm 3 has a stake in firm 2 only through firm 1, so $l_{32}^{-1} = 0$; hence, $l_{32} = l_{31}l_{12}/l_{11}$, which again can be verified to hold in the example.

Now assume that firm 1 increases its stake in firm 2 by $\omega > 0$ (hence $r = 1$ and $s = 2$). Then simple algebra shows that

$$\mathbf{Z}^{\omega} - \mathbf{Z} = \begin{pmatrix} 0 & \omega & \beta\omega \\ 0 & 0 & 0 \\ \frac{\alpha\eta\omega}{F} & \frac{\eta\omega(1+\alpha^2-\alpha\beta)}{F} & 0 \end{pmatrix},$$

where $F \equiv (1-\alpha\beta)(1-\alpha(\beta+\omega)) > 0$. Note that the second row in the matrix contains zeros. Hence, z_{2j} for all j does not change, implying that the collusive incentive of the target firm (firm 2) is not affected by the increase in firm 1's stake in firm 2 (case (i) of Theorem 4.3). Moreover, if firm 3 does not have a stake in firm 1 (the acquirer), i.e., $\eta = 0$, then $l_{31} = 0$, and z_{3j} does not change as well for all j (case (ii) of Theorem 4.3). Finally, so long as $\eta > 0$ and $l_{i1} > 0$ for $i = 1, 3$ (i.e., $l_{ir} > 0$ for $i = 1, 3$), then the above result for $\mathbf{Z}^{\omega} - \mathbf{Z}$ shows that z_{12}, z_{13}, z_{31}, and z_{32} increase as case (iii) of Theorem 4.3 predicts.

4.4.4 A firm increases its stake in a rival firm by buying shares from another rival firm

Theorem 4.3 assumes implicitly that when firm r increases its stake in firm s, it buys additional shares from the outside investors or the controller of firm s. However, cases exist in which one firm buys shares in a rival firm from another rival. A case in point is a recent transaction in the global steel industry, where Luxemburg-based Arcelor has increased its stake in the Brazilian steelmaker CST from 18.6% to 27.95% by buying shares from Acesita which is also based in Brazil.[16] To examine the effect of such ownership transfers on the incentives to collude, suppose that firm r increases its stake in firm s by buying an ownership stake ϕ from firm k. The resulting PCO matrix $\mathbf{W}^{\phi}$ is obtained from the original PCO matrix $\mathbf{W}$ by increasing the rs-th entry in $\mathbf{W}$ by ϕ and lowering the ks-th entry by ϕ. Equation (2) in Zeng (2001) shows that in this case,

$$z_{ij}^{\phi} \equiv \frac{l_{ij}^{\phi}}{l_{ii}^{\phi}} = \frac{l_{ij} + \varepsilon_i^{\phi} l_{sj}}{l_{ii} + \varepsilon_i^{\phi} l_{si}}, \qquad \varepsilon_i^{\phi} \equiv \frac{\phi(l_{ir} - l_{ik})}{1 - \phi(l_{sr} - l_{sk})}. \tag{4.21}$$

Note that (4.21) is somewhat similar to the expression we used earlier for z_{ij}^{ω} in

[16] Acesita sold its entire 18.7% stake in CST to Arcelor and to CVRD which is a large Brazilian miner of iron and ore. In addition to its stake in CST, Arcelor also owns stakes in Acesita and in Belgo-Mineira, which is another Brazilian steelmaker (see "CVRD, Arcelor Team up for CST", *The Daily Deal*, December 28, 2002, M&A; "Minister: Steel Duties Still Under Study - Brazil", *Business News Americas*, April 8, 2002.)

(4.18). The main difference is that while $\varepsilon_i \geq 0$, now $\varepsilon_i^{\phi} \gtreqless 0$ as $l_{ir} \gtreqless l_{ik}$.

Using (4.21) yields

$$\frac{\partial z_{ij}^{\phi}}{\partial \phi} = \frac{l_{ii}l_{sj} - l_{si}l_{ij}}{\left(l_{ii} + \varepsilon_i^{\phi} l_{si}\right)^2} \times \frac{l_{ir} - l_{ik}}{\left(1 - \phi(l_{sr} - l_{sk})\right)^2}. \tag{4.22}$$

Repeating the same steps as in Theorem 4.3, we obtain the following result.

Theorem 4.4. *Starting with a PCO matrix* $\mathbf{W}$*, suppose that firm r buys a stake* ϕ *in firm s from firm k, so that the new PCO matrix* $\mathbf{W}^{\phi}$ *is obtained from* $\mathbf{W}$ *by increasing the rs-th entry by* ϕ *and decreasing the ks-th by* ϕ*. Then,*

(i) $\widehat{\delta}_s(\mathbf{W}^{\phi}) = \widehat{\delta}_s(\mathbf{W})$*,*

(ii) $\widehat{\delta}_i(\mathbf{W}^{\phi}) = \widehat{\delta}_i(\mathbf{W})$ *if* $l_{ir} = l_{ik}$ *(firm i has the same imputed share in firms r and k), and*

(iii) $\widehat{\delta}_i(\mathbf{W}^{\phi}) \lesseqgtr \widehat{\delta}_i(\mathbf{W})$ *for all* $i \neq s$ *as* $l_{ir} \gtreqless l_{ik}$*.*

Theorem 4.4 implies the following result:

Corollary 4.2. *A transfer of partial cross ownership in firm s from firm k to firm r does not affect tacit collusion if the industry maverick is firm s or if, at the outset, the industry maverick has the same imputed share in firms k and r. Otherwise, the transfer of partial cross ownership facilitates tacit collusion if the industry maverick has a larger imputed share in firm r (the acquirer) than in firm k (the seller) but hinders tacit collusion if the reverse holds.*

Proposition 3 in Gilo et al. (2006) also considered the effects of a transfer of partial cross ownership in firm s from one firm to another but under the special assumption that at the outset all firms hold the exact same ownership stakes in one another. In this case, the matrix $\mathbf{L}$ is symmetric in the sense that its diagonal terms are all the same and its off-diagonal terms are all equal to each other. In particular, $l_{ir} = l_{ik}$ for all $i \neq r, k$, so part (ii) of Theorem 4.4 shows that $\widehat{\delta}_i(\mathbf{W}^{\phi}) = \widehat{\delta}_i(\mathbf{W})$ for all $i \neq r, k$ (which includes part (i) of Theorem 4.4 if $i = s$). As for firms r and k, then part (i) of Lemma 4.1 implies that $l_{rr} > l_{rk}$ and $l_{kr} < l_{kk}$. Hence, equation (4.22) shows that $\partial z_{rj}^{\phi}/\partial \phi \geq 0$ and $\partial z_{kj}^{\phi}/\partial \phi \leq 0$ for all j with strict inequality for $j = s$. Hence, by Lemma 4.3, $\widehat{\delta}_r(\mathbf{W}^{\phi}) < \widehat{\delta}_r(\mathbf{W})$ and $\widehat{\delta}_k(\mathbf{W}^{\phi}) > \widehat{\delta}_k(\mathbf{W})$, implying that the transfer of partial cross ownership in firm s from firm r to firm k strengthens the incentive of firm r to collude, weakens the incentive of firm k to collude and has no effect on the incentives of other firms to collude. In the symmetric case considered by Gilo et al. (2006), $\widehat{\delta}_1(\mathbf{W}) = \ldots = \widehat{\delta}_n(\mathbf{W})$, so the incentives of all firms to collude before the transfer of ownership are the same. Hence, the transfer of partial ownership turns firm k (the seller) into a maverick firm and since $\widehat{\delta}_k(\mathbf{W}^{\phi}) > \widehat{\delta}_k(\mathbf{W})$, tacit collusion is hindered.

In the present case where firms have asymmetric marginal costs, any firm can potentially be the maverick firm. In particular, Corollary 4.2 shows that collusion is hindered when the maverick is firm k and is facilitated if the maverick is firm r.

4.4.5 Conditions for firm 1 to be the maverick

Recall from Section 4.3 that when only firm 1 invests in a rival, firm 1 is the industry maverick. In the following theorem, we provide sufficient (but not necessary) conditions that ensure that firm 1 continues to be the industry maverick even in the presence of multilateral PCO arrangements (in the sense that $\widehat{\delta}_1(\mathbf{W}) > \widehat{\delta}_i(\mathbf{W})$ for all $i \neq 1$).

Theorem 4.5. *Sufficient (but not necessary) conditions for firm* 1 *(the most efficient firm in the industry) to be the industry maverick is that (i)* $z_{1j} \leq z_{ij}$ *for all* $i, j \neq 1$*, and (ii)* $l_{ii}\widehat{y}_i \leq l_{i1}y_1(c_2)$ *for all* $i \neq 1$*.*

Recall from the profits definitions in (4.12) that for all $i \neq 1$ we have $\pi_i^{d_i}(\widehat{p}; \mathbf{W}) = l_{ii}\widehat{y}_i$ and $\pi_i^f(c_2; \mathbf{W}) = l_{i1}y_1(c_2)$. Hence, Theorem 4.5 ensures that the most efficient firm (firm 1) is the industry maverick in the multilateral PCO setting if (i) its relative imputed share in each other firm j ($\neq 1$) is no greater than the relative share of any other firm in firm j, and (ii) the deviation profit of any other firm i ($\neq 1$) is no greater than i's profit after the failure of collusion.

4.5 Conclusion

Acquisitions of one firm's stock by a rival firm have been traditionally treated under Section 7 of the Clayton Act which condemns such acquisitions when their effect "*may be substantially to lessen competition*." However, the third paragraph of this section effectively exempts passive investments made "*solely for investment*." As argued in Gilo (2000), antitrust agencies and courts, when applying this exemption, did not conduct full-blown examinations as to whether such passive investments may substantially lessen competition.[17]

[17] We are aware of only two cases in which the ability of passive investments to lessen competition was acknowledged: the FTC's decision in *Golden Grain Macaroni Co.* (78 F.T.C. 63, 1971), and the consent decree reached with the DOJ regarding US West's acquisition of Continental Cablevision (this decree was approved by the district court in United states v. US West Inc., 1997-1 Trade cases (CCH), 71,767, D.C., 1997).

In this study we showed that although there are cases in which passive investments in rivals (both at the expense of outside shareholders and through ownership transfer among rivals) have no effect on the ability of firms to engage in tacit collusion, an across the board lenient approach towards such investments may be misguided. This is because passive investments in rivals may well facilitate tacit collusion, especially when these investments are multilateral and in firms that are not industry mavericks. We believe that antitrust courts and agencies should take account of these factors when considering cases involving passive investments among rivals.

Throughout the paper we have focused exclusively on the effect of PCO on the ability of firms to engage in (tacit) price fixing. However, if in addition to price fixing firms can also divide the market among themselves, then they would clearly be able to sustain collusion for a larger set of discount factors since they would have more instruments (the collusive price and the market shares). In particular, it would be possible to relax the incentive constraints of maverick firms by increasing their market shares at the expense of firms with nonbinding incentive constraints. This suggests in turn that in the presence of market sharing schemes, firms may have an incentive to become industry mavericks in order to receive a larger share of the market. As our analysis shows, one way to become an industry maverick is to avoid investing in rivals. Interestingly, this implies that besides the fact that market sharing schemes are harder to enforce (firms need to commit to ration their sales) and are more susceptible to antitrust scrutiny, they have another drawback, which is that they provide firms with a disincentive to invest in rivals and thereby facilitate tacit collusion.

4.A Proofs

Proof of Theorem 4.2. (i) Firm 1 chooses $\widehat{p}$ to maximize the left-hand side of (4.6). Assume that only $w_{1i} > 0$ for some $i \neq 1$. Then the corresponding first order condition (FOC) is $\partial\widehat{y}_1/\partial\widehat{p} + w_{1i}\partial\widehat{y}_i/\partial\widehat{p} = 0$. Since $\partial\widehat{y}_k/\partial\widehat{p} = (\partial Q(\widehat{p})/\partial\widehat{p})(\widehat{p} - c_k) + Q(\widehat{p})$, the last FOC can be rewritten as

$$w_{1i} = -\frac{\zeta\frac{\widehat{p}-c_1}{\widehat{p}} + 1}{\zeta\frac{\widehat{p}-c_i}{\widehat{p}} + 1} \equiv \Upsilon(\widehat{p}, c_1, c_i), \qquad (4.A.1)$$

where the price elasticity of demand is $\zeta = (\partial Q/\partial\widehat{p})(\widehat{p}/Q) < 0$. Hence, if w_{1i} changes, the right-hand side (rhs) of (4.A.1), $\Upsilon(\widehat{p}, c_1, c_i)$, also has to change. Thus, for given c_1 and c_i the collusive price must change. Using the fact that $\partial\zeta/\partial\widehat{p} = \zeta(1-\zeta)/\widehat{p} + (\widehat{p}/Q)(\partial^2 Q/\partial\widehat{p}^2)$, the derivative of the rhs of (4.A.1) with respect to the collusive price after some simple mathematical transformations can be shown to be given by

$$\frac{\partial\Upsilon(\widehat{p}, c_1, c_i)}{\partial\widehat{p}} = \frac{(c_i - c_1)\left(\frac{2\zeta^2}{\widehat{p}} - \frac{\partial^2 Q}{\partial\widehat{p}^2}\frac{\widehat{p}}{Q}\right)}{\left(\zeta\frac{\widehat{p}-c_i}{\widehat{p}} + 1\right)^2} \qquad \text{for all } i = 2, \ldots, n. \qquad (4.A.2)$$

Assumption 1 implies that the rhs of (4.A.2) is positive: for concave demand functions $\partial^2 Q/\partial\widehat{p}^2 \leq 0$. Note that Assumption 1 also allows for not too convex demand functions (with $\partial^2 Q/\partial\widehat{p}^2 \geq 0$), which we interpret here as $\frac{2\zeta^2}{\widehat{p}} - \frac{\partial^2 Q}{\partial\widehat{p}^2}\frac{\widehat{p}}{Q} > 0$. Thus from (4.A.1) and (4.A.2) it follows that $\widehat{p}^*$ is increasing with w_{1i} and is above p_1^m recalling that $p_1^m < p_2^m < \ldots < p_n^m$. The economic intuition is simple: when firm 1 gives a positive (or more) weight to the direct profits of other firms that call for higher prices, the collusive price must go up.

(ii) Absent PCO, the critical discount factor above which collusion can be sustained is $\widehat{\delta}_1(p_1^m)$. Using (4.5) and (4.7) it is clear that,

$$\widehat{\delta}_1(p_1^m) > \frac{y_1^m - \left(y_1^m + \sum_{i\neq 1} w_{1i} y_i(p_1^m)\right)/n}{y_1^m - y_1(c_2)} \geq \frac{y_1^m - \left(\widehat{y}_1^* + \sum_{i\neq 1} w_{1i}\widehat{y}_i^*\right)/n}{y_1^m - y_1(c_2)} \equiv \widehat{\delta}_1^{po}(\widehat{p}^*),$$

where $\widehat{y}_i^* = Q(\widehat{p}^*)(\widehat{p}^* - c_i)$ and the weak inequality follows because $\widehat{p}^*$ maximizes $\widehat{y}_1 + \sum_{i\neq 1} w_{1i}\widehat{y}_i$. To complete the proof, note that by the envelope theorem,

$$\frac{d\widehat{\delta}_1^{po}(\widehat{p}^*)}{dw_{1i}} = -\frac{\widehat{y}_i/n}{y_1^m - y_1(c_2)} < 0.$$

(iii) Since $c_2 < \ldots < c_n$, it follows that $\widehat{y}_2^* > \ldots > \widehat{y}_n^*$, implying that PCO by firm 1 in an efficient rival raises $\widehat{p}^*$ by less and lowers $\widehat{\delta}_1^{po}(\widehat{p}^*)$ by more than does a similar investment in a less efficient rival. The (more) formal proof of the first statement is the fact that one can easily show that the rhs of (4.A.2) is increasing in c_i, i.e., $\frac{\partial Y(\widehat{p},c_1,c_i)^2}{\partial \widehat{p} \partial c_i} > 0$. That is, the higher c_i (the less efficient is firm i), the more price change is needed for the FOC in (4.A.1) to hold. ■

Proof of Theorem 4.3*.* (i) Equation (4.19) implies that if $i = s$ (firm i is the target firm s), then $\partial z_{sj}^{\omega}/\partial \omega = 0$ for all j. Hence, by Lemma 4.3, $\widehat{\delta}_s(\mathbf{W}^{\omega}) = \widehat{\delta}_s(\mathbf{W})$.

(ii) Equation (4.19) implies that if $l_{ir} = 0$ (firm i has no direct and indirect stake in the investing firm r), then $\partial z_{ij}^{\omega}/\partial \omega = 0$ for all j. Again, by Lemma 4.3, $\widehat{\delta}_i(\mathbf{W}^{\omega}) = \widehat{\delta}_i(\mathbf{W})$.

(iii) Now suppose that $i \neq s$ and $l_{ir} > 0$. Theorem 1 in Zeng (2001) ensures that $l_{ii}l_{sj} - l_{si}l_{ij} \geq 0$ for all j. When $j = s$, the inequality is strict since then $l_{ii}l_{sj} - l_{si}l_{ij} = l_{ii}l_{ss} - l_{si}l_{is} > 0$, where the inequality follows because part (i) of Lemma 4.1 establishes that $l_{ij} < l_{ii}$ for all $j \neq i$. Together with the fact that $l_{ir} \geq 0$, this implies that $\partial z_{ij}^{\omega}/\partial \omega \geq 0$ for all i and all j, with a strict inequality for $j = s$. Hence, by Lemma 4.3, $\widehat{\delta}_i(\mathbf{W}^{\omega}) < \widehat{\delta}_i(\mathbf{W})$ for all $i \neq s$. ■

Proof of Theorem 4.5*.* Using (4.16) and (4.17) we obtain

$$
\begin{aligned}
&\widehat{\delta}_1(\mathbf{W}) - \widehat{\delta}_i(\mathbf{W}) \\
&= \frac{\left(y_1^m - \frac{1}{n}\sum_{j=1}^n z_{1j}\widehat{y}_j\right)\left(\widehat{y}_i - z_{i1}y_1(c_2)\right) - \left(\widehat{y}_i - \frac{1}{n}\sum_{j=1}^n z_{ij}\widehat{y}_j\right)\left(y_1^m - y_1(c_2)\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)\left(y_1^m - y_1(c_2)\right)} \\
&= \frac{\left(\widehat{y}_i - z_{i1}y_1^m\right)y_1(c_2) - \frac{1}{n}\sum_{j=1}^n z_{1j}\widehat{y}_j\left(\widehat{y}_i - z_{i1}y_1(c_2)\right) + \frac{1}{n}\sum_{j=1}^n z_{ij}\widehat{y}_j\left(y_1^m - y_1(c_2)\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)\left(y_1^m - y_1(c_2)\right)} \\
&= \frac{-\left(\widehat{y}_i - z_{i1}y_1^m\right)\left(\frac{\widehat{y}_1}{n} - y_1(c_2)\right) - \frac{1}{n}\sum_{j\neq 1} z_{1j}\widehat{y}_j\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)\left(y_1^m - y_1(c_2)\right)} \\
&\quad + \frac{\frac{1}{n}\sum_{j\neq 1} z_{ij}\widehat{y}_j\left(y_1^m - y_1(c_2)\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)\left(y_1^m - y_1(c_2)\right)},
\end{aligned}
$$

where the last equality follows because by definition, $z_{11} = 1$. Adding and subtracting $\frac{1}{n}\sum_{j\neq 1}^n z_{ij}\widehat{y}_j\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)$ and $\frac{1}{n}\sum_{j\neq 1} z_{ij}\widehat{y}_j\left(\widehat{y}_1 - ny_1(c_2)\right)$ to the numerator and rearranging terms yields

$$
\begin{aligned}
&\widehat{\delta}_1(\mathbf{W}) - \widehat{\delta}_i(\mathbf{W}) \\
&\quad = \frac{\left(z_{i1}y_1^m + \sum_{j\neq 1} z_{ij}\widehat{y}_j - \widehat{y}_i\right)\left(\frac{\widehat{y}_1}{n} - y_1(c_2)\right) + \frac{1}{n}\sum_{j\neq 1}\left(z_{ij} - z_{1j}\right)\widehat{y}_j\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)\left(y_1^m - y_1(c_2)\right)} \\
&\qquad + \frac{\frac{1}{n}\sum_{j\neq 1} z_{ij}\widehat{y}_j\left(\left(y_1^m - y_1(c_2)\right) - \left(\widehat{y}_i - z_{i1}y_1(c_2)\right) - \left(\widehat{y}_1 - ny_1(c_2)\right)\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)\left(y_1^m - y_1(c_2)\right)} \\
&\quad = \frac{\left(z_{i1}y_1^m + \sum_{j\neq 1} z_{ij}\widehat{y}_j - \widehat{y}_i\right)\left(\frac{\widehat{y}_1}{n} - y_1(c_2)\right) + \frac{1}{n}\sum_{j\neq 1}\left(z_{ij} - z_{1j}\right)\widehat{y}_j\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)\left(y_1^m - y_1(c_2)\right)} \\
&\qquad + \frac{\sum_{j\neq 1} z_{ij}\widehat{y}_j\left(\frac{y_1^m - \widehat{y}_1}{n} + \frac{(n-1)y_1(c_2)}{n} + \frac{l_{i1}y_1(c_2) - l_{ii}\widehat{y}_i}{l_{ii}n}\right)}{\left(\widehat{y}_i - z_{i1}y_1(c_2)\right)\left(y_1^m - y_1(c_2)\right)},
\end{aligned}
$$

Recalling that $z_{ii} = 1$, it follows that $\sum_{j\neq 1} z_{ij}\widehat{y}_j > \widehat{y}_i$. Moreover, $\widehat{y}_1/n \geq y_1(c_2)$ (see Figure 4.1). Hence, the first term is positive. Therefore, firm 1 is the maverick firm in the industry in the sense that $\widehat{\delta}_1(\mathbf{W}) > \widehat{\delta}_i(\mathbf{W})$ for all $i \neq 1$ if $z_{1j} \leq z_{ij}$ for all $i, j \neq 1$, and $l_{ii}\widehat{y}_i \leq l_{i1}y_1(c_2)$ for all $i \neq 1$. This completes the proof. ■

CHAPTER 5

Key groups in networks and their optimal size*

5.1 Introduction

One of the important topics in the sociology literature is the problem of identification of *key actor(s)* in social networks. So-called "centrality" measures have been proposed for this purpose that identify how "central" or powerful (on the basis of different criteria) each actor is in a network. These measures include centralities of degree, closeness, betweenness, and information (see e.g., Sabidussi, 1966; Freeman, 1977, 1979; Stephenson and Zelen, 1989). Other often used centralities are the status measure also known as the rank prestige index (Katz, 1953), an eigenvector based centrality measure (Bonacich, 1972, 1991), and the related centrality in Bonacich (1987). (A thorough discussion of centrality and many more references can be found in Wasserman and Faust, 1994, pp. 169-219.) The idea of finding the "most important" actors in social networks has been applied to a large number of cases across different disciplines.

Recently, however, Everett and Borgatti (1999, 2005) proposed new measures of a network's *group centrality* to account for the fact that the optimal selection of a set of k (> 1) actors is quite different from selecting the k actors with the largest individual centralities. These are the so-called group degree, group closeness, and group betweenness centralities. The inconsistency of the individual and group centralities is termed an "ensemble issue" in Borgatti (2006), who interprets this by a

* An earlier and much reduced version of this chapter appeared as the *NET Institute Working Paper*, 08-08, September 2008.

redundancy principle inherent to the majority of real-life networks (see also Burt, 1992). That is, for example, two actors with the largest individual centralities cannot be optimal set (target) of two individuals if they "... are redundant with respect to their liaising role – they are equivalent in that they connect the same third parties to each other" (p. 24), or these actors are structurally equivalent, meaning that they are connected to the same third parties. In both cases, the two actors have similar patterns of connections, and thus have exactly the same impact on the network. Hence, choosing one of them as a member of the target set implies that the second actor is redundant and should not be included in the optimal set.

Borgatti (2006) shows that depending on the situation, one needs to use certain measures of centrality. For instance, he distinguishes between a "Key Player Problem/Negative" (KPP-Neg) and a "Key Player Problem/Positive" (KPP-Pos). Given a social network, the aim of the KPP-Neg is finding a set of k actors who, if removed, would maximally disrupt the network, while that of the KPP-Pos is finding a set of k actors that is maximally connected to all other parties. In practice KPP-Neg, for example, arises whenever it is needed to immunize or quarantine a subset of the population in order to optimally contain an epidemic, or in a military context, to neutralize a small number of actors in a criminal network in order to maximally disrupt its functioning. KPP-Pos arises, for instance, in a public health context, when a health agency wishes to optimally spread information about health promoting practices and attitudes using a small subset of the population, or in a military context, when one needs to select an optimal set of actors to quickly diffuse (mis)information to all criminals. A similar optimization problem, termed KPP-Com, is defined in Puzis et al. (2007), and searches for the group with the maximal potential of controlling traffic in communication networks. All in all, the existence of such a large number of "importance" indicators implies that there is *not* a systematic criterion for choosing the "right" measure of network centrality in each particular situation.

In economics the impact and implications of the actors' networks of connections are usually studied using modern game theoretic tools. For example, the important feature of *network games* is that actors' payoffs depend on each other through network embeddedness (structure). In such games, each player chooses a level of some activity in an environment with negative global externalities (e.g., competition) and local positive externalities (e.g., learning, collaboration) that come through the network. This system has feedback effects, which are taken into account in the Nash equilibrium activity levels that are dependent on the underlying network topology. Recently, such a network game was analyzed by Ballester et al. (2006), who show that the *individual* equilibrium levels of agents are proportional to the so-called Katz-Bonacich centralities of the actors. Hence this study provides a behavioral foundation to the status measure of Katz (1953) and the network centrality measure of Bonacich (1987), "singling [them] out from the vast catalogue of network measures" (p. 1404).[1] Ballester et al. (2006) also propose a new measure of network centrality, named the *intercentrality measure*, that finds a *key player* from a social planner's perspective, i.e., the player with the maximum influence on *overall* activity (e.g., social welfare, aggregate crime level). In particular, it is shown that the key player is not necessarily the player with the highest KB centrality.

In this chapter we consider a more general setting of finding a *key group* in such network games. In the key group problem, the planner targets a certain number of players by removing them from the network of local interactions, which causes a complete modification of the distribution of individual outcomes. Further, the assumption of ex ante identical players used in Ballester et al. (2006) is quite restrictive from a practical point of view, because in that case all observable differences between individuals are ignored. Such heterogeneities are, for example, with respect to the player's age, education, occupation, race, gender, etcetera. This study also explicitly takes into account these ex ante exogenous heterogeneities of players, and shows that the results of the key player/group problem dramatically change if compared to those based on the assumption of ex ante identical players. In searching for the key group we make use of *weighted* and unweighted Katz-Bonachich (KB) centralities and *group intercentrality* measures, where the weights are the observable differences of players. We should mention that the group intercentralities are derived in terms of *the initial network configuration* only. As a comparison, an unweighted group intercentrality, proposed in Ballester et al. (2004) in the framework of a crime network model, is defined in terms of k different networks with and without the group-members (to be discussed Section 5.2). For its calculation in empirical applications, it is required that players are deleted consecutively from the network(s). Then, the sum of the computed individual intercentralities in these networks is the group intercentrality measure. In contrast, our alternative expressions for the weighted and unweighted group intercentrality measures (defined within the more general framework of the network games in Ballester et al., 2006) do not require any extraction of players from the network.

The removal of more than one player from a network has two effects. First, less

[1] As will be shown later, the two centrality measures are affine transformations of each other.

players contribute to the aggregate equilibrium outcome (direct effect), and, second, the network geometry is modified, implying that the remaining players adopt different actions (indirect effect). These effects are fully taken into account in the group intercentrality measure with ex ante identical agents, which considers not only the individual KB centralities of the group-members, but also their contributions to the KB centralities of players outside the group. However, in a more general setting with ex ante heterogeneous agents, it is not only the weighted group intercentrality, but also its interaction with the unweighted group (inter)centrality that matters for the direct and indirect effects. Hence, with players' ex ante heterogeneity, the weighted and unweighted group intercentrality and KB centrality measures (together with other parameters of the model to be discussed later) jointly identify the key group.

The second contribution of this study is that we endogenize the size of the key group, which is important since targeting groups incurs costs, next to certain benefits. These gains and costs are largely dependent on the size of the key group. Hence, from the planner's point of view it is crucial to know what is the optimal size of the target group. It is shown that within the class of network games studied in Ballester et al. (2006), the optimal size of the key group is determined by the minimal *key group loss* measure that depends on players' weighted and unweighted KB centralities and key group intercentralities, and the costs of group targeting. Further, we provide a condition, which guarantees that the problem of choosing the optimal size of the key group has an interior solution.

The rest of this chapter is organized as follows. In Section 5.2 we characterize the optimal target selection tasks – both the key player and the key group problems with ex ante identical players. Some properties of the group intercentrality measure are discussed. Section 5.3 relaxes the homogeneity assumption, and shows that the solution of the key player/group problem depends on individuals' weighted and unweighted group intercentralities and KB centralities. We endogenize the size of the key group in Section 5.4. Section 5.5 applies the key group problem to an hypothetical example of the covert networks that characterize the organizational structure of large terrorist organizations. In particular, it is shown that once individuals' observable differences are taken into account, the results dramatically change if compared to those based on the assumption of ex ante identical agents. Section 5.6 contains some concluding remarks. All proofs are given in the Appendix.

5.2 The problem of selecting the appropriate target

In what follows, we (briefly) present the Ballester et al. (2006, henceforth BCZ) model and their proposed intercentrality measure in finding the key player in networks with ex ante identical agents. Then we extend the problem to a search of groups consisting of an arbitrary number of players that have the highest impact on the overall activity.

5.2.1 Key player search

Each player $i = 1, \ldots, n$ selects an effort $x_i \geq 0$ and gets the bilinear payoff

$$u_i(\mathbf{x}) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j, \tag{5.1}$$

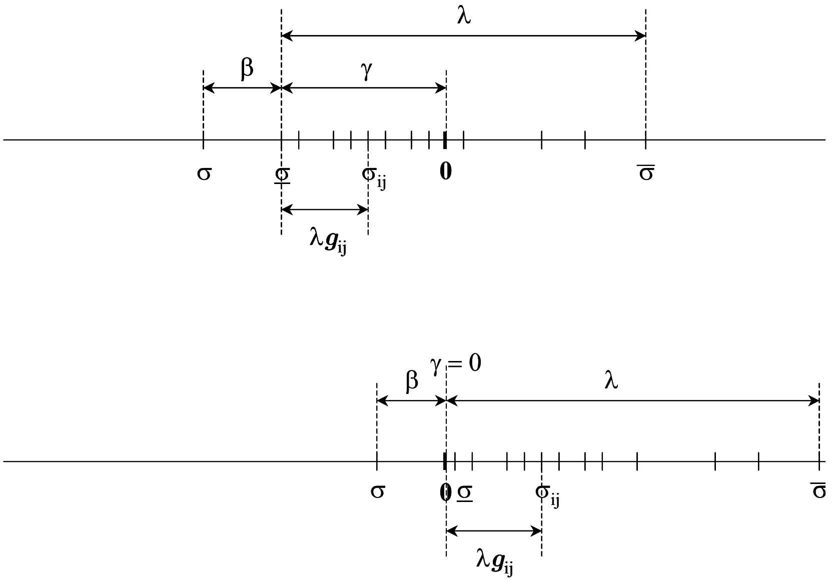
which is strictly concave in its own effort, $\partial^2 u_i / \partial x_i^2 = \sigma_{ii} < 0$, hence marginal utility of player i is decreasing in its own action. Further, we set $\alpha_i = \alpha > 0$ and $\sigma_{ii} = \sigma$ for all $i = 1, \ldots, n$.

The network payoff (relative) complementarities across pairs of actors are reflected by the cross-derivatives $\partial^2 u_i / \partial x_i \partial x_j = \sigma_{ij}$ for $i \neq j$. That is, the marginal utility of actor i is increasing in actor j's effort if $\sigma_{ij} > 0$, implying that the efforts by i and j are strategic complements. Similarly, $\sigma_{ij} < 0$ means that the actions of i and j are strategic substitutes from i's perspective.

Let $\underline{\sigma} = \min\{\sigma_{ij} \ | i \neq j\}$, $\overline{\sigma} = \max\{\sigma_{ij} \ | i \neq j\}$, $\sigma < \min\{\underline{\sigma}, 0\}$ and define $\gamma = -\min\{\underline{\sigma}, 0\} \geq 0$. If efforts are strategic substitutes for some pair of players, then $\gamma > 0$, otherwise, $\underline{\sigma} \geq 0$ implies $\gamma = 0$. Hence, as will be also shown below, the parameter γ reflects the global substitutability of efforts across all pairs of players. Let $\lambda = \overline{\sigma} + \gamma \geq 0$. Assuming $\underline{\sigma} \neq \overline{\sigma}$ implies $\lambda > 0$, which is a generic property. From the last definition, it follows that λ corresponds to the highest possible relative complementarity for all pairs of players.[2] Finally, let $g_{ij} = (\sigma_{ij} + \gamma)/\lambda$ for $i \neq j$, and $g_{ii} = 0$ for all $i = 1, \ldots, n$. By construction, $0 \leq g_{ij} \leq 1$. The parameter g_{ij} measures the relative complementarity in efforts from player i's perspective within the pair (i, j) with respect to the benchmark value $-\gamma \leq 0$. The above interpretations can be easily seen in Figure 5.1, where the figure in the upper panel shows the case of both strategic substitutability and complementarity of efforts (i.e., $\underline{\sigma} < 0$) and

[2] Note that λ is *not* the highest possible complementarity for all pairs of players (indicated by $\overline{\sigma}$), but the largest possible *relative* complementarity with respect to $-\gamma$. To see this difference, note that if $\underline{\sigma} < 0$, then $\gamma = -\min\{\underline{\sigma}, 0\} > 0$, thus $\overline{\sigma} + \gamma = \lambda > \overline{\sigma}$.

Figure 5.1: Decomposition of the cross effects



Source: Ballester et al. (2005).

the lower panel corresponds to the case of strategic complementarity of players' efforts only (i.e., $\underline{\sigma} > 0$). Note that given the assumption $\sigma < \min\{\underline{\sigma}, 0\}$, without loss of generality, the (common) second-order derivative in own efforts is set as $\partial^2 u_i / \partial x_i^2 = \sigma = -\beta - \gamma$, where $\beta > 0$.

The matrix $\mathbf{G} = [g_{ij}]$ is a zero-diagonal nonnegative n-square matrix, which is interpreted as the adjacency matrix of the network **g** of relative payoff complementarities across pairs. A particular case corresponds to the symmetric matrix **G** with $g_{ij} = g_{ji}$. In addition, if the cross effects take only two values, i.e., $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i \neq j$ with $\overline{\sigma} \leq 0$, then **G** is a symmetric (0,1) matrix, and thus **g** is an undirected and unweighted network.[3]

Given the definitions of β, γ, λ and **G**, the matrix of cross-effects $\boldsymbol{\Sigma} = [\sigma_{ij}]$ can be readily decomposed into three additive components (or sources of bilateral interactions) as $\sigma_{ij} = -\gamma + \lambda g_{ij}$ if $i \neq j$ and $\sigma_{ii} = -\beta - \gamma$ if $i = j$. That is,

[3] An unweighted network is represented by binary data that indicate only the presence or absence of ties between pairs of actors, while in a weighted network also the intensities or frequencies of such links are quantified. In a nondirected network the relation is mutual, while in a directed network the relation is not always reciprocal, hence the origin and the end of links are distinguished.

$$\boldsymbol{\Sigma} = -\beta \mathbf{I} - \gamma \mathbf{U} + \lambda \mathbf{G}, \tag{5.2}$$

where $\mathbf{I}$ is the n-square identity matrix and $\mathbf{U}$ is the n-square matrix of ones.

In (5.2) the idiosyncratic effect, $-\beta\mathbf{I}$, which is the same for each player, reflects (part of) the concavity in own efforts. The global interaction effect, $-\gamma\mathbf{U}$, gives a uniform substitutability in efforts across all pairs of players. The local interaction effect, $\lambda\mathbf{G}$, reflects a relative complementarity in efforts, which can be heterogeneous across different pairs of actors. In what follows, the strength of local interactions relative to own concavity is denoted by $a \equiv \lambda/\beta$.

Denote the largest (or dominant) eigenvalue of $\mathbf{G}$ by $\mu(\mathbf{G}) > 0$. Then if $a\mu(\mathbf{G}) < 1$, the matrix $\mathbf{B}(\mathbf{g}, a) = (\mathbf{I} - a\mathbf{G})^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k$ is well defined,[4] and its coefficients $b_{ij}(\mathbf{g}, a)$ count the number of paths in $\mathbf{g}$ starting in i and ending in j, where paths of length k are weighted by a^k. Hence, whenever $\lambda < \beta$, the parameter a in this interpretation is a decay factor that scales down the weight of longer paths. Denote the summation vector, i.e., the vector of ones, by $\boldsymbol{\imath}$. The vector of Katz-Bonacich (KB) centralities of parameter a in $\mathbf{g}$ is $\mathbf{b}(\mathbf{g}, a) = \mathbf{B}(\mathbf{g}, a)\boldsymbol{\imath}$, and its i-th component $b_i(\mathbf{g}, a) = \sum_{j=1}^{n} b_{ij}(\mathbf{g}, a)$ indicates the *total number of direct and indirect paths* in $\mathbf{g}$ that start from position i.[5] Note that, by definition, $b_{ii}(\mathbf{g}, a) \geq 1$, hence $b_i(\mathbf{g}, a) \geq 1$ with equality holding when i is an isolated actor, i.e., when $g_{ij} = g_{ji} = 0$ for all $j \neq i$.[6]

From Theorem 1 in BCZ (2006) follows that for $a\mu(\mathbf{G}) < 1$, the unique interior Nash equilibrium of the network game is

$$\mathbf{x}^*(\boldsymbol{\Sigma}) = \frac{\alpha}{\beta + \gamma b(\mathbf{g}, a)} \mathbf{b}(\mathbf{g}, a), \tag{5.3}$$

where $b(\mathbf{g}, a) = \sum_{i=1}^{n} b_i(\mathbf{g}, a)$.

Equation (5.3) shows that players' individual equilibrium outcomes are proportional to their KB centralities. The condition $a\mu(\mathbf{G}) < 1$ (or equivalently, $\lambda\mu(\mathbf{G}) < \beta$) for the equilibrium existence and uniqueness requires the payoff complementarity (reflecting size and pattern of positive synergies), $\lambda\mu(\mathbf{G})$, to be smaller than the own concavity, β. This interpretation holds because λ measures the level of

[4] This follows from Theorem III* in Debreu and Herstein (1953, p. 601).

[5] In fact, Bonacich (1987) defines the network centrality measure by the vector $\mathbf{h}(\mathbf{g}, a, b) = b(\mathbf{I} - a\mathbf{G})^{-1}\mathbf{G}\boldsymbol{\imath}$, where the parameter b "*affects only the length of the vector* $[\mathbf{h}(\mathbf{g}, a, b)]$" (p. 1173). It is not difficult to show that $\mathbf{b}(\mathbf{g}, a) = \boldsymbol{\imath} + a\mathbf{h}(\mathbf{g}, a, 1)$. This measure is directly related to the Katz (1953) network status measure $\mathbf{k}(\mathbf{g}, a) = a(\mathbf{I} - \mathbf{G})^{-1}\mathbf{G}\boldsymbol{\imath}$, since $\mathbf{k}(\mathbf{g}, a) = a\mathbf{h}(\mathbf{g}, a, 1) = \mathbf{b}(\mathbf{g}, a) - \boldsymbol{\imath}$.

[6] Or when $a = 0$, which is not allowed in this network game.

positive cross-effects, whereas $\mu(\mathbf{G})$ captures the population-wide pattern of these positive cross-effects. Or, in terms of the decomposition given in (5.2), the equilibrium exists, is unique and interior, only when the positive feed-back loops $+\lambda\mathbf{G}$ are dampened by own concavity $-\beta\mathbf{I}$.

Given the equilibrium efforts, it is clear that the planner can manipulate the network geometry by removing one or more players from the network $\mathbf{g}$, in which case the distribution of individual outcomes is completely modified. In this sense, the policy relevant issue (e.g., reducing crime rate) studied in BCZ (2006) is removing one player, and identifying the network's optimal target. The optimal target is the actor whose removal maximally reduces the aggregate equilibrium outcome. Denote by $\mathbf{G}^{-i}$ the new adjacency matrix derived from $\mathbf{G}$ by setting to zero all of its i-th row and column elements. The new matrix of cross-effects $\boldsymbol{\Sigma}^{-i}$ is similarly derived from $\boldsymbol{\Sigma}$. The resulting network is $\mathbf{g}^{-i}$. Then the planner's problem is picking the player i from the population, whose removal from the initial network $\mathbf{g}$ gives the highest possible reduction in the aggregate equilibrium level. Note from (5.3) that the aggregate equilibrium outcome is proportional to the total number of direct and indirect paths in $\mathbf{g}$ that stem from all players. Formally, the problem is $\max\{x^*(\boldsymbol{\Sigma}) - x^*(\boldsymbol{\Sigma}^{-i}) | i = 1, \ldots, n\}$, where $x^*(\boldsymbol{\Sigma}) = \boldsymbol{\iota}'\mathbf{x}^*(\boldsymbol{\Sigma})$. The planner's problem is thus equivalent to

$$\min\left\{x^*(\boldsymbol{\Sigma}^{-i}) \mid i = 1, \ldots, n\right\}. \tag{5.4}$$

The *key player* i^* is a solution to (5.4).

The *intercentrality* of player i in $\mathbf{g}$ is defined as[7]

$$c_i(\mathbf{g}, a) = b_i(\mathbf{g}, a) + \sum_{j \neq i}\left[b_j(\mathbf{g}, a) - b_j(\mathbf{g}^{-i}, a)\right] = \frac{b_i(\mathbf{g}, a)^2}{b_{ii}(\mathbf{g}, a)}. \tag{5.5}$$

While the KB centrality of actor i (i.e., $b_i(\mathbf{g}, a)$) counts the number of direct and indirect paths in $\mathbf{g}$ stemming from i, equation (5.5) clearly shows that the "intercentrality counts the total number of such paths that hit i; it is the sum of i's [Katz-] Bonacich centrality and i's contribution to every other player's [Katz-] Bonacich centrality" (BCZ, 2006, p. 1411). Theorem 3 in BCZ (2006) proves that the key player i^* has the highest intercentrality, i.e., $c_{i^*}(\mathbf{g}, a) \geq c_i(\mathbf{g}, a)$ for all $i = 1, \ldots, n$. In their Example 1, the authors further show that the most central player (according to the KB centrality measure) is not the key player for relatively large values of a.

[7] The last part of (5.5) follows from Lemma 1 in BCZ (2006).

This occurs since then indirect effects matter and, as the intercentrality takes into account both a player's centrality and his contribution to the centralities of the others, the key player (with the highest joint direct and indirect effect on the aggregate outcome) might well differ from the most central player.

Given the definition of the KB centrality, we have that the $n \times 1$ vector of KB centralities after the removal of player i is $\mathbf{b}(\mathbf{g}^{-i}, a) = \mathbf{B}(\mathbf{g}^{-i}, a)\boldsymbol{\imath}$. All the i-th row (and column) off-diagonal elements of the matrix $\mathbf{B}(\mathbf{g}^{-i}, a) = (\mathbf{I} - a\mathbf{G}^{-i})^{-1}$ are equal to zero, while its corresponding diagonal entry is positive and equals $b_{ii}(\mathbf{g}^{-i}, a) = 1$. Hence, the KB centrality of the eliminated player i is $b_i(\mathbf{g}^{-i}, a) = 1$. Therefore, the middle part in (5.5) gives an alternative expression for the intercentrality measure in terms of the sums of the KB centralities before and after elimination of player i from the network of local interactions as $c_i(\mathbf{g}, a) = b(\mathbf{g}, a) - b(\mathbf{g}^{-i}, a) + 1$.

5.2.2 Key group search

In this section we wish to generalize the key player problem studied in BCZ (2006) to a group target selection problem. Thus, the planner's objective is now optimally reducing the aggregate equilibrium outcome by picking k appropriate players $i_1, i_2, \ldots, i_k$ ($i_s \neq i_r$) from the population, where $1 \leq k \leq n$. Recall that the matrix of cross-effects, $\boldsymbol{\Sigma}$, can be both symmetric and asymmetric, hence the following key group search analysis can be applied to undirected, directed, binary and/or valued graphs that are characterized by the network $\mathbf{g}$ having a symmetric or asymmetric adjacency matrix $\mathbf{G}$, which is not necessarily a (0,1) matrix.

The planner is searching for k players from the population, such that the difference between the equilibrium aggregate outcomes before and after the removal of the appropriate players from the network is maximal. That is, formally, the planner solves $\max\left\{x^*(\boldsymbol{\Sigma}) - x^*(\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}}) | \{i_1, \ldots, i_k\} \subseteq N; i_r \neq i_s\right\}$, where the set of all players is $N = \{1, \ldots, n\}$. This is equivalent to

$$\min\left\{x^*(\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}}) \mid \{i_1, \ldots, i_k\} \subseteq N; i_r \neq i_s\right\}, \tag{5.6}$$

where $\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}}$ is the new matrix of cross-effects obtained from $\boldsymbol{\Sigma}$ by setting to zero all its i_1-th, i_2-th, …, i_k-th row and column elements. The resulting network and adjacency matrix are, respectively, $\mathbf{g}^{-\{i_1,\ldots,i_k\}}$ and $\mathbf{G}^{-\{i_1,\ldots,i_k\}}$. Problem (5.6), similar to the key player problem in (5.4), is a finite optimization problem, which admits, at least, one solution. Let $\{i_1^*, \ldots, i_k^*\}$ be a solution to the key group problem (5.6), which we call the *key group of size k*.

Let us first extend the individual intercentrality measure in (5.5) to the *group intercentrality*, which measures intercentrality of a group of players rather than of a single player. Analogously to (5.5), the k-th order group intercentrality of players $i_1, \ldots, i_k$ can be written as

$$\begin{aligned} c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, \mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) &= \sum_{r=i_1}^{i_k} b_r(\mathbf{g}, a) + \sum_{j \neq i_1,\ldots,i_k} \left[b_j(\mathbf{g}, a) - b_j(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) \right] \\ &= b(\mathbf{g}, a) - b(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) + k, \end{aligned} \tag{5.7}$$

where k appears in the expression because $b_{i_s}(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) = 1$ for all $s = 1, \ldots, k$ and these KB centralities of the eliminated players are not part of the group intercentrality measure above.[8] Thus, similar to the individual intercentrality measure, the group intercentrality $c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, \mathbf{g}^{-\{i_1,\ldots,i_k\}}, a)$ counts not only the total number of (weighted) paths in $\mathbf{g}$ that stem from positions $i_1, \ldots, i_k$ (i.e., the KB centralities of players $i_1, \ldots, i_k$), but also the total number of paths that hit these players. In other words, it is the sum of the KB centralities of all members of the group $\{i_1, \ldots, i_k\}$, and their contributions to every other player's KB centrality.

We can also express the group intercentrality in terms of the individual intercentralities as follows (suppressing a's):

$$\begin{aligned} c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, \mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) &= b(\mathbf{g}) - b(\mathbf{g}^{-\{i_1,\ldots,i_k\}}) + k \\ &= \left[b(\mathbf{g}) - b(\mathbf{g}^{-i_1}) + 1 \right] + \left[b(\mathbf{g}^{-i_1}) - b(\mathbf{g}^{-\{i_1,i_2\}}) + 1 \right] + \left[b(\mathbf{g}^{-\{i_1,i_2\}}) \right. \\ &\quad \left. - b(\mathbf{g}^{-\{i_1,i_2,i_3\}}) + 1 \right] + \cdots + \left[b(\mathbf{g}^{-\{i_1,\ldots,i_{k-1}\}}) - b(\mathbf{g}^{-\{i_1,\ldots,i_k\}}) + 1 \right] \\ &= c_{i_1}(\mathbf{g}) + c_{i_2}(\mathbf{g}^{-i_1}) + c_{i_3}(\mathbf{g}^{-\{i_1,i_2\}}) + \cdots + c_{i_k}(\mathbf{g}^{-\{i_1,\ldots,i_{k-1}\}}) \\ &\equiv c_{\{i_1,\ldots,i_k\}}\big(\mathbf{g}, \mathbf{g}^{-i_1}, \ldots, \mathbf{g}^{-\{i_1,\ldots,i_{k-1}\}}, a\big). \end{aligned} \tag{5.8}$$

The derived group intercentrality is exactly how Ballester et al. (2004) defined the group intercentrality measure in their study of a similar problem in the framework of crime networks. Their Proposition 7 states that the key group of size k, where $1 \leq k \leq n-1$, has the highest group intercentrality in $\mathbf{g}$. From (5.8) we see that $c_{\{i_1,\ldots,i_k\}}\big(\mathbf{g}, \mathbf{g}^{-i_1}, \ldots, \mathbf{g}^{-\{i_1,\ldots,i_{k-1}\}}, a\big)$ is explicitly dependent not only on the initial network $\mathbf{g}$, but also on $k-1$ extra networks, which are obtained by removing consecutively members of the group for which the value of intercentrality is computed. Similarly, $c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, \mathbf{g}^{-\{i_1,\ldots,i_k\}}, a)$ in (5.7) is defined in terms of the two networks

[8] That is, the i_1-th, i_2-th, ..., i_k-th row (and column) off-diagonal elements of the matrix $\mathbf{B}(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) = (\mathbf{I} - a\mathbf{G}^{-\{i_1,\ldots,i_k\}})^{-1}$ are all zeros, while the corresponding diagonal entries are unity, i.e., $b_{i_s i_s}(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) = 1$ for all $s = 1, \ldots, k$. Hence, $b_{i_s}(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) = 1$ for all $s = 1, \ldots, k$.

$\mathbf{g}$ and $\mathbf{g}^{-\{i_1,\ldots,i_k\}}$. Therefore, the obvious question arises whether this group intercentrality measure can be expressed in terms of *only* the initial network topology, similar to the solution of the key player problem given in (5.5). In what follows we give this alternative closed-form expression for the group intercentrality measure, which is expressed in terms of only the initial network $\mathbf{g}$, hence the removal of players as in (5.8) above is no longer required. This will also allow us to study some interesting properties of the group intercentrality measure.

Definition 5.1. *Consider a network* $\mathbf{g}$ *with adjacency matrix* $\mathbf{G}$ *and a scalar* a *such that* $\mathbf{B}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1}$ *exists and is nonnegative. The k-th order group intercentrality of players* $i_1, \ldots, i_k$ $(i_r \neq i_s)$ *in* $\mathbf{g}$ *is*

$$c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a) = \boldsymbol{\iota}'\mathbf{BE}\left(\mathbf{E}'\mathbf{BE}\right)^{-1}\mathbf{E}'\mathbf{b},$$

where $\mathbf{E}$ *be the* $n \times k$ *matrix defined as* $\mathbf{E} = \left(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}\right)$ *with* $\mathbf{e}_{i_r}$ *being the* i_r*-th column of the identity matrix, and* $1 \leq k \leq n$.

In the proof of Theorem 5.1 (given in Appendix 5.A) we show that $c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a) = c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, \mathbf{g}^{-\{i_1,\ldots,i_k\}}, a)$. Summarizing our findings so far, we now have three equivalent expressions for the k-th order group intercentrality. These are (5.7), (5.8), and the expression given in Definition 5.1. The interpretation of (5.7) thus applies also to the other two expressions.[9]

The following important identity characterizes all the path changes in a network when a group of k nodes is removed (see Appendix 5.A).[10]

Lemma 5.1. *Let* $\mathbf{B} = [\mathbf{I} - a\mathbf{G}]^{-1}$ *exists and be nonnegative. Let* $\mathbf{e}_{i_r}$ *be the* i_r*-th column of the identity matrix,* $\mathbf{E} = \left(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}\right)$*, and* $\mathbf{B}^{-\{i_1,\ldots,i_k\}} = [\mathbf{I} - a\mathbf{G}^{-\{i_1,\ldots,i_k\}}]^{-1}$*, where* $1 \leq k \leq n$*. Then the identity* $\mathbf{B} - \mathbf{B}^{-\{i_1,\ldots,i_k\}} = \mathbf{BE}\left(\mathbf{E}'\mathbf{BE}\right)^{-1}\mathbf{E}'\mathbf{B} - \mathbf{EE}'$ *always holds.*

Using Lemma 5.1 we establish the following result that gives the solution to the problem (5.6) in terms of the k-th order group intercentrality measure (see Appendix 5.A).

Theorem 5.1. *If* $a\mu(\mathbf{G}) < 1$*, the key group* $\{i_1^*, \ldots, i_k^*\}$ *of size k that solves the problem* $\min\{x^*(\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}}) \mid \{i_1, \ldots, i_k\} \subseteq N; i_r \neq i_s\}$ *has the highest k-th order group*

[9] Using the analytical formula of the inverse matrix, the closed-form second-order group intercentrality of players i and j $(\neq i)$ in $\mathbf{g}$ with symmetric $\mathbf{G}$ can be written as $c_{\{i,j\}}(\mathbf{g}, a) = (b_{jj}b_i^2 + b_{ii}b_j^2 - 2b_{ij}b_ib_j)/(b_{ii}b_{jj} - b_{ij}b_{ji})$.

[10] Lemma 1 in BCZ (2006) is a particular case of our Lemma 5.1 with $k = 1$ and $\mathbf{G}$ being a symmetric adjacency matrix.

intercentrality in $\mathbf{g}$*, where* $1 \leq k \leq n$*, i.e.,* $c_{\{i_1^*,\ldots,i_k^*\}}(\mathbf{g},a) \geq c_{\{i_1,\ldots,i_k\}}(\mathbf{g},a)$ *for all* $\{i_1,\ldots,i_k\} \subseteq N$ *and* $i_r \neq i_s$*.*

It is interesting to note that Theorem 3 and Remark 5 in BCZ (2006) are particular cases of our Theorem 5.1 when $k = 1$ and the matrix of cross-effects $\boldsymbol{\Sigma}$ is, respectively, symmetric and asymmetric. This follows since with $k = 1$ the group intercentrality in Definition 5.1 boils down to

$$c_i(\mathbf{g},a) = \boldsymbol{\iota}'\mathbf{B}\mathbf{e}_i \left(\mathbf{e}_i'\mathbf{B}\mathbf{e}_i\right)^{-1} \mathbf{e}_i'\mathbf{b} = \frac{\boldsymbol{\iota}'\mathbf{B}\mathbf{e}_i \cdot b_i(\mathbf{g},a)}{b_{ii}(\mathbf{g},a)},$$

which is the intercentrality of player i when $\boldsymbol{\Sigma}$ is not symmetric. Remark 5 in BCZ (2006) states that this last intercentrality defines the key player in the case of asymmetric matrix of cross-effects.

When the matrix of cross-effects is symmetric, then $b_{kj}(\mathbf{g},a) = b_{jk}(\mathbf{g},a)$ for all k and all j, i.e., $\mathbf{B} = \mathbf{B}'$. Hence, we have $\boldsymbol{\iota}'\mathbf{B} = \boldsymbol{\iota}'\mathbf{B}' = (\mathbf{B}\boldsymbol{\iota})' = \mathbf{b}'$, implying that for a symmetric adjacency matrix $\mathbf{G}$ the group intercentrality of players $i_1,\ldots,i_k$ in Definition 5.1 can be rewritten as

$$c^{\text{sym}}_{\{i_1,\ldots,i_k\}}(\mathbf{g},a) = \mathbf{b}'\mathbf{E}\left(\mathbf{E}'\mathbf{B}\mathbf{E}\right)^{-1}\mathbf{E}'\mathbf{b}. \tag{5.9}$$

Then it immediately follows that with $k = 1$ the above measure is simply the individual intercentrality measure given in (5.5).

If we set $k = n$ and choose the ordering of all removed players such that $\mathbf{E} = \mathbf{I}$, the group intercentrality in Definition 5.1 reduces to $c_{\{1,2,\ldots,n\}}(\mathbf{g},a) = \boldsymbol{\iota}'\mathbf{B}\mathbf{B}^{-1}\mathbf{b} = \boldsymbol{\iota}'\mathbf{b} = b(\mathbf{g},a)$, which is the sum of the KB centralities of all the n players.[11] This is not surprising, because if we are interested in a group of all players, there are no outside actors left. Consequently, there are no non-members on which agents can exert payoff externalities. Recall that such externalities are internalized by the intercentrality measure, which makes it different from the KB centrality measure. But for $k = n$ there are no other externalities to account for, hence the group intercentrality is nothing else than the sum of the KB centralities of all players.

The above observation also implies that if the network $\mathbf{g}$ consists of two separate (independent) subnetworks, then the group intercentrality of *all* players from one of the subnetworks is just equal to the sum of the KB centralities of the players

[11] Different ordering of the n players results in a different permutation matrix $\mathbf{E}$ of order n, but it will always give exactly the same outcome. Similarly, using the alternative formulation of the group intercentrality in (5.8), $c_{\{i_1,\ldots,i_k\}}(\mathbf{g},a) = b(\mathbf{g},a) - b(\mathbf{g}^{-\{i_1,\ldots,i_k\}},a) + k$, it is easy to confirm that for $k = n$ we have $c_{\{1,\ldots,n\}}(\mathbf{g},a) = b(\mathbf{g},a)$, since then $b(\mathbf{g}^{-\{1,\ldots,n\}},a) = \boldsymbol{\iota}'(\mathbf{I} - a\mathbf{O})^{-1}\boldsymbol{\iota} = n$, where $\mathbf{O}$ is the n-square null matrix.

from that group.[12] That is, as group outsiders do not have any link with the group insiders, there cannot be any kind of payoff externalities that group-members exert on outsiders. Of course, this result does not hold anymore, if the group consists of players from both subnetworks. Let us consider some other properties of the group intercentrality measure.

Positivity: The group intercentrality measure is always strictly positive because $c_{\{i_1,\dots,i_k\}}(\mathbf{g},a) \geq k$ for all $k = 1,\dots,n$. This trivially follows from $c_{\{i_1,\dots,i_k\}}(\mathbf{g},a) = b(\mathbf{g},a) - b(\mathbf{g}^{-\{i_1,\dots,i_k\}},a) + k$, noting that $b(\mathbf{g},a) \geq b(\mathbf{g}^{-\{i_1,\dots,i_k\}},a)$. The last inequality is due to the fact that $b(\mathbf{g},a) = \boldsymbol{\iota}'(\mathbf{I} + a\mathbf{G} + a^2\mathbf{G}^2 + \cdots)\boldsymbol{\iota}$ and $\mathbf{G} \geq \mathbf{G}^{-\{i_1,\dots,i_k\}}$.

Isolate group: An isolate group has the group intercentrality value equal to its size, i.e., $c_{\{i_1,\dots,i_k\}}(\mathbf{g},a) = k$ if $i_1,\dots,i_k$ are all isolates (i.e., $g_{i_s j} = g_{j i_s} = 0$ for all $s = 1,\dots,k$ and all j). In such cases, $b(\mathbf{g},a) = b(\mathbf{g}^{-\{i_1,\dots,i_k\}},a)$ as the two corresponding networks $\mathbf{g}$ and $\mathbf{g}^{-\{i_1,\dots,i_k\}}$ are exactly identical. Thus, $c_{\{i_1,\dots,i_k\}}(\mathbf{g},a) = b(\mathbf{g},a) - b(\mathbf{g}^{-\{i_1,\dots,i_k\}},a) + k = k$ for all $k = 1,\dots,n$.

Subadditivity (or redundancy): $\sum_{s=1}^{k} c_{i_s}(\mathbf{g},a) \geq c_{\{i_1,\dots,i_k\}}(\mathbf{g},a)$ for all $k = 1,\dots,n$, with equality holding if and only if players $i_1, i_2, \dots, i_k$ are all isolates (see Appendix 5.A). In words, the group intercentrality of order k never exceeds the sum of the individual intercentralities of the players composing the group. This is the outcome of the fact that in networks, players may be redundant with respect to adjacency, distance, and bridging (see e.g., Borgatti, 2006).

Symmetry: If $b(\mathbf{g}^{-\{i_1,\dots,i_k\}},a) = b(\mathbf{g}^{-\{j_1,\dots,j_k\}},a)$ for the two groups $\{i_1,\dots,i_k\}$ and $\{j_1,\dots,j_k\}$, then $c_{\{i_1,\dots,i_k\}}(\mathbf{g},a) = c_{\{j_1,\dots,j_k\}}(\mathbf{g},a)$. That is, two groups of the same size are symmetric, if they have an identical (marginal) contribution to the overall equilibrium activity. This interpretation follows, since in this case the Nash equi-

[12] Mathematically, this can be proved as follows. Let the network **g** consist of two clusters (I and II), and no player in cluster I has a link to any of the players in cluster II, *and* vice versa, no player in cluster II has a link to any player of cluster I. That is, in terms of partitioned matrices we have

$$\mathbf{B} = \left[\mathbf{I} - a\begin{pmatrix} \mathbf{G}_k & \mathbf{O}_{kt} \\ \mathbf{O}_{tk} & \mathbf{G}_t \end{pmatrix}\right]^{-1} = \begin{bmatrix} \mathbf{B}_k & \mathbf{O}_{kt} \\ \mathbf{O}_{tk} & \mathbf{B}_t \end{bmatrix},$$

where, for example, $\mathbf{G}_k$ is the k-square adjacency matrix of all k players in cluster I, $\mathbf{O}_{kt}$ is the $k \times t$ null matrix, $\mathbf{B}_k = (\mathbf{I}_k - a\mathbf{G}_k)^{-1}$, and $k + t = n$. To find the group intercentrality of all players in cluster I, take $\mathbf{E}' = [\mathbf{I}_k \ \mathbf{O}_{kt}]$. Note that in this case the vector of KB centralities is equal to

$$\mathbf{b} = \mathbf{B}\boldsymbol{\iota} = \begin{bmatrix} \mathbf{b}_k \\ \mathbf{b}_t \end{bmatrix},$$

where, for example, $\mathbf{b}_t = \mathbf{B}_t\boldsymbol{\iota}_t$ is the vector of KB centralities of all players in cluster II. Then using the group intercentrality formula in Definition 5.1 and the above partitioned matrix **B**, one can by simple matrix multiplication easily verify that the group intercentrality of all k players from cluster I is $c_{\{i_1,\dots,i_k\}} = \boldsymbol{\iota}_k'\mathbf{b}_k$. Similarly, the t-th order group intercentrality of all t players from cluster II is equal to $c_{\{i_{k+1},\dots,i_n\}} = \boldsymbol{\iota}_t'\mathbf{b}_t$, where we have to redefine **E** now such that 1's appear in rows corresponding to the players of cluster II only.

librium efforts (5.3) of the network game imply $x^*(\mathbf{\Sigma}) - x^*(\mathbf{\Sigma}^{-\{i_1,\dots,i_k\}}) = x^*(\mathbf{\Sigma}) - x^*(\mathbf{\Sigma}^{-\{j_1,\dots,j_k\}})$.

Strict monotonicity: The group intercentrality is strictly monotonically increasing in the size of the group, i.e., $c_{\{i_1,\dots,i_{k+1}\}}(\mathbf{g}, a) > c_{\{i_1,\dots,i_k\}}(\mathbf{g}, a)$ for all $k = 1, \dots, n-1$. This holds as $c_{\{i_1,\dots,i_{k+1}\}}(\mathbf{g}, a) - c_{\{i_1,\dots,i_k\}}(\mathbf{g}, a) = b(\mathbf{g}^{-\{i_1,\dots,i_k\}}, a) - b(\mathbf{g}^{-\{i_1,\dots,i_{k+1}\}}, a) + 1 = c_{i_{k+1}}(\mathbf{g}^{-\{i_1,\dots,i_k\}}, a) \geq 1$, where we have used the positivity property of the intercentrality measure.

Constant difference: If eliminating two groups $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_{k+r}\}$ with $r \geq 1$ totally nullify the new adjacency matrices of the resulting networks, then $c_{\{j_1,\dots,j_{k+r}\}}(\mathbf{g}, a) - c_{\{i_1,\dots,i_k\}}(\mathbf{g}, a) = r$. This is because then $\mathbf{G}^{-\{i_1,\dots,i_k\}} = \mathbf{G}^{-\{j_1,\dots,j_{k+r}\}} = \mathbf{O}$, implying $b(\mathbf{g}^{-\{i_1,\dots,i_k\}}, a) = b(\mathbf{g}^{-\{j_1,\dots,j_{k+r}\}}, a)$. Using then the expression (5.7) for the intercentrality measure gives the result. Note that this outcome (similar to the symmetry property) implies that the two groups have equal contributions to the aggregate equilibrium efforts, since then it follows from (5.3) that $x^*(\mathbf{\Sigma}^{-\{j_1,\dots,j_{k+r}\}}) = x^*(\mathbf{\Sigma}^{-\{i_1,\dots,i_k\}}) = \alpha n/(\beta + \gamma n)$.

It might happen that the planner is interested in a group of players whose removal from the initial network $\mathbf{g}$ gives the *lowest* possible reduction in the aggregate equilibrium outcome, i.e., instead of (5.6) the planner's problem now is $\max\left\{x^*(\mathbf{\Sigma}^{-\{i_1,\dots,i_k\}}) \mid \{i_1, \dots, i_k\} \subseteq N; i_r \neq i_s\right\}$. Since the last problem and (5.6) are mirror reflections of each other, Theorem 5.1 implies the following.

Corollary 5.1. *If $a\mu(\mathbf{G}) < 1$, the key group $\{i_1^*, \dots, i_k^*\}$ of size k that solves the problem* $\max\left\{x^*(\mathbf{\Sigma}^{-\{i_1,\dots,i_k\}}) \mid \{i_1, \dots, i_k\} \subseteq N; i_r \neq i_s\right\}$ *has the lowest k-th order group intercentrality of parameter a in $\mathbf{g}$, where $1 \leq k \leq n$, i.e., $c_{\{i_1^*,\dots,i_k^*\}}(\mathbf{g}, a) \leq c_{\{i_1,\dots,i_k\}}(\mathbf{g}, a)$ for all $\{i_1, \dots, i_k\} \subseteq N$ with $i_r \neq i_s$.*

5.3 Key player/group search: accounting for players' exogenous heterogeneity

The entire discussion in the previous section was based on the assumption that players are ex ante exogenously identical. In terms of the model, we have assumed that $\alpha_i = \alpha$ for all $i = 1, \dots, n$ in the utility function (5.1). However, from a practical point of view this assumption is quite restrictive, since ignoring exogenous heterogeneity, captured by different values of α_i, implies that *observable* differences between individuals are neglected. For player i, for example, these heterogeneities include (among other factors) gender, age, race, motivation, education, parents' characteristics (education, occupation, age, etc.), household size, residential neighborhood factors, and also the average levels of all these mentioned factors for all players that are linked to player i. Evidently, in general, not accounting for such important differences gives totally biased results in empirical work. A recent study of Calvó-Armengol et al. (2009) takes such exogenous heterogeneity into account in studying the effect of social networks on pupils' school performance in case of an adolescent friendship network in the US.

Before considering the key player/group problem in the presence of ex ante heterogeneity, we first introduce the following definition, following BCZ (2006) and Calvó-Armengol et al. (2009).

Definition 5.2. *Given a positive vector* $\mathbf{u}$ *and a small enough scalar* $a \geq 0$, *the vector of* $\mathbf{u}$*-weighted Katz-Bonacich centrality of parameter* a *in the network* $\mathbf{g}$ *is* $\mathbf{b}_{\mathbf{u}}(\mathbf{g}, a) = (\mathbf{I} - a\mathbf{G})^{-1}\mathbf{u} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k \mathbf{u}$.

Note that the "standard" unweighted KB centrality $\mathbf{b}(\mathbf{g}, a)$ corresponds to the $\mathbf{u}$-weighted KB centrality when $\mathbf{u} = \boldsymbol{\iota}$. Calvó-Armengol et al. (2009) show that the previous condition of $\lambda\mu(\mathbf{G}) < \beta$ does not anymore guarantee the interiority of the equilibrium, although it suffices for the equilibrium existence and uniqueness in case of $\boldsymbol{\alpha} \neq \alpha\boldsymbol{\iota}$. Instead the corresponding sufficient condition becomes $\lambda\mu(\mathbf{G}) + n\gamma(\overline{\alpha}/\underline{\alpha} - 1) < \beta$, where $\overline{\alpha} = \max\{\alpha_i \,|\, i = 1, \ldots, n\}$, $\underline{\alpha} = \min\{\alpha_i \,|\, i = 1, \ldots, n\}$, and thus $n\gamma(\overline{\alpha}/\underline{\alpha} - 1) \geq 0$. Its economic interpretation is somewhat similar to that of the earlier condition, but now it also depends on the size of the global level of substitutabilities, γ. The new condition imposes more stringent requirements on $\lambda\mu(\mathbf{G})$ (local complementarities), γ (global substitutabilities) and $\overline{\alpha}/\underline{\alpha}$ (marginal payoff differences) such that "players have no incentives to increase their effort level without bound [in the presence of payoff complementarities], neither to free-ride on their network peers by decreasing them to zero [in the presence of payoff substitutabilities]" (Calvó-Armengol et al., 2008, p. 39).

Theorem 1 in Calvó-Armengol et al. (2009) shows that once different values of α_i across players in the objective function (5.1) are allowed, then the Nash equilibrium effort levels are different from those in (5.3) and are instead given by

$$\mathbf{x}_{\boldsymbol{\alpha}}^*(\boldsymbol{\Sigma}) = \frac{1}{\beta}\left[\mathbf{b}_{\boldsymbol{\alpha}}(\mathbf{g}, a) - \frac{\gamma b_{\boldsymbol{\alpha}}(\mathbf{g}, a)}{\beta + \gamma b(\mathbf{g}, a)}\mathbf{b}(\mathbf{g}, a)\right], \tag{5.10}$$

where $b_{\boldsymbol{\alpha}}(\mathbf{g}, a) = \boldsymbol{\iota}'\mathbf{b}_{\boldsymbol{\alpha}}(\mathbf{g}, a)$, and the equilibrium efforts are denoted by the superscript $\boldsymbol{\alpha}$ as well to indicate the fact that compared to (5.3) now players' ex ante heterogeneity is taken into account.

Note that when $\boldsymbol{\alpha} = \alpha\boldsymbol{\imath}$, we have $b_{\boldsymbol{\alpha}}(\mathbf{g}, a) = \alpha b(\mathbf{g}, a)$ and the equilibrium closed-form expression in (5.10) reduces to (5.3). Now we first wish to establish the relationship between the overall equilibrium outcome, $x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) = \boldsymbol{\imath}'\mathbf{x}^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma})$, and the density of the adjacency matrix $\mathbf{G}$.[13] Consider two adjacency matrices $\widetilde{\mathbf{G}}$ and $\mathbf{G}$, not necessarily symmetric, such that $\widetilde{g}_{ij} \geq g_{ij}$ for all i and all j ($\neq i$), with at least one strict inequality (in matrix notation, $\widetilde{\mathbf{G}} > \mathbf{G}$). It follows from (5.2) that $\widetilde{\boldsymbol{\Sigma}} = -\beta\mathbf{I} - \gamma\mathbf{U} + \lambda\widetilde{\mathbf{G}}$. The following result is proved in the Appendix.

Lemma 5.2. *Let $\widetilde{\mathbf{G}}$ and $\mathbf{G}$ be two adjacency matrices such that $\widetilde{\mathbf{G}} > \mathbf{G}$. For given $\boldsymbol{\alpha}$, β, γ and λ, if $\lambda\mu(\widetilde{\mathbf{G}}) + n\gamma(\overline{\alpha}/\underline{\alpha} - 1) < \beta$, then $x^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}}) > x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma})$.*

The lemma above shows that a network with more links has a strictly larger overall equilibrium outcome than the network with fewer connections. This is because a higher density of $\mathbf{G}$ implies more complementarities between players that lead to an increase in the number of un- and $\boldsymbol{\alpha}$-weighted direct and indirect paths, which ultimately increase the overall equilibrium activity.

As in the previous section the superscript $-\{i_1, \ldots, i_k\}$ to the adjacency matrix $\mathbf{G}$ indicates that its corresponding rows and columns elements are all set to zero. The planner is searching for k players from the population that have the maximal simultaneous impact on the overall equilibrium efforts. Similar to the players' ex ante homogeneity case discussed in Section 5.2, the planner's problem of maximizing the difference between the aggregate outputs before and after removal of individuals $i_1, \ldots, i_k$ from the network is equivalent to

$$\min\left\{x^*_{\boldsymbol{\alpha}}\left(\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}}\right) \mid \{i_1, \ldots, i_k\} \subseteq N; i_r \neq i_s\right\}. \tag{5.11}$$

Evidently, given the exogenous heterogeneity of players captured by $\boldsymbol{\alpha}$, in general, the solutions (i.e., key groups of size k) of (5.6) and (5.11) do *not* correspond to each other. Similar to Definition 5.2, we next define a weighted version of the group intercentrality measure as follows.

Definition 5.3. *Given a positive vector $\mathbf{u}$ and a small enough scalar $a \geq 0$, the $\mathbf{u}$-weighted group intercentrality of order k of players $i_1, \ldots, i_k$ ($i_r \neq i_s$) of parameter a in $\mathbf{g}$ is $c^{\mathbf{u}}_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a) = \boldsymbol{\imath}'\mathbf{BE}\left(\mathbf{E}'\mathbf{BE}\right)^{-1}\mathbf{E}'\mathbf{b}_{\mathbf{u}}$, where $\mathbf{E} = \left(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}\right)$ and $\mathbf{b}_{\mathbf{u}}$ is the $\mathbf{u}$-weighted KB centrality vector.*

Notice that when $\mathbf{u} = \boldsymbol{\imath}$, the $\mathbf{u}$-weighted group intercentrality defined above boils down to the k-order group intercentrality measure in Definition 5.1. For a symmet-

[13] Theorem 2 in BCZ (2006) considers this link for the case of players' ex ante homogeneity (i.e., $\boldsymbol{\alpha} = \alpha\boldsymbol{\imath}$) and symmetric matrix of cross-effects (i.e., $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}'$).

ric adjacency matrix $\mathbf{G}$, the $\mathbf{u}$-weighted group intercentrality becomes (compare to (5.9))

$$c^{\mathbf{u},\text{sym}}_{\{i_1,\ldots,i_k\}}(\mathbf{g},a)=\mathbf{b}'\mathbf{E}(\mathbf{E}'\mathbf{B}\mathbf{E})^{-1}\mathbf{E}'\mathbf{b}_{\mathbf{u}}.$$

When $k=1$, the $\mathbf{u}$-weighted group intercentrality in Definition 5.3 reduces to $c^{\mathbf{u}}_i(\mathbf{g},a)=\boldsymbol{\iota}'\mathbf{B}\mathbf{e}_i\cdot b^{\mathbf{u}}_i/b_{ii}$, where $b^{\mathbf{u}}_i$ is the i-th element of $\mathbf{b}_{\mathbf{u}}(\mathbf{g},a)$. Further for a symmetric matrix of cross-effects $\boldsymbol{\Sigma}$ (or, equivalently, symmetric $\mathbf{G}$) we will have $c^{\mathbf{u},\text{sym}}_i(\mathbf{g},a)=b_ib^{\mathbf{u}}_i/b_{ii}$.

Similar to the unweighted intercentrality, the alternative formulation of the $\mathbf{u}$-weighted group intercentrality measure can be written as

$$c^{\mathbf{u}}_{\{i_1,\ldots,i_k\}}(\mathbf{g},a)=b_{\mathbf{u}}(\mathbf{g},a)-b_{\mathbf{u}}(\mathbf{g}^{-\{i_1,\ldots,i_k\}},a)+\sum_{s=1}^{k}u_{i_s}, \tag{5.12}$$

where $b_{\mathbf{u}}(\mathbf{g}^{-\{i_1,\ldots,i_k\}},a)=\boldsymbol{\iota}'\mathbf{B}(\mathbf{g}^{-\{i_1,\ldots,i_k\}},a)\mathbf{u}$. The KB centralities of the removed players are $b^{\mathbf{u}}_{i_s}(g^{-\{i_1,\ldots,i_k\}},a)=u_{i_s}$ for all $s=1,\ldots,k$ (see footnote 8) and should not be part of the group intercentrality measure, hence the sum $\sum_{s=1}^{k}u_{i_s}$ is added.[14]

It follows from (5.10) that $x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma})=b_{\boldsymbol{\alpha}}(\mathbf{g},a)/[\beta+\gamma b(\mathbf{g},a)]$. Thus, using (5.12) and the identity $\sum_{s=1}^{k}\alpha_{i_s}=\boldsymbol{\iota}'\mathbf{E}\mathbf{E}'\boldsymbol{\alpha}$, the overall equilibrium efforts when players $i_1,\ldots,i_k$ are removed from the network can be expressed in terms of *only* the initial network $\mathbf{g}$ as follows

$$\begin{aligned}
x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}}) &= \frac{b_{\boldsymbol{\alpha}}(\mathbf{g}^{-\{i_1,\ldots,i_k\}},a)}{\beta+\gamma b(\mathbf{g}^{-\{i_1,\ldots,i_k\}},a)}\\
&= \frac{b_{\boldsymbol{\alpha}}(\mathbf{g},a)+\sum_{s=1}^{k}\alpha_{i_s}-c^{\boldsymbol{\alpha}}_{\{i_1,\ldots,i_k\}}(\mathbf{g},a)}{\beta+\gamma\big(b(\mathbf{g},a)+k-c_{\{i_1,\ldots,i_k\}}(\mathbf{g},a)\big)}\\
&= \frac{\boldsymbol{\iota}'\left[\mathbf{B}+\mathbf{E}\mathbf{E}'-\mathbf{B}\mathbf{E}(\mathbf{E}'\mathbf{B}\mathbf{E})^{-1}\mathbf{E}'\mathbf{B}\right]\boldsymbol{\alpha}}{\beta+\gamma\boldsymbol{\iota}'\left[\mathbf{B}+\mathbf{E}\mathbf{E}'-\mathbf{B}\mathbf{E}(\mathbf{E}'\mathbf{B}\mathbf{E})^{-1}\mathbf{E}'\mathbf{B}\right]\boldsymbol{\iota}}\equiv r^{\boldsymbol{\alpha}}_{\{i_1,\ldots,i_k\}}(\mathbf{g},a,\gamma).
\end{aligned} \tag{5.13}$$

Equation (5.13) shows that the *residual* aggregate equilibrium activity without players $i_1,\ldots,i_k$, $r^{\boldsymbol{\alpha}}_{\{i_1,\ldots,i_k\}}(\mathbf{g},a,\gamma)$, besides depending on the main parameters of the model (i.e., $a=\lambda/\beta$ and γ), also depends on $\boldsymbol{\alpha}$, and on the unweighted and $\boldsymbol{\alpha}$-weighted KB centrality and group intercentrality measures. Hence,

Theorem 5.2. *If $\lambda\mu(\mathbf{G})+n\gamma(\overline{\alpha}/\underline{\alpha}-1)<\beta$, the key group $\{i^*_1,\ldots,i^*_k\}$ of size k that solves the problem $\min\left\{x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}})\,|\,\{i_1,\ldots,i_k\}\subseteq N; i_r\neq i_s\right\}$ matches the lowest*

[14] Thus, in case of players' exogenous homogeneity, $\mathbf{u}=\boldsymbol{\iota}$, we have $\sum_{s=1}^{k}u_{i_s}=k$, so that $c^{\mathbf{u}}_{\{i_1,\ldots,i_k\}}(\mathbf{g},a)$ reduces to the standard unweighted group intercentrality given in (5.7).

residual aggregate equilibrium activity of parameters a and γ *in the network* $\mathbf{g}$*, where* $1 \leq k \leq n$*, i.e.,* $r^{\boldsymbol{\alpha}}_{\{i_1^*,\ldots,i_k^*\}}(\mathbf{g}, a, \gamma) \leq r^{\boldsymbol{\alpha}}_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a, \gamma)$ *for all* $\{i_1, \ldots, i_k\} \subseteq N$ *and* $i_r \neq i_s$*.*

Note that in case of homogeneous observable characteristics of players (i.e., $\boldsymbol{\alpha} = \alpha\boldsymbol{\imath}$), the residual aggregate equilibrium activity boils down to (suppressing $(\mathbf{g}, a)$'s)

$$r_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a, \gamma) = \frac{\alpha\big(b + k - c_{\{i_1,\ldots,i_k\}}\big)}{\beta + \gamma\big(b + k - c_{\{i_1,\ldots,i_k\}}\big)} = \alpha\left[\gamma + \frac{\beta}{b + k - c_{\{i_1,\ldots,i_k\}}}\right]^{-1}, \quad (5.14)$$

hence no longer depends on the $\boldsymbol{\alpha}$-weighted group intercentralities and KB centralities. One can easily observe that in (5.14) for given k it is only the group intercentrality $c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a)$ that is changing when the identities of players $i_1, \ldots, i_k$ change, and since $\partial r_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a, \gamma)/\partial c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a) < 0$, minimization of the residual overall efforts with respect to the group of players $i_1, \ldots, i_k$ is exactly equivalent to such maximization of the group intercentrality measure. That is why Theorem 5.1 followed. But in the presence of players' ex ante heterogeneity, the overall residual equilibrium output is also a function of the $\boldsymbol{\alpha}$-weighted (inter)centralities, and, moreover, it depends in a *non*-additive way on $\boldsymbol{\alpha}$-weighted and unweighted group intercentralities and KB centralities. Hence it is the interaction of both $\boldsymbol{\alpha}$-weighted and standard KB centralities and group intercentralities together with given levels of concavity, global substitutability and local complementarity parameters that identify the key group (or the key player).

As a final remark, note that since $\alpha_i > 0$ for all $i = 1, \ldots, n$, all the properties of the unweighted intercentrality measure discussed in Section 5.2.2 also hold for the $\boldsymbol{\alpha}$-weighted intercentrality measure, except for the properties of isolate group and constant difference. The isolate group $i_1, \ldots, i_k$ has the group intercentrality equal to $c^{\boldsymbol{\alpha}}_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a) = \sum_{s=1}^{k} \alpha_{i_s}$. The difference in two $\boldsymbol{\alpha}$-weighted group intercentralities of orders $k + r$ and k, when removal of the corresponding groups totally nullify the new adjacency matrices, is no longer equal to r, and using (5.12) instead is $c^{\boldsymbol{\alpha}}_{\{j_1,\ldots,j_{k+r}\}}(\mathbf{g}, a) - c^{\boldsymbol{\alpha}}_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a) = \sum_{s=1}^{k+r} \alpha_{j_s} - \sum_{s=1}^{k} \alpha_{i_s}$, hence is not constant for different pairs of groups. This difference boils down to r if $\alpha_i = 1$ for all i. Further, the positivity property of the $\boldsymbol{\alpha}$-weighted group intercentrality holds because $c^{\boldsymbol{\alpha}}_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a) \geq \sum_{s=1}^{k} \alpha_{i_s}$ for all $k = 1, \ldots, n$.

Finally, Theorem 5.2 implies the following result if the planner's problem is $\max\big\{x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}}) \mid \{i_1, \ldots, i_k\} \subseteq N; i_r \neq i_s\big\}$ instead of (5.11).

Corollary 5.2. *If* $\lambda\mu(\mathbf{G}) + n\gamma(\overline{\alpha}/\underline{\alpha} - 1) < \beta$*, the key group* $\{i_1^*, \ldots, i_k^*\}$ *of size k that solves the problem* $\max\big\{x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}}) | \{i_1, \ldots, i_k\} \subseteq N; i_r \neq i_s\big\}$ *matches the highest* *residual aggregate equilibrium activity of parameters a and γ in $\mathbf{g}$, where $1 \leq k \leq n$, i.e., $r^{\boldsymbol{\alpha}}_{\{i^*_1,\ldots,i^*_k\}}(\mathbf{g}, a, \gamma) \geq r^{\boldsymbol{\alpha}}_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a, \gamma)$ for all $\{i_1, \ldots, i_k\} \subseteq N$.*

5.4 Optimal size of the key group

Up to this point the size of the group was assumed to be exogenously given. However, targeting a group of certain size besides bringing benefits, in general, also costs a planner time, money, energy, etc., the overall extent of which depends on the size of the group. For instance, if the aim of the planner is to maximally disrupt a criminal network by neutralizing a small subset of criminals, then the benefit of obtaining lower criminal activity in a society comes at the price of all kinds of costs related to planning and implementing the annihilating aim (i.e., information, time, people, instruments, etc.), and it is clear that both benefits and costs are largely dependent on the group size. Hence, since targeting groups of different sizes yields different benefits and costs, one may wish to study what is the optimal size of the key group that the planner should target on.

We assume that the planner, while targeting a *key* group of size k, receives a total benefit proportional to the group's contribution to the overall equilibrium activity, $\phi\left[x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) - x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_k\}})\right]$ with $\phi > 0$, and incurs the total cost of $f(k)$ for $k = 1, \ldots, n$. The cost function may be linear in the group's size, such as $f(k) = \nu k$ with $\nu > 0$, or we can have a convex cost function, say, given by $f(k) = \nu k^2$ so that the pattern of increasing marginal cost in the group size is captured. We assume that $\partial f(k)/\partial k > 0$. The planner optimally chooses the size of the group such that it maximizes the net gain, i.e.,

$$\max_k \phi\left[x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) - x^*_{\boldsymbol{\alpha}}\left(\boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_k\}}\right)\right] - f(k). \tag{5.15}$$

As long as $\mathbf{g} \neq \varnothing$, the key group will have members active in the network (non-isolates), hence $x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) > x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_k\}})$ as $\boldsymbol{\Sigma} > \boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_k\}}$ for all $k = 1, \ldots, n$ (this follows from Lemma 5.2). However, it follows from Lemma 5.2 that the benefit (the first part in (5.15)) is monotonically increasing in the key group size, since $x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_k\}}) \geq x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_{k+1}\}})$. This last inequality is not strict as it might very well happen that $\boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_k\}} = \boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_{k+1}\}}$ for some (large) k. Since both the benefit and the cost of group targeting are monotonically increasing in the size of the (key) group, it might very well happen that the net gain is always negative for all ranges of $k \in [1, n]$, thus the optimal size of the key group is zero. To avoid this uninteresting case of $k^* = 0$, we assume that there exists at least one $k > 0$ such that the condition $\phi\left[x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) - x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_k\}})\right] - f(k) > 0$ holds. Using (5.13) it can be shown that the last condition may be rewritten as

$$\frac{\phi}{f(k)} > \frac{(\beta+\gamma b)(\beta+\gamma(b+k-c_{i_1,\ldots,i_k}))}{(\beta+\gamma b)(c^{\boldsymbol{\alpha}}_{i_1,\ldots,i_k} - \sum_{s=1}^{k}\alpha_{i_s}) - \gamma b_{\boldsymbol{\alpha}}(c_{i_1,\ldots,i_k} - k)}, \tag{5.16}$$

where for simplicity we have suppressed the expression $(\mathbf{g}, a)$, and (5.16) in case of $\boldsymbol{\alpha} = \alpha \boldsymbol{\imath}$ reduces to

$$\frac{\phi}{f(k)} > \frac{(\beta+\gamma b)(\beta+\gamma(b+k-c_{i_1,\ldots,i_k}))}{\alpha\beta(c_{i_1,\ldots,i_k} - k)}. \tag{5.17}$$

The isolate group property implies that $c^{\boldsymbol{\alpha}}_{i^*_1,\ldots,i^*_k}(\mathbf{g}, a) = \sum_{s=1}^{k}\alpha_{i_s}$ and $c_{i^*_1,\ldots,i^*_k}(\mathbf{g}, a) = k$ if and only if each of $i^*_1, \ldots, i^*_k$ is an isolate player. However, with $\mathbf{g} \neq \varnothing$ it is always true that not all the members of the key group are isolate, hence the right-hand sides of (5.16) and (5.17) are well-defined. Problem (5.15) is equivalent to $\min\left\{\ell(\boldsymbol{\alpha}, \boldsymbol{\Sigma}, k) \mid k = 1, \ldots, n\right\}$, where a *loss of the key group of size k* is defined as $\ell(\boldsymbol{\alpha}, \boldsymbol{\Sigma}, k) \equiv \phi x^*_{\boldsymbol{\alpha}}\left(\boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_k\}}\right) + f(k)$. Hence, the loss of the key group of size k is equal to

$$\ell(\boldsymbol{\alpha}, \boldsymbol{\Sigma}, k) = \phi \cdot r^{\boldsymbol{\alpha}}_{\{i^*_1,\ldots,i^*_k\}}(\mathbf{g}, a, \gamma) + f(k),$$

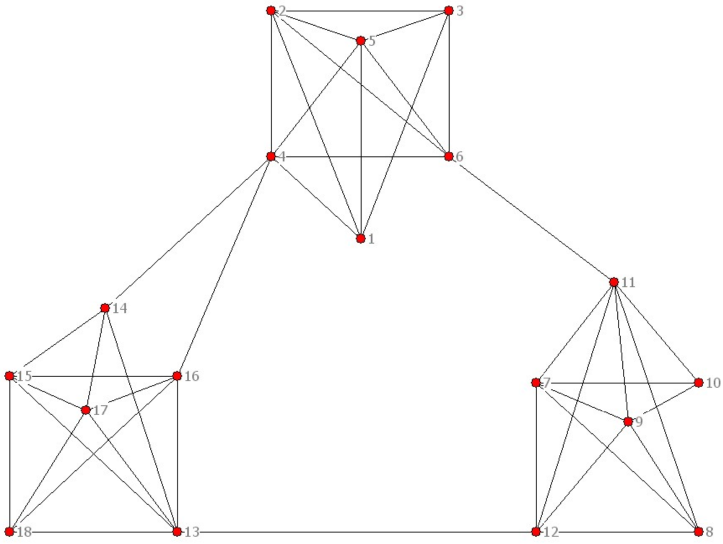
where we use the expressions in (5.13) and (5.14) for the residual aggregate equilibrium activity, $r^{\boldsymbol{\alpha}}_{\{i^*_1,\ldots,i^*_k\}}(\mathbf{g}, a, \gamma)$, with and without considering players' ex ante exogenous heterogeneity, respectively. Note also that now we are restricting our focus on *key* groups only.

The key group loss thus, in general, depends in a complex way on the $\boldsymbol{\alpha}$-weighted and unweighted (inter)centralities. It shows that an increase in k has two opposing effects on the key group loss. First, the loss decreases since the residual aggregate activity is monotonically decreasing in the group size (which follows from Lemma 5.2). Second, the loss of the key group goes up due to an increase in costs of targeting larger sized groups.[15] The following result is then obvious.

Theorem 5.3. *Assume that the net gain of targeting the key group of size k is given by* $\phi\left[x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) - x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}^{-\{i^*_1,\ldots,i^*_k\}})\right] - f(k)$, *where* $\phi > 0$, *and the condition in* (5.16) *holds*

[15] This resembles the well-known Baumol-Tobin model of money demand, where the total cost of holding money consists of: (*i*) forgone interests that decrease in the number of money withdrawals from a bank, and (*ii*) cost of withdrawals that is increasing in the number of trips to the bank.

Figure 5.2: A hypothetical example of the covert network



for at least one k. Then the optimal key group size k^* has the lowest key group loss, i.e., $\ell(\boldsymbol{\alpha}, \boldsymbol{\Sigma}, k^*) \leq \ell(\boldsymbol{\alpha}, \boldsymbol{\Sigma}, k)$ *for all* $k = 1, \ldots, n$.

5.5 Application to a covert network example

In this section we apply the key player/group problem and the key group loss measure for identifying the optimal group size to a hypothetical example of a covert network. The *covert network*, or the *red ream* consists of a set of small, but largely interconnected sets of agents with little links between the sets, which mimics the organizational structure of large terrorist organizations (see e.g., Krebs, 2002). An example of (a small part of) such a network is given in Figure 5.2, which consists of three densely intraconnected groups of six players each that are also weakly connected to each other.

Let us first consider the case of $\boldsymbol{\alpha} = \alpha\boldsymbol{\imath}$. Table 5.1 gives the KB centrality, individual and group intercentrality measures for $k \in [1, 4]$ and $a = 0.1$.[16] Since the graph of the network is undirected, we use the symmetric group intercentrality measure given in (5.9). Although players 4, 11 and 13 have the highest number of direct links

[16] The largest eigenvalue of the network in Figure 5.2 is equal to 4.894, hence the values of $a \in (0, 0.204)$ result in a well-defined and nonnegative matrix $\mathbf{B}$. The MATLAB program for computation of the group intercentrality measures is given in the appendix of Temurshoev (2008).

Table 5.1: Centrality and intercentrality measures

Rank	Player	b_i	Player	c_i^{sym}	Group of size 2	$c_{\{i_1,i_2\}}^{\text{sym}}$
1 (key)	13	2.161	4	4.282	{4,11}	8.307
2	4	2.156	13	4.269	{4,13}	8.297
3	11	2.130	11	4.152	{11,13}	8.284
4	16	2.009	16	3.748	{4,12}	7.954
	Group of size 3					$c_{\{i_1,i_2,i_3\}}^{\text{sym}}$
1 (key)	{4,11,13}					12.196
2	{2,11,13}, {5,11,13}					11.716
3	{4,11,15}, {4,11,17}					11.679
4	{4,7,13}, {4,9,13}					11.671
	Group of size 4					$c_{\{i_1,i_2,i_3,i_4\}}^{\text{sym}}$
1 (key)	{2,4,11,13}, {3,4,11,13}, {4,5,11,13}					14.685
2	{4,7,11,13}, {4,9,11,13}, {4,11,13,15}, {4,11,13,17}					14.575
3	{1,4,11,13}, {4,6,11,13}, {4,6,7,13}, {4,6,9,13}					14.320
4	{2,11,13,16}, {5,11,13,16}					14.311

Note: The intercentralities of all possible groups of size $k \in [1,4]$ were computed, which mathematically amount to the combinations of $n = 18$ players taken k at a time, $C_k^n = n!/(k!(n-k)!)$. Hence, all 18, 153, 816, and 3060 groups of size $k = 1,\ldots,4$ were considered, respectively.

(i.e., six direct contacts each), player 13 is the most central player (it has the highest KB centrality), while player 4 is the key player (it has the highest intercentrality). This outcome was already shown in a different example in BCZ (2006, Table 1), which implies that the most central player is not necessarily an optimal target for the social planner who seeks the key player - i.e., the player with the highest joint direct and indirect impact on aggregate equilibrium outcome.

Turning our attention to the key group problem, Table 5.1 clearly demonstrates that the key group of size 2 consists of actors 4 and 11, and that does not match with the top 2 players with the highest individual intercentralities (i.e, actors 4 and 13). Note that they are also the most central players indicated by their respective KB centralities. In this example, the key group of size 3 includes the three players with the largest individual intercentrality (and KB centrality) measures. This is, in fact, an expectable outcome because these three actors (i.e., 4, 11 and 13) play liaising roles in connecting the three subsets of the network in Figure 5.2, besides having the largest number of direct links. Note also that each actor belongs to a different subset. However, in the case of group size 4, the key group is again not comprised of players with the highest individual intercentralities (and KB centralities). Together with the players from the key group of size 3, the fourth actor is player 2 (3, and 5, respectively, because there are three key groups of size 4 with equal group intercentrality values) and not player 16. Moreover, the set of four players with the largest individual intercentralities appears only in the fifth rank with $c^{\text{sym}}_{\{4,11,13,16\}} = 14.254$ (not shown in Table 5.1).

Observe that in Table 5.1 all the members of the key group of size k are also included in the key group of size $k+1$. This is, however, a mere coincidence, and is not true in general. That is, the key group selection problem is *not* identical to a sequential key player problem.[17]

The lack of coincidence between the composition of the key group and the ranking based on the key player problem is due to a *redundancy principle* inherent to the majority of real life networks. Arguing that the information and control benefits of a large and *diverse* network are more than those of a small and homogeneous network, Burt (1992, p.17), for example, states: "What matters is the number of nonredundant contacts. Contacts are redundant to the extent that they lead to the same people, and so provide the same information benefits." In general, redundancy of players in a network may be with respect to adjacency, distance, and bridging (see e.g., Borgatti, 2006). One of the measures of redundancy is the notion of structural equivalence of nodes that reflects agents' similarity in terms of their linkages to third parties. In our case, some actors might be quite similar to each other in terms of their linking structure, and thus it is expected that the key group members consist of players that are relatively nonredundant.

To compare our results of the key group problem (with ex ante identical actors) with the notion of structural equivalence of players, we use a *hierarchical agglomerative cluster analysis* to identify groups of players that are similar in their patterns of ties to all other players (see e.g., Lattin et al., 2003, Chapter 8). Cluster analysis partitions actors to subgroups of perfectly or approximately structurally equivalent members. Each actor is initially considered as a singleton cluster, and then clusters are successively joined until all players merge into a single cluster. The process starts with constructing a so-called similarity matrix of players. We measure similarity of a pair of players by counting the proportion of their matches to all other actors.[18] The resulting similarity matrix is given in Table 5.2. The number 0.938 in the cell (2,1), for example, means that actors 1 and 2 have the same tie (present or absent) to other actors 93.8% of the time. That is, actors 1 and 2 have the same tie with actor $k \neq 1,2$ if $g_{1k} = g_{2k}$, where we have set $g_{ij} = g_{ji} = 1$ if there is a direct

[17] In fact, the point that the key group problem and the sequential key player problem are not equivalent is shown in Ballester et al. (2004, pp. 19-20).

[18] We also used a matrix of Euclidian distances to measure the "distance" or "dissimilarity" between the tie profiles of each pair of actors. The outcome of the cluster analysis *totally* coincides with that based on the similarity matrix of proportions of matches.

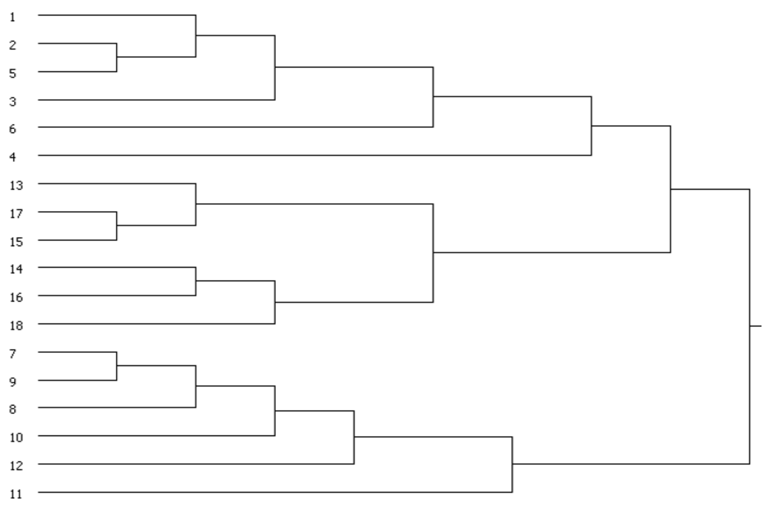
Table 5.2: Similarity matrix of the covert network in Figure 5.2

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1.000																	
2	0.938	1.000																
3	0.875	0.938	1.000															
4	0.750	0.813	0.875	1.000														
5	0.938	1.000	0.938	0.813	1.000													
6	0.938	0.875	0.813	0.688	0.875	1.000												
7	0.438	0.375	0.438	0.313	0.375	0.500	1.000											
8	0.500	0.438	0.500	0.375	0.438	0.563	0.938	1.000										
9	0.438	0.375	0.438	0.313	0.375	0.500	1.000	0.938	1.000									
10	0.563	0.500	0.563	0.438	0.500	0.625	0.875	0.938	0.875	1.000								
11	0.375	0.438	0.500	0.375	0.438	0.438	0.938	0.875	0.938	0.813	1.000							
12	0.438	0.375	0.438	0.313	0.375	0.500	0.875	0.938	0.875	0.875	0.813	1.000						
13	0.375	0.313	0.375	0.500	0.313	0.313	0.438	0.500	0.438	0.438	0.375	0.438	1.000					
14	0.625	0.563	0.500	0.500	0.563	0.563	0.438	0.500	0.438	0.563	0.375	0.563	0.750	1.000				
15	0.438	0.375	0.438	0.563	0.375	0.375	0.375	0.438	0.375	0.500	0.313	0.500	0.938	0.813	1.000			
16	0.563	0.500	0.438	0.438	0.500	0.500	0.375	0.438	0.375	0.500	0.313	0.500	0.813	0.938	0.875	1.000		
17	0.438	0.375	0.438	0.563	0.375	0.375	0.375	0.438	0.375	0.500	0.313	0.500	0.938	0.813	1.000	0.875	1.000	
18	0.500	0.438	0.500	0.500	0.438	0.438	0.438	0.500	0.438	0.563	0.375	0.563	0.875	0.875	0.938	0.938	0.938	1.000

connection between players i and j ($\neq i$). So in the example considered, there are 15 out of 16 links (absent and present) that are the same for players 1 and 2 (thus, $15/16 = 0.9375$). Hence, the higher this score the more similar are the players in a particular pair.

Next the agglomerative cluster analysis starts from n clusters of size 1 and at each stage of the process finds the two "closest" (most homogeneous) clusters and joins them together. For example, the pairs of players $\{2, 5\}$, $\{17, 15\}$ and $\{7, 9\}$ are perfectly structurally equivalent (i.e., the corresponding cells are equal to 1.000 in Table 5.2). So these pairs are joined together to make the first three clusters of two actors each. On the basis of new similarity matrices this process continues until only one cluster of size n remains, and this hierarchical sequence of merging clusters is visually depicted by a tree diagram, also called a *dendrogram*. We have used the *average link* criteria for forming clusters, which computes the similarity of the average scores in the newly formed cluster to all other partitions.[19] The resulting tree diagram is given in Figure 5.3.

Figure 5.3: Hierarchical dendrogram of the covert network in Figure 5.2



[19] There are two other basic criteria for forming clusters: *single link* and *complete link*. The single (complete) method computes similarities on the basis of the similarity of the member of the new cluster that is most (least) similar to each other cluster. While the single link approach is too myopic, the complete link method tends to give highly separated diagrams with tightly bound clusters. Hence, the average linkage is a sort of compromise between the single and the complete linkages. "Some authors prefer this method [average approach] because it comes closest to fitting a tree that satisfies a least squares minimization criterion" (Lattin et al., 2003, p. 282).

As can be seen from Figure 5.3, nonoverlapping clusters are a product of the hierarchical agglomerative cluster analysis, i.e., the smaller clusters are subsumed within successively larger clusters at higher levels of agglomeration. It is clear that higher values of agglomeration indicate lower structural equivalence, less similarity, or greater within-cluster "distance". However, for our purposes we are not interested in choosing the level of agglomeration that provides the "best" representation of the number of structurally equivalent positions in the network. Instead we aim at confirming or rejecting our conjecture that the key group includes less structurally equivalent (nonredundant) players that are exogenously identical.

The dendrogram in Figure 5.3 identifies two clusters at relatively high agglomeration level: $\{7, 8, 9, 10, 11, 12\}$ and the rest subsuming the second cluster. Note that the two actors with the highest individual intercentrality measures (i.e., players 4 and 13) are both members of the second cluster, hence are more homogeneous in their tieing structure than a pair of players from the two different clusters. As we expected, the key group of size 2 consists of players 4 and 11 that are part of the two different clusters, thus being less redundant with respect to each other than the pair $\{4, 13\}$. Moreover, within these two clusters, respectively, actors 4 and 11 are less similar to all other members, which is shown by the fact that they join their clusters only at the highest level of agglomeration. Similarly, the relatively higher level of similarity produces three clusters: $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{7, 8, 9, 10, 11, 12\}$, and $C = \{13, 14, 15, 16, 17, 18\}$. Again, the members of the key group of size 3 (i.e., players 4, 11, and 13) are each part of one of these three different clusters. A higher similarity level (i.e., lower agglomeration level) disaggregates cluster A into two sets of relatively homogeneous players by identifying actor 4 as one cluster and the rest as the second (while clusters B and C remain unchanged). Hence, our key group of size 4 besides actors 4, 11, 13 picks one player from the remaining part of cluster A, i.e., from $\{1, 2, 3, 5, 6\}$.

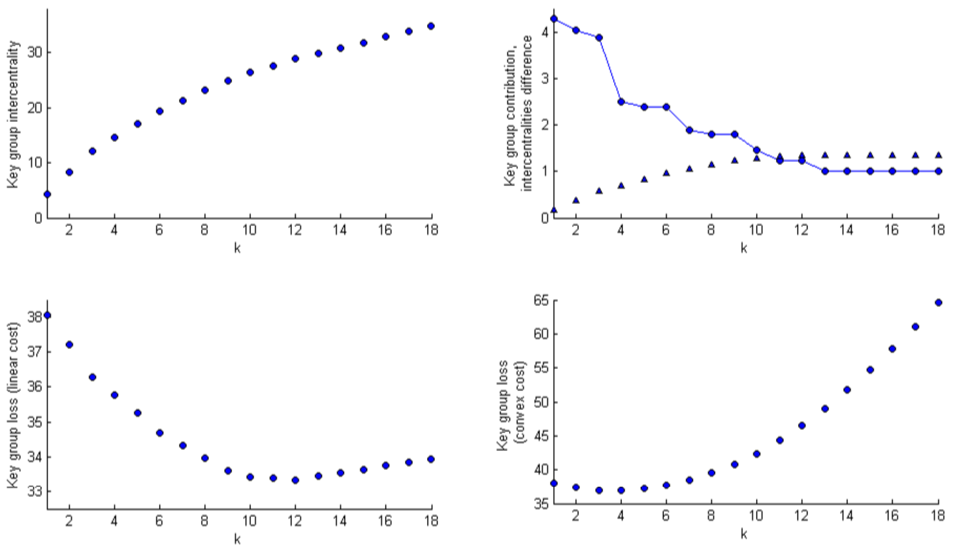
However, this coincidence of the key group problem and the cluster analysis does *not* hold in general. In particular, for a larger size of the key group and lower levels of agglomeration the two approaches yield different results. First, observe from Figure 5.3 that one *cannot* disaggregate the players into 6, 9, 10, 12, 13, 14, 16 and 17 clusters as certain agglomeration (similarity) levels of the partitions are equal. Second, not all non-similar actors (i.e., actors from different clusters) comprise the key group of a larger size. For example, let us take the key group of size 7. There are twelve key groups of size 7 with group intercentralities equal to 21.321. One of these groups is $\{2, 3, 4, 7, 11, 13, 15\}$. From Figure 5.3 we have the following seven clusters: $\{1,2,3,5\}$, $\{4\}$, $\{6\}$, $\{13,15,17\}$, $\{14,16,18\}$, $\{7,8,9,10,12\}$ and $\{11\}$. Hence, four members of the given key group of size 7 are part of only two clusters (players 2 and 3 are in one cluster, 13 and 15 are in another), while there is no key group member from clusters $\{6\}$ and $\{14,16,18\}$. In fact, these last two clusters are not represented in any of the other eleven key groups of size 7. These findings clearly show that using cluster analysis for finding key groups in networks would be misleading, since the right candidate for the key group should not only have a diverse *direct* linking structure, but also a diverse *indirect* impact on the rest of the system. Another reason why cluster analysis cannot exactly determine the key group members is that different criteria for forming clusters may very well give different outcomes. All in all, however, despite these inconsistencies, the key group problem and cluster analysis are related in the sense that the key group members are *to some extent* less structurally equivalent. In particular, members of the key group of a smaller size are rather nonredundant with respect to each other in terms of their linking patterns in the network.[20]

In our next step, the *key* group intercentralities for all $k = 1, \ldots, 18$ of our covert network example are graphed in Figure 5.4 (upper left figure), where it is assumed that $\alpha = \beta = 1$ and $\gamma = 0.1$. Key group intercentrality sequentially increases from $c_4(\mathbf{g}, 0.1) = 4.282$ up to $c_{\{1,\ldots,18\}}(\mathbf{g}, 0.1) = b(\mathbf{g}, 0.1) = 34.848$, which reflects the strict monotonicity property of the group intercentrality measure. In the upper right figure we graph the contribution of the key group to the overall equilibrium activity (depicted by triangles), i.e., $x^*(\boldsymbol{\Sigma}) - x^*(\boldsymbol{\Sigma}^{-\{i_1^*,\ldots,i_k^*\}})$ for all $k = 1, \ldots, n$, and the difference in the key group intercentralities (depicted by a line). This difference, $c_{\{i_1^*,\ldots,i_k^*\}}(\mathbf{g}, 0.1) - c_{\{i_1^*,\ldots,i_{k-1}^*\}}(\mathbf{g}, 0.1)$, is decreasing for $k = 1, \ldots, 12$, and remains constant at the value of one afterwards. This is because removing key group of size $k = 12, \ldots, 18$ totally nullifies the new adjacency matrix in our example, which is what we have called the constant difference property of the group intercentrality measure in Section 5.2.2. Note that this also implies that all the key groups of size $k \in [12, 18]$ have equal contributions to the aggregate equilibrium output, which is equal to 1.342 (i.e., $x^*(\boldsymbol{\Sigma}) - x^*(\boldsymbol{\Sigma}^{-\{i_1^*,\ldots,i_k^*\}}) = \alpha b(\mathbf{g}, 0.1)/(\beta + \gamma b(\mathbf{g}, 0.1)) - \alpha n/(\beta + \gamma n)$).

To find out what is the optimal size of the key group, we determine the value

[20] Another main reason of (possible) different results of the key group problem and cluster analysis for groups of larger size, is that, in general, the key group selection problem is not identical to a sequential key player problem. Thus, comparing the key group outcomes to those from the hierarchical agglomerative cluster analysis when the size of the considered group is rather large, in general, makes little sense. This is because in cluster analysis once actors are part of a group they will never leave it and only additional actors join the group at a higher level of agglomeration.

Figure 5.4: Key group intercentrality and losses ($\alpha = \beta = 1$, $\gamma = \nu = 0.1$ and $\phi = 5$)



for k that gives the minimum key group loss (Theorem 5.3). The bottom part of Figure 5.4 graphs the key group losses based on the cost functions of $f(k) = \nu k$ and $f(k) = \nu k^2$, respectively, where $\nu = 0.1$ and $\phi = 5$. It shows that in case of the linear cost function, the optimal size of the key group is $k_l^* = 12$, while that in case of the convex cost function is $k_c^* = 3$. Hence, the form of the cost function $f(k)$ plays a crucial role in identifying the optimal size of the key group. This is an expectable outcome, since with convex cost functions marginal cost is increasing in k, thus targeting higher order key groups might easily lead to negative net gains for the planner.

We should also note that when targeting groups of players is costless, i.e., $f(k) = 0$ for all $k = 1, \ldots, n$, then the smallest size k that results in the largest contribution to the overall activity level depends on whether there are isolate players in the network present or not. For a connected network, the easiest optimal choice is simply the group consisting of all players, i.e., $k^* = n$. If, for example, in our example with linear cost function we would have $\nu = 0$, then the optimal group size is $k^* = 12, \ldots, 18$. All the key groups of size $k = 12, \ldots, 18$ give an equal net benefit (of 6.038) because, as already mentioned above, they all contribute equally to the overall equilibrium outcome.

Finally, let us consider the case of ex ante exogenous heterogeneity of players, i.e., $\boldsymbol{\alpha} \neq \alpha\boldsymbol{\iota}$, and assume that $\alpha_i = 1.27$ for $i = 7, 8, 9, 10$ and unity otherwise. From Figure 5.2 it is clear that we assume that the four players in the bottom right subset of the covert network (i.e., players 7, 8, 9 and 10) are given more weight in their exogenous heterogeneity part, while the weights of the rest remain unchanged and identical across players. The four largest $\boldsymbol{\alpha}$-weighted KB centralities and smallest residual equilibrium outputs with corresponding players or group of players for $k \in [1, 4]$ and $a = 0.1$ are reported in Table 5.2. The $\boldsymbol{\alpha}$-weighted KB centrality identifies players 7 and 9 to be the most central actors. Comparing this with the results of the unweighted KB centralities given in Table 5.1 suggests a big difference in the outcomes. In particular, players 7 and 9 (together with 2 and 5) have only the 8-th largest unweighted KB centrality of $b_i = 1.948$ (not shown in Table 5.1).

Table 5.3: The $\boldsymbol{\alpha}$-weighted centrality and residual aggregate activity

R.	Player	$b_i^{\boldsymbol{\alpha}}$	Player	r_i	Group of size 2	$r_{\{i_1,i_2\}}$
1	7; 9	2.3676	7; 9	7.9888	{4,9}, {4,7}	7.8200
2	11	2.3056	11	7.9924	{7,13}, {9,13}	7.8230
3	8	2.1881	12	8.0320	{11,13}	7.8231
4	13	2.1759	8	8.0342	{4,11}	7.8279

R.	Group of size 3	$r_{\{i_1,i_2,i_3\}}$	Group of size 4	$r_{\{i_1,\ldots,i_4\}}$
1	{4,7,13}, {4,9,13}	7.6409	{4,7,11,13}, {4,9,11,13}	7.4577
2	{4,11,13}	7.6452	{4,7,9,13}	7.4652
3	{4,7,11}, {4,9,11}	7.6537	{2,7,11,13}, {5,7,11,13}, {2,9,11,13}, {5,9,11,13}	7.4929
4	{7,11,13}, {9,11,13}	7.6568	{4,7,11,15}, {4,9,11,15}, {4,9,11,17}, {4,7,11,17}	7.4937

Note: "R." stands for rank. As before it is assumed that $\alpha = \beta = 1$, $a = \gamma = \nu = 0.1$ and $\phi = 5$. Given these values, the sufficiency condition in Theorem 5.2 for the equilibrium uniqueness and interiority is satisfied. See also notes to Table 5.1.

From Theorem 5.2 it follows the the key player/group has the *lowest* residual overall equilibrium output, $r^{\boldsymbol{\alpha}}_{\{i_1,\ldots,\}}(\mathbf{g}, a, \gamma)$, given in (5.13). Table 5.3 shows that the key player, when exogenous heterogeneity of all actors is taken into account, is player 7 (or, 9), and *not* player 4 as identified in Table 5.1. This is an expectable outcome, since from the "heavy" participants with the largest heterogeneity values (i.e., from players 7, 8, 9 and 10) actors 7 and 9 have the largest number of direct contacts (i.e., players 7 and 9 both have 5 direct contacts, while 9 and 10 have only 3 connections). The key group of size two consists of pairs $\{4, 7\}$ and $\{4, 9\}$. Again the corresponding key group $\{4, 11\}$ from Table 5.1 takes now the 4-th rank in Table 5.3 (note that the same rank may share more than one group). Further, Table 5.2 shows that the similarity score is 0.313 for both pairs $\{4, 7\}$ and $\{4, 9\}$, and is 0.375 for $\{4, 11\}$. This means that actors 4 and 7 (or 4 and 9) are less similar in their linking patterns to their players than actors 4 and 11. Hence, with ex ante heterogeneity, both the size of exogenous weights and linking patterns of players determine the key group members. Note also that the number of key groups can be different with and without considering individuals' heterogeneity. For example, there are two key groups of size 4 in Table 5.3, while neglecting exogenous heterogeneity identifies three key groups of that size in Table 5.1. Similarly, comparing members of the key groups of larger sizes confirms that taking exogenous characteristics of individuals into account results in entirely different outcomes if compared to those based on the homogeneity assumption.

5.6 Concluding comments

In this chapter we focused on a network game studied by Ballester et al. (2004), where a *group intercentrality measure* identifies the *key group*. That is, a set of players which, once removed, has the largest (or smallest) impact on the overall activity level. We derived an alternative closed-form expression for the group intercentrality measure that depends only on the initial network configuration. This generalizes the key player problem in Ballester et al. (2006) from a search of a single player to a group selection problem targeting an arbitrary number of players. Further, unlike the mentioned studies, we consider the key player/group problem taking into account players' ex ante exogenous heterogeneity. It is shown that the results may change dramatically if such heterogeneity is neglected. From a practical perspective, this suggests that individual, observable differences must be taken into account, otherwise the outcome may be considerably biased. Finally, we endogenize the size of the key group by taking into account the benefits and costs of a planner in targeting key groups of different sizes. The optimal size of the key group gives the lowest key group loss (or, equivalently, the highest net benefit) from the social planner's perspective.

Since the equilibrium efforts depend on *weighted* Katz-Bonacich centrality and weighted group intercentrality measures, and since the weights represent observable characteristics of players (and their peers), the question arises how in practice one can deal with the multidimensionality of players' ex ante heterogeneity (e.g., education, age, gender, etcetera). One solution would be to identify as many key players/groups as the number of observable characteristics of individuals. This would be helpful in cases where the individuals are targeted only on the basis of the *specific* characteristics that concern the planner. On the other hand, if the analyst were interested in a general indicator of the players' importance in networks, the obvious solution would be to transform all the observable characteristics into one (or a few) weight variable(s), such that it accounts for as much variability in the original weights as possible. For reducing the number of exogenous heterogeneity dimensions one could, for example, use principal component analysis. Once such weight is (or, few weights are) generated, finding the key player/group in networks is straightforward as discussed in this chapter.

Possible empirical applications of the key group problem with endogenous group size depend on the research question and the network content. In any case they will aim at finding a group of players with the largest (or smallest) influence over the aggregate activity level, which is a target to be optimized by the social planner. It is particularly useful for addressing such kind of issues in economics, because the notions of competition and complementarity due to the network embeddedness are explicitly taken into account. Examples include the analysis of crime networks (Ballester et al., 2004; Calvó-Armengol and Zenou, 2004), conformism and social norms (Bernheim, 1994; Akerlof, 1997), firms' collaboration networks (Goyal and Moraga-González, 2001; Goyal and Joshi, 2003), networks of interlocking directorates (Dooley, 1969; Mizruchi, 1996; Heemskerk and Schnyder, 2008), and coauthor networks (Goyal et al., 2006).

A final remark is that our analysis is not restricted to linear-quadratic utilities that incorporate the externalities of players' actions linearly. For a general utility function that captures nonlinear externalities, a decomposition similar to (5.2) can be made. This in turn implies that the first-order approximation of the levels of players' actions will correspond to the Katz-Bonacich centrality measures. Also, the entire analysis was done for a given network. Endogenizing the network decision is possible in a two-stage game, where in the first stage players decide whether to stay in the network or leave it for some outside option. In the second stage the network game is played by the remaining actors. This is particularly useful for the analysis of the effects of different policies in addressing the same issue. Such a study was undertaken by Ballester et al. (2004), who showed that the policy of increasing wages raises the effectiveness of the key player policy in reducing crime.

5.A Proofs

Proof of Lemma 5.1. From the monotonicity of the largest eigenvalue with respect to the coefficients of the matrix it follows that $\mu(\mathbf{G}) \geq \mu(\mathbf{G}^{-\{i_1,\ldots,i_k\}})$.[21] Thus, if $\mathbf{B}$ exists and is nonnegative, so is $\mathbf{B}^{-\{i_1,\ldots,i_k\}}$ for all $\{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\}$ $(i_r \neq i_s)$. Define $\mathbf{A} \equiv a\mathbf{G}$, hence $\mathbf{A}^{-\{i_1,\ldots,i_k\}} = a\mathbf{G}^{-\{i_1,\ldots,i_k\}}$. Without loss of generality, we partition the matrices $\mathbf{A}$ and $\mathbf{A}^{-\{i_1,\ldots,i_k\}}$ (or, equivalently, $\mathbf{B}$ and $\mathbf{B}^{-\{i_1,\ldots,i_k\}}$) in such a way that the k removed players constitute their upper left submatrices. Then from the theory of partitioned matrices it follows that

$$\mathbf{B}^{-\{i_1,\ldots,i_k\}} = \begin{bmatrix} \mathbf{I}_k & \mathbf{O}_{kt} \\ \mathbf{O}_{tk} & \mathbf{I}_t - \mathbf{A}_{tt} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I}_k & \mathbf{O}_{kt} \\ \mathbf{O}_{tk} & (\mathbf{I}_t - \mathbf{A}_{tt})^{-1} \end{bmatrix}, \tag{5.A.1}$$

where, for example, $\mathbf{I}_t$ is the t-dimensional identity matrix, $\mathbf{O}_{kt}$ is the $k \times t$ null matrix, and $k + t = n$.

We know that $\mathbf{B} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots$, hence $\mathbf{B} - \mathbf{I} = \mathbf{A} + \mathbf{A}^2 + \cdots = \mathbf{AB}$, which in terms of the above-defined partitioned matrices yields the following identity

$$\begin{aligned} \begin{bmatrix} \mathbf{B}_{kk} - \mathbf{I}_k & \mathbf{B}_{kt} \\ \mathbf{B}_{tk} & \mathbf{B}_{tt} - \mathbf{I}_t \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_{kk} & \mathbf{A}_{kt} \\ \mathbf{A}_{tk} & \mathbf{A}_{tt} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{kk} & \mathbf{B}_{kt} \\ \mathbf{B}_{tk} & \mathbf{B}_{tt} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}_{kk}\mathbf{B}_{kk} + \mathbf{A}_{kt}\mathbf{B}_{tk} & \mathbf{A}_{kk}\mathbf{B}_{kt} + \mathbf{A}_{kt}\mathbf{B}_{tt} \\ \mathbf{A}_{tk}\mathbf{B}_{kk} + \mathbf{A}_{tt}\mathbf{B}_{tk} & \mathbf{A}_{tk}\mathbf{B}_{kt} + \mathbf{A}_{tt}\mathbf{B}_{tt} \end{bmatrix}. \end{aligned}$$

The identities in the second row blocks above result in $\mathbf{B}_{tk} = (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}\mathbf{A}_{tk}\mathbf{B}_{kk}$ and $\mathbf{B}_{tt} = (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}(\mathbf{I}_t + \mathbf{A}_{tk}\mathbf{B}_{kt})$, which are, respectively, equivalent to $\mathbf{B}_{tk}\mathbf{B}_{kk}^{-1} = (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}\mathbf{A}_{tk}$ and $\mathbf{B}_{tt} - (\mathbf{I}_t - \mathbf{A}_{tt})^{-1} = (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}\mathbf{A}_{tk}\mathbf{B}_{kt}$. Thus using the first equation in the second yields $\mathbf{B}_{tt} - (\mathbf{I}_t - \mathbf{A}_{tt})^{-1} = \mathbf{B}_{tk}\mathbf{B}_{kk}^{-1}\mathbf{B}_{kt}$. This together with (5.A.1) imply

$$\begin{aligned} \mathbf{B} - \mathbf{B}^{-\{i_1,\ldots,i_k\}} &= \begin{bmatrix} \mathbf{B}_{kk} - \mathbf{I}_k & \mathbf{B}_{kt} \\ \mathbf{B}_{tk} & \mathbf{B}_{tk}\mathbf{B}_{kk}^{-1}\mathbf{B}_{kt} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{B}_{kk}\mathbf{B}_{kk}^{-1}\mathbf{B}_{kk} - \mathbf{I}_k & \mathbf{B}_{kk}\mathbf{B}_{kk}^{-1}\mathbf{B}_{kt} \\ \mathbf{B}_{tk}\mathbf{B}_{kk}^{-1}\mathbf{B}_{kk} & \mathbf{B}_{tk}\mathbf{B}_{kk}^{-1}\mathbf{B}_{kt} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{B}_{kk} \\ \mathbf{B}_{tk} \end{bmatrix} \mathbf{B}_{kk}^{-1} \begin{bmatrix} \mathbf{B}_{kk} & \mathbf{B}_{kt} \end{bmatrix} - \begin{bmatrix} \mathbf{I}_k & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \\ &= \mathbf{BE}(\mathbf{E}'\mathbf{BE})^{-1}\mathbf{E}'\mathbf{B} - \mathbf{EE}', \end{aligned} \tag{5.A.2}$$

[21] This follows from Theorem I* in Debreu and Herstein (1953, p. 600).

where $\mathbf{E}$ is the $n \times k$ matrix defined as $\mathbf{E} = (\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k})$ with $\mathbf{e}_i$ being the i-th identity column.

The final result in (5.A.2) shows that the partitioning (i.e., having the k eliminated players in the upper left block diagonal matrix) is quite arbitrary, hence the result holds for any non-ordered matrix $\mathbf{B}$. Moreover, the set of players $i_1, \ldots, i_k$ can be arbitrarily ordered in the matrix $\mathbf{B}_{kk}$ as well. Equation (5.A.2) proves that for all h and all l we have

$$b_{hl} - b_{hl}^{-\{i_1,\ldots,i_k\}} = \mathbf{e}_h'\mathbf{B}\mathbf{E}(\mathbf{E}'\mathbf{B}\mathbf{E})^{-1}\mathbf{E}'\mathbf{B}\mathbf{e}_l - \mathbf{e}_h'\mathbf{e}_l, \tag{5.A.3}$$

where, the matrix $\mathbf{B}$ is *not* necessarily partitioned between $\{i_1, \ldots, i_k\}$ and the rest of the players (as, for instance, in (5.A.2)). This completes the proof of Lemma 5.1. ■

Proof of Theorem 5.1. From (5.3) it follows that the aggregate equilibrium activity is equal to $x^*(\boldsymbol{\Sigma}) = \alpha b(\mathbf{g}, a)/(\beta + \gamma b(\mathbf{g}, a))$. Hence, for $\alpha > 0$ we have that

$$\frac{\partial x^*(\boldsymbol{\Sigma}^{-\{i_1,\ldots,i_k\}})}{\partial b(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a)} = \frac{\alpha\beta}{\left(\beta + \gamma b(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a)\right)^2} > 0.$$

This in turn implies that the key group problem given in (5.6) is exactly equivalent to the problem $\min\{b(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) \mid \{i_1, \ldots, i_k\} \subseteq N; i_r \neq i_s\}$, which has the same solution as $\max\left\{b(\mathbf{g}, a) - b(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) \mid \{i_1, \ldots, i_k\} \subseteq N; i_r \neq i_s\right\}$. Using the definition of the KB centrality, Lemma 5.1, $\mathbf{B}_{kk}^{-1} = (\mathbf{E}'\mathbf{B}\mathbf{E})^{-1}$, and the fact that $\boldsymbol{\iota}'\mathbf{E}\mathbf{E}'\boldsymbol{\iota} = k$, we have

$$\begin{aligned} b(\mathbf{g}, a) - b(\mathbf{g}^{-\{i_1,\ldots,i_k\}}, a) &= \boldsymbol{\iota}'\left(\mathbf{B} - \mathbf{B}^{-\{i_1,\ldots,i_k\}}\right)\boldsymbol{\iota} \\ &= \boldsymbol{\iota}'\left(\mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{E}'\mathbf{B} - \mathbf{E}\mathbf{E}'\right)\boldsymbol{\iota} = \boldsymbol{\iota}'\mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{E}'\mathbf{b} - k. \end{aligned} \tag{5.A.4}$$

For fixed k, players $i_1, \ldots, i_k$ that maximize (5.A.4), also maximize $\boldsymbol{\iota}'\mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{E}'\mathbf{b}$, which is exactly the intercentrality measure $c_{\{i_1,\ldots,i_k\}}(\mathbf{g}, a)$ in Definition 5.1. This completes the proof. ■

Proof of the subadditivity property. It is not difficult to show that $\sum_{s=1}^{k} \frac{1}{b_{i_s i_s}} \mathbf{e}_{i_s}\mathbf{e}_{i_s}' = \mathbf{E}\mathbf{E}'\widehat{\mathbf{B}}^{-1}\mathbf{E}\mathbf{E}'$, where $\widehat{\mathbf{B}}$ is the diagonal matrix with b_{ii} on its main diagonal and zeros elsewhere. Thus, the sum of the individual intercentralities of k players is $\sum_{s=1}^{k} c_{i_s}(\mathbf{g}, a) = \sum_{s=1}^{k} \frac{1}{b_{i_s i_s}} \boldsymbol{\iota}'\mathbf{B}\mathbf{e}_{i_s}\mathbf{e}_{i_s}'\mathbf{b} = \boldsymbol{\iota}'\mathbf{B}\left(\sum_{s=1}^{k} \frac{1}{b_{i_s i_s}} \mathbf{e}_{i_s}\mathbf{e}_{i_s}'\right)\mathbf{b} = \boldsymbol{\iota}'\mathbf{B}\mathbf{E}\mathbf{E}'\widehat{\mathbf{B}}^{-1}\mathbf{E}\mathbf{E}'\mathbf{B}\boldsymbol{\iota}$. Further, using the definition of the group intercentrality, the identity $\mathbf{E}'\widehat{\mathbf{B}}^{-1}\mathbf{E}\mathbf{E}' = \mathbf{E}'\widehat{\mathbf{B}}^{-1}$, and Lemma 5.1, we obtain

$$
\begin{aligned}
\Delta \equiv \sum_{s=1}^{k} c_{i_s}(\mathbf{g}, a) - c_{i_1,i_2,\ldots,i_k}(\mathbf{g}, a) &= \boldsymbol{\iota}' \left[\mathbf{BEE}'\widehat{\mathbf{B}}^{-1}\mathbf{EE}'\mathbf{B} - \mathbf{BE}(\mathbf{E}'\mathbf{BE})^{-1}\mathbf{E}'\mathbf{B}\right]\boldsymbol{\iota} \\
&= \boldsymbol{\iota}' \left[\mathbf{BEE}'\widehat{\mathbf{B}}^{-1}\mathbf{B} - \mathbf{B} + \mathbf{B}^{-\{i_1,\ldots,i_k\}} - \mathbf{EE}'\right]\boldsymbol{\iota} \\
&= \boldsymbol{\iota}' \left[\mathbf{B}(\mathbf{EE}'\mathbf{C} - \mathbf{I}) + \mathbf{B}^{-\{i_1,\ldots,i_k\}} - \mathbf{EE}'\right]\boldsymbol{\iota},
\end{aligned}
\tag{5.A.5}
$$

where $\mathbf{C} \equiv \widehat{\mathbf{B}}^{-1}\mathbf{B}$.

Now as in the proof of Lemma 5.1 above, we partition, without loss of generality, the matrices $\mathbf{A}$ and $\mathbf{A}^{-\{i_1,\ldots,i_k\}}$ (equivalently, $\mathbf{B}$ and $\mathbf{B}^{-\{i_1,\ldots,i_k\}}$) in such a way that the k removed players constitute their upper left submatrices. Then from the theory of partitioned matrices it readily follows that

$$
\begin{aligned}
\mathbf{EE}'\mathbf{C} &= \begin{bmatrix} \mathbf{I}_k & \mathbf{O}_{kt} \\ \mathbf{O}_{tk} & \mathbf{O}_{tt} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{kk} & \mathbf{C}_{kt} \\ \mathbf{C}_{tk} & \mathbf{C}_{tt} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{kk} & \mathbf{C}_{kt} \\ \mathbf{O}_{tk} & \mathbf{O}_{tt} \end{bmatrix}, \\
\mathbf{B}(\mathbf{EE}'\mathbf{C} - \mathbf{I}) &= \begin{bmatrix} \mathbf{B}_{kk}(\mathbf{C}_{kk} - \mathbf{I}_k) & \mathbf{B}_{kk}\mathbf{C}_{kt} - \mathbf{B}_{kt} \\ \mathbf{B}_{tk}(\mathbf{C}_{kk} - \mathbf{I}_k) & \mathbf{B}_{tk}\mathbf{C}_{kt} - \mathbf{B}_{tt} \end{bmatrix}, \\
\mathbf{B}^{-\{i_1,\ldots,i_k\}} - \mathbf{EE}' &= \begin{bmatrix} \mathbf{O}_{kk} & \mathbf{O}_{kt} \\ \mathbf{O}_{tk} & (\mathbf{I}_t - \mathbf{A}_{tt})^{-1} \end{bmatrix},
\end{aligned}
$$

where in the last expression we have used (5.A.1). Thus, using the above partition, (5.A.5) can be rewritten as

$$
\Delta = \boldsymbol{\iota}' \begin{bmatrix} \mathbf{B}_{kk}(\mathbf{C}_{kk} - \mathbf{I}_k) & \mathbf{B}_{kk}\mathbf{C}_{kt} - \mathbf{B}_{kt} \\ \mathbf{B}_{tk}(\mathbf{C}_{kk} - \mathbf{I}_k) & \mathbf{B}_{tk}\mathbf{C}_{kt} - \mathbf{B}_{tt} + (\mathbf{I}_t - \mathbf{A}_{tt})^{-1} \end{bmatrix} \boldsymbol{\iota}. \tag{5.A.6}
$$

The matrix in the upper right block of (5.A.6) may be written as $\mathbf{B}_{kk}\mathbf{C}_{kt} - \mathbf{B}_{kt} = (\mathbf{B}_{kk} - \widehat{\mathbf{B}}_{kk})\mathbf{C}_{kt}$. Further, to "simplify" the matrix in the bottom right block of (5.A.6) we make use of the identities $\mathbf{B}_{tk} = (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}\mathbf{A}_{tk}\mathbf{B}_{kk}$ and $\mathbf{B}_{tt} = (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}(\mathbf{I}_t + \mathbf{A}_{tk}\mathbf{B}_{kt})$ derived in the proof of Lemma 5.1 above. Thus, $\mathbf{B}_{tk}\mathbf{C}_{kt} - \mathbf{B}_{tt} + (\mathbf{I}_t - \mathbf{A}_{tt})^{-1} = (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}\mathbf{A}_{tk}\mathbf{B}_{kk}\mathbf{C}_{kt} - (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}\mathbf{A}_{tk}\mathbf{B}_{kt} = (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}\mathbf{A}_{tk}(\mathbf{B}_{kk} - \widehat{\mathbf{B}}_{kk})\mathbf{C}_{kt}$. Further, $\mathbf{C}_{kk} - \mathbf{I}_k = \widehat{\mathbf{B}}_{kk}^{-1}(\mathbf{B}_{kk} - \widehat{\mathbf{B}}_{kk})$. Therefore, (5.A.6) is equal to

$$
\Delta = \boldsymbol{\iota}' \begin{bmatrix} \mathbf{B}_{kk}\widehat{\mathbf{B}}_{kk}^{-1}(\mathbf{B}_{kk} - \widehat{\mathbf{B}}_{kk}) & (\mathbf{B}_{kk} - \widehat{\mathbf{B}}_{kk})\mathbf{C}_{kt} \\ \mathbf{B}_{tk}\widehat{\mathbf{B}}_{kk}^{-1}(\mathbf{B}_{kk} - \widehat{\mathbf{B}}_{kk}) & (\mathbf{I}_t - \mathbf{A}_{tt})^{-1}\mathbf{A}_{tk}(\mathbf{B}_{kk} - \widehat{\mathbf{B}}_{kk})\mathbf{C}_{kt} \end{bmatrix} \boldsymbol{\iota} \geq 0, \tag{5.A.7}
$$

where nonnegativity follows because $\mathbf{B}_{kk} \geq \widehat{\mathbf{B}}_{kk}$ and the other submatrices in (5.A.7) are all nonnegative. It is now clear that $\Delta = 0$ *if and only if* the whole partitioned matrix in (5.A.7) is a null matrix, which is the case only when $\mathbf{B}_{kk} = \widehat{\mathbf{B}}_{kk}$. This happens only if players $i_1, i_2, \ldots, i_k$ are all isolates, i.e., $g_{i_s j} = g_{j i_s} = 0$ for all j and all $s = 1, \ldots, k$, in which case $\mathbf{B}_{kk} = \widehat{\mathbf{B}}_{kk} = \mathbf{I}_k$. This completes the proof. ■

Proof of Lemma 5.2. From the objective in (5.1) the first-order condition is $\partial u_i / \partial x_i = \alpha_i + \sum_{j=1}^{n} \sigma_{ij} x_j = 0$. Using the decomposition of $\boldsymbol{\Sigma}$ in (5.2), this can be written in matrix form as $-\boldsymbol{\Sigma}\mathbf{x}^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) = (\beta\mathbf{I} + \gamma\mathbf{U} - \lambda\mathbf{G})\mathbf{x}^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) = \boldsymbol{\alpha}$, or, equivalently, as

$$\beta(\mathbf{I} - a\mathbf{G})\mathbf{x}^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) + \gamma x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma})\boldsymbol{\imath} = \boldsymbol{\alpha}, \tag{5.A.8}$$

where we have used the fact that $\mathbf{U}\mathbf{x}^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) = x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma})\boldsymbol{\imath}$ and $a = \lambda/\beta$. Premultiplying (5.A.8) by the vector $\boldsymbol{\imath}'\mathbf{B} = \boldsymbol{\imath}'(\mathbf{I} - a\mathbf{G})^{-1}$ and using the definitions of the KB centrality measures yields $[\beta + \gamma b(\mathbf{g}, a)]x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) = b_{\boldsymbol{\alpha}}(\mathbf{g}, a)$, or $x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) = b_{\boldsymbol{\alpha}}(\mathbf{g}, a)/[\beta + \gamma b(\mathbf{g}, a)]$.

Now in (5.A.8) instead of $\mathbf{G}$ we use the denser adjacency matrix $\widetilde{\mathbf{G}} = \mathbf{G} + \mathbf{D}$, where $\mathbf{D}$ is a (semi)positive matrix with at least one positive off-diagonal element. Then (5.A.8) becomes

$$\beta(\mathbf{I} - a\mathbf{G})\mathbf{x}^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}}) - \lambda\mathbf{D}\mathbf{x}^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}}) + \gamma x^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}})\boldsymbol{\imath} = \boldsymbol{\alpha}.$$

Premultiplying the last equation with the vector $\boldsymbol{\imath}'\mathbf{B}$, we obtain $[\beta + \gamma b(\mathbf{g}, a)]x^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}}) = b_{\boldsymbol{\alpha}}(\mathbf{g}, a) + \lambda\boldsymbol{\imath}'\mathbf{B}\mathbf{D}\mathbf{x}^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}})$, or, equivalently,

$$x^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}}) = x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma}) + \frac{\lambda\boldsymbol{\imath}'\mathbf{B}\mathbf{D}\mathbf{x}^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}})}{\beta + \gamma b(\mathbf{g}, a)}. \tag{5.A.9}$$

From Theorem 1 in Calvó-Armengol et al. (2009) it follows that the unique and interior equilibrium efforts, $\mathbf{x}^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}})$, are guaranteed if $\lambda\mu(\widetilde{\mathbf{G}}) + n\gamma(\overline{\alpha}/\underline{\alpha} - 1) < \beta$. This condition in turn implies $\lambda\mu(\mathbf{G}) + n\gamma(\overline{\alpha}/\underline{\alpha} - 1) < \beta$, which is the sufficient condition for the interiority of $\mathbf{x}^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma})$. This implication is due to the monotonicity of the largest eigenvalue with the coefficients of the matrix $\mathbf{G}$ (see Theorem I* in Debreu and Herstein, 1953, p. 600), i.e., $\mu(\widetilde{\mathbf{G}}) \geq \mu(\mathbf{G})$. Since $\lambda\boldsymbol{\imath}'\mathbf{B}\mathbf{D}\mathbf{x}^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}}) > 0$, equation (5.A.9) implies $x^*_{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Sigma}}) > x^*_{\boldsymbol{\alpha}}(\boldsymbol{\Sigma})$ as long as $\widetilde{\mathbf{G}} > \mathbf{G}$. ■

CHAPTER 6

Identifying optimal sector groupings with the hypothetical extraction method*

6.1 Introduction

There are ample studies within the input-output (IO) framework that investigate the issue of the identification of so-called "key sectors". These are the sectors with the largest potential of spreading growth impulses throughout the economy. The issue of key sector determination is seen to be useful for economic planning, in particular, in developing countries. From a development strategy point of view, it is reasonable for a country with a limited amount of financial resources to invest in those few industries that have the largest impact on the whole economy through their buying and selling linkages with other production sectors. It is also true that the overall economic growth depends on the sectoral growth rates, which are in turn dependent on the linkages between the sectors. Moreover, strong linkages provide an opportunity for industries to gain a competitive advantage. For instance, if a sector successfully enters a foreign market, it will be easier for industries (firms) that have high linkages with this sector to gain access to the foreign market as well (Porter, 1990; Hoen, 2002). The key sectors targeting approach, pioneered by Rasmussen (1956) and Hirschman (1958), was followed by a vast number of theoretical and empirical studies, and still constitutes one of the main areas in

* A shorter version of this chapter is forthcoming in the *Journal of Regional Science*, 2009b.

IO and regional economics (see e.g., Strassert, 1968; Yotopoulos and Nugent, 1973; Jones, 1976; Schultz, 1977; Cella, 1984; Hewings et al., 1989; Heimler, 1991; Dietzenbacher, 1992; Sonis et al., 1995; Dietzenbacher and van der Linden, 1997; Cai and Leung, 2004; Cardenete and Sancho, 2006; Midmore et al., 2006; Beynon and Munday, 2008; Magtibay-Ramos et al., 2008).

The application of key sector determination goes beyond examining only pure production linkages. For example, since (according to the classical development economics) economic growth for developing countries is intrinsically linked to changes in the structure of production, many studies applied the notion of key sectors for the analysis of structural change (see e.g., Hewings et al., 1989; Sonis et al., 1995; Roberts, 1995). Analogously, Diamond (1975), Meller and Marfán (1981), Groenewold et al. (1987, 1993) and Kol (1991) analyze employment linkages for Turkey, Chile, Australia, and for Indonesia, South Korea, Mexico and Pakistan, respectively. It should be mentioned that IO linkage analysis is, in particular, extensively used nowadays in addressing the growing environmental concerns, e.g., with regard to emissions of greenhouse gases and the depletion of natural resources. For example, Lenzen (2003) focuses on the economic structure of Australia by identifying key sectors and linkages that have large environmental impacts on the consumption of energy and water, on land disturbance, and on the generation of emissions of CO_2, NO_x and SO_2. Similarly, Sánchez-Chóliz and Duarte (2003), extending Rasmussen-type linkages, identify the key sectors in generating water pollution in the Aragonese economy.

In the current chapter we focus on the linkage analysis based on the *hypothetical extraction method* (HEM), which has become increasingly popular (Miller and Lahr, 2001). Just to mention a few recent studies, the HEM has been applied in the analysis of water use (Duarte et al., 2002), for the key sector identification (Andreosso-O'Callaghan and Yue, 2004), in the analysis of the economy-wide roles of separate sectors, such as the agriculture sector (Cai and Leung, 2004), the construction sector (Song et al., 2006) and the real estate sector (Song and Liu, 2007). Los (2004) proposes to identify strategic industries using the HEM in a dynamic IO growth model. The HEM is also a useful tool to evaluate the significance of a sector in cases of crises-driven threats of industry shutdowns, which may help governments to decide whether to support financially the sector under threat or not.[1] The main

[1] The threat of downfall in the US car industry in the current financial crisis and the debates on providing massive public spending to the industry can serve as an example. Other examples, are the downfall of the only Dutch aircraft manufacturer Fokker in 1995-96, and the disappearance of the Belgian national airline Sabena in 2001, both of which resulted in the shutdown of an entire national industry (Los, 2004).

contribution of this chapter to the literature on key sector identification from the HEM perspective is that we distinguish between and *explicitly* formulate the (optimization) problems of finding a *key sector* and a *key group of sectors*, and derive their closed-form solutions that are termed industries' *factor worths*. The term "factor" refers to any indicator that an analyst uses in identifying the most important industries. This might be a social factor, such as employment, income, government revenue; or an environmental factor, such as primary energy consumption, greenhouse gas emissions, water use, land disturbance; or an economic/financial factor, such as GDP, gross operating surplus, export/import propensity; or any combination of these factors.

Our formulation of the HEM has the following implications. Firstly, given that we have found simple closed-form expressions for quantifying industries' importance, an analyst does *not* have to perform a three-step procedure of the HEM (to be explained in Section 6.2), which becomes a rather formidable task, in particular, when the number of industries is rather large (say, 100 or more). Secondly, and more importantly, we distinguish between a *key sector problem* and a *key group problem* and show that the key group of $k \geq 2$ sectors is, in general, *different* from the set of top k sectors selected on the basis of the key sector problem. This is important, since up to date, to the best of our knowledge, the linkage literature (implicitly) accepted the top k sectors from the ranking of individual sector contributions to the economy-wide output as the key group. This incongruence is due to the fact that while the key sector problem looks for the effect of the extraction of one sector, the key group problem considers the effect of a *simultaneous* extraction of $k \geq 2$ sectors that takes differently into account the cross-contributions of the extracted industries to total factor arising within and outside the group. This impact is largely dependent on the (dis)similarity of the linkage patterns of sectors to each other and of their final demand and factor production/consumption structures. Thirdly, we show that the HEM is directly related to the fields of influence approach (Sonis and Hewings, 1989, 1992), which gives an alternative economic interpretation of the HEM in terms of the overall impact on aggregate factor due to an incremental change in sectors' input self-dependencies. Fourthly, our formulation of the HEM allows to examine a *combined* key sector/group problem, where the objective is a combination of several factors. For instance, one may wish to identify a key sector that has simultaneously the largest total (direct and indirect) contribution to economy-wide employment *and* the smallest total impact on carbon emissions generation. Finally, it is shown that the related problems of finding a key *region* and key group of regions in an interregional IO framework can be investigated in a similar way.

We also examine the effect of a change in an input coefficient on the factor importance of an industry. It is shown that a positive (negative) change in a direct input coefficient never decreases (increases) the factor generating importance of any sector, and we provide necessary and sufficient conditions for a strict change. The economic interpretations of such a change include, for example, an increase in complexity of technological links between sectors (or a rise in the density of the input matrix), an increase in sectoral interdependence, innovation or technological progress.

The rest of this chapter proceeds as follows. In Section 6.2.1 we present the optimization problem of finding a key sector, and examine how a change in a direct input coefficient affects the factor generating importance of industries. Section 6.2.2 generalizes the key sector problem to a key group identification problem, the solution of which is defined in terms of a *group factor worth* of industries. In Section 6.2.3 it is shown that the key group problem is not equivalent to the sequential key sector problem. The related problems of finding a key region and a key group of regions are briefly examined in Section 6.2.4. Further, the key sector/group problem in a net IO setting is discussed in Section 6.2.5. In Section 6.3 the link between the HEM and the fields of influence methods is explored. In Section 6.4 we discuss the connection between the HEM approach of finding the key sector/group and the game theoretic literature on social networks and allocation of gains from cooperation. Section 6.5 contains results from the empirical application of the key sector and key group problems to the Australian economy. Section 6.6 concludes. All proofs are relegated to the Appendix.

6.2 Formalizing the hypothetical extraction problems

In this section, taking the hypothetical extraction method perspective, we formalize the optimization problems of finding the key sector and the key group of sectors of the economy. Moreover, the analytical closed-form solutions of these problems are derived, which helps to identify the key sector/group in empirical work.

6.2.1 Finding the key sector

The main point of departure is the open static Leontief model (see e.g., Miller and Blair, 2009), given by $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}$, where $\mathbf{x}$ is the $n \times 1$ endogenous vector of gross outputs of n sectors, $\mathbf{A}$ is the n-square direct input requirements matrix, and $\mathbf{f}$ is the $n \times 1$ exogenous vector of final demands (including consumption, investments, exports, and government expenditures). The domestic input coefficients a_{ij} denote the output in industry i directly required as input for one unit of output in industry j, hence the i-th element of the vector $\mathbf{Ax}$ gives total amount of *intermediate* inputs of good i, required for production of output $\mathbf{x}$. That is, the basic equation of the open Leontief system states that gross output, $\mathbf{x}$, is the sum of all intermediate demand, $\mathbf{Ax}$, and final demand, $\mathbf{f}$. The reduced form of the model is

$$\mathbf{x} = \mathbf{Lf}, \tag{6.1}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ is the Leontief inverse.

The typical element of the Leontief inverse, l_{ij}, denotes the output in industry i directly and indirectly required to satisfy one unit of final demand in industry j. The row vector of *output multipliers* is defined as $\mathbf{m}_o' = \boldsymbol{\imath}'\mathbf{L}$, where $\boldsymbol{\imath}$ is the summation vector of ones. Its j-th element $m_j^o = \sum_{k=1}^{n} l_{kj}$ indicates the increase of total output in all industries per unit increase of final demand in industry j.

For the purpose of identification of important sectors we adopt the *hypothetical extraction method* (HEM) originally developed and used by Paelinck et al. (1965), Strassert (1968) and Schultz (1977), the central idea of which is briefly as follows. To estimate the importance of sector i to the economy, delete the i-th row and column of the input matrix $\mathbf{A}$, and then using the basic Leontief equation (6.1) compute the reduced outputs in this hypothetical case (the final demand vector also excludes its ith component, f_i). The difference between *total* outputs of the economy before and after the extraction (called "total linkage") measures the relative stimulative importance of sector i to the economy.[2]

However, unlike the traditional HEM approach, we allow for a rather general definition of importance, which may be used to address various economic, social, and/or environmental issues.[3] For instance, key sectors may be determined ac-

[2] This method was criticized for the reason that it does not distinguish the total linkages into backward and forward linkages (see e.g., Meller and Marfán, 1981; Cella, 1984; Clements, 1990; Dietzenbacher and van der Linden, 1997). However, we believe that for measuring a sector's economy-wide impact it is the most adequate HEM, since setting to zero only a column (row) to compute the backward (forward) linkages in the non-complete HEM takes only a one-sided impact into account. Moreover, it is difficult to entirely separate backward and forward effects from each other, since there are always forward-links present in the backward linkage measures, and vice versa (see e.g., Yotopoulos and Nugent, 1973; Cai and Leung, 2004). See Miller and Lahr (2001) for an excellent discussion on all possible extractions, who state that for the purpose of finding a key sector "... we believe the original hypothetical extraction approach ... is totally adequate - Meller and Marfán and other modifications notwithstanding" (p. 429).

[3] For example, ten Raa (2005, p. 26) states: "Output increases induced by a final demand stimulus are of little interest in themselves. What matters is the income generated by the additional economic activity."

cording to their potential of generating income, emission of greenhouse gases, creating jobs, or resource use. For the purpose of a general exposition of the HEM problem, we refer to the various policy-relevant indicators as *factors*. Let the vector of *direct factor coefficients* $\boldsymbol{\pi}$ denote the sectoral factor usage/generation per unit of total output, hence the row vector of *factor multipliers* is $\mathbf{m}'_{\pi} = \boldsymbol{\pi}'\mathbf{L}$, and its j-th element $m_j^{\pi} = \sum_{k=1}^{n} \pi_k l_{kj}$ indicates the economy-wide increase of factor usage/production per unit increase of final demand in industry j.

We are now in a position to address the key sector identification problem. Let us first denote by $\mathbf{A}^{-i}$ the new input matrix derived from $\mathbf{A}$ by setting to zero all elements in the i-th row and column. The crucial assumption made (which is usual for all the HEM approaches) is that in a new system without sector i the input structure of all sectors $j \neq i$ remains unchanged. From an economic point of view, this implies that foreign (external) industries substitute the domestic sector i in providing the inputs in order to satisfy the intermediate demands of the remaining industries and the final demand for commodity i. Although at first glance this assumption seems restrictive, in fact it is not, given our main aim of identifying the importance of sector i. The point is that by taking all other input coefficients fixed, we explicitly allow the resulting outcome to depend only on the elimination of sector i, which is now not participating in the "roundabout" of the production process. The vector of total outputs after extracting sector i is $\mathbf{x}^{-i} = \mathbf{L}^{-i}\mathbf{f}^{-i}$, where $\mathbf{L}^{-i} = (\mathbf{I} - \mathbf{A}^{-i})^{-1}$, and $\mathbf{f}^{-i}$ is the same as $\mathbf{f}$ except its ith entry that is set to zero. The reason for excluding f_i in the final demand vector $\mathbf{f}^{-i}$ is that when sector i ceases to exist, its (domestic) output should be zero, which from (6.1) is equivalent to $f_i = 0$ (see also e.g., Schultz, 1977; Miller and Lahr, 2001).

The objective is selecting sector i, such that its extraction from the system generates the largest possible reduction in the factor of interest (say, total income). Formally, the problem is

$$\max\{\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-i} \mid i = 1, \ldots, n\}. \tag{6.2}$$

This is a finite optimization problem, which has at least one solution. A solution to (6.2) is denoted by i^* and is called a *key sector*. Removing i^* from the initial production structure has the largest overall impact on the factor generation. To solve (6.2) we use the following result due to Zeng (2001, Theorem 1, p. 304).[4]

[4] Independently, also Ballester et al. (2006, Lemma 1, p. 1411) establish the same result in a social network framework. We should note that their Lemma 1 is given for a *symmetric* adjacency matrix, and does not consider the ii-th element of the difference $\mathbf{L} - \mathbf{L}^{-i}$. For the asymmetric case, change $m_{ij}(\mathbf{g}, a)$ to $m_{ji}(\mathbf{g}, a)$ in their Lemma 1. The problems of finding the key players in social networks (Ballester et

Lemma 6.1. *Let* $\mathbf{L}$ *and* $\mathbf{L}^{-i}$ *be, respectively, the Leontief inverses before and after extraction of sector i from the production system, and* $\mathbf{e}_i$ *be the i-th column of the identity matrix. Then* $\mathbf{L}-\mathbf{L}^{-i}=\frac{1}{l_{ii}}\mathbf{L}\mathbf{e}_i\mathbf{e}_i'\mathbf{L}-\mathbf{e}_i\mathbf{e}_i'$.

Using Lemma 6.1, problem (6.2) can after some mathematical transformations be rewritten as (see Appendix 6.A):

$$\max\left\{\frac{1}{l_{ii}}\mathbf{m}_\pi'\mathbf{e}_i\mathbf{e}_i'\mathbf{x}\,\middle|\, i=1,\ldots,n\right\}=\max\left\{\frac{m_i^\pi x_i}{l_{ii}}\,\middle|\, i=1,\ldots,n\right\}. \tag{6.3}$$

The problem in (6.2) is equivalent to $\min\{\boldsymbol{\pi}'\mathbf{x}^{-i}\ |i=1,\ldots,n\}$. However, a direct approach to solve (6.2) forces an analyst to extract each sector separately, and compute and compare the required objective. This becomes a formidable task when the number of sectors, n, is large, although modern technology has reduced the problem. Nevertheless, the closed form expression in (6.3) shows that there exists a much simpler (and elegant) way to get the desired outcome.

Definition 6.1. *Consider the open Leontief model* $\mathbf{x}=\mathbf{L}\mathbf{f}$*, where* $\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}$*, and let the row vector of factor multipliers be* $\mathbf{m}_\pi'=\boldsymbol{\pi}'\mathbf{L}$*, where* $\boldsymbol{\pi}$ *is the direct factor coefficient vector. The factor worth of sector i is* $\omega_i^\pi(\mathbf{A},\mathbf{f},\boldsymbol{\pi})=\left(m_i^\pi x_i\right)/l_{ii}$.

Therefore, given Definition 6.1 and the objective in (6.3) we have established the following result.

Theorem 6.1. *The key sector* i^* *that solves* $\max\{\boldsymbol{\pi}'\mathbf{x}-\boldsymbol{\pi}'\mathbf{x}^{-i}\,|\,i=1,\ldots,n\}$ *has the highest factor worth, i.e.,* $\omega_{i^*}^\pi(\mathbf{A},\mathbf{f},\boldsymbol{\pi})\geq\omega_i^\pi(\mathbf{A},\mathbf{f},\boldsymbol{\pi})$ *for all* $i=1,\ldots,n$.

From Theorem 6.1 it follows that the standard measure of a high factor multiplier m_i^π is not sufficient for sector i to be an optimal target, say, for investments. For that, besides m_i^π, the size of the sector's output x_i and its total self-dependency on inputs as indicated by l_{ii} are equally important, where the first has a positive effect, while the second has an inverse effect on the worth of sector i.

The traditional gross output approach of the HEM corresponds to problem (6.2) or (6.3) where the summation vector $\boldsymbol{\imath}$ is substituted for the vector of factor coefficients $\boldsymbol{\pi}$. The following result is then an immediate implication of Theorem 6.1.

Corollary 6.1. *The key sector* i^* *that solves* $\max\{\boldsymbol{\imath}'\mathbf{x}-\boldsymbol{\imath}'\mathbf{x}^{-i}\ |i=1,\ldots,n\}$ *has the largest gross output worth, i.e.,* $\omega_{i^*}^o(\mathbf{A},\mathbf{f})\geq\omega_i^o(\mathbf{A},\mathbf{f})$ *for all* $i=1,\ldots,n$*, where* $\omega_i^o(\mathbf{A},\mathbf{f})=m_i^o x_i/b_{ii}$ *is the gross output worth of sector i.*

al., 2006, and Chapter 5 of this thesis) and key sectors in the economy are closely related. Lemma 6.1 is a particular case of our Lemma 6.2, hence its proof is skipped as it directly follows from the proof of our second lemma.

Notice that the gross output factor worth of sector i is nothing else than the "total linkage" of a sector as defined in the classical HEM approach.

Next we examine how stronger interdependence of sectors affects the factor worth of sector i. Let the input matrix $\widetilde{\mathbf{A}}$ represent the alternative input structure, and, without loss of generality, assume that $\widetilde{\mathbf{A}}$ differs from $\mathbf{A}$ only with respect to the rc-th element that is increased by $\alpha > 0$. Then it is apparent that $\widetilde{\mathbf{L}} = \mathbf{I} + \widetilde{\mathbf{A}} + \widetilde{\mathbf{A}}^2 + \cdots > \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots = \mathbf{L}$,[5] which in turn implies that, given $\mathbf{f}$ and $\boldsymbol{\pi}$, both the numerator and denominator in the definition of the factor worth of sector i might increase, hence it is not clear whether $\omega_i^{\pi}(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi})$ is larger or smaller than $\omega_i^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$. In Theorem 6.2 below we show that a rise in the direct input interdependence between two sectors never decreases sector i's factor worth, and, moreover, we establish necessary and sufficient condition(s) under which such a change surely increases $\omega_i^{\pi}(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi})$.

Theorem 6.2. *Let the input matrix $\widetilde{\mathbf{A}}$ differ from $\mathbf{A}$ only with respect to the rc-th entry, which has been changed by $\alpha \neq 0$. Given $\boldsymbol{\pi}$ and the nonnegative final demand $\mathbf{f}$ with $f_c > 0$, if $\alpha \gtrless 0$ then*

(a) $\omega_i^{\pi}(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi}) \gtrless \omega_i^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ *for $i = r, c$;*

(b) $\omega_i^{\pi}(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi}) \gtreqless \omega_i^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ *for all $i \neq r, c$, with equality holding if and only if $l_{ir} = l_{ci} = 0$.*

One implication of Theorem 6.2 is that when domestic industries become more interdependent on each other, then the factor generating importance falls for no sector and surely increases for sectors directly involved in this higher input dependencies (i.e., for sectors r and c). Moreover, any other sector i's worth increases as well if $l_{ir} > 0$ and/or $l_{ci} > 0$. The second implication is a reduction in the use of domestically produced inputs increases the factor worth of no sector for the same vectors of final demand and factor coefficients. In particular, if, say, due to innovation a_{rc} decreases, then sectors r and c's factor worths strictly decrease and any other sector i's importance also weakens whenever $l_{ir} > 0$ and/or $l_{ci} > 0$. These two conditions (i.e., l_{ir} and/or l_{ci} being positive) imply that sector i should either provide (directly and/or indirectly) inputs to industry r and/or uses inputs (directly and/or indirectly) from sector c.

The straightforward special case of Theorem 6.2 is when $\boldsymbol{\pi} = \boldsymbol{\imath}$, which shows that the gross output worth of sector i increases (decreases) if the input coefficient a_{rc} increases (decreases) and sector i provides inputs (directly and/or indirectly) to sector r and/or uses inputs from industry c.

[5] Matrix inequality notations are given in the subsection "General notations" in Chapter 1.

Corollary 6.2. *Assume that the input coefficient a_{rc} changes by $\alpha \neq 0$, i.e., $\tilde{a}_{rc} = a_{rc} + \alpha$. Then, given nonnegative $\mathbf{f}$ with $f_c > 0$, $\omega_i^o(\widetilde{\mathbf{A}}, \mathbf{f}) \gtrless \omega_i^o(\mathbf{A}, \mathbf{f})$ for $i = r, c$, and $\omega_i^o(\widetilde{\mathbf{A}}, \mathbf{f}) \gtreqless \omega_i^o(\mathbf{A}, \mathbf{f})$ for all $i \neq r, c$ whenever $\alpha \gtrless 0$, with equality holding if and only if $l_{ir} = l_{ci} = 0$.*

6.2.2 From individual key sector to key group

Although the linkage literature using the HEM acknowledges the possibility of extraction of several industries, the theoretical analysis does not go beyond describing it using partitioned matrices to the reduced form of the Leontief model (see e.g., Miller and Lahr, 2001). This, however, is quite cumbersome to implement empirically because one has to consider all possible combinations of a certain number of industries in order to determine the most important group of sectors. This may explain the lack of empirical studies on the role of a group of industries. Hence, in all studies, to the best of our knowledge, the HEM was applied to only one sector, and the most important industries were defined to be those with the largest individual contributions to total output (or any other factor).[6]

In this section we wish to fill this gap in the literature, generalizing the key sector problem from the previous section to the *key group problem*. Similar to the notion of individual key sector, a *key group* of $k \geq 2$ sectors is defined as the group of industries, whose removal from the production system has the largest impact on the overall factor consumption/generation.[7] Since the two problems are inherently different, we expect that, in general, the top k sectors with the largest factor worths do *not* compose the key group, which is also confirmed in the empirical application in Section 6.5. The underlying reason for this outcome is that industries can be *redundant* (or, equivalently, similar to each other) with respect to their linkage patterns to other sectors and their structures of final demand and factor production. Hence, targeting industries with very similar linkage characteristics might not be an optimal policy strategy. Instead choosing sectors with different patterns of (significant) production linkages, and higher values of final demands and factor usage will induce the largest impact on the factor consumption/production.

[6] Dietzenbacher et al. (1993) extended the notion of extracting individual sectors to the *non-complete* extraction of *individual* regions in an interregional setting. The related key region problem is discussed in Section 6.2.4. We should, however, note that the approach in Dietzenbacher et al. (1993) is not equivalent to ours because they consider the extraction of only *inter*regional linkages of a region, and not the entire region. Moreover, we extend the problem to finding a key group of several regions and give its closed-form solution.

[7] Note that if the factor generation is unfavorable from a societal point of view (e.g., an increase in CO_2 emissions has detrimental consequences) and the policy-makers want to find the *least* harmful industries to target on, then the key group will be defined as the set of industries that has the *smallest* impact on the factor generation.

The objective is now picking k ($1 \leq k \leq n$) sectors $i_1, i_2, \ldots, i_k$ ($i_s \neq i_r$) such that their extraction from the production structure generates the largest impact on the overall factor usage/production, i.e.,

$$\max \left\{ \boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-\{i_1,\ldots,i_k\}} \mid \{i_1,\ldots,i_k\} \subseteq \{1,2,\ldots,n\}; i_s \neq i_r \right\}, \tag{6.4}$$

where $\mathbf{x}^{-\{i_1,\ldots,i_k\}} = \mathbf{L}^{-\{i_1,\ldots,i_k\}}\mathbf{f}^{-\{i_1,\ldots,i_k\}}$, and the superscript $-\{i_1,\ldots,i_k\}$ refers to the situation where sectors $i_1, i_2, \ldots, i_k$ are hypothetically extracted from the economy.

The new Leontief inverse is $\mathbf{L}^{-\{i_1,\ldots,i_k\}} = (\mathbf{I} - \mathbf{A}^{-\{i_1,\ldots,i_k\}})^{-1}$, where all the elements of the new input matrix corresponding to the extracted sectors are nullified. These sectors in the hypothetical case should have zero (domestic) outputs, hence $\mathbf{f}^{-\{i_1,\ldots,i_k\}}$ is exactly the same as $\mathbf{f}$ but with $f_{i_s} = 0$ for all $s = 1,\ldots,k$. The solution to (6.4) is denoted by $\{i_1^*, i_2^*, \ldots, i_k^*\}$ and is called the *key group of size k*.

The following important identity characterizes the changes in the elements of the Leontief inverse when a group of k sectors is hypothetically extracted from the production system (see Appendix 6.A).

Lemma 6.2. *Let $\mathbf{L}^{-\{i_1,\ldots,i_k\}}$ be the Leontief inverse after extraction of sectors $i_1, i_2, \ldots, i_k$ from the production system, where $1 \leq k \leq n$, and $\mathbf{e}_i$ be the i-th column of the identity matrix. Then the identity $\mathbf{L} - \mathbf{L}^{-\{i_1,\ldots,i_k\}} = \mathbf{L}\mathbf{E}(\mathbf{E}'\mathbf{L}\mathbf{E})^{-1}\mathbf{E}'\mathbf{L} - \mathbf{E}\mathbf{E}'$ always holds, where $\mathbf{E}$ is the $n \times k$ matrix defined as $\mathbf{E} = \left(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k}\right)$.*

Note that Lemma 6.1 is just a special case of Lemma 6.2 with $k = 1$. We should also note that the k extracted sectors can be ordered arbitrarily, hence the matrix $\mathbf{E}$ can have any ordering of the identity columns corresponding to the extracted sectors.[8] Using Lemma 6.2 it can be shown that the problem (6.4) is exactly equivalent to (see Appendix 6.A)

$$\max \left\{ \mathbf{m}_\pi'\mathbf{E}(\mathbf{E}'\mathbf{L}\mathbf{E})^{-1}\mathbf{E}'\mathbf{x} \mid \{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\}; i_s \neq i_r \right\}. \tag{6.5}$$

Note that in the maximization process the vectors of factor multipliers and gross outputs and the Leontief inverse matrix (i.e., $\mathbf{m}_\pi$, $\mathbf{x}$ and $\mathbf{L}$) are all given, and only the k identity columns in $\mathbf{E}$ are changed in order to consider all possible combinations of k sectors from the totality of n industries.

[8] If $k = n$ and $\mathbf{E} = \mathbf{I}$, then $\mathbf{L} - \mathbf{L}^{-\{i_1,\ldots,i_k\}} = \mathbf{L} - \mathbf{I}$, which would have been expected. However, in case $k = n$, $\mathbf{E}$ does not have to be an identity matrix, but $\mathbf{E}$ may be any permutation matrix of order n.

Definition 6.2. *Consider the open Leontief model* $\mathbf{x} = \mathbf{Lf}$*, where* $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$*. Let the row vector of factor multipliers be* $\mathbf{m}'_\pi = \boldsymbol{\pi}'\mathbf{L}$*, where* $\boldsymbol{\pi}$ *is the direct factor coefficient vector, and* $\mathbf{e}_i$ *be the i-th column of the identity matrix. The group factor worth of sectors* $i_1, \ldots, i_k$ $(i_r \neq i_s)$ *is* $\omega^{\pi}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \mathbf{m}'_\pi \mathbf{E}(\mathbf{E}'\mathbf{LE})^{-1}\mathbf{E}'\mathbf{x}$*, where* $\mathbf{E} = \left(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k}\right)$*.*

The matrix $\mathbf{L}_{kk} \equiv \mathbf{E}'\mathbf{LE}$ includes all the elements of the Leontief inverse $\mathbf{L}$ that are directly related to the extracted sectors. Given the key group problem (6.5) and Definition 6.2, we thus have the following result.

Theorem 6.3. *For* $k \in [1, n]$ *the key group of size k* $\{i^*_1, i^*_2, \ldots, i^*_k\}$ *that solves* $\max\left\{\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-\{i_1,\ldots,i_k\}} \mid \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\};\ i_s \neq i_r\right\}$ *has the highest group factor worth, i.e.,* $\omega^{\pi}_{i^*_1,\ldots,i^*_k}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) \geq \omega^{\pi}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ *for all* $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ *with* $i_s \neq i_r$*.*

Note that the key group problem in (6.5) with $k = 1$ boils down to the key sector problem (6.3). Hence, given the group factor worth in Definition 6.2, Theorem 6.1 is also a particular case of Theorem 6.3 when the target is only one sector (i.e., $k = 1$).

When the key group of size k is searched in the spirit of the traditional HEM approach, the immediate outcome of Theorem 6.3 is the following corollary.

Corollary 6.3. *For* $k \in [1, n]$ *the key group of size k* $\{i^*_1, \ldots, i^*_k\}$ *that solves* $\max\left\{\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}^{-\{i_1,\ldots,i_k\}} \mid \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}; i_s \neq i_r\right\}$ *has the highest group (gross) output worth, i.e.,* $\omega^{o}_{i^*_1,\ldots,i^*_k}(\mathbf{A}, \mathbf{f}) \geq \omega^{o}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f})$ *for all* $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ *with* $i_s \neq i_r$*, where* $\omega^{o}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f}) = \mathbf{m}'_o \mathbf{E}(\mathbf{E}'\mathbf{LE})^{-1}\mathbf{E}'\mathbf{x}$*.*

While the key sector problem looks for the effect of the extraction of one sector, the key group problem considers the effect of a *simultaneous* extraction of $k \geq 2$ sectors. Hence, the two problems are not equivalent. If two industries are perfectly identical with respect to their linkages patterns (including input coefficients' sizes) and more or less have similar values of their final demands and factor production, then their group factor worth is expected to be less than that of another group, which consists of one of the mentioned sectors and an industry that has a quite different pattern of interindustry linkages and factor generation ability. This indicates the importance of the *redundancy principle* in the IO framework. This principle is well-known in the sociology literature on social networks and emphasizes the redundancy of actors with respect to adjacency, distance, and bridging (see e.g., Burt, 1992; Borgatti, 2006). In particular, Ronald Burt is well-known for his notion of *structural holes* that is used as an empirical measure of (non)redundancy. "Nonredundant contacts are disconnected in some way – either directly, in the sense that they have no direct contacts with one another, or indirectly, in the sense that one has contacts that exclude the others (Burt, 1992, p. 18). Structural holes are the gaps between nonredundant contacts, and as such there are many structural holes in networks that are rich in nonredundant contacts. Taking redundancy into account is crucial in determining the most important group in social networks (see Everett and Borgatti, 1999, 2005; Temurshoev, 2008), which was also discussed in Chapter 5, where the redundancy of links among players was found to be an important factor for key group identification in network games. However, in the IO framework it is not only the redundancy of sectors with respect to their production linkages that matters, but also the similarity of the structures of sectors' final demands and factor production is important in determining the key group (see the next section).

In general, within the IO framework, we expect that k (≥ 2) sectors with the largest individual factor worths will not be much different from the key group of size k if the IO tables are highly aggregated because then much information on sectors' heterogeneity is lost (which has been confirmed in our experimental simulations). Otherwise, the difference should be in place, and will largely depend on the structures and sizes of the production system, direct factor coefficients and final demands.[9]

6.2.3 The key group problem is not equivalent to the sequential key sector problem

In this section we want to illustrate that the members of the key group of size k are *not* necessarily included in the key group of size $k+1$. Let us consider a simple example with seven sectors. The corresponding IO table and the results of the key group problem, where the objective is gross output, are given in Table 6.1. Group output worths are given relative to the total output before the extraction (in per-

[9] In this generalized HEM setting, one can also focus on several objectives simultaneously. If, for instance, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{c}$ denote, respectively, the direct value-added, labor and CO_2 coefficients vectors, then the *combined* key sector (resp. group) problem is given by (6.2) (resp. (6.4)) with the direct factor coefficients defined as $\boldsymbol{\pi} = \mathbf{v} + \mathbf{w} - \mathbf{c}$. Note that since CO_2 generation is unfavorable, its direct coefficients are entered with a minus sign in the definition of $\boldsymbol{\pi}$. Also notice that factors written is this form can have an economic meaning only if they are all expressed in the same measurement unit. This can be done, for example, by multiplying the number of jobs by a price so that employment is expressed in some common for all factors currency term (like in the index number literature). Or, one might assign appropriate weights to each factor that is included in $\boldsymbol{\pi}$. For instance, we may write $\mathbf{w} = t_v\mathbf{j}$, where the vector of the (number of) jobs direct coefficients $\mathbf{j}$ is expressed in terms of currency using the weight $t_v = \mathbf{v}'\mathbf{x}/\mathbf{j}'\mathbf{x}$ that indicates the value of income per one (full-time) job. We should, however, mention that the disadvantage of this approach is that the outcome is weight dependent. That is, a different weighting scheme might very well give a different result. The generalized HEM can also be used in terms of the specific categories of the final demand. For instance, from a trade policy perspective it might be interesting to find out the key contributor(s) to some factor in the trade process of a country, in which case instead of $\mathbf{f}$ one uses the vector of exports (minus imports) only.

Table 6.1: A hypothetical IO table and the relative group output worths

Sector	Interindustry transaction matrix							Final demand	Total output
	1	2	3	4	5	6	7		
1	0	0	10	10	0	0	0	50	70
2	0	0	10	10	0	0	0	50	70
3	10	10	0	0	10	10	0	30	70
4	10	10	0	0	10	10	0	30	70
5	0	0	10	10	0	0	0	50	70
6	0	0	10	10	0	0	10	50	80
7	0	0	0	0	0	10	0	50	60

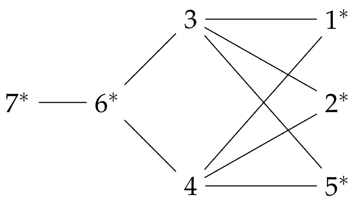
Rank	Group of size k and its relative gross output worth (*RW* %)			
	$k = 1$	*RW*	$k = 2$	*RW*
1 (key)	6	25.0	{3,4}	45.5
2	3; 4	24.7	{1,6}, {2,6}, {5,6}	44.1
3	1; 2; 5	21.0	{3,6}, {4,6}	43.2
	$k = 3$	*RW*	$k = 4$	*RW*
1 (key)	{1,2,6}, {1,5,6}, {2,5,6}	61.5	{1,2,5,6}	77.6
2	{3,4,6}, {3,4,7}	59.2	{1,2,5,7}	72.3
3	{1,2,5}	57.7	{1,2,6,7},{1,5,6,7},{2,5,6,7}	71.7

centages) of the corresponding groups.

Note that for simplicity we have assumed that the matrix of interindustry transactions is symmetric and all its positive elements are identical (i.e., 10), which will make the interpretation easier. The results in the table show that the key sector in this hypothetical economy is sector 6, whose extraction from the production system causes a reduction in the overall gross output by 25%. However, it is not a member of the key group of size 2, which is represented by sectors 3 and 4 with the relative group output worth of 45.5%. Notice that these sectors have the second largest *individual* impact on the overall output. Similarly, the key group of size 2, i.e., $\{3, 4\}$, does not show up at all within the key groups of size 3 and 4.

Figure 6.1 graphs all the direct intersectoral links represented by the interindustry transaction matrix in Table 6.1. From Table 6.1 we see that there are two types of sectors in terms of final demand. Sectors 3 and 4 have the lowest final demands, i.e., 30, while the final demands for the other industries are all equal to 50. In the graph we show this difference by a star superscript for sectors with the largest final demand. Now it is clear from Figure 6.1 that sectors 1, 2 and 5 are not only identical in terms of their sizes of final demands, but they are also perfectly *structurally equivalent* to each other (or, are redundant with respect to their linkages) because they are connected to the same third parties, namely to sector 3 and to sector 4. In

Figure 6.1: The network of interindustry transactions



their turn, sectors 3 and 4 also have similar patterns of ties to other industries and the same size of final demands.

Sector 3 (or sector 4) with the largest number of direct intersectoral links cannot be the key sector because of its low value of final demand. Instead, sector 6 is the key sector because it has (i) a relatively large number of direct and indirect connections, and (ii) a large value of final demand, both of which results in its high contribution to the overall gross output.

However, sector 6 cannot be a part of the key group of size 2 as its joint impact on the system with some other industry is not maximal. To see this from the network disruption perspective, suppose we eliminate sector 3 and sector 6. Then in Figure 6.1 out of 9 lines, representing mutual interdependence of sectors, three lines remain: those connecting sector 4 (with lowest final demand) to each of sectors 1, 2 and 5 (with largest final demands). However, extraction of sector 3 and sector 4 results in the maximal disruption of the above network of production linkages, i.e., only one line between sector 6 and sector 7 will remain.

However, the key group problem is not only about the maximal disruption of the network of production linkages. It may very well happen that sectors whose joint elimination results in the maximal disruption of the production network do not compose the key group. If, for example, sectors 3, 4 and 6, were eliminated from Figure 6.1, the resulting network of production linkages would be empty. Hence, by removing these sectors the network is totally disrupted, but they do *not* compose the key group of size 3 as shown in Table 6.1. For key groups, also the scale of final demand satisfaction plays a crucial role, hence three groups of sectors with largest final demands, i.e., {1,2,6}, {1,5,6} and {2,5,6}, are chosen as the key groups of size 3 even though extraction of these sectors will not give an empty production network.

Hence, within an IO framework it is not only the redundancy (or, equivalently, similarity) of sectors with respect to their linkages patterns that matter, but also the joint contribution of industries to final demand categories and overall factor generation (which in this example was gross output) are crucial in determining the key group members. This is also the reason why the simple *sequential* key sector problem is not adequate to totally address the key group problem. By sequential search we mean that once the key group of size k has been identified, one needs only to add an extra sector from all possible $n-k$ remaining industries in order to identify the key group of size $k+1$.

6.2.4 The key group problem in an interregional setting

The key sector/group problem can be easily applied in an interregional (or multi-regional) IO setting. A related issue is to find a key *region* or key group of regions in such a setting.[10] The basic open Leontief model (6.1) in an interregional framework with p (≥ 2) regions can be written as

$$\begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^p \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n_1} - \mathbf{A}^{11} & -\mathbf{A}^{12} & \cdots & -\mathbf{A}^{1p} \\ -\mathbf{A}^{21} & \mathbf{I}_{n_2} - \mathbf{A}^{22} & \cdots & -\mathbf{A}^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{A}^{p1} & -\mathbf{A}^{p2} & \cdots & \mathbf{I}_{n_p} - \mathbf{A}^{pp} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \vdots \\ \mathbf{f}^p \end{bmatrix}, \tag{6.6}$$

where $\mathbf{A}^{rr}$ is the (intra)regional input coefficients matrix for region r $(= 1, \ldots, p)$, $\mathbf{A}^{rs}$ is the matrix of interregional input (trade) coefficients with deliveries from region r to region s ($r \neq s$), $\mathbf{f}^r$ and $\mathbf{x}^r$ are, respectively, the vectors of final demand and gross output for region r, and $\mathbf{I}_{n_r}$ is the identity matrix of dimension n_r (i.e., the number of sectors in region r; hence each region may have a different number of industries). The problem of finding the most important region or groups of regions with respect to some factor is also given by (6.4), where the extended vectors of gross output and factor coefficients have dimensions corresponding to the p-region model in (6.6). A group factor worth of *regions* $r_1, \ldots, r_k$ can be defined as $\omega^{\pi}_{r_1,\ldots,r_k}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \mathbf{m}'_{\pi}\mathbf{E}(\mathbf{E}'\mathbf{L}\mathbf{E})^{-1}\mathbf{E}'\mathbf{x}$, where now $\mathbf{E} = (\mathbf{I}_{r_1}, \ldots, \mathbf{I}_{r_k})$ and $\mathbf{I}_{r_k} = (\mathbf{O}\ \mathbf{I}_{n_k}\ \mathbf{O})'$ is the $(\sum_{j=1}^{p} n_j) \times n_k$ matrix with the identity matrix $\mathbf{I}_{n_k}$ placed in a position corresponding to that of region k, and $\mathbf{O}$ is a zero matrix of appropriate size. Let $\boldsymbol{\pi}^r$ be the direct factor coefficient vector for region r and $\mathbf{L}^{rs}$ be a submatrix of the partitioned Leontief inverse in (6.6) corresponding to regions r and s (note that, in general, $\mathbf{L}^{rr} \neq (\mathbf{I} - \mathbf{A}^{rr})^{-1}$). Then the factor worth of region r is $\omega^{\pi}_r(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \mathbf{m}'_{\pi,r}(\mathbf{L}^{rr})^{-1}\mathbf{x}^r$, where the factor multiplier vector of region r

[10] Examining this extension was suggested by one of the anonymous referees.

is $\mathbf{m}_{\pi,r} = \sum_{j=1}^{p} \boldsymbol{\pi}^{j}\mathbf{L}^{jr}$. The j-th element of $\mathbf{m}_{\pi,r}$ gives the nation-wide increase of some factor π due to one unit increase in final demand of product j in region r. Note that the factor multipliers $\mathbf{m}'_{\pi} = (\mathbf{m}'_{\pi,1}, \ldots, \mathbf{m}'_{\pi,p})$ in this setting represent the nation-wide effects and are not region-specific (e.g., $\boldsymbol{\pi}^{r}\mathbf{L}^{rr}$) as they are based on the extended $\mathbf{L}$ from the interregional framework. Finally, Theorem 6.3 can be readily used, i.e., the key group of regions of size $k \in [1, p]$ has the highest group factor worth.

6.2.5 The key sector/group problem in a net IO setting

Earlier works of Leontief were formulated in terms of "net" accounts, i.e., internal sectoral flows were set to zero. Hence, in the so-called *net* IO framework intrasectoral transactions are excluded from the IO tables (see e.g., Parikh, 1975; Jensen, 1978). In this section we explore the key sector/group problem within the net IO setting, and show that the results (in a standard or gross IO setting) are invariant to the netting out of intrasectoral transactions for any factor other than gross output.[11]

Let us denote by $\mathbf{Z}$ the $n \times n$ interindustry transaction matrix and by $\hat{\mathbf{Z}}$ the diagonal matrix containing z_{ii} along its main diagonal and zero otherwise. Then in the net IO model $\mathbf{Z}_N = \mathbf{Z} - \hat{\mathbf{Z}}$ and $\hat{\mathbf{x}}_N = \hat{\mathbf{x}} - \hat{\mathbf{Z}}$, where N stands for "net". Also the corresponding input matrix becomes $\mathbf{A}_N = \mathbf{Z}_N\hat{\mathbf{x}}_N^{-1}$

Definition 6.3. *Consider the net open Leontief model $\mathbf{x}_N = \mathbf{L}_N\mathbf{f}$, where $\mathbf{L}_N = (\mathbf{I} - \mathbf{A}_N)^{-1}$. Let the row vector of factor net multipliers be $\mathbf{m}'_{\pi_N} = \boldsymbol{\pi}'_N\mathbf{L}_N$, where $\boldsymbol{\pi}_N$ is the direct factor coefficient vector, and $\mathbf{e}_i$ be the i-th identity column. The group factor net worth of sectors (or regions) $i_1, \ldots, i_k$ is $\omega^{\pi_N}_{i_1,\ldots,i_k}(\mathbf{A}_N, \mathbf{f}, \boldsymbol{\pi}_N) = \mathbf{m}'_{\pi_N}\mathbf{E}(\mathbf{E}'\mathbf{L}_N\mathbf{E})^{-1}\mathbf{E}'\mathbf{x}_N$, where $\mathbf{E} = \left(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}\right)$.*

It is obvious that Theorem 6.3 can be readily used in this setting as well, i.e., the highest group factor net worth determines the key group. Now we explore the difference between the group factor worth and group factor net worth measures, which will shed light on the difference between the key group problems within the gross and net IO settings.

Let $\hat{\mathbf{A}}$ be the diagonal matrix containing direct input coefficients a_{ii} along its main diagonal and zero elsewhere, thus $\hat{\mathbf{A}} = \hat{\mathbf{Z}}\hat{\mathbf{x}}^{-1}$, or $\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{Z}}$. Therefore, $\hat{\mathbf{x}}_N = \hat{\mathbf{x}} - \hat{\mathbf{Z}} = (\mathbf{I} - \hat{\mathbf{A}})\hat{\mathbf{x}}$, or, equivalently, $\mathbf{x}_N = (\mathbf{I} - \hat{\mathbf{A}})\mathbf{x}$. Using this together with $\mathbf{Z}_N = \mathbf{Z} - \hat{\mathbf{Z}}$ and $\hat{\mathbf{x}}_N = \hat{\mathbf{x}} - \hat{\mathbf{Z}}$ imply that $\mathbf{L}_N = (\mathbf{I} - \mathbf{A}_N)^{-1} = \left((\hat{\mathbf{x}}_N - \mathbf{Z}_N)\hat{\mathbf{x}}_N^{-1}\right)^{-1} =$

[11] We are grateful to one of the referees, who suggested to "revise some results and even concepts to render them invariant with respect to the netting out. ... Factor usage better be independent of the reporting or nonreporting of own use ... ."

$\left((\hat{\mathbf{x}}_N - \mathbf{Z} + \hat{\mathbf{Z}})\hat{\mathbf{x}}_N^{-1}\right)^{-1} = \left((\hat{\mathbf{x}} - \mathbf{Z})\hat{\mathbf{x}}_N^{-1}\right)^{-1} = \left((\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}\hat{\mathbf{x}}_N^{-1}\right)^{-1} = \hat{\mathbf{x}}_N\hat{\mathbf{x}}^{-1}\mathbf{L} = (\mathbf{I} - \hat{\mathbf{A}})\mathbf{L}$. Thus, we have proved that

$$l_{ij}^N = (1 - a_{ii})l_{ij} \quad \text{and} \quad x_i^N = (1 - a_{ii})x_i \;\; \text{for all } i \text{ and all } j, \tag{6.7}$$

where l_{ij}^N is the ij-th element of $\mathbf{L}_N$ (Leontief inverse in the net IO model) and x_i^N is the total output of sector i excluding intrasectoral transaction z_{ii}.

Let us denote the sectoral factor production/usage by the vector $\mathbf{u}$, hence the direct factor coefficient vectors are $\boldsymbol{\pi}' = \mathbf{u}'\hat{\mathbf{x}}^{-1}$ and $\boldsymbol{\pi}_N' = \mathbf{u}'\hat{\mathbf{x}}_N^{-1}$. In what follows we make a distinction between the cases when $\boldsymbol{\pi} \neq \boldsymbol{\imath}$ and $\boldsymbol{\pi}_N \neq \boldsymbol{\imath}$ (i.e., the factor coefficients are determined endogenously for given $\mathbf{u}$), and the cases when $\boldsymbol{\pi} = \boldsymbol{\pi}_N = \boldsymbol{\imath}$ (i.e., the factor coefficients are exogenous and set to unity). The importance of this distinction will become clear shortly. Using the above derived relation of $\mathbf{L}_N = \hat{\mathbf{x}}_N\hat{\mathbf{x}}^{-1}\mathbf{L}$, we also get $\mathbf{m}_{\pi_N} = \boldsymbol{\pi}_N'\mathbf{L}_N = \mathbf{u}'\hat{\mathbf{x}}_N^{-1}\hat{\mathbf{x}}_N\hat{\mathbf{x}}^{-1}\mathbf{L} = \boldsymbol{\pi}'\mathbf{L} = \mathbf{m}_\pi'$ whenever $\boldsymbol{\pi} \neq \boldsymbol{\imath} \neq \boldsymbol{\pi}_N$. Hence, for any factor other than total output, the factor multipliers of the gross and net IO models are exactly equal to each other.[12] Using all these results we obtain that for any $\boldsymbol{\pi} \neq \boldsymbol{\imath}$

$$\omega_i^{\pi_N}(\mathbf{A}_N, \mathbf{f}, \boldsymbol{\pi}_N) = \frac{m_i^{\pi_N} x_i^N}{l_{ii}^N} = \frac{m_i^{\pi}(1 - a_{ii})x_i}{(1 - a_{ii})l_{ii}} = \frac{m_i^{\pi} x_i}{l_{ii}} = \omega_i^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}),$$

that is, the factor net worth of sector i is nothing else than sector i's factor worth (whenever the objective of the key sector problem is not total output).

Next, it can be easily shown that the equality $\mathbf{E}'(\mathbf{I} - \hat{\mathbf{A}}) = \mathbf{E}'(\mathbf{I} - \hat{\mathbf{A}})\mathbf{E}\mathbf{E}'$ always holds. Using this equality, (6.7), and $\mathbf{m}_{\pi_N} = \mathbf{m}_\pi$, the group factor net worth for any $\boldsymbol{\pi} \neq \boldsymbol{\imath}$ can be rewritten as

$$\begin{aligned}
\omega_{i_1,\dots,i_k}^{\pi_N}(\mathbf{A}_N, \mathbf{f}, \boldsymbol{\pi}_N) &= \mathbf{m}_{\pi_N}'\mathbf{E}(\mathbf{E}'\mathbf{L}_N\mathbf{E})^{-1}\mathbf{E}'\mathbf{x}_N \\
&= \mathbf{m}_\pi'\mathbf{E}\big(\mathbf{E}'(\mathbf{I} - \hat{\mathbf{A}})\mathbf{L}\mathbf{E}\big)^{-1}\mathbf{E}'(\mathbf{I} - \hat{\mathbf{A}})\mathbf{x} \\
&= \mathbf{m}_\pi'\mathbf{E}\big(\mathbf{E}'(\mathbf{I} - \hat{\mathbf{A}})\mathbf{E}\mathbf{E}'\mathbf{L}\mathbf{E}\big)^{-1}\mathbf{E}'(\mathbf{I} - \hat{\mathbf{A}})\mathbf{E}\mathbf{E}'\mathbf{x} \\
&= \mathbf{m}_\pi'\mathbf{E}(\mathbf{E}'\mathbf{L}\mathbf{E})^{-1}(\mathbf{E}'(\mathbf{I} - \hat{\mathbf{A}})\mathbf{E})^{-1}\mathbf{E}'(\mathbf{I} - \hat{\mathbf{A}})\mathbf{E}\mathbf{E}'\mathbf{x} \\
&= \mathbf{m}_\pi'\mathbf{E}(\mathbf{E}'\mathbf{L}\mathbf{E})^{-1}\mathbf{E}'\mathbf{x} = \omega_{i_1,\dots,i_k}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}),
\end{aligned}$$

thus the group worth of sectors (regions) in the standard and the net IO frameworks are exactly equal to each other for any factor other than total output. We have

[12] Output multipliers in these settings do not equal each other simply because in the relation $\mathbf{L}_N = \hat{\mathbf{x}}_N\hat{\mathbf{x}}^{-1}\mathbf{L}$, in general, $\hat{\mathbf{x}}_N\hat{\mathbf{x}}^{-1} \neq \boldsymbol{\imath}$ as usually $z_{ii} > 0$, thus $x_i^N < x_i$.

established the following result.

Theorem 6.4. *The group factor worth of sectors (regions) equals their group factor net worth for any factor other than total output, i.e.,* $\omega^{\pi}_{i_1,\ldots,i_k}(\mathbf{A},\mathbf{f},\boldsymbol{\pi}) = \omega^{\pi_N}_{i_1,\ldots,i_k}(\mathbf{A}_N,\mathbf{f},\boldsymbol{\pi}_N)$ *for all* $k = 1,\ldots,n$ *and* $\boldsymbol{\pi} \neq \boldsymbol{\imath} \neq \boldsymbol{\pi}_N$.

The consequence of Theorem 6.4 is that the composition of the key sector and the key group in generating/using some factor (other than total output) is invariant to inclusion or exclusion of intrasectoral transactions in the IO data. Moreover, the rankings of all groups are exactly the same under the two IO frameworks with and without internal sectoral flows provided that the objective of the key sector/group problem is not total output. This is, of course, a desirable property of the group factor worth measure, since some analysts might prefer using the net IO model.

When, however, $\boldsymbol{\pi} = \boldsymbol{\imath}$, the group output worth is, in general, *not* invariant with respect to the netting out of intrasectoral transactions. This follows since for all $k = 1,\ldots,n$ (using the above results)

$$\begin{aligned}\omega^{o}_{i_1,\ldots,i_k}(\mathbf{A},\mathbf{f}) - \omega^{o_N}_{i_1,\ldots,i_k}(\mathbf{A}_N,\mathbf{f}) &= (\mathbf{m}'_o - \mathbf{m}'_{o_N})\mathbf{E}(\mathbf{E}'\mathbf{L}\mathbf{E})^{-1}\mathbf{E}'\mathbf{x} \\ &= \boldsymbol{\imath}'(\mathbf{I} - \mathbf{I} + \hat{\mathbf{A}})\mathbf{L}\mathbf{E}(\mathbf{E}'\mathbf{L}\mathbf{E})^{-1}\mathbf{E}'\mathbf{x} \\ &= \boldsymbol{\imath}'\hat{\mathbf{A}}\mathbf{L}\mathbf{E}(\mathbf{E}'\mathbf{L}\mathbf{E})^{-1}\mathbf{E}'\mathbf{x},\end{aligned} \tag{6.8}$$

where the vector of the so-called net output multipliers is $\mathbf{m}'_{o_N} = \boldsymbol{\imath}'\mathbf{L}_N = \boldsymbol{\imath}'(\mathbf{I} - \hat{\mathbf{A}})\mathbf{L}$. Hence, equation (6.8) shows that when $a_{ii} = a$ for all i, the key sector/group problem with total output as the objective in the gross and net IO models give identical results. This is because then (6.8) boils down to $\omega^{o_N}_{i_1,\ldots,i_k}(\mathbf{A}_N,\mathbf{f}) = (1 - a) \times \omega^{o}_{i_1,\ldots,i_k}(\mathbf{A},\mathbf{f})$, which implies that the key sector/group composition and the rankings of all other groups in the two settings are identical when $\hat{\mathbf{A}} = a\mathbf{I}$ in case of total output as the objective of the key sector/group problem.[13]

The above findings are illustrated by a simple example of a three-sector economy given in Table 6.2. Group factor and output worths are given in relative terms (i.e., relative to, respectively, total factor and total output before the extraction). It shows that relative factor worths both with and without intrasectoral transactions are equal to each other for factor usage (say, water), which is an expectable result due to Theorem 6.4. However, when the objective of the key sector/group problem

[13] The fact that the invariance property of the group output worth does not hold in general is not a big issue, since many would agree that gross output is a rather uninteresting indicator. One of the referees stated that "Of course, gross output is a largely uninteresting measure. This is why national accounts were built! They were designed to make better estimates of what is interesting – gross domestic product, which avoids the double counting inherent to gross output."

Table 6.2: Relative group factor worths in the gross and net IO settings

	Agr	Mng	Mnf	**f**	**x**
Agr	6	3	3	5	17
Mng	2	2	1	5	10
Mnf	2	2	10	5	19
Factor	5	4	8		
$k = 1$	*RFW/RFNW*		*ROW*	*RONW*	
Agr	49.6		55.0	56.6*	
Mng	48.1		47.3	52.2	
Mnf	61.6*		58.0*	51.0	
	Groups of size 2				
Rank	(*RFW/RFNW*)		(*ROW*)	(*RONW*)	
1	Mng, Mnf (86.6)		Agr, Mnf (86.4)	{Agr, Mng}, {Agr, Mnf}, {Mng, Mnf} (82.1)	
2	Agr, Mnf (85.3)		Mng, Mnf (83.2)	-	
3	Agr, Mng (73.9)		Agr, Mng (77.1)	-	

Note: Agr, Mng and Mnf denote, respectively, Agriculture, Mining and Manufacturing. Other abbreviations are: *RFW* – relative factor worth, *RFNW* – relative factor net worth, *ROW* – relative output worth, *RONW* – relative output net worth.

is total output, the results with intrasectoral transactions largely differ from those based on the net IO model. For example, the highest output worth has Manufacturing (Mnf) with its relative gross output worth of 58.0%, while in the net IO framework Agriculture (Agr) is the key sector with the relative total output net worth of 56.6%. Further, all possible groups of size 2 (i.e., 3 groups) have equal group output worths in the net IO model (i.e., the relative group output net worths of all groups of size 2 is 82.1%), while that is not the case when intrasectoral transactions are accounted for.

6.3 The link to the fields of influence approach

Another well-known technique for evaluating sectors' influence on the rest of the economy is Sonis and Hewings' notion of a *field of influence* method (see e.g., Sonis and Hewings, 1989, 1992). This methodology answers the question of how changes in some elements of the input matrix affect the rest of the system by examining the impact on the elements of the Leontief inverse, and is general enough to handle changes in one direct coefficient, in all elements of a row or column of the input matrix, or in all coefficients simultaneously. From an economic point of view this enables one to analyze, for example, the effects of technological change, improvements in efficiency, changes in product lines, changes in the structure and complexity of an economy over time, or changes in trade dependency of a country.

To briefly introduce this method, let us consider a change of $\alpha \neq 0$ in only one coefficient a_{rc}, with all other input coefficients being fixed. Then the Leontief inverse after the change is[14]

$$\widetilde{\mathbf{L}} = \mathbf{L} + \frac{\alpha}{1 - \alpha l_{cr}} \mathbf{F}(r,c), \tag{6.9}$$

where $\mathbf{F}(r,c) = \mathbf{L}\mathbf{e}_r\mathbf{e}_c'\mathbf{L}$ is the *first-order field of influence* matrix of the coefficient a_{rc}.

The sum of all elements of the first-order field of influence matrix, $\boldsymbol{\imath}'\mathbf{F}(r,c)\boldsymbol{\imath}$, gives the *first-order intensity field of influence* of the direct input a_{rc}. In Sonis and Hewings (1989) this concept was introduced in order to measure the *inverse importance* of direct inputs. Consequently, those elements of $\mathbf{A}$ whose changes lead to the largest impact on the system are called the *inverse-important coefficients*.

Unlike the standard first-order intensity $\boldsymbol{\imath}'\mathbf{F}(r,c)\boldsymbol{\imath}$, the scalar $\boldsymbol{\imath}'\mathbf{F}(r,c)\mathbf{f}$ weights every purchasing sector in the sum according to the size of its final demand, hence can be called as the *output first-order intensity weighted field of influence* of a_{rc}. This makes more sense in computing the global intensity since sectors are not given an equal importance, but rather their scale of final demand satisfaction is taken into account. More generally, we term the scalar $\boldsymbol{\pi}'\mathbf{F}(r,c)\mathbf{f}$ as a *factor first-order intensity weighted field of influence* of the coefficient a_{rc}, since it measures the effect of the input coefficient change on total factor generation rather than on gross output. Having defined this intensity measure, we can rewrite the factor worth of sector i from Definition 6.1 as

$$\omega_i^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \frac{m_i^{\pi} x_i}{l_{ii}} = \frac{\boldsymbol{\pi}'\mathbf{L}\mathbf{e}_i\mathbf{e}_i'\mathbf{L}\mathbf{f}}{l_{ii}} = \frac{\boldsymbol{\pi}'\mathbf{F}(i,i)\mathbf{f}}{l_{ii}},$$

which clearly shows that the key sector problem (6.2) searches for the sector i that, on the one hand, has a large economy-wide impact on the total factor usage/generation due to (incremental) change in its *direct input self-dependency* (i.e., due to change in a_{ii}), and on the other hand, is less input dependent on itself directly and indirectly. The first statement is true since the effect of a change in direct input self-dependency of sector i on the overall factor consumption/generation is given by the factor first-order intensity weighted field of influence of the input coefficient a_{ii}, $\boldsymbol{\pi}'\mathbf{F}(i,i)\mathbf{f}$.

From the theory of partitioned matrices it follows that for a nonsingular matrix

[14] Notice that $\frac{\partial \tilde{l}_{ij}}{\partial \alpha}\big|_{\alpha=0} = f_{ij}(r,c) = l_{ir}l_{cj} = f_{cr}(j,i)$. Equation (6.9) follows from the well-known Sherman and Morrison (1950) formula of the inverse change given by $\tilde{l}_{ij} = l_{ij} + \alpha l_{ir}l_{cj}/(1 - \alpha l_{cr})$.

$\mathbf{X}$ the identity

$$\begin{vmatrix} \mathbf{X} & \mathbf{b} \\ \mathbf{c}' & \delta \end{vmatrix} = |\mathbf{X}|(\delta - \mathbf{c}'\mathbf{X}^{-1}\mathbf{b}) \tag{6.10}$$

holds, where $|\mathbf{X}|$ is the determinant of $\mathbf{X}$.

Using (6.10) with $\delta = 0$ and $\mathbf{X} = \mathbf{L}_{kk}$ (recall that $\mathbf{L}_{kk} = \mathbf{E}'\mathbf{L}\mathbf{E}$), we can write the ij-th element of the matrix $\mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\mathbf{L}$ as follows

$$\mathbf{l}_{i\bullet}'\mathbf{L}_{kk}^{-1}\mathbf{l}_{\bullet j} = \frac{-\begin{vmatrix} \mathbf{L}_{kk} & \mathbf{l}_{\bullet j} \\ \mathbf{l}_{i\bullet}' & 0 \end{vmatrix}}{|\mathbf{L}_{kk}|}, \tag{6.11}$$

where $\mathbf{l}_{i\bullet}'$ is the i-th row of the matrix $\mathbf{L}\mathbf{E}$ and $\mathbf{l}_{\bullet j}$ is the j-th column of $\mathbf{E}'\mathbf{L}$.

But the numerator in the last equation is nothing else than the ij-th element of the *matrix field of influence of order k* of the direct input coefficients $a_{i_1 i_1}, a_{i_2 i_2}, \ldots, a_{i_k i_k}$, $\mathbf{F}[(i_1, i_1), (i_2, i_2), \ldots, (i_k, i_k)]$ (see e.g., Fritz et al., 2002).[15] That is, this matrix quantifies the effect of an infinitesimal change in the coefficients $a_{i_1 i_1}, a_{i_2 i_2}, \ldots, a_{i_k i_k}$ on the elements of the Leontief inverse.

Let us now take $k = 2$. Then the denominator in (6.11) can be rewritten as

$$|\mathbf{L}_{22}| = \begin{vmatrix} l_{ii} & l_{ij} \\ l_{ji} & l_{jj} \end{vmatrix} = l_{ii}l_{jj} - l_{ij}l_{ji} = l_{ii}\left(l_{jj} - \frac{l_{ji}l_{ij}}{l_{ii}}\right) = l_{ii}l_{jj}^{-i}, \text{ or}$$
$$|\mathbf{L}_{22}| = l_{ii}l_{jj} - l_{ij}l_{ji} = l_{jj}\left(l_{ii} - \frac{l_{ij}l_{ji}}{l_{jj}}\right) = l_{jj}l_{ii}^{-j},$$

where we have used Lemma 6.1. That is, for example, l_{jj}^{-i} is the jj-th element of the Leontief inverse after sector i has been removed from the production system, $\mathbf{L}^{-i}$ (note that $i \neq j$). Hence, we now may expect that, in general, $|\mathbf{L}_{kk}| = l_{i_1 i_1} l_{i_2 i_2}^{-i_1} l_{i_3 i_3}^{-\{i_1, i_2\}} \cdots l_{i_k i_k}^{-\{i_1, \ldots, i_{k-1}\}}$ for all $k = 1, \ldots, n$, where $l_{i_k i_k}^{-\{i_1, \ldots, i_{k-1}\}}$ is the $i_k i_k$-th element of $\mathbf{L}^{-\{i_1, \ldots, i_{k-1}\}}$, i.e., the Leontief inverse after the (hypothetical) extraction of sectors $i_1, \ldots, i_{k-1}$ from the economy. Let us prove this by mathematical induction. Assume that the last expression holds for $k - 1$, i.e., $|\mathbf{L}_{(k-1),(k-1)}| = l_{i_1 i_1} l_{i_2 i_2}^{-i_1} l_{i_3 i_3}^{-\{i_1, i_2\}} \cdots l_{i_{k-1} i_{k-1}}^{-\{i_1, \ldots, i_{k-2}\}}$. Then using (6.10) and Lemma 6.2 we derive (note

[15] We should note that the only difference comes in signs when k is even, i.e., in the fields of influence approach the determinant in the numerator of the last equation is multiplied by $(-1)^k$. However, in our setting there is no sign change of the determinant considered: it cannot be negative, since then it has to be the case that $\mathbf{L}^{-\{i_1, \ldots, i_k\}} > \mathbf{L}$, which contradicts the Leontief inverse property.

that $i_r \neq i_s$)

$$|\mathbf{L}_{kk}| = \begin{vmatrix} \mathbf{L}_{(k-1),(k-1)} & \mathbf{l}_{\bullet i_k} \\ \mathbf{l}'_{i_k\bullet} & l_{i_k i_k} \end{vmatrix} = |\mathbf{L}_{(k-1),(k-1)}| \left(l_{i_k i_k} - \mathbf{l}'_{i_k\bullet} \mathbf{L}^{-1}_{(k-1),(k-1)} \mathbf{l}_{\bullet i_k} \right)$$
$$= |\mathbf{L}_{(k-1),(k-1)}| \cdot l^{-\{i_1,\ldots,i_{k-1}\}}_{i_k i_k} = l_{i_1 i_1} l^{-i_1}_{i_2 i_2} l^{-\{i_1,i_2\}}_{i_3 i_3} \cdots l^{-\{i_1,\ldots,i_{k-1}\}}_{i_k i_k}.$$

Employing the above results in (6.11) implies that the group factor worth of sectors $i_1, \ldots, i_k$ ($i_r \neq i_s$) from Definition 6.2 can be rewritten as

$$\omega^{\pi}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \mathbf{m}'_{\pi} \mathbf{E} \mathbf{L}^{-1}_{kk} \mathbf{E}' \mathbf{x} = \boldsymbol{\pi} \mathbf{L} \mathbf{E} \mathbf{L}^{-1}_{kk} \mathbf{E}' \mathbf{L} \mathbf{f} = \frac{\boldsymbol{\pi}' \mathbf{F}[(i_1, i_1), \ldots, (i_k, i_k)] \mathbf{f}}{l_{i_1 i_1} l^{-i_1}_{i_2 i_2} l^{-\{i_1,i_2\}}_{i_3 i_3} \cdots l^{-\{i_1,\ldots,i_{k-1}\}}_{i_k i_k}}.$$

This implies that the key group problem (6.4) searches for a group of k sectors with the highest group factor worth, which is directly proportional to the impact on overall factor generation of incremental changes in *direct input self-dependencies* of sectors comprising the group,[16] and inversely related to the size of their unit *own total input dependence* that does not overstate the intermediate role of the group members. To see the interpretation of the second effect, let us consider the group of size two. Then $|\mathbf{L}_{22}| = l_{ii} l^{-i}_{jj}$ is the product of the unit (i.e., per one unit of final demand) *own* total input dependence of sector i and the unit own total input dependence of sector j without any (intermediate) role for sector i (since it has been removed from the system). This gives the size of the own total input dependence of the group $\{i, j\}$. Hence, the own input dependence $l_{ii} l^{-i}_{jj}$ (or, equivalently, $l_{jj} l^{-j}_{ii}$) does not include the contribution of the group-member whose own input dependence was already accounted for. Exactly the same interpretation holds for the general case of $|\mathbf{L}_{i_k i_k}| = l_{i_1 i_1} l^{-i_1}_{i_2 i_2} l^{-\{i_1,i_2\}}_{i_3 i_3} \cdots l^{-\{i_1,\ldots,i_{k-1}\}}_{i_k i_k}$ (note that the order of $i_1, \ldots, i_k$ can be changed without affecting the result and its economic interpretation). The above mentioned interpretation of the group factor worth makes sense, because it is reasonable to consider a group of industries as the key group if a (incremental) change in their internal input structure has the maximum impact on the factor production/consumption (the first effect), *and* the group-members are less input dependent on themselves directly and indirectly (the second effect).

All in all, we have shown that the (generalized) HEM and the fields of influence approach are closely related, which is, in fact, not surprising since both methods deal with the same issue of the impact of a change in input coefficients on the entire

[16] This interpretation is due to the economic meaning of $\boldsymbol{\pi}' \mathbf{F}[(i_1, i_1), \ldots, (i_k, i_k)] \mathbf{f}$, which we might similarly term as a *factor intensity weighted field of influence of order k* of input coefficients $a_{i_1 i_1}, \ldots, a_{i_k i_k}$.

economic system within the IO framework.

6.4 Connection to game theory

There is a link between the key sector/group identification problem discussed in Section 6.2 and the game theoretic literature on finding the key players in social networks, on the one hand, and on finding a fair allocation of gains from cooperation among coalition participants, on the other hand. The connection between Chapter 5 on social networks and the previous sections of this chapter is due to the similar mathematical structure of the problems posed in Section 5.2 and Section 6.2. That is, comparing the key group problems in the framework of social networks with ex ante identical individuals and interindustry relations, one can easily observe the mathematical similarity of their solutions.[17] This is not surprising since these problems address conceptually the same issue, finding sectors or actors that have the largest overall impact on the aggregate outcome. Of course, their interpretations are totally different given their different underlying theoretical frameworks.

In what follows we discuss an additional link of the key sector/group problem, namely to the coalitional game literature on measuring players' power. The question of a *fair* allocation of gains obtained from cooperation among several actors was one of the main points of focus at the outset of game theory. The setup is as follows. A cooperation of actors results in a certain overall gain that has to be divided among the actors within the coalition. However, this is not a trivial issue given that actors have different contributions to the coalition. The legitimate question then is how to allocate "fairly" the gain from cooperation to its participants. Or in other words, how important is each actor to the coalition, and what payoff does (s)he deserve? One approach is to use the *Shapley value*, named in honor of Lloyd Shapley, who introduced it in his classical 1953 paper "A value for n-person games". Using an axiomatic approach, Shapley constructed a solution remarkable for its intuitive definition and unique characterization by a set of reasonable axioms. The specialization of the Shapley value to simple games[18] is often used as an index of voting power and is known as the Shapley-Shubik power index (Shapley and Shubik, 1954). An other related indicator is the Banzhaf power index proposed

[17] Note, however, that this is no longer true once the exogenous heterogeneity of players is taken into account (see Theorem 5.2).

[18] A game is simple if payoffs are either 1 or 0, i.e., coalitions are either "winning" or "losing". See Shapley (1962) for details on simple games.

in Banzhaf (1965).[19] The generalization of Shapley and Banzhaf values to a *coalitional structure*, where the interaction between players is *not* symmetric in the sense that actors may be part of different groups, which might make negotiations between groups impossible, is studied, in particular, by Aumann and Drèze (1974) and Owen (1977, 1981).[20] A *share function* solution of van der Laan and van den Brink (1998) assigns to every player its share in the worth of the grand coalition, and contains the Shapley share function and the Banzhaf share function as special cases. A solution in terms of share functions for games with a coalitional structure is introduced in van der Laan and van den Brink (2002). Since all these values are closely related to the original contribution of Shapley (1953), we will in what follows only discuss the link of our factor worth measure to the Shapley value.

Formally, a *coalitional form game* on a finite set of players $N = \{1, 2, \ldots, n\}$ is a function v from the set of all coalitions 2^N to the set of real numbers $\mathbb{R}$, with the properties

1. $v(\varnothing) = 0$,
2. $v(S \cup T) \geq v(S) + v(T)$, whenever $S \cap T = \varnothing$.

The interpretation of $v(S)$ is the *expected total payoff* (gain or rent) that the coalition S can get in the game v. The second property, the so-called superadditivity condition, implies that cooperation can only benefit players, and never makes them worse off. The Shapley value (ϕ) is one way to distribute the total gain to all players, which assigns to each game v a vector of payoffs $\boldsymbol{\phi}(v)' = (\phi_1, \phi_2, \ldots, \phi_n)$ in $\mathbb{R}^n$. Alternatively, one can think of $\phi_i(v)$ as the measure of i's power in the game v. For all $S \subseteq N$ and all $i \in S$, the marginal contribution of player i to coalition S in game v is defined by $v(S) - v(S \setminus \{i\})$. Shapley constructed the following value that assigns an expected marginal contribution of each player in the game with respect to a uniform distribution over the set of all permutations on the set of players:

$$\phi_i(v) = \sum_{S \subseteq N,\, i \in S} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v(S \setminus \{i\}) \right], \tag{6.12}$$

where s is the cardinality of S (or number of players in S).

In words, $\phi_i(v)$ is essentially the weighted average of player i's marginal con-

[19] Straffin (1977, 1988) interprets the Shapley-Shubik and Banzhaf indices as the probabilities of affecting the voting outcome. In this sense, he shows that the Shapley-Shubik index is more appropriate when voters' decisions are correlated (e.g., a society judging welfare by common standards), while the Banzhaf index is more appropriate if voters behave independently of each other.

[20] Cooperative games with a coalitional structure imply a two-level interaction between the players (see eg., Hart and Kurz, 1983). Firstly, the value of the grand coalition is distributed amongst the coalitions, and secondly, the worth of each coalition is allocated amongst the players within this coalition. See also, Winter (1989, 1992); Owen and Winter (1992).

tributions $v(S) - v(S \setminus \{i\})$ with corresponding weights $(s-1)!(n-s)!/n!$, where the sum is taken over all the coalitions S to which player i belongs. Note that the weight is equal to the product of the $(s-1)!$ different permutations of the members of coalition S aside from player i and the $(n-s)!$ different permutations of players outside the coalition S, and then divided by the $n!$ different permutations of all the players in the grand coalition N.

The Shapley value satisfies the following four axioms.

Efficiency: $\sum_{i \in N} \phi_i(v) = v(N)$, i.e., the resources available to the grand coalition are precisely distributed amongst all the players.

Symmetry: If $v(S \cup \{i\}) = v(S \cup \{j\})$ for every subset $S \subset N$ with $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$. That is, if players $i, j \in N$ make the same marginal contribution to any coalition S that contains neither i nor j, then i and j are symmetric with respect to game v, and have equal shares.

Dummy: If i is a dummy (or null) player, i.e., $v(S \cup \{i\}) = v(S)$ for all $S \subset N$, then $\phi_i(v) = 0$. This axiom requires that players with a zero marginal contribution to every coalition are given zero payoffs.

Additivity: For any two games v and w on a set N of players, $\phi_i(v+w) = \phi_i(v) + \phi_i(w)$ for all $i \in N$, where $v+w$ is the game defined by $(v+w)(S) = v(S) + w(S)$. This axiom requires that the value is an additive operator on the space of all games.

The remarkable finding of Shapley (1953) is that there exists a *unique* value that satisfies these four simple axioms, and it is the Shapley value given in (6.12).[21]

Now we are in a position to compare the factor worth of sector i, $\omega_i^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = m_i^{\pi} x_i / l_{ii}$, with the Shapley value. These two indicators are similar in the sense that both assess the power of an agent on the base of its marginal contribution. The difference, however, is that the factor worth focuses on the marginal contribution of a sector to the total factor generated by production sectors taken *altogether* (see (6.2)), while the Shapley value takes into account the marginal contributions of a player to *all permutations* on the set of players.

To see clearly the similarities and distinctions, we will check whether the factor worth satisfies the above mentioned four axioms. In the framework of the IO analysis the value of all industries is the resulting aggregate factor, $\boldsymbol{\pi}'\mathbf{x}$. It can be shown that for any nonnegative final demand and direct factor coefficients vectors $\mathbf{f}$ and $\boldsymbol{\pi}$ the inequality $\sum_{i=1}^{n} \omega_i^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) \geq \boldsymbol{\pi}'\mathbf{x}$ always holds, where the inequality is strict when $\mathbf{f}$ and $\boldsymbol{\pi}$ are strictly positive vectors (see Appendix 6.A). Thus, the sum of the individual factor worths of all sectors is at least as large

[21] For more details see e.g., Roth (1988); Winter (2002).

as the total factor that all the industries generate/use. In other words, the factor worth does *not* satisfy the efficiency axiom in the context of the coalitional game. The symmetry property, however, holds in the key sector framework, which is an expectable outcome. Two sectors i and j with the same individual contributions to overall factor have identical factor worths, which follows from the fact that $\omega_i^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = m_i^{\pi} x_i / l_{ii} = \boldsymbol{\pi}'(\mathbf{x} - \mathbf{x}^{-i}) = \boldsymbol{\pi}'(\mathbf{x} - \mathbf{x}^{-j}) = m_j^{\pi} x_j / l_{jj} = \omega_j^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$. The factor worth measure also satisfies the third axiom of dummy, when the null (or dummy) sector has $\omega_i^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = 0$. There are many dummy possibilities. The definition of the factor worth implies that the null sector has zero output, $x_i = 0$. For example, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{24} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} 0 \\ f_2 \\ \vdots \\ f_n \end{bmatrix},$$

then $x_1 = 0$ implying that sector 1 can be viewed as a dummy sector. Hence, hypothetically extracting sector 1 makes no difference.

Finally, the combined key sector problem briefly discussed in footnote 9 implies that the additivity property of the Shapley value is also satisfied in case of the factor worth. If we would define the combined factors worth by $\omega_i^{v,w}(\mathbf{A}, \mathbf{f}, \mathbf{v}, \mathbf{w})$, where $\mathbf{v}$ and $\mathbf{w}$ are the direct sectoral value-added and labor coefficients, respectively, then from the factor worth definition it follows that $\omega_i^{v,w}(\mathbf{A}, \mathbf{f}, \mathbf{v}, \mathbf{w}) = \omega_i^{v}(\mathbf{A}, \mathbf{f}, \mathbf{v}) + \omega_i^{w}(\mathbf{A}, \mathbf{f}, \mathbf{w})$, which is the additivity condition in the context of the coalitional game literature. All in all, we have shown that the Shapley value and the factor worth measures are intuitively closely related. Evidently, their applications are quite different given that these measures are the outcome of two entirely different frameworks.

6.5 Application to the Australian economy

We have already noted that the input-output linkage studies (implicitly) accepted the k sectors (where $1 < k < n$) with the largest individual factor worths as the *key group* of k sectors. In this section, using an example of the Australian economy we show that this is not true as long as the HEM approach is concerned, i.e., the k sectors with the highest factor worths, in general, do *not* compose the *key group* of size k.

We have used data from Foran et al. (2005) and Centre for Integrated Sustainability Analysis (2005) that include the 1994-1995 Australian IO tables and satellite accounts at a 136 industry-level classification.[22] For simplicity, the industries were codified, and the list of codes is given in Appendix 6.B. The key sector/group problem is performed for two environmental, one financial and one social factor. These are, respectively, water use, carbon dioxide (CO_2) emissions, gross operating surplus, and wages and salaries. The results are reported in the first five columns of Table 6.3 in terms of *relative* group factor worths, i.e., the group factor worths as a percentage of the overall factor use/generation before the extraction of sectors comprising the group. For instance, the relative profits (gross operating surplus) worth of sectors i and j ($\neq i$) equals $(\omega^{p}_{i,j}(\mathbf{A}, \mathbf{f}, \mathbf{p})/\mathbf{p}'\mathbf{x}) \times 100$, where $\mathbf{p}$ is the vector of sectoral direct profits coefficients, thus $\mathbf{p}'\mathbf{x}$ is the total gross operating surplus in the economy. Hence, these relative measures refer to the percentage decrease in economy-wide factor use/generation caused by the extraction. We only report the top 5 groups of size $k \in [1, 4]$, and, obviously, the group with rank 1 in each list is the corresponding *key group*.

Several observations can be made from Table 6.3. The first and most obvious observation is that different objectives give a different composition of the key group of a certain size and different rankings of sectors or group of sectors. This is totally expectable, as different sectors perform different functions in the economy, thus should not be equivalent in terms of consumption/production of various factors.

Second, the composition of the key group of size k is, in general, different from the k sectors with the largest (individual) factor worths, which confirms our expectation that the key sector problem is not equivalent to the key group problem. For example, let us look at the key group problem in terms of water use. The second column of Table 6.3 shows that Dairy cattle & milk (Dc) is the key sector in water use with a relative water consumption worth of 19.5%.[23] The key group of size two consists of the key sector Dc and Beef cattle (Bc) jointly accounting for 37.6% of the economy-wide water consumption, which, however, does *not* include Diary products (Dp) that has the second largest water (usage) worth. Further, the key group of size 3 includes, besides Dc and Bc, Water supply, sewerage and drainage services (Wa), which has only the sixth rank according to the key sector problem with water worth of 10.6% (not shown in Table 6.3). The traditional "top-list" ap-

[22] Foran et al. (2005) give a detailed description of the data sources and their construction.

[23] In the language of the HEM problem, if Dairy cattle & milk (Dc) sector would be eliminated from the economy, the overall use of water would be reduced by 19.5%.

Table 6.3: Relative group factor worths of Australian industries, 1994-1995

Rank	Group of size k and its relative factor worth (%)				Factor multipliers	Factor use/ generation	Factor responsibility
	$k=1$	$k=2$	$k=3$	$k=4$			
	Objective: Water use				(ths.l/A$)	(Tl)	(Tl)
1 (key)	Dc (19.5)	Bc, Dc (37.6)	Bc, Dc, Wa (48.1)	Bc, Dc, Vf, Wa (58.0)	Ri (7.47)	Dc (3.54)	Dp (2.89)
2	Dp (18.6)	Dc, Mp (37.3)	Dc, Mp, Wa (47.7)	Dc, Mp, Vf, Wa (57.5)	Sc (1.64)	Bc (3.23)	Mp (2.68)
3	Bc (18.2)	Bc, Dp (36.8)	Bc, Dc, Vf (47.6)	Bc, Dp, Vf, Wa (57.1)	Dc (1.48)	Wa (2.02)	Fd (1.35)
4	Mp (18.1)	Dp, Mp (36.4)	Bc, Dp, Wa (47.3)	Dp, Mp, Vf, Wa (56.7)	Su (1.26)	Vf (1.80)	Ho (1.13)
5	Vf (10.7)	Dc, Wa (30.0)	Dc, Mp, Vf (47.2)	Bc, Dc, Fd, Wa (55.9)	Bc (0.73)	Ri (1.43)	Wa (1.12)
	Objective: CO_2 emissions				(kg/A$)	(Mtonnes)	(Mtonnes)
1 (key)	El (32.8)	Bc, El (52.9)	Bc, Fr, El (64.4)	Bc, Fr, El, Is (69.0)	Fr (98.3)	El (136.6)	Mp (59.7)
2	Bc (20.3)	El, Mp (50.6)	Fr, Mp, El (62.1)	Bc, Fr, El, Wt (67.7)	Sw (25.2)	Bc (81.2)	El (53.8)
3	Mp (18.3)	El, Fr (44.9)	Bc, Is, El (57.6)	Bc, Fr, El, Rb (67.5)	Bc (17.9)	Fr (50.9)	Fr (38.0)
4	Fr (12.3)	El, Is (37.5)	Bc, El, Wt (56.4)	Bc, Fr, El, At (67.2)	Hw (15.4)	Is (17.9)	Rt (21.7)
5	Is (5.5)	El, Wt (36.4)	Bc, El, Rb (56.3)	Bc, Fd, Fr, El (67.1)	Lm (14.8)	At (10.1)	Rb (16.8)
	Objective: Gross operating surplus (profits)				(A$/A$)	(A$ Bln)	(A$ Bln)
1 (key)	Dw (21.7)	Dw, Wt (31.2)	Dw, Rb, Wt (37.1)	Dw, Rb, Rt, Wt (42.5)	Dw (0.84)	Dw (38.7)	Dw (41.6)
2	Wt (9.7)	Dw, Rb (27.9)	Dw, Rt, Wt (36.7)	Dw, Nb, Rb, Wt (41.7)	Si (0.68)	Wt (7.5)	Rb (11.9)
3	Rb (6.6)	Dw, Rt (27.5)	Dw, Nb, Wt (35.9)	Dw, Nb, Rt, Wt (41.3)	Bl (0.63)	Rb (7.1)	Rt (11.0)
4	Rt (5.9)	Dw, Ms (26.9)	Dw, Ms, Wt (35.1)	Dw, Ho, Rb, Wt (40.8)	Br (0.622)	St (6.44)	Wt (9.4)
5	Ms (5.3)	Dw, Nb (26.7)	Dw, Ho, Wt (35.0)	Dw, Ms, Rb, Wt (40.8)	Ng (0.62)	Ms (6.39)	Nb (9.0)
	Objective: Net wages and salaries				(A$/A$)	(A$ Bln)	(A$ Bln)
1 (key)	Wt (12.4)	Rt, Wt (22.8)	Hs, Rt, Wt (31.7)	Ed, Hs, Rt, Wt (40.5)	Ed (0.61)	Ed (14.6)	Rt (18.3)
2	Rt (10.9)	Hs, Wt (21.3)	Ed, Rt, Wt (31.6)	Gv, Hs, Rt, Wt (39.0)	Gd (0.58)	Hs (14.2)	Hs (15.5)
3	Hs (9.1)	Ed, Wt (21.24)	Ed, Hs, Wt (30.2)	Gv, Ed, Rt, Wt (38.8)	Hs (0.533)	Rt (11.7)	Ed (14.5)
4	Ed (9.09)	Hs, Rt (20.1)	Gv, Rt, Wt (30.1)	Hs, Nb, Rt, Wt (38.0)	Os (0.53)	Wt (11.6)	Gv (11.5)
5	Gv (7.8)	Ed, Rt (20.0)	Ed, Hs, Rt (29.1)	Ed, Nb, Rt, Wt (37.9)	Gv (0.50)	Gv (10.0)	Nb (10.8)
Total	136	9,180	410,040	13,633,830	136	136	136

Note: "Total" is the total number of all possible groups of size k. Mathematically, it is equal to the combinations of $n = 136$ sectors taken k at a time, $C_k^n = n!/(k!(n-k)!)$. One teraliter (Tl) is equivalent to 10^{12} litres. The source of the seventh column "Factor use/generation" is the satellite accounts in Foran et al. (2005) and Centre for Integrated Sustainability Analysis (2005), while the rest are own computations based on these data. One megatonne (Mtonne) equals 10^6 tonnes. Sectors' abbreviations are listed in Appendix 6.B.

proach would consider the "key" group of size 4 consisting of dairy and beef cattle, and dairy and meet products (i.e., Dc, Dp, Bc and Mp as the top 4 sectors with the largest individual water usage worths), while the formal key group problem finds beef and diary cattle (Bc, Dc), Vegetable and fruit growing (Vf), and Water supply, sewerage & drainage (Wa) to be the part of the key group. The legitimate question is why the "top-list" approach does not give the true outcome identified by the key group problem.[24] The group factor worth of sectors $i_1, \ldots, i_k$ can be rewritten as

$$\omega^{\pi}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \sum_{s=i_1}^{i_k} \pi_s x_s + \sum_{j \neq i_1,\ldots,i_k} \pi_j \left(x_j - x_j^{-\{i_1,\ldots,i_k\}} \right),$$

which shows that the factor worth of the extracted sectors includes not only their own contributions to factor usage/generation (the first sum), but also their contributions to the factor consumed/produced via every other sector outside the group (the second sum).[25] Hence, with inherently different structures and sizes of inter-sectoral links, and intermediate and final demands, the group of k sectors will play quite a different role in overall factor usage/generation process than a single industry, in particular, through its indirect channel.

This result wedges a bridge between the IO linkage analysis and the sociology literature on actors' importance in social networks. This link has to do with what sociologists call the *redundancy principle* (see e.g., Burt, 1992; Borgatti, 2006), which in our framework means that sectors may be redundant with respect to their linkage patterns, factor generation abilities and final demand structures. For example, two sectors with approximately the same sizes of inter-industry transactions, factor generation, and final demands are redundant when they connect the same third industries to each other, or when they are connected to the same third parties. In the sociological terminology, such redundant sectors are called to be *structurally equivalent*. In the framework of social networks, Temurshoev (2008) showed that there is a link between the key group members and clusters of similar agents (see also Chapter 5 in this thesis). That is, the key group generally contains members from

[24] Note that in our example these two approaches give identical results for $k \in [1, 4]$ when the objectives are profits, and wages and salaries. We should, however, stress that these observations by no means subside the existence of the difference between the two approaches, and thus the key group problem should always be given preference over the "top-list" approach whenever the HEM is a study methodology. For example, the application to the Kyrgyzstan economy for value-added and gross output resulted in a dramatic difference between the "top-list" and the key group problem approaches in defining the key group (these results are not shown here as we have decided to focus only on the Australian economy).

[25] Note that in case of gross output being the objective, i.e., when $\pi_i = 1$ for all i, the group output worth equals the sum of gross outputs of the extracted sectors and their contributions via every other sector's gross output.

different clusters, i.e., key group members are rather nonredundant. Applying this redundancy principle to the IO framework may explain the fact that Dairy products (Dp) that ranks high in the key sector problem (i.e., for $k = 1$) is not contained in key groups of size $k > 1$ in Table 6.3 in the case of water usage. For example, the key group of size 2 contains Dairy cattle & milk and Beef cattle (Dc and Bc) and not Dc in combination with the second largest consumer of water - Dairy products (Dp). This is because Dc and Dp have rather similar patterns (and sizes) of production linkages, water usage and final demands, whereas Bc and Dc are not similar.[26] A partial proof for this argument are the correlation coefficients of -0.008 and 0.706 of the input matrix rows (sales structures) corresponding to Bc and Dc, and to Dc and Dp, respectively. Since the HEM extracts sectors entirely from the system, the correlation values for the vectors of both input and sales coefficients are 0.046 for Bc and Dc, and 0.278 for Dc and Dp. Hence, the key group members (i.e., Bc and Dc) have a much lower similarity in terms of their direct production (buying and selling) linkages than Dc and Dp.

The third observation from Table 6.3 is that sectors in the key group of size k are also part of the key group of size $k + 1$, which raises the question whether this is a general property or whether it is a mere coincidence. We have already shown in Section 6.2.3 that this is *not* true in general, i.e., the group target selection problem is not equivalent to a *sequential* key sector problem. One might (rightly) think that this fact is unfortunate from a computational perspective, since this urges an analyst to compute the factor worths for *all* possible combinations of k from all n sectors, which, for instance, in our case with groups of size 4 required to consider more than 13.6 million combinations,[27] and that search process would be significantly reduced (i.e., to only 133 cases) if the key group problem and the sequential key sector problem would be equivalent. Given that we have conjectured that the key

[26] This can be showed formally using cluster analysis, which is, however, beyond the scope of this chapter. In this respect, our study has a link to Hoen (2002), who analyzes the groups of sectors with strong connections using different cluster identification methods and ends at choosing a *block diagonalization method* to suit best for clustering purposes. This method rearranges sectors in such a way that the important linkages (bigger than some specified threshold level) of a matrix (such as the intermediate values, input coefficients, Leontief inverse) appear in blocks along the main diagonal, and thus sectors in one block comprise one separate cluster. However, a word of caution is in place with respect to the diagonalization method: it does *not* allow for "cluster switching". For instance, Howe and Stabler (1989) showed that an object may be assigned to totally different cluster if the number of identified clusters changes. In fact, this property of block diagonalization Hoen (2002) considers positively as other "cluster methods ... did not show this phenomenon [i.e., cluster switching] for sectors" (p. 139). However, the HEM allows for sector switching if one interprets the key group members in terms of different clusters' membership, at least, theoretically (see the next observation).

[27] This computation on a PC with a memory (RAM) of 4 GB and a Windows Experience Index base score of 4.6 took overall 17 minutes and 44 seconds. The MATLAB program can be provided by the author upon request.

group members are rather nonredundant with respect to their patterns and sizes of production linkages, factor generation and final demands, they should be part of different clusters of similar industries. Thus, the HEM allows, at least theoretically, for "cluster switching" of sectors once the number of (identified) clusters changes.[28] The phenomenon of "cluster switching" has been found, for example, in Howe and Stabler (1989). In fact, because "cluster switching" exists and because of the redundancy principle it follows that the key group problem is not equivalent to the sequential key sector approach. Hence, the fact that the key group problem requires to search for all possible combinations is, in fact, advantageous as taking into account "cluster switching" possibilities of industries and their redundancy, it determines the appropriate (right) key group members.

The fourth observation is that a group of a few industries accounts for the majority of the environmental factors, while generation of profits and salaries is relatively dispersed among sectors. So 58% and 69% of, respectively, water (direct and indirect) consumption and CO_2 emissions are due to the key groups of size 4 from the total of 136 sectors. This has, for example, the following policy implication: focusing on a very few industries would give quite a big impact in terms of, say, CO_2 emissions, but in order to have a large effect on social factors generation many more industries should be given policy priority. More specifically, we can see that Electricity supply (El) alone accounts for 32.8% of the Australian carbon dioxide emissions, while other factor worths (i.e., for water use, profits and wages generation) of key sectors are much smaller. Beef cattle (Bc) and Electricity supply (El) only (members of the key group of size 2) are responsible for 52.9% of Australian CO_2 emissions, hence any attempt to reduce carbon emissions should target these industries in the first place. For example, as suggested by Daniels (1992) in order to avoid long-term losses of productivity, biodiversity and real income, Australia has to re-direct its production from these high emissions-intensive industries towards more value-adding sectors. In case of water use, the key sector is Dairy cattle & milk (Dc), while Beef cattle (Bc), Dc and Water supply, sewerage and drainage services (Wa) jointly account for 48.1% of water consumption. Hence, again any policy towards more efficient use of water must consider these mentioned industries in the first place. Comparing our results given in Table 6.3 to those found in Lenzen (2003,

[28] To see this consider the following hypothetical case with four sectors. Suppose there are three clusters: {1,2}, {3} and {4} and the key group of size three is {1,3,4}. It might very well happen that in reducing the number of clusters we get the following two clusters: {1,3} and {2,4}, in which case sector 2 "switches" from its original cluster {1,2} to the cluster {4}. Then, in this "cluster switching" case the key group of size two can be, for example, {1,4}. Note that in this case the key group of size 2 is a part of the key group of size 3, which as shown in Section 6.2.3 does not have to be true in general.

Table 3) with a different IO apparatus of structural path analysis reveals that the two methodologies give similar results. As a policy recommendation Lenzen (2003) analogously mentions that "... in order to reduce the irrigation-induced stress on the Murray-Darling river system in South-Eastern Australia, shifts in production from water-intensive industries towards more value-adding sectors have been recommended" (p. 29). Analogous conclusions can be made with respect to the two other factors of profits and wages.

The last observation from Table 6.3 is that the percentage decrease in overall factor usage/ production upon extraction of groups is always smaller than the sum of the individual relative factor worths of sectors comprising the group. In fact, it is shown in Appendix 6.A that for any nonnegative final demand vector it is always true that $\sum_{s=1}^{k} \omega_{i_s}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) \geq \omega_{i_1,i_2,\ldots,i_k}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ for all $k = 2, 3, \ldots, n$. This inequality reflects the redundancy principle in the IO framework discussed earlier. For example, the relative group water worth of Dc and Dp is 19.7% (not shown in Table 6.3), while the sum of their individual relative water worths is 38.1% ($= 19.5 + 18.6$). Hence, the big difference of $\omega_{Dc}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) + \omega_{Dp}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) - \omega_{Dc,Dp}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = 18.4\%$ (when π is water consumption) shows that indeed the two sectors (i.e., Dc and Dp) are largely redundant, hence cannot comprise together the key group of size 2. For the key groups, however, this difference is very small, indicating that there is very little redundancy between the key group members.

In order to compare the results of the generalized HEM to other indicators, we present in the last three columns of Table 6.3 the top 5 sectors with the largest factor multipliers $\mathbf{m}_{\pi}$, direct factor usage/production $\boldsymbol{\pi}'\hat{\mathbf{x}}$, and factor responsibility. The first two indicators do not need additional explanation, hence we briefly discuss the third one. Multiplying the diagonalized matrix of the factor coefficients by the Leontief inverse gives the matrix $\hat{\boldsymbol{\pi}}\mathbf{L}$, whose ij-th element shows the amount of factor used/produced by sector i per unit final demand of sector j. Hence, the ij-th entry of the matrix $\hat{\boldsymbol{\pi}}\mathbf{L}\hat{\mathbf{f}}$ is the amount of the factor used/generated by sector i due to final demand of sector j, or equivalently, how much factor was consumed/produced by sector i *for* final demand of sector j. Thus, summing over all i gives the amount of the factor consumed/produced by all industries *for* final demand of product j, which is the j-th element of the vector $\boldsymbol{\pi}'\mathbf{L}\hat{\mathbf{f}} = \mathbf{m}_{\pi}'\hat{\mathbf{f}}$. In other words, this is the amount of the factor that "consumers" of product j are responsible for, hence the term "responsibility".[29]

Multipliers are traditionally used to identify sectors with the largest backward

[29] See Hoen and Mulder (2003) for a similar computation in analyzing the Dutch CO_2 emissions.

linkages (if one wants to spend one extra dollar e.g., for investment purposes). Table 6.3 shows that factor multipliers can give quite different results than those based on the HEM. This is expectable since factor worths besides the size of multipliers also take into account sectors' gross output size and their own input dependencies. Rice (Ri) has the highest water use multiplier (7470 litres per A$ of its final demand), while it is not a member of the key groups of size $k \in [1,4]$, and, moreover, it does not show up in the list of top 5 groups at all. Rice (Ri) though is the 5-th largest direct consumer of water (1.43 Tl), but it is not in the list of the top 5 responsible sectors. In case of CO_2 emissions, Forestry (Fr) has the largest CO_2 multiplier, but it is not a member of the key group of size $k < 3$. For gross operating surplus all four indicators give quite close outcomes with Ownership of dwellings (Dw) being the most important sector in each respect. Education (Ed) has the largest wages multiplier, and becomes a member of the key group of size 4. An advantage of multipliers lies in the price evaluation of commodities because multipliers are expressed per unit of final demand. In other words, industries with high factor multipliers are sensitive to changes in the factor price (see e.g., Dietzenbacher and Velázquez, 2007). In our case, a pricing policy that tries to internalize the costs of using water and CO_2 emissions will have the largest impact on the prices of, respectively, Rice (Ri) and Forestry (Fr).[30]

Notice also that for water use and CO_2 emissions there is a good correspondence between the key group members and the list of sectors with the largest direct factor usage/generation in Table 6.3. But this is not always the case: the largest capacity of generating wages has Education (Ed, 14.6 Bln A$), which is not a member of the key group of size $k < 4$. Instead, Retail trade (Rt), which is *responsible* for the largest amount of wages (18.3 A$), is part of the key group of size $k \geq 2$. For water usage and CO_2 emissions dairy and meat products (Dp and Mp) are the most responsible sectors, while in both cases they do not show up as part of the key groups. However, these industries are members of groups that are second or third in the list. All in all, it seems that the HEM takes into account both sectors' direct factor consumption/generation and sectors' responsibility in using/producing the factor by other industries. This is, of course, the specific advantage of using the generalized HEM, which fully considers all kinds of interlinkages associated with the hypothetically extracted sector(s).

[30] In this respect for Australian case, Foran et al. (2005) regarding agricultural, forestry and food products state: "... the prices we pay for the products reflect the marginal cost of production, rather than the full resource and environmental costs of production. ... Moves to internalize the full costs of production in the final price of the market product may mean substantial price increases" (p. 1).

6.6 Conclusion

In this chapter we have investigated the issue of the identification of a key sector and a key group of sectors in the economy by the hypothetical extraction method (HEM). These two issues are formalized in terms of optimization problems and their closed-form solutions were derived, which, to the best of our knowledge, have never been done in the literature. We show that for this purpose the analyst does not have to perform the three step procedure of the HEM. That is, delete the corresponding row(s) and column(s) of the input matrix, calculate the overall factor usage/production in the hypothetical case, and find the difference between the actual and hypothetical objectives. These steps are rather excessive given that we have found simple analytical formulas (measures of industries' factor worths) for the desired outcome, which make the calculation quite easy.

We showed that the top k (> 1) sectors in the key sector problem do not comprise the key group of size k. This is demonstrated in the empirical application to the Australian economy for four factors (i.e., water use, CO_2 emissions, profits, and wages and salaries). It always holds that the key group has the highest group factor worth. The last is directly related to the overall impact on aggregate factor usage/generation of an incremental change in direct input self-dependencies of the group-members, and inversely related to their own total input dependence. This interpretation is the outcome of linking the HEM to the fields of influence method. The key sector/group problem can be easily used to address several policy issues simultaneously, for instance, finding key sectors in terms of increasing employment and decreasing emissions of greenhouse gases.

It is further shown that the related problems of finding a key region and key group of regions within the interregional IO framework can be investigated similarly. We show that industries' factor worth are invariant to the netting out of intrasectoral transactions for any factor other than gross output. Hence, the outcomes of the key group problems in the standard and the so-called net IO frameworks are exactly the same for any factor other than total output. Also the connection of the factor worth measure to the Shapley value from coalitional game literature is discussed. Finally, it is proved that a positive (negative) change in a direct input coefficient never decreases (increases) the factor generating importance of any sector, and the necessary and sufficient conditions for a subsequent change are provided.

In the empirical application of the key group problem, we, for example, found that in Australia in 1994-1995 out of all possible combinations of groups of size 2 (i.e., from $C_2^{136} = 9180$ groups) Beef cattle and Electricity supply had the highest

CO_2 emissions worth: "closing down" these sectors would reduce carbon emissions by 52.9%. In the case of water consumption, Beef cattle, Dairy cattle, and Water supply, sewerage and drainage services have the highest relative water usage worth of 48.1% (among all groups of size 3). Hence any policy attempt to reduce carbon emissions and more efficient use of water in Australia should consider the fact that the above mentioned industries are heavy carbon- and water-intensive sectors.

6.A Proofs

Derivation of problem (6.3). The objective function in problem (6.2) is $\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-i} = \boldsymbol{\pi}'(\mathbf{L}\mathbf{f} - \mathbf{L}^{-i}\mathbf{f}^{-i})$. Adding and subtracting $\mathbf{L}^{-i}\mathbf{f}$ to the expression in parentheses gives $\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-i} = \boldsymbol{\pi}'(\mathbf{L} - \mathbf{L}^{-i})\mathbf{f} + \boldsymbol{\pi}'\mathbf{L}^{-i}(\mathbf{f} - \mathbf{f}^{-i})$. It is apparent that $\mathbf{f} - \mathbf{f}^{-i} = f_i\mathbf{e}_i$. This together with Lemma 6.1 yields

$$\begin{aligned}
\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-i} &= \boldsymbol{\pi}'\left(\frac{1}{l_{ii}}\mathbf{L}\mathbf{e}_i\mathbf{e}_i'\mathbf{L} - \mathbf{e}_i\mathbf{e}_i'\right)\mathbf{f} + f_i\boldsymbol{\pi}'\left(\mathbf{L} - \frac{1}{l_{ii}}\mathbf{L}\mathbf{e}_i\mathbf{e}_i'\mathbf{L} + \mathbf{e}_i\mathbf{e}_i'\right)\mathbf{e}_i \\
&= \frac{1}{l_{ii}}\boldsymbol{\pi}'\mathbf{L}\mathbf{e}_i\mathbf{e}_i'\mathbf{L}\mathbf{f} - f_i\pi_i + f_i\boldsymbol{\pi}'\mathbf{L}\mathbf{e}_i - \frac{f_i}{l_{ii}}\boldsymbol{\pi}'\mathbf{L}\mathbf{e}_i\mathbf{e}_i'\mathbf{L}\mathbf{e}_i + f_i\pi_i \\
&= \frac{1}{l_{ii}}\boldsymbol{\pi}'\mathbf{L}\mathbf{e}_i\mathbf{e}_i'\mathbf{L}\mathbf{f} + f_i\boldsymbol{\pi}'\mathbf{L}\mathbf{e}_i - \frac{f_i}{l_{ii}}\boldsymbol{\pi}'\mathbf{L}\mathbf{e}_i\mathbf{e}_i'\mathbf{L}\mathbf{e}_i = \frac{1}{l_{ii}}\mathbf{m}_\pi'\mathbf{e}_i\mathbf{e}_i'\mathbf{x},
\end{aligned}$$

where the last term follows since $\mathbf{e}_i'\mathbf{L}\mathbf{e}_i = l_{ii}$. ■

Proof of Theorem 6.2. Using the definitions of the factor worth, factor multiplier and equation (6.1), we have $\omega_i^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \frac{1}{l_{ii}} m_i^\pi x_i = \left(\sum_{j=1}^n \pi_j l_{ji}\right) \sum_{j=1}^n \frac{l_{ij}}{l_{ii}} f_j$. Then,

$$\Delta_i^\pi \equiv \omega_i^\pi(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi}) - \omega_i^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \left(\sum_{j=1}^n \pi_j \tilde{l}_{ji}\right) \sum_{j=1}^n \frac{\tilde{l}_{ij}}{\tilde{l}_{ii}} f_j - \left(\sum_{j=1}^n \pi_j l_{ji}\right) \sum_{j=1}^n \frac{l_{ij}}{l_{ii}} f_j,$$

where $\tilde{l}_{ij}$ is a generic element of the Leontief inverse $\widetilde{\mathbf{L}} = (\mathbf{I} - \widetilde{\mathbf{A}})^{-1}$. Adding and subtracting $\left(\sum_j \pi_j \tilde{l}_{ji}\right) \sum_j \frac{l_{ij}}{l_{ii}} f_j$ to the last expression and noting that $\tilde{m}_i^\pi = \sum_{j=1}^n \pi_j \tilde{l}_{ji}$ yields

$$\Delta_i^\pi = \tilde{m}_i^\pi \sum_{j=1}^n \left(\frac{\tilde{l}_{ij}}{\tilde{l}_{ii}} - \frac{l_{ij}}{l_{ii}}\right) f_j + \left(\sum_{j=1}^n \pi_j(\tilde{l}_{ji} - l_{ji})\right) \frac{x_i}{l_{ii}}. \tag{6.A.1}$$

From Sherman and Morrison (1950) it follows that $\tilde{l}_{ij} = l_{ij} + \epsilon_i l_{cj}$, where $\epsilon_i = \alpha l_{ir}/(1 - \alpha l_{cr})$. Therefore,

$$\frac{\tilde{l}_{ij}}{\tilde{l}_{ii}} - \frac{l_{ij}}{l_{ii}} = \frac{l_{ij} + \epsilon_i l_{cj}}{l_{ii} + \epsilon_i l_{ci}} - \frac{l_{ij}}{l_{ii}} = \frac{\epsilon_i(l_{ii}l_{cj} - l_{ci}l_{ij})}{l_{ii}\tilde{l}_{ii}}.$$

Plugging the last expression in (6.A.1) and using Sherman and Morrison's formula again yields

$$\Delta_i^\pi = \tilde{m}_i^\pi \sum_{j=1}^n \frac{\epsilon_i(l_{ii}l_{cj} - l_{ci}l_{ij})f_j}{l_{ii}\tilde{l}_{ii}} + \left(\sum_{j=1}^n \pi_j \epsilon_j l_{ci}\right) \frac{x_i}{l_{ii}}$$

$$= \frac{\alpha}{l_{ii}(1-\alpha l_{cr})} \left[l_{ir} \frac{\tilde{m}_i^\pi}{\tilde{l}_{ii}} \sum_{j=1}^{n} \left(l_{ii} l_{cj} - l_{ci} l_{ij} \right) f_j + l_{ci} \tilde{m}_r^\pi x_i \right]. \qquad (6.A.2)$$

The well-known property of the Leontief inverse is that $l_{ii} \geq 1$ and $l_{ii} > l_{ij} \geq 0$ for all i and all $j \neq i$ given that the column sums of $\mathbf{A}$ are less than one (see e.g., Gilo et al., 2006, Lemma 1, p. 85). Theorem 1 in Zeng (2001) shows that $l_{ii}l_{cj} \geq l_{ci}l_{ij}$, with strict inequality holding when $j = c \neq i$. Hence, $\sum_j (l_{ii}l_{cj} - l_{ci}l_{ij}) f_j > 0$ for all $i \neq c$ (assuming that $f_j \geq 0$, and at least $f_c > 0$). It is not difficult to see that for $i = c$ every term in this sum is zero, hence the first term of Δ_c^π in (6.A.2) (when $i = c$) vanishes, however, its second term is positive as $l_{cc} \geq 1$. So it always holds that $\Delta_c^\pi \gtrless 0$ if $\alpha \gtrless 0$. This is also always the case when $i = r$, i.e., $\Delta_r^\pi \gtrless 0$ for $\alpha \gtrless 0$ because then the first term in (6.A.2) is positive due to $l_{rr} \geq 1$. However, for all other $i \neq r, c$ the expression within the square brackets in (6.A.2) is not always positive, and becomes zero whenever $l_{ir} = l_{ci} = 0$ in which case $\Delta_i^\pi = 0$ with $i \neq r, c$. Otherwise, if $l_{ir} > 0$ and/or $l_{ci} > 0$ the sign of Δ_i^π for $i \neq r, c$ will depend only on α, and is positive (resp. negative) if $\alpha > 0$ (resp. $\alpha < 0$). This completes the proof. ■

Proof of Lemma 6.2. Lemma 5.1 in Chapter 5 of this thesis in the framework of social network analysis is mathematically equivalent to Lemma 6.2 in this chapter. Hence, see the proof of Lemma 5.1, where instead of matrix $\mathbf{B}$ now we have the Leontief inverse matrix $\mathbf{L}$. Note, however, that unlike $\mathbf{B}$ the Leontief inverse always exists. ■

Derivation of problem (6.5). As in the derivation of problem (6.3), the objective function in (6.4) can be rewritten as $\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-\{i_1,\ldots,i_k\}} = \boldsymbol{\pi}'(\mathbf{L} - \mathbf{L}^{-\{i_1,\ldots,i_k\}})\mathbf{f} + \boldsymbol{\pi}'\mathbf{L}^{-\{i_1,\ldots,i_k\}}(\mathbf{f} - \mathbf{f}^{-\{i_1,\ldots,i_k\}})$, where $\mathbf{f} - \mathbf{f}^{-\{i_1,\ldots,i_k\}} = \sum_{s=1}^{k} f_{i_s}\mathbf{e}_{i_s} = \mathbf{E}\mathbf{E}'\mathbf{f}$. This, together with Lemma 6.2 and the fact that $\mathbf{E}\mathbf{E}'\mathbf{E}\mathbf{E}' = \mathbf{E}\mathbf{E}'$, gives (defining $\mathbf{L}_{kk} \equiv \mathbf{E}'\mathbf{L}\mathbf{E}$)

$$\begin{aligned}
\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-\{i_1,\ldots,i_k\}} &= \boldsymbol{\pi}'\left[\mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\mathbf{L} - \mathbf{E}\mathbf{E}'\right]\mathbf{f} + \boldsymbol{\pi}'\left[\mathbf{L} - \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\mathbf{L} + \mathbf{E}\mathbf{E}'\right]\mathbf{E}\mathbf{E}'\mathbf{f} \\
&= \boldsymbol{\pi}'\mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\mathbf{L}\mathbf{f} - \boldsymbol{\pi}'\mathbf{E}\mathbf{E}'\mathbf{f} + \boldsymbol{\pi}'\mathbf{L}\mathbf{E}\mathbf{E}'\mathbf{f} - \boldsymbol{\pi}'\mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\mathbf{L}\mathbf{E}\mathbf{E}'\mathbf{f} + \boldsymbol{\pi}'\mathbf{E}\mathbf{E}'\mathbf{f} \\
&= \boldsymbol{\pi}'\mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\mathbf{L}\mathbf{f} + \boldsymbol{\pi}'\mathbf{L}\mathbf{E}\mathbf{E}'\mathbf{f} - \boldsymbol{\pi}'\mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{L}_{kk}\mathbf{E}'\mathbf{f} = \boldsymbol{\pi}'\mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\mathbf{L}\mathbf{f},
\end{aligned}$$

which is exactly the objective of the key group problem (6.5), using $\mathbf{m}'_\pi = \boldsymbol{\pi}'\mathbf{L}$ and $\mathbf{x} = \mathbf{L}\mathbf{f}$. ■

Proof of the inequality $\sum_{s=1}^{k} \omega_{i_s}^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) \geq \omega_{i_1,i_2,\ldots,i_k}^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$. This inequality is comparable to the subadditivity property of the group intercentrality measure proved in Appendix 5.A from Chapter 5, where now instead of matrix **B** we have the Leontief inverse matrix **L**. Hence, given the definitions of the group intercentrality and group factor worth measures in Definition 5.1 and Definition 6.2, respectively, (5.A.7) in the context of the Leontief model can be written as

$$\begin{aligned}&\sum_{s=1}^{k}\omega_{i_s}^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})-\omega_{i_1,i_2,\ldots,i_k}^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})\\&\quad=\boldsymbol{\pi}'\begin{bmatrix}\mathbf{L}_{kk}\widehat{\mathbf{L}}_{kk}^{-1}(\mathbf{L}_{kk}-\widehat{\mathbf{L}}_{kk}) & (\mathbf{L}_{kk}-\widehat{\mathbf{L}}_{kk})\mathbf{C}_{kt}\\ \mathbf{L}_{tk}\widehat{\mathbf{L}}_{kk}^{-1}(\mathbf{L}_{kk}-\widehat{\mathbf{L}}_{kk}) & (\mathbf{I}_t-\mathbf{A}_{tt})^{-1}\mathbf{A}_{tk}(\mathbf{L}_{kk}-\widehat{\mathbf{L}}_{kk})\mathbf{C}_{kt}\end{bmatrix}\mathbf{f},\end{aligned} \tag{6.A.3}$$

where $\mathbf{L}_{kk}=\mathbf{E}'\mathbf{L}\mathbf{E}$, $\widehat{\mathbf{L}}_{kk}=\mathbf{E}'\widehat{\mathbf{L}}\mathbf{E}$ with $\widehat{\mathbf{L}}$ being the diagonal matrix with l_{ii} on its main diagonal and zeros elsewhere, $\mathbf{A}_{tt}$ is the input submatrix that contains input coefficients of all t sectors not in the group $\{i_1,\ldots,i_k\}$ (where $t+k=n$), $\mathbf{A}_{tk}$ is the input submatrix containing the input coefficients with deliveries from non-members to the members of the group $\{i_1,\ldots,i_k\}$, and $\mathbf{C}=\widehat{\mathbf{L}}^{-1}\mathbf{L}$. All these mentioned matrices are non-negative, and $\mathbf{L}_{kk}\geq\widehat{\mathbf{L}}_{kk}$. Therefore, for any nonnegative vectors $\boldsymbol{\pi}$ and $\mathbf{f}$, it always holds that $\sum_{s=1}^{k}\omega_{i_s}^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})\geq\omega_{i_1,i_2,\ldots,i_k}^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})$, which is strict if $\boldsymbol{\pi}$ and $\mathbf{f}$ are strictly positive vectors (given that $\mathbf{A}$ is not a null matrix).

If $k=n$, then (6.A.3) becomes $\sum_{i=1}^{n}\omega_{i}^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})-\omega_{1,\ldots,n}^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})=\boldsymbol{\pi}'\big[\mathbf{L}\widehat{\mathbf{L}}^{-1}(\mathbf{L}-\widehat{\mathbf{L}})\big]\mathbf{f}=\boldsymbol{\pi}'\big[\mathbf{L}\widehat{\mathbf{L}}^{-1}-\mathbf{I}\big]\mathbf{x}\geq 0$ provided that the final demand vector is nonnegative. Since $\omega_{1,\ldots,n}^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})=\boldsymbol{\pi}'\mathbf{x}$, it follows that $\sum_{i=1}^{n}\omega_{i}^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})\geq\boldsymbol{\pi}'\mathbf{x}$, i.e., the sum of individual factor worths of all sectors is at least as large as the size of the overall factor production/consumption. ■

6.B Codes assigned to 136 Australian sectors

We mainly adopt the sectoral codes of Foran et al. (2005), which are listed below in alphabetical order.

Sym.	Industry	Sym.	Industry
Ac	Insecticides, pesticides and other agricultural chemicals	Lm	Lime
Ai	Aircraft	Lp	Leather and leather products
Al	Aluminium alloys and aluminium recovery	Ma	Agricultural, mining and construction machinery
Ao	Alumina	Mi	Mineral and glass wool and other non-metallic mineral products
Ap	Automotive petrol	Mn	Exploration and services to mining
At	Air and space transport	Mp	Meat and meat products
Ba	Barley, unmilled	Ms	Legal, accounting, marketing and business management services
Bc	Beef cattle	Mv	Motor vehicles and parts, other transport equipment
Bk	Banking	Nb	Non-residential buildings, roads, bridges and other construction
Bl	Black coal	Ne	Newspapers, books, recorded media and other publishing
Bm	Beer and malt	Nf	Non-ferrous metal recovery and basic products
Bp	Bread, cakes, biscuits and other bakery products	Ng	Natural gas
Br	Brown coal, lignite	Oc	Adhesives, inks, polishes and other chemical products
Bs	Typing, copying, staff placement and other business services	Oe	Photographic, optical, medical and radio equipment, watches
Bt	Bus and tramway transport services	Of	Oils and fats
Bu	Prefabricated buildings	Oi	Crude oil
Bv	Soft drinks, cordials and syrups	Om	Coins, jewellery, sporting goods and other manufacturing
Bx	Bauxite	Os	Police, interest groups, fire brigade and other services
Cc	Concrete and mortar	Ot	Cable car, chair lift, monorail and over-snow transport
Ce	Cement	Pa	Paper containers and products
Cg	Services to agriculture, ginned cotton, shearing and hunting	Pc	Petroleum bitumen, refinery LPG and other refinery products
Ch	Basic chemicals	Pd	Property developer, real estate and other property services
Cl	Clothing	Pe	Poultry and eggs
Cm	Communication services	Pg	Pigs
Cn	Confectionery	Ph	Pharmaceutical goods for human use
Co	Copper	Pi	Pipeline transport services
Cp	Plaster and other concrete products	Pl	Plastic products
Cr	Bricks and other ceramic products	Pp	Pulp, paper and paperboard
Cs	Childminding and other community care services	Pr	Printing, stationery and services to printing
Ct	Cosmetics and toiletry preparations	Ps	Hairdressing, goods hiring, laundry and other personal services
Cu	Libraries, parks, museums and the arts	Pt	Paints
Dc	Dairy cattle and untreated whole milk	Rb	Residential building, construction, repair and maintenance
De	Soap and other detergents	Rd	Road freight transport services
Df	Defence	Rf	Railway freight transport services
Dp	Dairy products	Rh	Repairs of household and business equipment
Dw	Ownership of dwellings	Ri	Rice, in the husk
Ed	Education	Rp	Railway passenger transport services
Ee	Cable, wire, batteries, lights and other electrical equipment	Rs	Sport, gambling and recreational services
El	Electricity supply	Rt	Retail trade
En	Electronic equipment, photocopying, gaming machines	Ru	Rubber products
Eq	Pumps, bearings, air conditioning and other equipment	Rv	Repairs of motor vehicles, agricultural and other machinery
Et	Motion picture, radio and television services	Rw	Railway equipment
Fc	Flour, cereal foods, rice, pasta and other flour mill products	Sb	Ships and boats
Fd	Raw sugar, animal feeds, seafoods, coffee and other foods	Sc	Seed cotton
Fe	Mixed fertilisers	Sf	Security broking and dealing and other services to finance
Fi	Commercial fishing	Sg	Sand, gravel and other construction materials mining
Fm	Nuts, bolts, tools and other fabricated metal products	Sh	Sheet containers and other sheet metal products
Fn	Money market corporation and other non-bank finance	Si	Financial asset investors and holding company services
Fo	Gas oil, fuel oil	Sm	Frames, mesh and other structural metal products
Fp	Vegetables, fruit, juices and other fruit and vegetable products	Sp	Water transport
Fr	Forestry and services to forestry	St	Travel agencies, forwarding and other services to transport
Fu	Furniture	Su	Sugar cane
Fw	Footwear	Sw	Softwoods, conifers
Ga	Gas production and distribution	Sz	Silver and zinc ores
Gd	Sanitary and garbage disposal services	Ta	Taxi and hired car with driver
Gl	Gold and lead	Ti	Sawn timer, woodchips and other sawmill products
Gp	Glass and glass products	To	Tobacco products
Gv	Government administration	Tp	Carpets, curtains, tarpaulins, sails, tents and other textiles
Hh	Household appliances and hot water systems	Ts	Scientific research, technical and computer services
Ho	Accommodation, cafes and restaurants	Tx	Processed wool, textile fibres, yarns and woven fabrics
Hs	Health services	Uo	Uranium, nickel, tin, manganese and other non-ferrous metal ores
Hw	Hardwoods, brushwoods, scrubwoods, hewn and other timber	Vf	Vegetable and fruit growing, hay, plant nurseries, flowers
In	Insurance	Wa	Water supply, sewerage and drainage services
Io	Iron ores	Wh	Wheat, legumes for grain, oilseeds, oats and other grains
Is	Basic iron and steel, pipes, tubes, sheets, rods, bars, rails, fittings	Wo	Sheep and shorn wool
Ke	Kerosene and aviation jet fuel	Wp	Plywood, window frames, doors and other wood products
Kn	Knitting mill products	Ws	Wine and spirits
Lg	Liquefied natural gas, liquefied natural petrol	Wt	Wholesale trade

CHAPTER 7

Epilogue

7.1 Introduction

This thesis has focused on three types of interdependencies: at the levels of firms, individuals and economic sectors. These were firms' shareholding interlocks, social networks of people, and production linkages of industries. Given that these interrelationships have their own distinguishing features, it is obvious that the analytical frameworks used in their analysis were (quite) different as well. This explains why this study did not focus on one specific field, but instead investigated topics from several subfields of economics and sociology, such as Finance, Industrial Organization, Input-Output Economics, Network Economics, and Social Network Analysis. However, on the other hand, the issues considered in this thesis are not at all independent of each other, in contrast to what might appear at first glance. The analyses of the complex webs of interrelations have a lot in common. In some sense they adopt a unified analytical framework, thus extending the frontiers of common interests in the above-mentioned fields.

Some of the main questions in this study aimed at the following. How to quantify ownership complexity caused by cross ownership by both individuals and companies? What is the appropriate measure of separation of ownership and control rights due to firms' cross-shareholdings? Does a firm with passive stockholdings in its rivals exert strictly higher market power than a firm without any shareholdings? Do interfirm shareholding interlocks matter in the empirical study of market performance? What is the effect of partial cross ownership on the incentives of asymmetric (in terms of costs) firms to collude and on the collusive price? How to find the group of individuals with the maximum impact on the overall equilibrium outcome in (social) networks? How to incorporate individuals' exogenous heterogeneity into the analysis of key players search? Is the key sector problem equivalent to the problem of identification of key group of sectors? If no, what are the underlying reasons?

These questions were thoroughly addressed in the previous chapters. In the next section we give a brief summary of the obtained results. The last section discusses three directions for future research, each based on the findings in this thesis.

7.2 Summary of results

Chapter 2 proposed new measures of network complexity due to the existence of cross ownership links among firms. The measures called "weighted average distance of indirect linkages" (WADIL) and "weighted average distance of total linkages" (WADTL) quantify the complexity of an ownership structure that is characterized by crossholdings of stocks. The proposed measures consider both the sizes of direct and indirect shareholdings, and the average distance between the owners and the owned firms. We say that owner (or firm) i has an *indirect* stake in firm r if it has a stake in a firm that has a stake in firm r, or if it has a stake in a firm that has a stake in a firm that has a stake in firm r, and so on. The average distance was obtained from the average number of intermediate firms via whom the ownership link between i and r runs. The values of WADILs and WADTLs indicate whether a certain link is of a direct nature only or whether indirect shareholdings also play a role in the link. The larger values of WADILs and WADTLs indicate a more complex network involving a larger number of different ownership paths. Combining the linkage size and the distance allowed us to visualize the cross-shareholding interlocks and the true ownership relations. The methodology was applied to the Czech banking sector in 1997. It was found that there is ample evidence that indirect ownership relations play a crucial role in the banking sector in the Czech Republic. Further, the link between the proposed measures of network complexity and the degree of separation of dividend and control rights due to cross-shareholdings was explored. It was suggested that the WADILs and the WADTLs may serve as alternative measures for the degree of separation. That is, the more complex the network of non-negligible ownership relations is, the larger is the degree of control enhancement due to cross-shareholding links among firms. As a consequence, also the gap between the control and ownership stakes of owners in firms is larger. This was confirmed by the empirical results for the Czech banking sector. The obtained

WADILs and WADTLs were also compared to the wedges between ownership and control rights, where the last were quantified by well-known methodologies from finance, namely the "weakest link" and the "dominant shareholder" approach.

The effect of disregarding partial cross ownership (PCO) (i.e., shares that do not give control power to their owners) in empirical studies of market performance and firms' market power was investigated in Chapter 3.[1] For this purpose the well-known framework of the "structure-conduct-performance school" in industrial organization was used. For the estimation of firms' market power and the tacit collusion that is inherent to an industry, the framework was modified by including both direct and indirect shareholdings. It was proved that, unlike in the no-PCO case, the link between firms' price-cost margins and the degree of market competitiveness is nonlinear in the presence of PCO. Thus, ignoring PCO in an analysis of an industry with extensive shareholdings between firms, will most likely lead to biased results due to model misspecification. In an empirical application, it was found that Japanese commercial banks in 2003 were competing in a modest collusive environment. However, if PCO was disregarded, the results were different and indicated a Cournot oligopoly. It was further found that banks with PCO in their rivals exert a strictly larger market power than those without any shareholdings. In particular, city banks with many shareholdings were found to exercise a much larger market power than regional banks with none or few stockholdings. Hence, the hypothesis that acquiring shares in rivals for a firm is one of the means of enhancing its market power was confirmed in Chapter 3.

Chapter 4 adopted an infinitely repeated Bertrand oligopoly model to investigate the effect of partial ownership arrangements of firms under cost asymmetries on their incentives to collude. We first considered the case where only the most efficient firm in the industry invests in rivals. It was shown that a unilateral partial ownership by this firm may facilitate a market-sharing scheme in which all firms charge the same collusive price and divide the market between them. We showed that when the most efficient firm invests in rivals, the collusive price, which is a compromise between the monopoly prices of the different firms, increases relative to the case where there are no partial ownership arrangements. Further, we focused on the effect of a change in the PCO structure on tacit collusion. It was shown that when the stake that firm r has in firm s increases at the expense of outside shareholders collusion is never hindered. It will even be strictly facilitated if and only

[1] In Chapter 4 we also consider the case when only one firms invests in rivals. We call this a partial ownership (PO) case. PCO, instead, reflects the fact that in the presence of multilateral ownership arrangements, cross-shareholdings by firms are possible.

if (i) the industry maverick (the firm with the strongest incentive to deviate from a collusive agreement) has a direct or indirect stake in firm *r*, and (ii) firm *s* is not the industry maverick. When (i) and/or (ii) fail to hold, the increase in firm *r*'s stake in firm *s* does not affect tacit collusion. These results extend the earlier findings in Gilo et al. (2006) and show that the results for firms with symmetric cost functions generalize to the asymmetric costs case. Chapter 4 also considered the case of such an ownership change due to transfer of ownership between firms. It was shown that a transfer of partial cross ownership in firm *s* from firm *k* to firm *r* does not affect tacit collusion if the industry maverick is firm *s* or if, at the outset, the industry maverick has the same total (direct and indirect) share in firms *k* and *r*. Otherwise, the transfer of partial cross ownership facilitates tacit collusion if the industry maverick has a larger total share in firm *r* (the acquirer) than in firm *k* (the seller) but hinders tacit collusion if the reverse holds.

Chapter 5 extended the problem of finding the key player in a network game studied by Ballester et al. (2006) to the search of the key group, where players' exogenous heterogeneity was taken into account. The key group is the group of players that has the maximum (or minimum) possible impact on the overall equilibrium activity level of the network. We derived a closed-form expression of the so-called group intercentrality measure, which is used to identify the key group in networks, and explored some of its properties. Further, the measures of weighted and unweighted group intercentralities that depend only on the initial network configuration were shown to be useful for the identification of the key group of heterogeneous players. The weights are based on observable differences of players, such as age, education, occupation, race, religion, family size, or parents' education. It was shown that once these observable differences are accounted for, the results of the key player/group problem may significantly change when compared to the results based on the assumption of homogeneous players. Finally, the size of the key group was endogenized, which is an important issue since targeting groups of different sizes incurs different benefits and costs. Hence, from the planner's perspective it is essential to get an idea of what is the optimal size of the key group, i.e., what size yields the largest net benefit.

Chapter 6 investigated the issue of finding key sectors of an economy, that is, sectors with the maximum potential of spreading growth impulses throughout the economy and thus impacting output or some other factor (such as value added, employment, or CO_2 emissions). For this purpose, the hypothetical extraction method (HEM) from input-output analysis was adopted, which measures the contribution of each sector to the overall gross output or any other factor by comparing the original result with the result that is obtained from omitting one sector (or a group of sectors) from the model. The reduction in, for example, output is due to this omission and thus reflects the role of the hypothetically extracted (group of) sector(s). Explicit formulations of the optimization problems of finding a key sector and a key group of sectors from the HEM perspective were given, and their analytical solutions (called industries' factor worths) were derived. It was shown that the key group of $k \geq 2$ sectors is, in general, different from the k sectors with the largest individual contributions to the overall factor production/consumption, which was confirmed in an example of the Australian economy in case of water use and CO_2 emissions in the mid of 1990s. This outcome has to do with the fact that in reality sectors may be redundant with respect to each other if they have similar patterns and sizes of production linkages, final demands and factor production capabilities. The related issues of finding a key region and key group of regions in an interregional input-output (IO) framework were investigated similarly. Further, we showed that the factor worth measure is invariant to the netting out of intrasectoral transactions for any factor other than gross output. Hence, the outcomes of the key sector/group problems in the standard and the so-called net input-output settings are exactly identical so long as the factor is not total output. The link of the HEM problems to the fields of influence approach was pointed out, which gives an alternative economic interpretation of these problems in terms of the economy-wide effects of an incremental change in sectors' input self-dependencies. Finally, it was proved that an increase (decrease) in an input coefficient never decreases (increases) the factor worth/importance of any sector, and the conditions for a subsequent strict change were derived.

7.3 Related future research

Often, doing research raises new issues. In this respect I absolutely agree with my supervisor Erik Dietzenbacher, who in the final chapter of his dissertation states: "Answers raise new questions, solutions define new problems, results call for a generalization or a sharpening, assumptions for a relaxation, gaps need to be filled up, and loose ends are to be tied up" (Dietzenbacher, 1991, p. 267). Hence, in what follows I will present three directions for future research, the basis of which is essentially the current study.

7.3.1 Engines of growth: a hypothetical extraction approach

In Chapter 6, the problem of the identification of the key sectors for generating some economic, social, and/or environmental factor was discussed. A similar approach can be applied to the identification of "key sectors" for generating economy-wide total factor productivity (TFP) growth. In the literature, such sectors are called the *engines of growth*. However, the generalized hypothetical extraction method (HEM) as discussed thoroughly in Chapter 6 is not adequate to deal with finding the engines of growth, because TFP growth cannot be directly incorporated into the input-output framework and needs a somewhat different setting. Such a framework will be discussed below after we briefly present the analysis of productivity spillovers.

7.3.1.1 Productivity analysis of spillovers

Ten Raa and Wolff (2000) propose to identify the engines of growth as follows. The departing point is the Solow (1957) residual definition of total factor productivity (TFP) growth, g:

$$g = \frac{\mathbf{p}'d\mathbf{f} - wdL - rdK}{\mathbf{p}'\mathbf{f}}, \tag{7.1}$$

where $\mathbf{f}$ is the final demand vector (also termed net output vector), L and K are, respectively, labor and capital inputs, w and r are their respective prices, and $\mathbf{p}$ is the vector of production prices. These prices reflect zero profits since

$$\mathbf{p}'(\mathbf{I} - \mathbf{A}) = \mathbf{v}' = w\mathbf{l}' + r\mathbf{k}', \tag{7.2}$$

where $\mathbf{A}$ is the input matrix, and $\mathbf{v}$, $\mathbf{l}$ and $\mathbf{k}$ are the vectors of direct value-added, labor and capital coefficients.

Using the balancing equation of the open Leontief model $\mathbf{f} = (\mathbf{I} - \mathbf{A})\mathbf{x}$, where $\mathbf{x}$ is the vector of total (or gross) outputs, the numerator of (7.1) can be written as

$$\begin{aligned} \mathbf{p}'d\mathbf{f} - wdL - rdL &= \mathbf{p}'d(\mathbf{I} - \mathbf{A})\mathbf{x} - wd(\mathbf{l}'\mathbf{x}) - rd(\mathbf{k}'\mathbf{x}) \\ &= (-\mathbf{p}'d\mathbf{A} - wd\mathbf{l}' - rd\mathbf{k}')\mathbf{x} + (\mathbf{p}'(\mathbf{I} - \mathbf{A}) - w\mathbf{l}' - r\mathbf{k}')d\mathbf{x}, \end{aligned} \tag{7.3}$$

where the last term vanishes if we use the production prices in (7.2). Using (7.3), (7.1) reduces to

$$g = \frac{-(\mathbf{p}'d\mathbf{A} + wd\mathbf{l}' + rd\mathbf{k}')\mathbf{x}}{\mathbf{p}'\mathbf{f}} = \frac{\boldsymbol{\pi}'\hat{\mathbf{p}}\mathbf{x}}{\mathbf{p}'\mathbf{f}}, \tag{7.4}$$

where $\boldsymbol{\pi}' = -(\mathbf{p}'d\mathbf{A} + wd\mathbf{l}' + rd\mathbf{k}')\hat{\mathbf{p}}^{-1}$ is the row vector of sectoral TFP growth rates, and $\hat{\mathbf{p}}\mathbf{x}/(\mathbf{p}'\mathbf{f})$ is the vector of so-called Domar weights.

Spillovers are measured as a weighted average of the TFP growth in supplying sectors. Four explanatory variables for the TFP growth rate of sector j, π_j, are distinguished: (1) an autonomous source, α, (2) R&D in sector j per dollar of gross output, $\rho = RD_j/(p_j x_j)$, (3) a direct productivity spillover, $\sum_i (p_i a_{ij}/p_j)\pi_i$, and (4) a capital embodied spillover, $\sum_i (p_i b_{ij}/p_j)\pi_i$, where b_{ij} is the capital stock coefficient of capital good i in sector j. This yields the following regression equation (Wolff, 1997):

$$\boldsymbol{\pi}' = \alpha\boldsymbol{\imath}' + \beta_1\boldsymbol{\rho}' + \beta_2\boldsymbol{\pi}'\hat{\mathbf{p}}\mathbf{A}\hat{\mathbf{p}}^{-1} + \beta_3\boldsymbol{\pi}'\hat{\mathbf{p}}\mathbf{B}\hat{\mathbf{p}}^{-1} + \boldsymbol{\varepsilon}', \tag{7.5}$$

where $\boldsymbol{\varepsilon}$ is the vector of error terms.

Let us denote the spillover matrix by

$$\mathbf{C} \equiv \beta_2\hat{\mathbf{p}}\mathbf{A}\hat{\mathbf{p}}^{-1} + \beta_3\hat{\mathbf{p}}\mathbf{B}\hat{\mathbf{p}}^{-1} = \hat{\mathbf{p}}\left[\beta_2\mathbf{A} + \beta_3\mathbf{B}\right]\hat{\mathbf{p}}^{-1}, \tag{7.6}$$

then, ignoring the error term, (7.5) can be rewritten as

$$\boldsymbol{\pi}' = \alpha\boldsymbol{\imath}' + \beta_1\boldsymbol{\rho}' + \boldsymbol{\pi}'\mathbf{C}. \tag{7.7}$$

Plugging (7.7) back in (7.4) yields

$$g = \frac{[\alpha\boldsymbol{\imath}' + \beta_1\boldsymbol{\rho}' + \boldsymbol{\pi}'\mathbf{C}]\hat{\mathbf{p}}\mathbf{x}}{\mathbf{p}'\mathbf{f}} = \alpha DR + \beta_1\frac{RD'\boldsymbol{\imath}}{\mathbf{p}'\mathbf{f}} + \frac{\boldsymbol{\pi}'\mathbf{C}\hat{\mathbf{p}}\mathbf{x}}{\mathbf{p}'\mathbf{f}}, \tag{7.8}$$

where RD is the vector of sectoral R&Ds, $\boldsymbol{\imath}$ is the summation vector, and $DR = \mathbf{p}'\mathbf{x}/\mathbf{p}'\mathbf{f}$ is the Domar ratio.

Equation (7.8) gives the *direct* effect of R&D on TFP growth. β_1 measures the *direct rate of return to R&D intensity* or, equivalently, the *direct return to R&D*, in terms of output value per dollar expenditure, because the denominator in the definition g in (7.1) is also the dollar value of expenditures, i.e., $\mathbf{p}'\mathbf{f}$.

The *total* returns to R&D, however, are obtained by taking into account the spillover effects, captured by the last term in (7.8). Define $\mathbf{M} \equiv (\mathbf{I} - \mathbf{C})^{-1} =$

$\mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \cdots$. Solving (7.7) for the vector of TFP growth rates gives

$$\boldsymbol{\pi}' = (\alpha \boldsymbol{\imath}' + \beta_1 \boldsymbol{\rho}')\mathbf{M}, \tag{7.9}$$

hence, using (7.4)

$$g = \frac{\boldsymbol{\pi}'\hat{\mathbf{p}}\mathbf{x}}{\mathbf{p}'\mathbf{f}} = \frac{(\alpha \boldsymbol{\imath}' + \beta_1 \boldsymbol{\rho}')\mathbf{M}\hat{\mathbf{p}}\mathbf{x}}{\mathbf{p}'\mathbf{f}}. \tag{7.10}$$

Therefore the *total rate of return to R&D intensity* ρ_i amounts to $\beta_1(\sum_j m_{ij}p_jx_j)/\mathbf{p}'\mathbf{f}$. Here, β_1 is inflated by multipliers m_{ij} because of spillover effects and also by gross/net output ratios as the sectoral R&D intensities ρ_i are defined as the R&D/gross output ratios. Note that while the first decomposition in (7.4) is a TFP growth accounting identity, the second decomposition in (7.10) attributes TFP growth to sources of growth taking into account the spillover effects.

Since $\boldsymbol{\rho}' = RD'(\hat{\mathbf{p}}\hat{\mathbf{x}})^{-1}$, we have that

$$g = \alpha \frac{\boldsymbol{\imath}'\mathbf{M}\hat{\mathbf{p}}\mathbf{x}}{\mathbf{p}'\mathbf{f}} + \beta_1 \frac{RD'(\hat{\mathbf{p}}\hat{\mathbf{x}})^{-1}\mathbf{M}\hat{\mathbf{p}}\mathbf{x}}{\mathbf{p}'\mathbf{f}},$$

hence the *total return to R&D*, in terms of output value per dollar expenditure in sector i, amounts to $\beta_1(\sum_j m_{ij}p_jx_j)/(p_ix_i)$. So the direct return to β_1 is inflated by the factor $(\sum_j m_{ij}p_jx_j)/(p_ix_i)$ because of spillover effects stemming from sector i. Since the factors $(\sum_j m_{ij}p_jx_j)/(p_ix_i)$ reinforce the returns to R&D, they are *spillover multipliers*. Hence, the vector of spillover multipliers is given by $(\hat{\mathbf{p}}\hat{\mathbf{x}})^{-1}\mathbf{M}\hat{\mathbf{p}}\mathbf{x}$. Spillover multipliers are equal to the ratio of the total to the direct return to R&D, thus measure the external effects of sectoral R&D.

From (7.10) it follows that the overall TFP growth g is decomposed into sources of growth $\alpha + \beta_1\rho_i$ aggregated by the linkages $\sum_j m_{ij}p_jx_j$ for sector $i = 1, \ldots, n$. Sectors that contribute much to overall TFP growth in this decomposition are the *engines of growth*. For example, the largest engine of growth is the sector, say, i, with the largest value of $(\alpha + \beta_1\rho_i)\sum_j m_{ij}p_jx_j$.

7.3.1.2 Engines of growth from a hypothetical extraction perspective

For simplicity, let us denote the vector of *spillover linkages* by $\mathbf{s} \equiv \mathbf{M}\hat{\mathbf{p}}\mathbf{x}$. The sectoral direct and indirect productivity gains are thus given by the vector $(\alpha\mathbf{I} + \beta_1\hat{\boldsymbol{\rho}})\mathbf{s}$ as derived in the previous section. ten Raa and Wolff (2000) define the engines of growth as the 10 sectors with the largest values in the last vector.

Let us now consider the problem of identifying the engines of growth from the HEM approach, i.e., sectors whose elimination from the systems of economy-wide production and spillover interrelations causes the largest reduction in the overall TFP growth rate. Consider the identification of $k \in [1, n-1]$ engines of growth. Denote by $\mathbf{C}^{-\{i_1,\ldots,i_k\}}$ the new spillover matrix derived from $\mathbf{C}$ by setting to zero all its i_s-th rows and columns elements, where $s = 1, \ldots, k$. From (7.6), it follows that this is equivalent to nullifying all rows and columns entries corresponding to $i_1, \ldots, i_k$ of the input matrix $\mathbf{A}$ and the capital stock coefficient matrix $\mathbf{B}$. The assumption therefore is that in the new system without sectors $i_1, \ldots, i_k$ the production and capital stock structures of other active sectors $j \notin \{i_1, \ldots, i_k\}$ remain unchanged.[2] Although at first glance this assumption seems restrictive, in fact it is not, given our main aim of identifying the importance of sectors $i_1, \ldots, i_k$ in generating nation-wide growth considering the spillover effects. The point is that by taking all other input and capital stock coefficients fixed, we explicitly allow the resulting outcome to depend only on the extraction of sectors $i_1, \ldots, i_k$, which are now not participating in the "roundabout" of the production process, hence not contributing to the TFP growth either. The vector of spillover linkages after extracting sectors $i_1, \ldots, i_k$ is $\mathbf{s}^{-\{i_1,\ldots,i_k\}} = \mathbf{M}^{-\{i_1,\ldots,i_k\}}\hat{\mathbf{p}}\mathbf{x}^{-\{i_1,\ldots,i_k\}}$, where $\mathbf{M}^{-\{i_1,\ldots,i_k\}} = (\mathbf{I} - \mathbf{C}^{-\{i_1,\ldots,i_k\}})^{-1}$, and $\mathbf{x}^{-\{i_1,\ldots,i_k\}} = (\mathbf{I} - \mathbf{A}^{-\{i_1,\ldots,i_k\}})^{-1}\mathbf{f}^{-\{i_1,\ldots,i_k\}}$. The new net output vector $\mathbf{f}^{-\{i_1,\ldots,i_k\}}$ is the same as $\mathbf{f}$ except its i_1-th, ..., i_k-th entries that are all set to zero. The reason for setting $f_{i_s} = 0$ for all $s = 1, \ldots, k$ is that when sectors $i_1, \ldots, i_k$ cease to exist, their (domestic) gross outputs should be zero.

We further denote the sum of autonomous source and R&D intensities by $\boldsymbol{\lambda} \equiv \alpha\boldsymbol{\imath} + \beta_1\boldsymbol{\rho}$. Given the vectors of sources of growth $\boldsymbol{\lambda}$ and production prices $\mathbf{p}$, the objective is picking k ($1 \leq k \leq n-1$) sectors $i_1, i_2, \ldots, i_k$ ($i_s \neq i_r$) such that their extraction from the economy generates the highest possible reduction in the overall TFP growth rate, $g = \boldsymbol{\lambda}'\mathbf{s}/\mathbf{p}'\mathbf{f}$. Formally, the problem is

$$\max\left\{\frac{\boldsymbol{\lambda}'\mathbf{s}}{\mathbf{p}'\mathbf{f}} - \frac{\boldsymbol{\lambda}'\mathbf{s}^{-\{i_1,\ldots,i_k\}}}{\mathbf{p}'\mathbf{f}^{-\{i_1,\ldots,i_k\}}} \;\middle|\; \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}; i_s \neq i_r\right\}. \tag{7.11}$$

This is a finite optimization problem, which admits at least one solution. The solution to (7.11) is denoted by $\{i_1^*, i_2^*, \ldots, i_k^*\}$ and is called the *k engines of growth*. Removing these industries from the initial input and capital stock structures have the largest impact on the overall TFP growth rate.

[2] This is usual for all the HEM approaches, the only difference now is that besides production we also consider the capital stock structure.

Before giving an explicit solution to the problem (7.11), first, recall that $\mathbf{f} = \mathbf{x} - \mathbf{Ax}$. Its premultiplication by the diagonal matrix of the (fixed) production prices yields $\hat{\mathbf{p}}\mathbf{f} = \hat{\mathbf{p}}\mathbf{x} - \hat{\mathbf{p}}\mathbf{A}\hat{\mathbf{p}}^{-1}\hat{\mathbf{p}}\mathbf{x}$, hence $\hat{\mathbf{p}}\mathbf{x} = (\mathbf{I} - \hat{\mathbf{p}}\mathbf{A}\hat{\mathbf{p}}^{-1})^{-1}\hat{\mathbf{p}}\mathbf{f}$. Thus, in what follows in this section the Leontief inverse is defined as $\mathbf{L} \equiv (\mathbf{I} - \hat{\mathbf{p}}\mathbf{A}\hat{\mathbf{p}}^{-1})^{-1}$.

Let $\mathbf{E}$ be the $n \times k$ matrix defined as $\mathbf{E} = \left(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}\right)$, where $\mathbf{e}_i$ is the i-th column of the identity matrix. Each column of $\mathbf{E}$ has all zeros except one positive number being unity that corresponds to one of the extracted sectors. Hence, the reduced multiplier matrix $\mathbf{M}_{kk} = \mathbf{E}'\mathbf{ME}$ includes all the elements of the original multiplier matrix $\mathbf{M}$ that are directly related to the extracted sectors $i_1, \ldots, i_k$. Similarly, the reduced Leontief inverse is $\mathbf{L}_{kk} = \mathbf{E}'\mathbf{LE}$. Recall from (7.9) that the vector of sectoral TFP growth rates is $\boldsymbol{\pi}' = \boldsymbol{\lambda}'\mathbf{M}$. Next note that the problem in (7.11) is equivalent to

$$\min \left\{ \frac{\boldsymbol{\lambda}'\mathbf{s}^{-\{i_1,\ldots,i_k\}}}{\mathbf{p}'\mathbf{f}^{-\{i_1,\ldots,i_k\}}} \;\middle|\; \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}; i_s \neq i_r \right\}.$$

The "residual" TFP growth rate in the objective above is defined as the *reduced TFP growth* due to extraction of sectors $i_1, \ldots, i_k$ ($i_r \neq i_s$), and it can be shown to be equal to (see Appendix 7.A)

$$g^r_{i_1,\ldots,i_k} = \frac{\boldsymbol{\pi}'\left(\mathbf{I} - \mathbf{EM}_{kk}^{-1}\mathbf{E}'\mathbf{M}\right)\left(\mathbf{I} - \mathbf{LEL}_{kk}^{-1}\mathbf{E}'\right)\hat{\mathbf{p}}\mathbf{x}}{\mathbf{p}'(\mathbf{I} - \mathbf{EE}')\mathbf{f}}. \tag{7.12}$$

Notice that, disregarding the denominator in (7.12), when $k = n$ and $\mathbf{E} = \mathbf{I}$ the numerator of the reduced TFP growth due to extraction of *all* sectors in (7.12) becomes zero, which is entirely expectable because without an industry, the hypothetical total (direct and indirect) productivity gains should be zero, i.e., $\boldsymbol{\lambda}'\mathbf{s}^{-\{1,\ldots,n\}} = 0$. We have established the following result that expresses the solution of the engines of growth identification problem (7.11) in terms of the reduced TFP growth rate given in (7.12).[3]

Theorem 7.1. *For* $1 \leq k \leq n-1$ *the k engines of growth* $\{i_1^*, i_2^*, \ldots, i_k^*\}$ *that solve the problem* (7.11) *give the lowest reduced TFP growth rate, i.e.,* $g^r_{i_1^*,\ldots,i_k^*} \leq g^r_{i_1,\ldots,i_k}$ *for all* $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ *with* $i_s \neq i_r$.

First, note that in finding the engines of growth from the HEM perspective, given the reduced TFP growth measure in (7.12), performing the traditional procedure of the HEM approach (which includes deleting certain rows and columns of the

[3] Alternatively, given the problem (7.11) we could define the *group TFP growth worth* of sectors $i_1, \ldots, i_k$ as $\omega^g_{i_1,\ldots,i_k} = g - \ell^g_{i_1,\ldots,i_k}$. Hence, in this interpretation, the k engines of growth have the *largest* group TFP growth worth, i.e., $\omega^g_{i_1^*,\ldots,i_k^*} \geq \omega^g_{i_1,\ldots,i_k}$ for all $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

matrices **A** and **B**) is not needed at all. Matrices **A**, **B**, **L** and **M** together with the vectors $\boldsymbol{\pi}$, **p** and **x** are all given, and only the k identity columns in **E** are changed in order to consider all possible combinations of k sectors from the totality of n industries in a search for the k engines of growth. Second, apparently, for fixed k the group of sectors that constitutes the engines of growth depends on the joint sectoral interactions of TFP growth rates, spillover linkages, multiplier effects, and sizes of the gross and net outputs in a complex way as is captured by the reduced TFP growth measure. Let us see this in a simple case when $k = 1$, which implies that one is looking for the *largest engine of growth*. Then $\mathbf{E} = \mathbf{e}_i$, $\mathbf{M}_{kk} = \mathbf{E}'\mathbf{M}\mathbf{E} = \mathbf{e}_i'\mathbf{M}\mathbf{e}_i = m_{ii}$, and $\mathbf{L}_{kk} = l_{ii}$, thus it can be shown that (7.12) reduces to

$$g_i^r = \frac{1}{\mathbf{p}'\mathbf{f} - p_i f_i}\left[\boldsymbol{\lambda}'\mathbf{s} - \frac{\pi_i s_i}{m_{ii}} - \frac{p_i x_i}{m_{ii} l_{ii}}\sum_{k \neq i}(m_{ii}\pi_k - m_{ik}\pi_i) l_{ki}\right], \tag{7.13}$$

which is the *reduced TFP growth* due to extraction of sector i. Then a corollary to Theorem 7.1 is that the single engine of growth, i^*, gives the smallest reduced TFP growth rate, i.e., $g_{i^*}^r \leq g_i^r$ for all $i = 1, \ldots, n$. So it is not only the TFP growth rate of sector i, π_i, that defines it to be the engine of growth, but also its spillover linkage, $s_i = \sum_j m_{ij} p_j x_j$, total (direct and indirect) input self-dependency, l_{ii}, total joint input and capital self-dependency, m_{ii}, and its values of gross and net outputs, $p_i x_i$ and $p_i f_i$, are all important. In particular, (7.13) shows that the engine of growth has a large TFP growth rate and spillover linkage, is less dependent on itself, and, more engaged in the "roundabout" of the production process, hence having higher gross and lower net outputs. But it is the joint relative importance of these factors that defines the engine of growth.

As in the case of key sectors' identification discussed in Chapter 6, it is important to understand that the problem of finding the single engine of growth (i.e., $k = 1$ in (7.11)) is *different* from problem (7.11) with $k > 1$. In other words, k (> 1) sectors, whose extraction results in the smallest reduced TFP growth rates, do *not* necessarily comprise the group of k engines of growth. While the single engine of growth search problem looks for the effect of the extraction of one sector, the more general problem in (7.11) considers the effect of a *simultaneous* extraction of $k \geq 2$ sectors. Hence, the last problem takes into full account all the cross-contributions of the extracted sectors to the nationwide TFP growth that is generated both within and outside the group of sectors. These effects are, of course, differently accounted for when $k = 1$.

Consider two industries that are largely identical with respect to their input and capital linkage patterns (including input and capital stock coefficients' sizes) and that are also similar in terms of their final demands, gross outputs and sources of growth, then their group contribution to the total TFP growth is expected to be less than that of the group consisting of two industries that have quite different patterns of (significant) linkages and TFP growth generation ability. In this case it is said that the first two industries are redundant with respect to each other, hence should not be included *both* in the group with 2 engines of growth. Thus, in general, the k (> 1) sectors, whose individual extraction yields the smallest reduced TFP growth, do *not* comprise the k engines of growth due to the redundancy principle inherent to the majority of real-life input and capital stock networks of interactions of industries.

Computerization is found to have a dramatic impact on growth and structural change by Wolff (2002). In ten Raa and Wolff's (2000) study, the computer and office equipment industry was found to be the largest engine of growth in the US economy for two subperiods of 1967-1977 and 1977-1987, while it was only at the 19-th position in 1958-1967. In the HEM approach discussed above, however, ranking of the individual sectors from the problem of identification of a single engine of growth does not tell us anything about the group of engines of growth. It is the *joint* contribution of sectors to the economy-wide TFP growth generation that makes them engines of growth. In this respect it would be, for example, interesting to find out what is the *minimum* value of k that allows the computers and office machinery industry to be a member of the group of k engines of growth. We plan to do an (extensive) empirical study of the engines of growth determination problem discussed above for several countries, and make a detailed comparison of the results both across countries and over time. This might shed more light on the sectoral analysis of structural change in different countries in terms of industries' contribution to the nation-wide TFP growth rates.

7.3.2 On interregional feedbacks in input-output models

Consider the following interregional input-output model with p regions:

$$\begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^p \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{11} & \mathbf{L}^{12} & \cdots & \mathbf{L}^{1p} \\ \mathbf{L}^{21} & \mathbf{L}^{22} & \cdots & \mathbf{L}^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{L}^{p1} & \mathbf{L}^{p2} & \cdots & \mathbf{L}^{pp} \end{bmatrix} \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \vdots \\ \mathbf{f}^p \end{bmatrix}, \tag{7.14}$$

where

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}^{11} & \mathbf{L}^{12} & \cdots & \mathbf{L}^{1p} \\ \mathbf{L}^{21} & \mathbf{L}^{22} & \cdots & \mathbf{L}^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{L}^{p1} & \mathbf{L}^{p2} & \cdots & \mathbf{L}^{pp} \end{bmatrix} = \begin{bmatrix} \mathbf{I}-\mathbf{A}^{11} & -\mathbf{A}^{12} & \cdots & -\mathbf{A}^{1p} \\ -\mathbf{A}^{21} & \mathbf{I}-\mathbf{A}^{22} & \cdots & -\mathbf{A}^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{A}^{p1} & -\mathbf{A}^{p2} & \cdots & \mathbf{I}-\mathbf{A}^{pp} \end{bmatrix}^{-1}$$

is the Leontief inverse in an interregional setting, $\mathbf{A}^{rr}$ is the (intra)regional input coefficients matrix for region r $(= 1, \ldots, p)$, $\mathbf{A}^{rs}$ is the matrix of interregional input (trade) coefficients with deliveries from region r to region s $(r \neq s)$, $\mathbf{f}^r$ and $\mathbf{x}^r$ are, respectively, the vectors of changes in final demand and gross output for region r, and $\mathbf{I}$ is the identity matrix with appropriate dimension.

The question is how a change of final demand in region, say, 1 (i.e., $\mathbf{f}^1 > \mathbf{0}$ and $\mathbf{f}^r = \mathbf{0}$ for all $r \neq 1$) affects the outputs in that region and what would be the bias if instead of the interregional framework in (7.14) only a single-region input-output framework of $\mathbf{x}_s^1 = (\mathbf{I} - \mathbf{A}^{11})^{-1}\mathbf{f}^1$ would have been used. That is, how big would be the bias in $\mathbf{x}^1$ if the so-called *interregional feedback effects* were totally ignored. The term "feedback" refers to the fact that an increase in final demand in region 1 causes more demand also for the intermediate inputs from other regions, but the production in these regions is in its turn, in general, dependent on the inputs from region 1 as well. Thus other regions will also demand more intermediate goods from region 1. Notice that the final demand in region 1 may also decrease, in which case the directions of all the above mentioned effects will be reversed. If, on the other hand, some components of the final demand in region 1 increase and others decrease, then obviously the impact of the feedback effects is analytically uncertain.

To tackle the above assigned question, for simplicity, the components of (7.14) are reexpressed in terms of two partitioned matrices as follows:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^\bullet \end{bmatrix} = \left[\begin{array}{c} \mathbf{x}^1 \\ \hline \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^p \end{array}\right], \quad \mathbf{L} = \begin{bmatrix} \mathbf{L}^{11} & \mathbf{L}^{1\bullet} \\ \mathbf{L}^{\bullet 1} & \mathbf{L}^{\bullet\bullet} \end{bmatrix} = \left[\begin{array}{c|ccc} \mathbf{L}^{11} & \mathbf{L}^{12} & \cdots & \mathbf{L}^{1p} \\ \hline \mathbf{L}^{21} & \mathbf{L}^{22} & \cdots & \mathbf{L}^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{L}^{p1} & \mathbf{L}^{p2} & \cdots & \mathbf{L}^{pp} \end{array}\right],$$

and the vector of changes in final demand is $\mathbf{f}' = \begin{bmatrix} (\mathbf{f}^1)' & \mathbf{0}' \end{bmatrix}$, where $\mathbf{f}^\bullet$ is set to zero since there is a change in the final demand for region 1 only, i.e., $\mathbf{f}^r = \mathbf{0}$ for all $r \neq 1$.

The single-region framework can be rewritten as

$$\begin{bmatrix} \mathbf{x}_s^1 \\ \mathbf{x}_s^\bullet \end{bmatrix} = \begin{bmatrix} \mathbf{I}-\mathbf{A}^{11} & -\mathbf{O} \\ -\mathbf{O} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} (\mathbf{I}-\mathbf{A}^{11})^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{0} \end{bmatrix}, \quad (7.15)$$

which means that in this case in the entire matrix of regional and trade coefficients $\mathbf{A}$, all elements corresponding to any region other than 1 are set to zero. That is, $\mathbf{A}^{1\bullet} = \mathbf{O}$, $\mathbf{A}^{\bullet 1} = \mathbf{O}$ and $\mathbf{A}^{\bullet\bullet} = \mathbf{O}$, where the null matrix $\mathbf{O}$ in each case is assumed to have the appropriate dimension. This nullification is exactly similar to the (generalized) hypothetical extraction method studied in Chapter 6. Hence, we can readily use Lemma 6.2, wherein the setting is now an interregional framework and $\mathbf{E}' = [\mathbf{O}_{\bullet 1} \ \ \mathbf{I}_\bullet]$, where $\mathbf{I}_\bullet$ is the identity matrix of dimension equal to the total number of industries in all regions except region 1, and $\mathbf{O}_{\bullet 1}$ is the null matrix of row dimension equal to the (row or column) dimension of $\mathbf{I}_\bullet$ and column dimension equal to the number of sectors in region 1.[4] Denoting the Leontief inverse in (7.15) by $\mathbf{L}^{-\{\bullet\}}$, we thus have

$$\begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^\bullet \end{bmatrix} - \begin{bmatrix} \mathbf{x}_s^1 \\ \mathbf{x}_s^\bullet \end{bmatrix} = \left[\mathbf{L} - \mathbf{L}^{-\{\bullet\}}\right] \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{0} \end{bmatrix} = \left[\mathbf{LE}(\mathbf{E}'\mathbf{LE})^{-1}\mathbf{E}'\mathbf{L} - \mathbf{EE}'\right] \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{0} \end{bmatrix}$$
$$= \left\{ \begin{bmatrix} \mathbf{L}^{1\bullet}(\mathbf{L}^{\bullet\bullet})^{-1}\mathbf{L}^{\bullet 1} & \mathbf{L}^{1\bullet} \\ \mathbf{L}^{\bullet 1} & \mathbf{L}^{\bullet\bullet} \end{bmatrix} - \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \right\} \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{0} \end{bmatrix}.$$

Since we are interested in the effect on outputs in region 1, the last equation yields

$$\mathbf{x}^1 - \mathbf{x}_s^1 = \mathbf{L}^{1\bullet}(\mathbf{L}^{\bullet\bullet})^{-1}\mathbf{L}^{\bullet 1}\mathbf{f}^1 = \mathbf{L}^{1\bullet}(\mathbf{L}^{\bullet\bullet})^{-1}\mathbf{L}^{\bullet 1}(\mathbf{L}^{11})^{-1}\mathbf{x}^1, \quad (7.16)$$

where we have used the fact that $\mathbf{x}^1 = \mathbf{L}^{11}\mathbf{f}^1$ in (7.14) given that the vectors of the change in final demands of all other regions are zero.

Equation (7.16) gives the bias of ignoring interregional feedbacks at the sectoral level of region 1. A widely used measure of an error when a single-region model is used instead of the full interregional framework is the *overall percentage error* (OPE), first employed by Miller (1969) as "*a summary measure of deviation*" (p. 41) and is defined as $OPE = \boldsymbol{\imath}'(\mathbf{x}^1 - \mathbf{x}_s^1)/\boldsymbol{\imath}'\mathbf{x}^1 \times 100$, where $\boldsymbol{\imath}$ is a summation vector of ones. Define the *norm* of any matrix $\mathbf{M}$ as the largest column sum of the absolute values of its elements, and denote it by $\|\mathbf{M}\|$. Therefore, OPE can alternatively be rewritten in terms of norms as $OPE = \|\mathbf{x}^1 - \mathbf{x}_s^1\|/\|\mathbf{x}^1\| \times 100$. For any two matrices $\mathbf{M}$ and

[4] If all regions have the same number of industries equal to n, then $\mathbf{I}_\bullet$ and $\mathbf{O}_{\bullet 1}$ have dimensions of, respectively, $(p-1)n \times (p-1)n$ and $(p-1)n \times n$.

$\mathbf{N}$ the so-called *submultiplicative* property of the matrix norm holds, i.e., $\|\mathbf{MN}\| \leq \|\mathbf{M}\| \, \|\mathbf{N}\|$. Employing this property in (7.16) gives the following proposition.

Theorem 7.2. *The overall percentage error caused by ignoring interregional feedbacks is bounded above by* $\|\mathbf{L}^{1\bullet}(\mathbf{L}^{\bullet\bullet})^{-1}\mathbf{L}^{\bullet 1}(\mathbf{L}^{11})^{-1}\| \times 100$.

From (7.14)-(7.16) it follows that $\mathbf{L}^{11} = (\mathbf{I} - \mathbf{A}^{11})^{-1} + \mathbf{L}^{1\bullet}(\mathbf{L}^{\bullet\bullet})^{-1}\mathbf{L}^{\bullet 1}$. Further, from the theory of (the inverse of) partitioned matrices one can write $(\mathbf{L}^{11})^{-1} = \mathbf{I} - \mathbf{A}^{11} - \mathbf{A}^{1\bullet}(\mathbf{I} - \mathbf{A}^{\bullet\bullet})^{-1}\mathbf{A}^{\bullet 1}$ (see e.g., Sydsæter et al., 2005, p. 140). Using these two identities we have $\mathbf{L}^{1\bullet}(\mathbf{L}^{\bullet\bullet})^{-1}\mathbf{L}^{\bullet 1}(\mathbf{L}^{11})^{-1} = \left[\mathbf{L}^{11} - (\mathbf{I} - \mathbf{A}^{11})^{-1}\right](\mathbf{L}^{11})^{-1} = (\mathbf{I} - \mathbf{A}^{11})^{-1}\mathbf{A}^{1\bullet}(\mathbf{I} - \mathbf{A}^{\bullet\bullet})^{-1}\mathbf{A}^{\bullet 1}$. Thus, the upper bound in Theorem 7.2 can also be rewritten as $\|\mathbf{L}^{1\bullet}(\mathbf{L}^{\bullet\bullet})^{-1}\mathbf{L}^{\bullet 1}(\mathbf{L}^{11})^{-1}\| \times 100 = \|(\mathbf{I} - \mathbf{A}^{11})^{-1}\mathbf{A}^{1\bullet}(\mathbf{I} - \mathbf{A}^{\bullet\bullet})^{-1}\mathbf{A}^{\bullet 1}\| \times 100$. The right-hand side of the last expression (without number 100) is exactly what Guccione et al. (1988) call the *least upper bound* (LUB) of the OPE.[5] Thus, in Theorem 7.2 we gave an alternative expression of the LUB in terms of the elements of the Leontief inverse. Note that there is *no* need to use more than a two "region" partition in the analysis of the upper bound for the OPE. This has been tried, for example, in Miller (1986) for three-region case, who then not surprisingly noticed that "[t]he algebra is considerably more complex" (p. 292), but more importantly because applying matrix properties to such partitioning might very well result in a looser bound that cannot be the least upper bound.

We should mention that the OPE for any other factor than gross output can be easily accommodated in this framework. For this, similar to the discussions in Chapter 6, one should consider also the direct coefficients of the factor of interest (e.g., employment, CO_2 emissions, etc.). In that case the expression for LUB within the norm has to be multiplied by the diagonal matrix of the factor direct coefficients.

Of course, given the ongoing globalization, countries are becoming more and more interdependent not only via trade of final goods, but also through trade of intermediate goods that has been risen steadily over the last several decades. Therefore, one might expect that the error of ignoring interregional feedbacks is much larger now than some 40-50 years ago. This trend of globalization is reflected by more positive and increasing elements in the interregional input coefficient matrices. As a consequence the LUB not surprisingly increases. However, certainly this differs from region to region (or country to country) depending on the self-sufficiencies of the regions. So, the question of how big is nowadays the bias caused by using a single-region framework instead of the multiregional setting is an em-

[5] See also Gillen and Guccione (1980) and Miller (1986) that use a *looser* upper bound.

pirical issue. In the near future, we plan to quantify this bias in the the empirical application part of this section.

7.3.3 Algorithmic considerations of the group intercentrality and group worth measures

In Chapters 5 and 6 we considered the problems of finding the key groups of, respectively, players in networks of social interactions and sectors in an economy. We have also briefly mentioned the complexity issue of finding the exact solutions to these problems for a large number of players/industries and a rather large size of the groups. This is because in order to find the key group of size k, one needs to consider all possible combinations of k players/sectors out of n players/sectors. The number of combination is $C_k^n = n!/(k!(n-k)!)$, which increases exponentially in k and n. For example, in Table 6.3 we have searched for the key groups of size 1 to 4 from a total of 136 sectors, which required to compute the group factor worths of, respectively, 136, 9.180, 410.040, and 13.633.830 different groups. This example clearly demonstrates the problem of the computational complexity inherent to the above mentioned problems. Hence, the cases of searching key group(s) of reasonable size among very large number of groups becomes potentially intractable from a computing point of view. Therefore, the question arises whether for a large n and a rather large k one can find the exact solutions of the key group problems in reasonable time. It turns out that the answer to this question is negative because the posed problems are in the class of the so-called *NP-hard* problems from a combinatorial perspective. NP-hardness implies that there is no possible sophisticated algorithm that will return the exact solution for large n and k in our case. "[N]early all computer scientists ... believe that there is no such algorithm for solving any NP-hard problem. A simple reason for this is that, after decades of continuous search, no one has found efficient algorithm for solving any NP-hard problem" (Ballester et al., 2009, footnote 15). In what follows we first prove that the discussed key group problems are indeed NP-hard problems, and then consider the possible efficient approximate solutions to these problems once the computing search becomes intractable.

Let $N = \{1, 2, \ldots, n\}$ and $z : 2^N \to \mathbb{R}$ be a set function. Nemhauser et al. (1978) considered the following problem:

$$\max_{S \subseteq N} \{z(S) : |S| \leq k, \;\; z(S) \text{ submodular}\}, \tag{7.17}$$

where $|S|$ is the cardinality (i.e., the number of players/sectors) in the set S, while submodularity of a set function is defined as follows.

Definition 7.1. *Given a finite set N, a real-valued function z on the set of subsets of N is called submodular if $z(A) + z(B) \geq z(A \cup B) + z(A \cap B)$ for all $A, B \subseteq N$.*

Without loss of generality z is normalized such that $z(\varnothing) = 0$. We consider non-decreasing set functions in the sense that $z(S) \leq z(T)$ for all $S \subseteq T \subseteq N$. Let us denote the individual contribution by $\rho_i(S) = z(S \cup \{i\}) - z(S)$, which represents the incremental value of adding player/sector i to the set S. Proposition 2.1 in Nemhauser et al. (1978) establishes that an equivalent statement to Definition 7.1 that defines a submodular set function is $\rho_i(S) \geq \rho_i(T)$ for all $S \subseteq T \subseteq N$ and all $i \in N \setminus T$.

From Theorem 5.1 and Theorem 6.3 it follows that the key group problems within the social network and input-output settings are equivalent to the maximization of, respectively, group intercentrality and group factor worth measures. Using the last definition of a submodular function in terms of the individual contributions, in Appendix 7.A we establish the following result.

Lemma 7.1. *The measures of group intercentrality $c_S(\mathbf{g}, a)$ and factor worth $\omega_S^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ are submodular set functions.*

The problem of maximizing a submodular function is NP-hard, in general. Therefore, Lemma 7.1 implies that the computational complexity becomes large when the number of players/sectors n and the group size k are large in the key group problems. Therefore, in such cases *algorithmic approximations* are used, which require much less time in the computation. Consider an *R-step greedy algorithm (heuristic)* that sequentially eliminates the sets of R players/sectors with the highest group intercentrality/worth. Formally, suppose that $k = qR - p$, where q is a positive *integer* and $0 \leq p \leq R - 1$. The R-step greedy heuristic for a set function z works are follows.

Initialization: Let $S^0 = \varnothing$, $S^t = \cup_{i=1}^{t} I^i$, and set $t = 1$.
Iterations: For $t = 1, \ldots, q-1$ select $I^t \subseteq N \setminus S^{t-1}$ with $|I^t| = R$ such that $\zeta_{t-1} = z(S^t) - z(S^{t-1})$ is maximized.
Final step: Choose $I^* \subseteq N \setminus S^{q-1}$ with $|I^*| = R - p$ so as to maximize $z(S^{q-1} \cup I^*) - z(S^{q-1})$.

Consider, for example, the case when we want to find the key group of size $k = 30$ from total $n = 1000$ players using the R-step greedy algorithm, and thus choose

$R = 4$ and $q = 8$. This means that we would like to find out the approximation of the exact key group of size $k = 30$ in $q = 8$ computing steps (iterations). The procedure first sequentially at seven $(q - 1)$ steps eliminates 4 players as a key group (which makes $4 \times 7 = 28$ players). The two remaining members of the key group (i.e., $R - p = k - (q - 1)R = 30 - 28 = 2$, hence $p = 2$) are found in the final stage of the 4-step greedy algorithm. Note that if k is a multiple of R, then $p = 0$.

Let us denote the value of an R-step greedy solution by $z(G^R)$, where the approximate solution set is $G^R = S^{q-1} \cup I^*$, and the exact solution of (7.17) is given by $z(S^*)$. Then, provided the normalization $z(\varnothing) = 0$, the following result is proved in Theorem 4.3 in Nemhauser et al. (1978, pp. 282-283).

Theorem 7.3. *Suppose z is nondecreasing and the R-step greedy heuristic is applied to problem* (7.17). *If $K = qR - p$, with q a positive integer and integer $p \in [0, R - 1]$, then the upper bound of the error of approximation is*

$$\frac{z(S^*) - z(S^{G^R})}{z(S^*)} \leq \left(\frac{q - \lambda}{q}\right)\left(\frac{q-1}{q}\right)^{q-1},$$

where $\lambda = (R - p)/R$.

The bound in Theorem 7.3 for $q > 1$ can be rewritten as (using $K = qR - p$)

$$\left(\frac{q-\lambda}{q}\right)\left(\frac{q-1}{q}\right)^{q-1} = \left(1 + \frac{p}{R(q-1)}\right)\left(\frac{q-1}{q}\right)^{q} < \left(1 + \frac{p}{R(q-1)}\right)\frac{1}{e},$$

where $e \approx 2.718$ is the base of the natural logarithm. The last inequality follows since q is finite and $1/e = \lim_{q\to\infty}(1 - 1/q)^q$. If $p = 0$, then the bound in Theorem 7.3 boils down to $[(q - 1)/q]^q < 1/e \approx 0.3679$. That is, with $p = 0$ the maximum possible error when the R-step greedy algorithm is used to approximate the solution of (7.17) is 36.79%.

Note that if $p = 0$ and $q = 1/R$, then the R-step heuristic is a simple greedy algorithm that selects (eliminates) only one member in each iteration. This is exactly the sequential key player/sector problem that we have discussed in Chapter 5 and Chapter 6. Note that in the input-output setting, we have already shown in Section 6.2.3 that the key group problem is not equivalent to the sequential key sector problem. Since both the group intercentrality and group factor worth are nondecreasing and submodular functions, the result of Theorem 7.3 can be readily used if one wants to approximate the solutions of the key group problems given in (5.6) and (6.4). In particular, it shows that the worst approximation through the sequential key player/sector problem is less than 36.79%. When instead the groups of size R are sequentially selected (i.e., the R-step heuristic with $R > 1$ is used) for the approximation, the same upper bound holds for $p = 0$ (see above), while the error might be larger than 36.79% whenever $p > 0$. These bounds admittedly are very high. Ballester et al. (2009) provide some numerical simulations for 100 different random networks with $n = 10$ and $n = 15$, where they found small approximation errors of at most 1.7% obtained by using a simple greedy algorithm (i.e., the sequential key player problem) in addressing the key group problem in the social network setting.[6] Whether it holds in general for large-sized networks, and for large input-output datasets is a matter that needs deeper investigation.

[6] We should, however, note that the random networks always tend to be more "symmetric", thus they are, in general, different from the real-life networks.

7.A Proofs

Proof of Theorem 7.1. We already know from Lemma 6.2 in Chapter 6 that $\mathbf{L} - \mathbf{L}^{\{i_1,\ldots,i_k\}} = \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\mathbf{L} - \mathbf{E}\mathbf{E}'$, where $\mathbf{L}_{kk}^{-1} = \left(\mathbf{E}'\mathbf{L}\mathbf{E}\right)^{-1}$. Note that the last identity holds also if we use the matrix $\mathbf{M}$ instead of $\mathbf{L}$. We further have $\mathbf{f}^{-\{i_1,\ldots,i_k\}} = \mathbf{f} - \mathbf{E}\mathbf{E}'\mathbf{f}$, and $\hat{\mathbf{p}}\mathbf{E}\mathbf{E}' = \mathbf{E}\mathbf{E}'\hat{\mathbf{p}}$ since $\mathbf{E}\mathbf{E}'$ is a diagonal matrix. Recalling that the Leontief inverse in the current setting is defined as $\mathbf{L} = (\mathbf{I} - \hat{\mathbf{p}}\mathbf{A}\hat{\mathbf{p}})^{-1}$, we have

$$\begin{aligned}
\hat{\mathbf{p}}\mathbf{x}^{-\{i_1,\ldots,i_k\}} &= \mathbf{L}^{-\{i_1,\ldots,i_k\}}\hat{\mathbf{p}}\mathbf{f}^{-\{i_1,\ldots,i_k\}} = \left(\mathbf{L} - \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\mathbf{L} + \mathbf{E}\mathbf{E}'\right)\left(\hat{\mathbf{p}}\mathbf{f} - \mathbf{E}\mathbf{E}'\hat{\mathbf{p}}\mathbf{f}\right) \\
&= \hat{\mathbf{p}}\mathbf{x} - \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\hat{\mathbf{p}}\mathbf{x} + \mathbf{E}\mathbf{E}'\hat{\mathbf{p}}\mathbf{f} - \mathbf{L}\mathbf{E}\mathbf{E}'\hat{\mathbf{p}}\mathbf{f} + \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{L}_{kk}\mathbf{E}'\hat{\mathbf{p}}\mathbf{f} - \mathbf{E}\mathbf{E}'\mathbf{E}\mathbf{E}'\hat{\mathbf{p}}\mathbf{f} \\
&= \hat{\mathbf{p}}\mathbf{x} - \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\hat{\mathbf{p}}\mathbf{x},
\end{aligned}$$

where the last four terms in the expression after the second equality cancel out since $\mathbf{L}_{kk}^{-1}\mathbf{L}_{kk} = \mathbf{I}$ and $\mathbf{E}\mathbf{E}'\mathbf{E}\mathbf{E}' = \mathbf{E}\mathbf{E}'$. Using $\mathbf{s}^{-\{i_1,\ldots,i_k\}} = \mathbf{M}^{-\{i_1,\ldots,i_k\}}\hat{\mathbf{p}}\mathbf{x}^{-\{i_1,\ldots,i_k\}}$, the above derived expression, and Lemma 6.2, one obtains

$$\begin{aligned}
\boldsymbol{\lambda}'\mathbf{s}^{-\{i_1,\ldots,i_k\}} &= \boldsymbol{\lambda}'\left(\mathbf{M} - \mathbf{M}\mathbf{E}\mathbf{M}_{kk}^{-1}\mathbf{E}'\mathbf{M} + \mathbf{E}\mathbf{E}'\right)\left(\mathbf{I} - \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\right)\hat{\mathbf{p}}\mathbf{x} \\
&= \boldsymbol{\pi}'\left(\mathbf{I} - \mathbf{E}\mathbf{M}_{kk}^{-1}\mathbf{E}'\mathbf{M}\right)\left(\mathbf{I} - \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\right)\hat{\mathbf{p}}\mathbf{x} + \boldsymbol{\lambda}'\mathbf{E}\mathbf{E}'\hat{\mathbf{p}}\mathbf{x} - \boldsymbol{\lambda}'\mathbf{E}\mathbf{L}_{kk}\mathbf{L}_{kk}^{-1}\mathbf{E}'\hat{\mathbf{p}}\mathbf{x} \\
&= \boldsymbol{\pi}'\left(\mathbf{I} - \mathbf{E}\mathbf{M}_{kk}^{-1}\mathbf{E}'\mathbf{M}\right)\left(\mathbf{I} - \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\right)\hat{\mathbf{p}}\mathbf{x}.
\end{aligned}$$

Given our objective in (7.11), we are seeking the group of sectors that minimizes

$$\frac{\boldsymbol{\lambda}'\mathbf{s}^{-\{i_1,\ldots,i_k\}}}{\mathbf{p}'\mathbf{f}^{-\{i_1,\ldots,i_k\}}} = \frac{\boldsymbol{\pi}'\left(\mathbf{I} - \mathbf{E}\mathbf{M}_{kk}^{-1}\mathbf{E}'\mathbf{M}\right)\left(\mathbf{I} - \mathbf{L}\mathbf{E}\mathbf{L}_{kk}^{-1}\mathbf{E}'\right)\hat{\mathbf{p}}\mathbf{x}}{\mathbf{p}'(\mathbf{I} - \mathbf{E}\mathbf{E}')\mathbf{f}}, \tag{7.A.1}$$

which is the definition of the *reduced TFP growth rate* due to extraction of sectors $i_1, \ldots, i_k$ in (7.12). ■

Proof of Lemma 7.1. Take $S \subseteq T \subseteq N$ and $i \in N \setminus T$. The strict monotonicity property of the group intercentrality measure discussed in Section 5.2.2 of Chapter 5 immediately implies that $c_{S\cup\{i\}}(\mathbf{g}, a) - c_S(\mathbf{g}, a) = c_i(\mathbf{g}^{-S}, a) \geq c_i(\mathbf{g}^{-T}, a) = c_{T\cup\{i\}}(\mathbf{g}, a) - c_T(\mathbf{g}, a)$, where $\mathbf{g}^{-S}$ denotes the network without all members of the set S. This is exactly the definition of a submodular function in terms of the individual contributions.

Using all the necessary definitions from Chapter 6 and the property of the Leontief inverse matrix, for the industries factor worth $\omega_S^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ we have

$$\omega_{S\cup\{i\}}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) - \omega_S^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$$

$$
\begin{aligned}
&= (\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-\{S\cup\{i\}\}}) - (\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-S}) \\
&= \boldsymbol{\pi}'\mathbf{x}^{-S} - \boldsymbol{\pi}'\mathbf{x}^{-\{S\cup\{i\}\}} = \omega_i^{\pi}(\mathbf{A}^{-S}, \mathbf{f}^{-S}, \boldsymbol{\pi}) \\
&\geq \omega_i^{\pi}(\mathbf{A}^{-T}, \mathbf{f}^{-T}, \boldsymbol{\pi}) = \boldsymbol{\pi}'\mathbf{x}^{-T} - \boldsymbol{\pi}'\mathbf{x}^{-\{T\cup\{i\}\}} \\
&= \omega_{T\cup\{i\}}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) - \omega_T^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}),
\end{aligned}
$$

which is again the definition of the submodular function. Hence, both the group intercentrality and group factor worth are submodular set functions. ■

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Samenvatting

Dit proefschrift heeft zich op drie soorten onderlinge afhankelijkheid geconcentreerd: op het niveau van firma's, dat van individuen en dat van economische sectoren. Dit waren aandeelhouder-interlocks (waarin bedrijven wederzijds aandelen in elkaar hebben, ook wel *cross-ownerships* genaamd), sociale netwerken van mensen en productieverbanden tussen bedrijfstakken. Omdat deze inter-relaties hun eigen onderscheidende kenmerken hebben verschillen de gebruikte analytische kaders ook (nogal) van elkaar. Dit verklaart waarom dit onderzoek zich niet op één specifiek gebied richt, maar in plaats daarvan onderwerpen uit verschillende deelgebieden binnen de economie en de sociologie heeft onderzocht, zoals financiering, industriële organisatie, input-output analyse, netwerkeconomie en sociale netwerkanalyse. Anderzijds zijn de kwesties die in dit proefschrift in overweging zijn genomen echter totaal niet onafhankelijk van elkaar, in tegenstelling tot wat op het eerste gezicht wel zo lijkt te zijn. De analyses van de complexe netwerken van inter-relaties hebben veel met elkaar gemeen. In zekere zin hebben ze een uniform raamwerk en breiden zo binnen de bovengenoemde gebieden de grenzen van de gemeenschappelijke belangen uit.

Een aantal van de hoofdpunten van dit onderzoek waren gericht op het beantwoorden van de volgende vragen: Hoe kan de complexiteit van de eigendomsstructuur, veroorzaakt door *cross-ownership* van zowel individuen als bedrijven, worden gemeten? Wat is een geschikte maatstaf voor het scheiden van eigendomsrechten en beslissingsbevoegdheden indien er sprake is van wederzijdse belangen in elkaar? Heeft een bedrijf met passieve aandeelhouderschappen in zijn concurrenten per definitie meer marktmacht dan een firma zonder aandeelhouderschappen? Zijn aandeelhouder-interlocks empirisch van belang voor marktgedrag? Wat is het effect van gedeeltelijke *cross-ownership* op de neiging van firma's met asymmetrische kosten tot het maken van heimelijke (prijs)afspraken? Hoe kan de sleutelgroep van individuen worden gevonden, d.w.z. de groep die de grootste invloed uitoefent op het algemene evenwichtsresultaat binnen (sociale) netwerken? Hoe kan exogene heterogeniteit van individuen worden verwerkt in de analyse van het zoeken naar sleutelspelers? Is het probleem van het vinden van sleutelsectoren gelijk aan het probleem van het identificeren van de sleutelgroep van sectoren? Zo niet, wat zijn hiervoor de achterliggende redenen? Deze vragen zijn uitvoerig behandeld in dit proefschrift. In het navolgende geven we een korte samenvatting van de verkregen resultaten.

In Hoofdstuk 2 werden nieuwe maatstaven voorgesteld voor netwerkcomplexiteit, zoals veroorzaakt door het bestaan van verbanden tussen bedrijven tengevolge van *cross-ownership*. Deze maatstaven, die 'gewogen gemiddelde afstand van indirecte verbanden' (WADIL, *weighted average distance of indirect linkages*) en 'gewogen gemiddelde afstand van totale verbanden' (WADTL, *weighted average distance of total linkages*) worden genoemd, meten de complexiteit van een eigendomsstructuur, die gekarakteriseerd wordt door wederzijdse aandelenparticipatie. De voorgestelde maatstaven houden rekening met de groottes van zowel direct als indirect aandelenbezit en de gemiddelde afstand tussen de eigenaren en hun eigendommen. We stellen dat eigenaar (of firma) i een indirect belang in firma r heeft als hij een belang heeft in een firma die een belang heeft in firma r, of als hij een belang heeft in een firma die een belang heeft in een firma die een belang heeft in firma r, enzovoorts. De gemiddelde afstand werd gemeten op basis van het gemiddelde aantal tussenfirma's, die het eigendomsverband vormen tussen i en r. De waarden van de WADILs en de WADTLs geven aan of een bepaald verband enkel van directe aard is of dat ook indirecte aandeelhoudershappen een rol spelen in het verband spelen. Grotere waarden van WADILs en WADTLs verwijzen naar een complexer netwerk, hetgeen een groter aantal verschillende eigenschapspaden met zich mee brengt. Het combineren van de grootte van de verbinding en de afstand maakte het mogelijk om aandeelhouder-interlocks en de werkelijke eigendomsrelaties te visualiseren. Deze methodologie werd toegepast voor de Tsjechische bankensector in 1997. Er bleek overvloedig bewijs te zijn dat indirecte eigendomsrelaties een cruciale rol spelen binnen de Tsjechische bankensector. Verder werd het verband onderzocht tussen de voorgestelde maatstaven voor netwerkcomplexiteit en de scheiding tussen dividendrechten en beslissingsbevoegdheden indien er sprake is van *cross-ownership*. Er werd betoogd dat de WADILs en de WADTLs als alternatieve maatstaven zouden kunnen dienen voor de discrepantie tussen eigendomsrechten en beslissingsbevoegdheden. Dat wil zeggen, hoe complexer het netwerk van nietverwaarloosbare eigendomsrelaties, hoe groter de beslissingsbevoegdheden, als gevolg van wederzijdse aandelenparticipatie van firma's. Daarom is ook het verschil tussen eigendomsrechten en beslissingsbevoegdheden van firma-eigenaren groter. Dit werd bevestigd door de empirische resultaten voor de Tsjechische bankensector. De verkregen WADILs en de WADTLs werden ook vergeleken met de discrepantie tussen eigendomsrechten en beslissingsbevoegdheden, zoals gemeten op basis van bekende financiële methodologieën, namelijk de 'zwakste schakel-' en de 'dominante aandeelhoudersbenadering'.

Het effect van het veronachtzamen van *partial cross-ownership* (PCO) (dat wil zeggen, aandelen die voor de eigenaar niet tot beslissingsbevoegdheid leiden) in empirisch onderzoek naar prestaties en macht van bedrijven in een markt werd in Hoofdstuk 3 onderzocht.[7] Hiertoe werd het bekende schema van de 'structuur-gedrag-prestatie school' uit de industriële organisatie gebruikt. Voor de schatting van de marktmacht van firma's en de heimelijke onderlinge afspraken in een bedrijfstak werd het model gewijzigd, door zowel directe als indirecte aandeelhouderschappen op te nemen. Bewezen werd dat, in tegenstelling tot het geen-PCO geval, het verband tussen de prijs-kostenmarges van firma's en de mate van concurrentie in de markt niet-lineair is als er sprake is van PCO. Het negeren van PCO in een analyse van een bedrijfstak waarbinnen op grote schaal aandelenbezit tussen firma's bestaat zal dus hoogstwaarschijnlijk leiden tot onjuiste resultaten, als gevolg van een misspecificatie van het model. Een empirische toepassing wees uit dat Japanse commerciële banken in 2003 concurreerden op een markt met kenmerken van heimelijke afspraken op bescheiden niveau. Als PCO echter buiten beschouwing werd gelaten leidde dit tot andere resultaten, die duidden op een Cournot-oligopolie. Verder werd gevonden dat banken met PCO in hun concurrenten zonder uitzondering een grotere marktmacht hebben dan banken zonder enige aandeelhouderschappen. In het bijzonder bleken stadsbanken met veel aandeelhouderschappen een veel grotere marktmacht te hebben dan regionale banken met geen of weinig aandeelhouderschappen. Daarom werd in Hoofdstuk 3 de hypothese bevestigd dat het verwerven van aandelen in concurrenten een manier is voor een firma om de marktmacht te versterken.

In hoofdstuk 4 werd een oneindig vaak herhaald oligopoliemodel van Bertrand gebruikt om, voor firma's met kostenasymmetrieën, het effect te onderzoeken van *partial ownership* op de drijfveren tot het maken van heimelijke afspraken. We

[7] In Hoofdstuk 4 bekijken we ook de zaak waarin slechts één firma in concurrenten investeert. We noemen dit *partial ownership* (PO). PCO, daarentegen, weerspiegelt het feit dat in de aanwezigheid van multilaterale eigendomsafspraken, bedrijven wederzijds belangen in elkaar hebben.

hebben eerst het geval bekeken waarin alleen de meest efficiënte firma in de bedrijfstak investeert in concurrenten. Aangetoond werd dat *partial ownership* (dat dus unilateraal is) door deze firma leidt tot een markt waarin alle firma's dezelfde prijs vragen en de markt onder elkaar verdelen. We lieten zien dat wanneer de meest efficiënte firma in concurrenten investeert de heimelijk afgesproken prijs, die een compromis is tussen de monopolieprijzen van de verschillende firma's, toeneemt in vergelijking met situaties waarin er geen *partial ownership* voorkomt. Verder hebben we ons gericht op het effect van een verandering in de PCO-structuur op stilzwijgende collusie. Aangetoond werd dat als het belang dat firma r heeft in firma s groter wordt ten koste van aandeelhouders van buitenaf, de neiging tot het maken van heimelijke afspraken nooit afneemt. Die neiging neemt toe als (en alleen als) (i) de *industry maverick* (de firma met de sterkste stimulans om af te wijken van een heimelijke overeenkomst) een direct of indirect belang heeft in firma r, en (ii) firma s niet de *industry maverick* is. Als (i) en/of (ii) niet gelden zal de toename van het belang van firma r in firma s geen invloed uitoefenen op de stilzwijgende collusie. Deze resultaten bouwen voort op eerdere conclusies van Gilo e.a. (2006) en laten zien dat de resultaten voor firma's met symmetrische kostenfuncties ook gelden voor firma's met asymmetrische kosten. Hoofdstuk 4 bekeek ook de gevolgen van een eigendomsoverdracht tussen twee firma's. Er werd aangetoond dat een overdracht van PCO in firma s van firma k naar firma r geen invloed heeft op stilzwijgende collusie als firma s de *industry maverick* is, of als de industry maverick vanaf het begin hetzelfde totale (directe en indirecte) belang heeft in firma's k en r. In alle andere gevallen wordt stilzwijgende collusie bevorderd door de overdracht van PCO als de *industry maverick* een groter totaal belang heeft in firma r (de koper) dan in firma k (de verkoper), maar wordt dit juist belemmerd in het tegenovergestelde geval.

Hoofdstuk 5 ging uit van het probleem van het vinden van de sleutelspeler in een netwerkspel dat Ballester e.a. (2006) heeft onderzocht, en breidde dit uit naar het zoeken naar de sleutelgroep, waarin exogene heterogeniteit van spelers in beschouwing werd genomen. De sleutelgroep is de groep spelers die de grootst (of kleinst) mogelijke impact heeft op het algemene evenwicht van het netwerk. We hebben een uitdrukking in gesloten-vorm afgeleid om de zogenoemde groepsintercentraliteit te meten, die gebruikt wordt om de sleutelgroep binnen netwerken te identificeren. Daarnaast hebben we enkele kenmerken ervan onderzocht. De maatstaven voor gewogen en niet-gewogen groepsintercentraliteit, die alleen afhankelijk blijken te zijn van de aanvankelijke netwerkconfiguratie, zijn verder van belang bij het identificeren van de sleutelgroep van heterogene spelers. De gewichten waren gebaseerd op waarneembare verschillen tussen spelers, zoals leeftijd, opleiding, beroep, ras, religie, familiegrootte of de opleiding van de ouders. Aangetoond werd dat zodra er rekening gehouden wordt met deze waarneembare verschillen de resultaten van het probleem van de sleutelspeler/groep wezenlijk kunnen veranderen als ze vergeleken worden met resultaten die gebaseerd zijn op de veronderstelling van homogene spelers. Tenslotte werd de grootte van de sleutelgroep endogeen genomen, hetgeen een belangrijke kwestie is aangezien voor groepen van verschillende groottes verschillende kosten en baten gelden. Daarom is het vanuit het gezichtspunt van de planner essentieel een idee te krijgen van de optimale grootte van de sleutelgroep, dat wil zeggen welke grootte tot de grootste nettowinst leidt.

Hoofdstuk 6 richtte zich op de kwestie van het vinden van sleutelsectoren binnen een economie. Dat wil zeggen, sectoren met een maximale potentie voor het verspreiden van groei-impulsen binnen de economie en op deze manier voor het uitoefenen van invloed op de bruto-opbrengst of een andere factor (zoals toegevoegde waarde, werkgelegenheid of CO_2-uitstoot). Hiertoe werd de hypothetische extractiemethode (HEM) uit de input-output analyse gebruikt, die de bijdrage van elke sector aan de totale bruto-opbrengst (of een andere factor) meet door het oorspronkelijke resultaat te vergelijken met het resultaat dat verkregen is door een sector (of groep sectoren) uit het model weg te laten. De vermindering in bijvoorbeeld de bruto-opbrengst is een gevolg van deze omissie, en reflecteert zo de rol van de hypothetisch geselecteerde sector of groep sectoren. De optimaliseringsproblemen die zich voordoen bij het vinden van een sleutelsector of een sleutelgroep van sectoren op basis van het HEM-perspectief werden expliciet geformuleerd, en analytische oplossingen (de 'factorwaarden' van de bedrijfstakken) werden afgeleid. Aangetoond werd dat de sleutelgroep van $k \geq 2$ sectoren over het algemeen verschilt van de k sectoren met de grootste individuele bijdragen aan de totale bruto-opbrengst (of een andere factor). Dit werd bevestigd door een toepassing voor de Australische economie halverwege de jaren '90, met betrekking tot watergebruik en CO_2-uitstoot. Dit resultaat komt door het feit dat sectoren eigenlijk overbodig kunnen zijn ten opzichte van elkaar als ze vergelijkbaar zijn (zowel qua patroon als qua grootte) in termen van productieverbanden, finale vraag en capaciteit van productiefactoren. Gerelateerde kwesties die op dezelfde manier werden onderzocht zijn het vinden van een sleutelregio en een sleutelgroep van regio's in een interregionaal input-output raamwerk. Verder lieten we zien dat de factorwaardemeting onveranderlijk is indien de intrasectorale transacties worden uitgefilterd (tenzij de totale bruto-opbrengst de gekozen factor is). De uitkomsten van de problemen van de sleutelsector/sleutelgroep in de standaard en de zogenoemde netto input-output modellen zijn dus identiek, zolang de factor niet de totale bruto-opbrengst is. Het verband tussen de HEM-problemen en de zogenaamde *fields of influence* werd belicht, wat leidde tot een alternatieve economische interpretatie van problemen op het gebied van de economiebrede effecten van een verandering in de mate waarin een sector afhankelijk is van zijn eigen product als input. Tenslotte werd bewezen dat een stijging (daling) van een inputcoëfficiënt nooit de factorwaarde van een sector laat dalen (stijgen), en werden de voorwaarden voor een strikte stijging (daling) afgeleid.