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Dijksma, Aad; Wanjala, Gerald

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Generalized Schur Functions and Augmented Schur Parameters

Aad Dijksma and Gerald Wanjala

Abstract. Every Schur function $s(z)$ is the uniform limit of a sequence of finite Blaschke products on compact subsets of the open unit disk. The Blaschke products in the sequence are defined inductively via the Schur parameters of $s(z)$. In this note we prove a similar result for generalized Schur functions.

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1. Introduction

A Schur function is a holomorphic function defined on the open unit disk $\mathbb{D}$, which is bounded by 1 there. We denote the class of such functions by $\mathbf{S}$. If $s(z) \in \mathbf{S}$ is not identically equal to a unimodular constant, then by Schwarz' Lemma (see, for example, [9, Theorem 6.1]) the function

$$\widehat{s}(z) = \frac{1}{z}\,\frac{s(z) - s(0)}{1 - s(0)^* s(z)}$$

is again in the class $\mathbf{S}$. The map $s(z) \mapsto \widehat{s}(z)$ is called the Schur transformation on $\mathbf{S}$ and $\widehat{s}(z)$ is called the Schur transform of $s(z)$. To a Schur function $s(z)$ which is not equal to a unimodular constant we can associate a sequence of Schur functions $(s_j(z))_{j\geq 0}$ by repeatedly applying the Schur transformation:

$$s_0(z) := s(z),\ s_1(z) = \widehat{s}_0(z),\ \ldots,\ s_j(z) = \widehat{s}_{j-1}(z), \ldots.$$

This repeated application of the Schur transformation is called the Schur algorithm. The sequence $(s_j(z))_{j\geq 0}$ is finite and terminates at the nth step of the algorithm with $s_n(z)$ if and only if $|s_n(0)| = 1$. For then, by the maximum modulus principle, $s_n(z) \equiv s_n(0)$ and the Schur transformation is not defined for $s_n(z)$.

This occurs if and only if $s(z)$ is a Blaschke product of order n, that is, of the form

$$s(z) = c\prod_{j=0}^{n} \frac{z-\alpha_j}{1-\alpha_j^* z}, \quad \alpha_j \in \mathbb{D},\ c \in \mathbb{T},$$

where $\mathbb{T}$ stands for the unit circle. The numbers $\gamma_j = s_j(0)$, $j = 0, 1, \ldots$, are called the Schur parameters associated with $s(z)$. If the sequence $(\gamma_j)_{j\geq 0}$ is infinite then $|\gamma_j| < 1$ for all $j = 0, 1, \ldots$; if it stops with γ_n then $|\gamma_j| < 1$ for $j = 0, 1, \ldots, n-1$ and $|\gamma_n| = 1$. A sequence of complex numbers with these properties will be called a Schur sequence.

The sequence of Schur parameters determines the function. To see this, let m be an integer ≥ 0 and define the rational functions

$$B_{m,0}(s;z) = \begin{cases} \dfrac{z+\gamma_m}{1+\gamma_m^* z} & \text{if } |\gamma_m| < 1, \\ \gamma_m & \text{if } |\gamma_m| = 1, \end{cases}$$

and

$$B_{m,j}(s;z) = \frac{zB_{m,j-1}(s;z) + \gamma_{m-j}}{1+\gamma_{m-j}^* zB_{m,j-1}(s;z)}, \quad j = 1, 2, \ldots, m.$$

Hence $B_{m,j-1}(s;z) = \widehat{B}_{m,j}(s;z)$. If $|\gamma_m| < 1$, then $B_{m,j}(s;z)$ is a Blaschke product of order $j+1$ for $j = 0, 1, \ldots, m$; if $|\gamma_m| = 1$, then $B_{m,j}(s;z)$ is a Blaschke product of order j for $j = 0, 1, \ldots, m$. Moreover, the sequence of Schur parameters associated with $B_{m,m}(s;z)$ is finite and given by $\gamma_0, \gamma_1, \ldots, \gamma_m, 1$ if $|\gamma_m| < 1$ and $\gamma_0, \gamma_1, \ldots, \gamma_m$ if $|\gamma_m| = 1$. Thus the first $m+1$ Schur parameters of $s(z)$ coincide with the first $m+1$ Schur parameters of $B_{m,m}(s;z)$. I. Schur showed that this implies that the difference $s(z) - B_{m,m}(s;z)$ has a zero at $z = 0$ of order $\geq m+1$. This can also be seen by proving by induction that for $j = 0, 1, \ldots, m$, the difference $s_{m-j}(z) - B_{m,j}(s;z)$ has a zero at $z = 0$ of order at least $j+1$; see [14, Theorem I.2.1]. Since $|s(z)| \leq 1$ and $|B_{m,m}(s;z)| \leq 1$ on $\mathbb{D}$, Schwarz' Lemma implies that

$$|s(z) - B_{m,m}(s;z)| \leq 2|z|^{m+1}, \quad z \in \mathbb{D}, \tag{1.1}$$

and hence Carathéodory's theorem holds: If the sequence of Schur parameters breaks up at γ_n with $|\gamma_n| = 1$ then $s(z) = B_{n,n}(s;z)$; otherwise

$$s(z) = \lim_{m\to\infty} B_{m,m}(s;z), \tag{1.2}$$

where the limit is uniform in z on compact subsets of $\mathbb{D}$.

If we introduce the Möbius transform

$$\tau_n(w) = \frac{zw+\gamma_n}{1+\gamma_n^* zw} = \gamma_n + \frac{(1-|\gamma_n|^2)z}{\gamma_n^* z + 1/w},$$

then the composition formulas

$$\begin{aligned} s(z) &= \tau_0 \circ \tau_1 \circ \cdots \circ \tau_n(s_{n+1}(z)), \\ B_{m,m}(s;z) &= \begin{cases} \tau_0 \circ \tau_1 \circ \cdots \circ \tau_m(1) & \text{if } |\gamma_m| < 1, \\ \tau_0 \circ \tau_1 \circ \cdots \circ \tau_{m-1}(\gamma_m) & \text{if } |\gamma_m| = 1, \end{cases} \end{aligned} \tag{1.3}$$

hold and they show the close relation between the Schur algorithm, the Schur parameters, and continued fractions. In this note we do not pursue this connection but refer to the recent paper [16]. We only recall that if $(\gamma_j)_{j\geq 0}$ is a Schur sequence of complex numbers, then $(B_m(z))_{m\geq 0}$ with $B_m(z)$ as in (1.3) is a Cauchy sequence of finite Blaschke products, which converges to a function $s(z)\in\mathbf{S}$ whose sequence of Schur parameters coincides with $(\gamma_j)_{j\geq 0}$. This follows from the arguments leading up to (1.1) which also imply that

$$|B_m(z)-B_n(z)|\leq 2|z|^{\min\{m,n\}+1},\quad z\in\mathbb{D}.$$

An excellent account of Schur's work on analysis, including the Schur algorithm, can be found in [13]; we refer to this paper for the complete list of works of I. Schur in this area.

If $s(z)$ is a generalized Schur function which is holomorphic in $z=0$ (see Section 2 below), then the Krein–Langer factorization $s(z)=B(z)^{-1}s_0(z)$, where $s_0(z)$ is a Schur function and $B(z)$ is a finite Blaschke product with $B(0)\neq 0$, implies

$$s(z)=\lim_{m\to\infty}B(z)^{-1}B_{m,m}(s_0;z), \tag{1.4}$$

where the convergence is uniform in z on compact subsets of $\mathbb{D}\setminus\{$ poles of $s(z)\}$. In fact, because of (1.1), for every compact set $K\in\mathbb{D}\setminus\{$ poles of $s(z)\}$, there is a real number $M\geq 0$ such that

$$|s(z)-B(z)^{-1}B_{m,m}(s_0;z)|\leq M|z|^{m+1},\quad z\in K. \tag{1.5}$$

However, the functions on the right-hand side of (1.4) are not related to the generalized Schur algorithm for $s(z)$ nor to any form of continued fractions similar to what we described in the foregoing paragraphs. In this note we prove a result for generalized Schur functions which is analogous to (1.2) and is related to the generalized Schur algorithm. For that we introduce the sequence of augmented Schur parameters (see Section 3), which plays the same role for generalized Schur functions as the sequence of Schur parameters does for Schur functions. In Section 2 we present the preliminaries: generalized Schur functions, the generalized Schur transformation, and related notions and results needed in the sequel. Section 3 contains the two theorems of this note.

2. Generalized Schur functions and the generalized Schur transformation

For any integer $\kappa\geq 0$, by $\mathbf{S}_\kappa$ we denote the set of complex-valued functions $s(z)$ which are meromorphic on $\mathbb{D}$ and have the following equivalent properties:

1. $s(z)$ has κ poles (counted according to their multiplicities) and

$$\limsup_{r\uparrow 1}|s(re^{it})|\leq 1,\ \text{for almost all } t\in[0,2\pi).$$

2. The kernel

$$K_s(z,w) = \frac{1 - s(z)s(w)^*}{1 - zw^*}, \quad z, w \in \Omega(s),$$

has κ negative squares, where $\Omega(s)$ is the domain of holomorphy of $s(z)$.

3. The 2×2 matrix kernel

$$D_s(z,w) = \begin{pmatrix} \dfrac{1 - s(z)s(w)^*}{1 - zw^*} & \dfrac{s(z) - s(w^*)}{z - w^*} \\ \dfrac{\widetilde{s}(z) - \widetilde{s}(w^*)}{z - w^*} & \dfrac{1 - \widetilde{s}(z)\widetilde{s}(w)^*}{1 - zw^*} \end{pmatrix}$$

has κ negative squares on $\Omega(s) \cap \Omega(\widetilde{s})$, where $\widetilde{s}(z) = s(z^*)^*$.

4. The function $s(z)$ admits the Krein–Langer factorization

$$s(z) = B(z)^{-1} s_0(z), \quad B(z) = \prod_{j=1}^{\kappa} \frac{z - \alpha_j}{1 - \alpha_j^* z},$$

where $s_0(z) \in \mathbf{S}$, $\alpha_j \in \mathbb{D}$ and $s_0(\alpha_j) \neq 0$, $j = 1, 2, \ldots, \kappa$.

Evidently, $\mathbf{S}_0 = \mathbf{S}$. The functions of the class $\mathbf{S}_\kappa$ are called generalized Schur functions with κ negative squares. They were introduced and studied in [15]. For the equivalence of these properties, see [15], [7], and [8, Section 3.4]. By $\mathbf{S}_\kappa^0$ we denote the set of functions $s(z) \in \mathbf{S}_\kappa$ which are holomorphic at $z = 0$ and we set $\mathbf{S}^0 = \cup_{\kappa \geq 0} \mathbf{S}_\kappa^0$. Consider a function $s(z) \in \mathbf{S}^0$ which is not identically equal to a unimodular constant and assume it has the Taylor expansion

$$s(z) = \sigma_0 + \sigma_1 z + \sigma_2 z^2 + \cdots + \sigma_k z^k + \sigma_{k+1} z^{k+1} + \cdots .$$

Then the generalized Schur transform $\widehat{s}(z)$ of $s(z)$ is defined as follows.

(1) If $|\sigma_0| < 1$, then

$$\widehat{s}(z) = \frac{1}{z} \frac{s(z) - \sigma_0}{1 - \sigma_0^* s(z)}.$$

This formula coincides with the "classical" formula in the Introduction.

(2) If $|\sigma_0| > 1$ then the case $s(z) \equiv \sigma_0$ does not arise since this implies that $s(z) \notin \mathbf{S}^0$. This means there exists an integer $k \geq 1$ such that $\sigma_1 = \sigma_2 = \cdots = \sigma_{k-1} = 0$ and $\sigma_k \neq 0$. In this case,

$$\widehat{s}(z) = z^k \frac{1 - \sigma_0^* s(z)}{s(z) - \sigma_0}.$$

Note: k is the order of the pole of the quotient on the right-hand side.

(3) If $|\sigma_0| = 1$ then there exists an integer $k \geq 1$, such that $\sigma_1 = \sigma_2 = \cdots = \sigma_{k-1} = 0$ and $\sigma_k \neq 0$ since we assume that $s(z)$ is not a unimodular constant. With this k, determine complex numbers c_j, $j = 0, 1, \ldots, k-1$, such that

$$(s(z) - \sigma_0)(c_0 + c_1 z + \cdots + c_j z^j + \cdots) = \sigma_0 z^k$$

(so that $c_0 \neq 0$), define the polynomial $p(z) = c_0 + c_1 z + \cdots + c_{k-1} z^{k-1}$, and, finally, set $Q(z) = p(z) - z^{2k} p(1/z^*)^*$. In this case,

$$\widehat{s}(z) = z^q \frac{(Q(z) - z^k)s(z) - \sigma_0 Q(z)}{\sigma_0^* Q(z) s(z) - (Q(z) + z^k)},$$

where $q \geq 0$ is the order of the pole of the quotient on the right-hand side. Note: (i) q is finite because $\sigma_0^* Q(z) s(z) - (Q(z) + z^k) \not\equiv 0$ (see [1, page 5]). (ii) For some complex number t_{2k}, we have

$$\begin{aligned} \sigma_0^* Q(z) s(z) - (Q(z) + z^k) &= t_{2k} z^{2k} + \cdots, \\ (Q(z) - z^k) s(z) - \sigma_0 Q(z) &= (\sigma_0 t_{2k} - \sigma_k) z^{2k} + \cdots . \end{aligned}$$

(see [8, Lemma 3.3.1 and its proof]), and hence if $q = 0$, then $t_{2k} \neq 0$ and so $\widehat{s}(0) \neq \sigma_0$.

In all these cases $\widehat{s}(z)$ belongs to the class $\mathbf{S}^0$; see [8, Lemma 3.4.4] and also [1, Theorem 3.1]. In fact, the following result holds.

Lemma 2.1. *If $s(z) \in \mathbf{S}^0_\kappa$, then $\widehat{s}(z) \in \mathbf{S}^0_{\widehat{\kappa}}$, where $\widehat{\kappa} = \kappa$, $\kappa - k$, and $\kappa - k - q$ in cases* (1), (2), *and* (3), *respectively.*

For a proof we refer to [1, Theorems 5.1, 6.1, and 8.1]. The definition of the generalized Schur transformation goes back to [10], [12], [11], and [8, Definition 3.3.1]. In [8] it is applied to solve the problem: When is a formal power series around $z = 0$ the Taylor expansion of a generalized Schur function. In [1, 2, 4, 6, 17] it is studied for its effect on the coisometric and unitary operator realizations of a generalized Schur function, including those whose state spaces are the reproducing kernel Pontryagin spaces with kernels $K_s(z, w)$ and $D_s(z, w)$; in [3] it is shown to provide an algorithm for the unique factorization of a 2×2 matrix polynomial which is J-unitary on $\mathbb{T}$ (for the definition, see below) in normalized elementary factors; and, finally, in [5] (see also [12]) it is used in solving a basic interpolation problem for generalized Schur functions.

The inverse of the generalized Schur transformation in each of these three cases can be written as

$$s(z) = \frac{a(z)\widehat{s}(z) + b(z)}{c(z)\widehat{s}(z) + d(z)},$$

where the coefficient matrix

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

can be chosen as

$$\Theta_1(z) = \frac{1}{\sqrt{1 - |\sigma_0|^2}} \begin{pmatrix} 1 & \sigma_0 \\ \sigma_0^* & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \text{ if } |\sigma_0| < 1, \tag{2.1}$$

$$\Theta_2(z) = \frac{1}{\sqrt{|\sigma_0|^2 - 1}} \begin{pmatrix} \sigma_0 & 1 \\ 1 & \sigma_0^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^k \end{pmatrix}, \text{ if } |\sigma_0| > 1, \tag{2.2}$$

$$\Theta_3(z) = \begin{pmatrix} Q(z) + z^k & -\sigma_0 z^q Q(z) \\ \sigma_0^* Q(z) & -z^q(Q(z) - z^k) \end{pmatrix}, \text{ if } |\sigma_0| = 1. \tag{2.3}$$

These 2×2 matrix polynomials are J-unitary on $\mathbb{T}$, that is, satisfy $\Theta(z)^* J \Theta(z) = J$, $|z| = 1$, where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. The augmented Schur parameters

To $s(z) \in \mathbf{S^0}$ which is not identically equal to a unimodular constant we can apply the generalized Schur algorithm:

$$s_0(z) := s(z),\ s_1(z) = \widehat{s}_0(z), \ldots,\ s_j(z) = \widehat{s}_{j-1}(z), \ldots,$$

where now $\widehat{s}_{j-1}(z)$ denotes the generalized Schur transform of $s_{j-1}(z)$ for $j = 1, 2, \ldots$. We set $\gamma_j = s_j(0)$, $j = 0, 1, \ldots$. The sequence of functions $(s_j(z))_{j\geq 0}$ terminates at $s_n(z)$ if $s_n(z) \equiv \gamma_n$ with $|\gamma_n| = 1$, because in that case the generalized Schur transform of $s_n(z)$ is not defined. The number γ_j if $|\gamma_j| < 1$, the pair (γ_j, k_j) if $|\gamma_j| > 1$, and the quadruple $(\gamma_j, k_j, q_j, Q_j(z))$ if $|\gamma_j| = 1$, which are defined in accordance with the definitions of the generalized Schur transformation (see Section 2), will be called the augmented Schur parameter and briefly denoted by $\widehat{\gamma}_j$. The sequence $(\widehat{\gamma}_j)_{j\geq 0}$ will be called the sequence of augmented Schur parameters. It is finite and stops at $\widehat{\gamma}_n$, when the sequence $(s_j(z))_{j\geq 0}$ terminates at $s_n(z)$; in this case, $\widehat{\gamma}_n$ carries no further information, that is, $\widehat{\gamma}_n = \gamma_n$ and $|\gamma_n| = 1$. From the definition of the generalized Schur transformation we see that for $j = 0, 1, \ldots$ (and up to $n - 1$ if the sequence $(\widehat{\gamma}_j)_{j\geq 0}$ stops at $\widehat{\gamma}_n$ with $n \geq 1$) the following implications hold:

$$\left\{ \begin{array}{lcl} |\gamma_j| > 1 & \Longrightarrow & \gamma_{j+1} \neq 0, \\ |\gamma_j| = 1, q_j > 0 & \Longrightarrow & \gamma_{j+1} \neq 0, \\ |\gamma_j| = 1, q_j = 0 & \Longrightarrow & \gamma_{j+1} \neq \gamma_j. \end{array} \right. \tag{3.1}$$

Moreover, by [8, Lemma 3.4.5], see also [1, Corollary 9], there is an integer $j_0 \geq 0$ such that $s_j(z) \in \mathbf{S}$ for all $j \geq j_0$ and hence $\widehat{\gamma}_j = \gamma_j$ with $|\gamma_j| \leq 1$, $j \geq j_0$.

With the sequence $(\widehat{\gamma}_j)_{j\geq 0}$ we define for $m \geq 0$,

$$B_{m,0}(s; z) = \gamma_m \quad \text{if } |\gamma_m| = 1 \text{ and } (\widehat{\gamma}_j)_{j\geq 0} \text{ stops with } \widehat{\gamma}_m, \tag{3.2}$$

otherwise,

$$B_{m,0}(s; z) = \begin{cases} \dfrac{z + \gamma_m}{1 + \gamma_m^* z} & \text{if } |\gamma_m| < 1, \\[2ex] \dfrac{z^{k_m} + \gamma_m}{\gamma_m^* z^{k_m} + 1} & \text{if } |\gamma_m| > 1, \\[2ex] \dfrac{(Q_m(z) + z^{k_m}) - \gamma_m z^{q_m} Q_m(z)}{\gamma_m^* Q_m(z) - z^{q_m}(Q_m(z) - z^{k_m})} & \text{if } |\gamma_m| = 1, \end{cases}$$

and for $j = 1, 2, \ldots, m$,

$$B_{m,j}(s;z) = \begin{cases} \dfrac{\gamma_{m-j} + zB_{m,j-1}(s;z)}{1 + \gamma^*_{m-j} zB_{m,j-1}(s;z)} & \text{if } |\gamma_{m-j}| < 1, \\ \dfrac{z^{k_{m-j}} + \gamma_{m-j} B_{m,j-1}(s;z)}{\gamma^*_{m-j} z^{k_{m-j}} + B_{m,j-1}(s;z)} & \text{if } |\gamma_{m-j}| > 1, \\ \dfrac{(Q_{m-j}(z) + z^{k_{m-j}})B_{m,j-1}(s;z) - \gamma_{m-j} z^{q_{m-j}} Q_{m-j}(z)}{\gamma^*_{m-j} Q_{m-j}(z) B_{m,j-1}(s;z) - z^{q_{m-j}}(Q_{m-j}(z) - z^{k_{m-j}})} & \text{if } |\gamma_{m-j}| = 1. \end{cases}$$

Each $B_{m,j}(s;z)$ is of the form $B_1(z)^{-1}B_2(z)$ where $B_1(z)$ and $B_2(z)$ are finite Blaschke products with $B_1(0) \neq 0$ that is, a rational generalized Schur function holomorphic at $z = 0$ and having unimodular values on $\mathbb{T}$ (in particular, it has no poles in $\mathbb{T}$). Clearly, $B_{m,j-1}(s;z) = \widehat{B}_{m,j}(s;z)$, $j = 1, 2, \ldots, m$. Similarly as above, these formulas can be expressed in terms of Möbius transformations and hence are related to continued fractions. The sequence of augmented Schur parameters for $B_{m,m}(s;z)$ is $\widehat{\gamma}_0, \widehat{\gamma}_1, \ldots, \widehat{\gamma}_m$ in case $B_{m,0}(s;z)$ is given by (3.2), otherwise it is $\widehat{\gamma}_0, \widehat{\gamma}_1, \ldots, \widehat{\gamma}_m, 1$.

Let $j_0 \geq 0$ be an integer such that $s_j(z) \in \mathbf{S}$ for all $j \geq j_0$. From the definition of the generalized Schur transformation in Subsection 2 we have

$$s(z) = \frac{\theta_{11}(z)s_{j_0}(z) + \theta_{12}(z)}{\theta_{21}(z)s_{j_0}(z) + \theta_{22}(z)}, \tag{3.3}$$

where the coefficient matrix can be written as the product

$$\Theta_s(z) := \begin{pmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix} = \Theta_{(0)}(z)\Theta_{(1)}(z)\cdots\Theta_{(j_0-1)}(z),$$

in which each factor $\Theta_{(i)}(z)$ is of one of the forms $\Theta_1(z)$, $\Theta_2(z)$, and $\Theta_3(z)$ given by (2.1), (2.2), and (2.3), respectively.

Thus $\Theta_s(z)$ is a 2×2 matrix polynomial which is J-unitary on $\mathbb{T}$. It follows from (2.1), (2.2), and (2.3) (see also [8, Lemma 3.4.2 v)]) that for some integer $\ell_0 \geq j_0$,

$$\det \Theta_s(z) = z^{\ell_0}. \tag{3.4}$$

It is easy to see that

$$\rho(s) := \ell_0 - j_0 = \sum_{j:\,\widehat{\gamma}_j \text{ is a pair}} (k_j - 1) \; + \sum_{j:\,\widehat{\gamma}_j \text{ is a quadruple}} (2k_j + q_j - 1).$$

In particular, if j_1 and ℓ_1 are defined in the same way as j_0 and ℓ_0, then $\ell_1 - j_1 = \ell_0 - j_0$.

Theorem 3.1. *Let $s(z) \in \mathbf{S}^0$ and let $\rho(s)$ be as defined above. Then for each compact subset $K \subset \mathbb{D} \setminus \{\text{poles of } s(z)\}$, there exist a real number $M > 0$ and an integer $m_0 \geq j_0$ such that for all $z \in K$ and all $m \geq m_0$,*

$$|s(z) - B_{m,m}(s;z)| \leq M|z|^{\rho(s)+m+1}.$$

The estimate in the theorem is an improvement of (1.5) by a factor $|z|^{\rho(s)}$, but only holds for sufficiently large m. The theorem implies that

$$s(z) = \lim_{m\to\infty} B_{m,m}(s;z)$$

uniformly in z on compact subsets of $\mathbb{D} \setminus \{\text{poles of } s(z)\}$.

Proof. If the sequence $(\widehat{\gamma}_j)_{j\geq 0}$ of augmented Schur parameters corresponding to $s(z)$ is finite and terminates with $|\gamma_n| = 1$, then $s(z) = B_{n,n}(s;z)$ and the theorem holds true. We now assume that the sequence $(\widehat{\gamma}_j)_{j\geq 0}$ is infinite. From (3.3),

$$B_{j_0+m,j_0+m}(s;z) = \frac{\theta_{11}(z)B_{j_0+m,m}(s;z) + \theta_{12}(z)}{\theta_{21}(z)B_{j_0+m,m}(s;z) + \theta_{22}(z)},$$

and

$$B_{j_0+m,m}(s;z) = B_{m,m}(s_{j_0};z)$$

we obtain

$$s(z) - B_{j_0+m,j_0+m}(s;z) = \frac{\det\Theta_s(z)(s_{j_0}(z) - B_{m,m}(s_{j_0};z))}{(\theta_{21}(z)s_{j_0}(z) + \theta_{22}(z))(\theta_{21}(z)B_{m,m}(s_{j_0};z) + \theta_{22}(z))}. \tag{3.5}$$

By (1.1) and (3.4), the numerator of the quotient on the right-hand side satisfies the inequality

$$|\det\Theta_s(z)(s_{j_0}(z) - B_{m,m}(s_{j_0};z))| \leq 2|z|^{\ell_0+m+1}, \quad z \in \mathbb{D}. \tag{3.6}$$

We claim that the factor $\theta_{21}(z)s_{j_0}(z) + \theta_{22}(z)$ in the denominator does not vanish in $\mathbb{D} \setminus \{\text{poles of } s(z)\}$. To see this, assume that for some $z_0 \in \mathbb{D} \setminus \{\text{poles of } s(z)\}$ we do have that

$$\theta_{21}(z_0)s_{j_0}(z_0) + \theta_{22}(z_0) = 0. \tag{3.7}$$

Then, by (3.3) and since $z = z_0$ is not a pole of $s(z)$, we also have

$$\theta_{11}(z_0)s_{j_0}(z_0) + \theta_{12}(z_0) = 0.$$

The last two equations can be written in matrix form:

$$\Theta_s(z_0)\begin{pmatrix} s_j(z_0) \\ 1 \end{pmatrix} = 0.$$

This implies $\det\Theta_s(z_0) = 0$ and so, on account of (3.4), $z_0 = 0$. However, from [8, Lemma 3.4.2 iii) and v)] (or from [3, Theorem 6.6]) it follows that there are complex numbers $k_0 \neq 0$, k_1, …, such that

$$\theta_{21}(z)s_{j_0}(z) + \theta_{22}(z) = \frac{\det\Theta_s(z)}{\theta_{11}(z) - \theta_{21}(z)s(z)} = k_0 + k_1 z + \cdots .$$

This contradicts (3.7) with $z_0 = 0$ and proves the claim. Let K be a compact subset of $\mathbb{D} \setminus \{\text{poles of } s(z)\}$ and let

$$\varepsilon = \min\nolimits_{z\in K}|\theta_{21}(z)s_{j_0}(z) + \theta_{22}(z)|. \tag{3.8}$$

Because of the claim just proved, $\varepsilon > 0$. Applying (1.1), we find that for all $z \in \mathbb{D}$,

$$|(\theta_{21}(z)s_{j_0}(z) + \theta_{22}(z)) - (\theta_{21}(z)B_{m,m}(s_{j_0};z) + \theta_{22}(z))| \leq 2|z|^{m+1}\max\nolimits_{z\in\mathbb{D}}|\theta_{21}(z)|$$

and hence for some integer $m_1 \geq 0$ we have that for all $m \geq m_1$,

$$|\theta_{21}(z)B_{m,m}(s_{j_0};z) + \theta_{22}(z)| \geq \frac{1}{2}\varepsilon, \quad z \in K. \tag{3.9}$$

Combining (3.5), (3.6), (3.8), and (3.9), we see that for $m \geq m_1$,

$$|s(z) - B_{j_0+m,j_0+m}(s;z)| \leq \frac{4}{\varepsilon^2}\,|z|^{\ell_0+m+1}, \quad z \in K.$$

This readily implies the theorem with $m_0 = m_1 + j_0$ and $M = 4/\varepsilon^2$. □

A sequence $(\widehat{\gamma}_j)_{j\geq 0}$ will be called an augmented Schur sequence if:

(a) except for at most finitely many values of j, $\widehat{\gamma}_j$ is a complex number γ_j with $|\gamma_j| < 1$;

(b) in the exceptional cases, $\widehat{\gamma}_j$ is either a pair (γ_j, k_j) consisting of a complex number γ_j with $|\gamma_j| > 1$ and an integer $k_j \geq 1$ or a quadruple $(\gamma_j, k_j, q_j, Q_j(z))$ consisting of a unimodular complex number γ_j, integers $k_j \geq 1$ and $q_j \geq 0$, and a polynomial $Q_j(z) = p_j(z) - z^{2k_j}p_j(1/z^*)^*$, where $p_j(z)$ is a polynomial of degree $< k_j$ and $p_j(0) \neq 0$;

(c) in case the sequence is finite and ends with $\widehat{\gamma}_n$, also $\widehat{\gamma}_n$ is exceptional: $\widehat{\gamma}_n = \gamma_n$ with $|\gamma_n| = 1$; and

(d) the implications (3.1) hold.

Theorem 3.2. *Let $(\widehat{\gamma}_j)_{j\geq 0}$ be an augmented Schur sequence. Then there is a unique $s(z) \in \mathbf{S}^0$ such that $(\widehat{\gamma}_j)_{j\geq 0}$ is the corresponding sequence of augmented Schur parameters. The number κ of negative squares of $s(z)$ is given by*

$$\kappa = \sum_{j:\,\widehat{\gamma}_j \text{ is a pair}} k_j \; + \sum_{j:\,\widehat{\gamma}_j \text{ is a quadruple}} k_j + q_j.$$

Proof. Let m be an integer ≥ 0 such that for all $j \geq m$, $\widehat{\gamma}_j$ is a complex number γ_j with $|\gamma_j| < 1$. Let $s_m(z)$ be the Schur function whose sequence of Schur parameters coincides with $(\gamma_{m+j})_{j\geq 0}$. Define $B_{m,0}(z) = s_m(z)$ and define $B_{m,j}(z)$ in the same inductive way as $B_{m,j}(s;z)$ is defined starting with $B_{m,0}(s;z)$, $j = 1, 2, \ldots, m$. Then $s(z) = B_{m,m}(z)$ has the desired properties. The formula for κ follows from Lemma 2.1. □

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Aad Dijksma
Department of Mathematics, University of Groningen
P.O. Box 800, 9700 AV Groningen, The Netherlands
e-mail: dijksma@math.rug.nl

Gerald Wanjala
Department of Mathematics, Mbarara University of Science and Technology
P.O. Box 1410, Mbarara, Uganda
e-mail: wanjalag@yahoo.com