The Schur transform of a generalized Schur function and operator realizations

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Chapter 2

Generalized Schur functions and Pontryagin spaces

2.1 Generalized Schur functions

Let n be an integer > 0. By an $n \times n$ *matrix kernel* we mean an $n \times n$ matrix function $K(z, w)$ defined on $\Omega \times \Omega$ for some open set $\Omega = \Omega(K) \subset \mathbb{C}$. We say that the kernel $K(z, w)$ is *Hermitian* if

$$K(z, w)^* = K(w, z), \qquad z, w \in \Omega.$$

Let $K(z, w)$ be a Hermitian kernel and let κ be an integer ≥ 0. We say that $K(z, w)$ has κ *negative squares*, and write $\mathrm{sq}_- K(z, w) = \kappa$, if for any integer $\ell > 0$, points $w_1, w_2, \ldots, w_\ell \in \Omega$, and vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_\ell \in \mathbb{C}^n$, the Hermitian matrix

$$(2.1.1) \qquad (\langle K(w_i, w_j)\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j\rangle_{\mathbb{C}^n})_{i,j=1}^{\ell}$$

has at most κ and at least one matrix of this form has exactly κ negative eigenvalues, counting multiplicities. If $\mathrm{sq}_- K(z, w) = 0$ then all matrices of the form (2.1.1) are nonnegative and in this case we say that the kernel $K(z, w)$ is nonnegative. That $K(z, w)$ has κ *positive squares* is defined similarly.

Let $S(z)$ be a meromorphic $n \times n$ matrix function defined on the open unit disk $\mathbb{D}$ in the complex plane and denote by $\mathrm{hol}\,(S)$ the set of points $z \in \mathbb{D}$ at which $S(z)$ is holomorphic. By $K_S(z, w)$, $K_{S^\#}(z, w)$, and $D_S(z, w)$ we shall mean the kernels

$$(2.1.2) \qquad K_S(z, w) := \frac{J - S(z)JS(w)^*}{1 - zw^*} : \mathbb{C}^n \to \mathbb{C}^n,$$

$$(2.1.3) \qquad K_{S^\#}(z, w) := \frac{J - S^\#(z)JS^\#(w)^*}{1 - zw^*} : \mathbb{C}^n \to \mathbb{C}^n,$$

and
(2.1.4)

$$D_S(z,w) = \begin{pmatrix} \dfrac{J - S(z)JS(w)^*}{1-zw^*} & \dfrac{S(z)-S(w^*)}{z-w^*} \\ \dfrac{S^\#(z)-S^\#(w^*)}{z-w^*} & \dfrac{J - S^\#(z)JS^\#(w)^*}{1-zw^*} \end{pmatrix} : \begin{pmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{pmatrix} \to \begin{pmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{pmatrix},$$

where $S^\#(z) = S(z^*)^*$, $z^* \in \mathrm{hol}\,(S)$ and J is an $n \times n$ signature matrix, that is, $J = J^* = J^{-1}$. Note that $\Omega(K_S) = \mathrm{hol}\,(S)$, $\Omega(K_{S\#}) = \{z \in \mathbb{C} \mid z^* \in \mathrm{hol}\,(S)\}$, and $\Omega(D_S) = \Omega(K_S) \cap \Omega(K_{S\#})$. These kernels play a key role in the sequel.

Theorem 2.1.1 *If one of the kernels $K_S(z,w)$, $K_{S\#}(z,w)$, or $D_S(z,w)$ has κ negative squares then so do the other two.*

In this case we write $\mathrm{sq}_-(S)$ instead of $\mathrm{sq}_- K_S(z,w)$, $\mathrm{sq}_- K_{S\#}(z,w)$, and $\mathrm{sq}_- D_S(z,w)$. We shall prove this theorem in Subsection 2.3.2. Assuming it holds we now present the definitions of Schur and generalized Schur functions. First we consider the case $J = I$, the identity matrix. In this case the functions $S(z)$ for which the kernels $K_S(z,w)$, $K_{S\#}(z,w)$, and $D_S(z,w)$ are nonnegative are called *Schur functions*, and we denote the class of all such functions by $\mathbf{S}(\mathbb{C}^n)$. It is well known that $S(z) \in \mathbf{S}(\mathbb{C}^n)$ if and only if $S(z)$ is holomorphic on $\mathbb{D}$ and for each $z \in \mathbb{D}$, $||S(z)|| \leq 1$. In the scalar case, that is, the case where $n = 1$, we write $\mathbf{S}$ instead of $\mathbf{S}(\mathbb{C})$. Hence $s(z) \in \mathbf{S}$ if and only if $s(z)$ is holomorphic on $\mathbb{D}$ and bounded by 1 there. Still for the case $J = I$, the functions $S(z)$ for which the kernels $K_S(z,w)$, $K_{S\#}(z,w)$, and $D_S(z,w)$ have κ negative squares are called *generalized Schur functions with κ negative squares*. We denote the class of all such functions by $\mathbf{S}_\kappa(\mathbb{C}^n)$. If $S(z) \in \mathbf{S}_\kappa(\mathbb{C}^n)$ then $S(z)$ has κ poles in $\mathbb{D}$. In the sequel we only consider scalar generalized Schur functions. The class of such functions with κ negative squares will be denoted by $\mathbf{S}_\kappa$ instead of $\mathbf{S}_\kappa(\mathbb{C})$. It is well know that $s(z) \in \mathbf{S}_\kappa$ if and only if it has one of the following equivalent properties (see, for example, [10] and [30]):

1. $s(z)$ is meromorphic on $\mathbb{D}$, has κ poles (counting order), and

$$\limsup_{r\uparrow 1} |s(re^{it})| \leq 1 \text{ for almost all } t \in [0, 2\pi].$$

2. $s(z)$ admits the representation

$$s(z) = B(z)^{-1} s_0(z), \quad B(z) = \prod_{j=1}^{\kappa} \frac{z - z_j}{1 - zz_j^*}, \tag{2.1.5}$$

where $z_j \in \mathbb{D}$ and $s_0(z)$ is a Schur function with $s_0(z_j) \neq 0$, $j = 1, 2, \ldots, \kappa$.

The function $B(z)$ is called a Blaschke product of order κ. Generalized Schur functions were first considered and studied by H. Langer and M.G. Krein (see [30]). The representation (2.1.5) is called the *Krein-Langer factorization*. A similar representation holds in the matrix case but we shall not use this in the sequel. Nonnegative kernels of the form (2.1.2), (2.1.3), and (2.1.4) with an arbitrary signature matrix J have been studied by V.P. Potapov (see [32]). Kernels of the form (2.1.2) and (2.1.3) with κ negative squares appear in [11] and [12]. In the sequel we denote by $\mathbf{S}_\kappa(\mathbb{C}^n, J)$ the class of meromorphic functions $S(z)$ on $\mathbb{D}$ for which these kernels have κ negative squares. We call a function from this class a *J-generalized Schur function with κ negative squares.* We are mainly interested in the case where $n = 2$ and

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1.6}$$

In this special case we shall write $\Theta(z)$ instead of $S(z)$.

2.2 Pontryagin spaces

By an *inner product space* $(\mathcal{H}, \langle\, \cdot\,,\,\cdot\,\rangle_{\mathcal{H}})$ we shall mean a complex linear space $\mathcal{H}$ with an inner product $\langle\, \cdot\,,\,\cdot\,\rangle_{\mathcal{H}}$ defined on it. By an *inner product* here we mean a complex valued function $\langle\, \cdot\,,\,\cdot\,\rangle_{\mathcal{H}}$ on $\mathcal{H} \times \mathcal{H}$ which satisfies the following axioms:

(i) $\langle ax + by, z\rangle_{\mathcal{H}} \;=\; a\langle x, z\rangle_{\mathcal{H}} + b\langle y, z\rangle_{\mathcal{H}}, \quad x, y, z \in \mathcal{H}$ and $a, b \in \mathbb{C}$.

(ii) $\langle x, y\rangle_{\mathcal{H}} \;=\; \langle y, x\rangle_{\mathcal{H}}^*$.

In Section 1.3 we encountered the positive definite inner product on $\mathbb{C}^n$. According to the above definition an inner product need not be positive definite. If $(\mathcal{H}, \langle\, \cdot\,,\cdot\,\rangle_{\mathcal{H}})$ is an inner product space, its *anti-space* is the space $(\mathcal{H}, -\langle\, \cdot\,,\,\cdot\,\rangle_{\mathcal{H}})$, which as a linear space coincides with $\mathcal{H}$ but with the sign of the inner product reversed.

By a *Krein space* we mean an inner product space $(\mathcal{K}, \langle\, \cdot\,,\,\cdot\,\rangle_{\mathcal{K}})$ which can be expressed as an orthogonal direct sum

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-, \tag{2.2.1}$$

in which $(\mathcal{K}_+, \langle\, \cdot\,,\,\cdot\,\rangle_{\mathcal{K}})$ and $(\mathcal{K}_-, -\langle\, \cdot\,,\,\cdot\,\rangle_{\mathcal{K}})$ are Hilbert spaces. The representation (2.2.1) is called a *fundamental decomposition* of the Krein space $\mathcal{K}$. The linear operator $J_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ defined by

$$J_{\mathcal{K}}(k_+ + k_-) = k_+ - k_-, \quad k_\pm \in \mathcal{K}_\pm,$$

is called the *fundamental symmetry* associated with decomposition (2.2.1). In general a decomposition of the form (2.2.1) is not unique. The numbers

$\text{ind}_\pm\mathcal{K} = \dim\mathcal{K}_\pm$, which are either finite or ∞ are called the *positive* and *negative indices* of $\mathcal{K}$ and are independent of the decompositions. By a *regular subspace* of a Krein space $\mathcal{K}$ we mean a closed subspace $\mathcal{M}$ of $\mathcal{K}$ which is a Krein space in the inner product of $\mathcal{K}$. In this thesis, a *Pontryagin space* $\mathcal{P}$ is a Krein space with $\text{ind}_-\mathcal{P} < \infty$.

A fundamental decomposition (2.2.1) induces a topology on the Krein space $\mathcal{K}$. First, one forms the *associated Hilbert space* $|\mathcal{K}| = \mathcal{K}_+ \oplus |\mathcal{K}_-|$ by replacing $\mathcal{K}_-$ by its anti-space $|\mathcal{K}_-|$, which is a Hilbert space. The Hilbert space $|\mathcal{K}|$ has an *associated norm* $||\cdot||$ and it can be shown that two norms arising from different fundamental decompositions are equivalent and therefore define the same norm topology. The notions of continuity and convergence in Krein spaces are understood to be with respect to this norm topology.

By $\mathrm{B}(\mathcal{K},\mathcal{H})$ we denote the set of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$. If $A \in \mathrm{B}(\mathcal{K},\mathcal{H})$, its *adjoint* is the unique operator $A^* \in \mathrm{B}(\mathcal{H},\mathcal{K})$ such that

$$\langle Ak, h\rangle_{\mathcal{H}} = \langle k, A^*h\rangle_{\mathcal{K}}, \qquad k \in \mathcal{K},\ h \in \mathcal{H}.$$

The existence of the adjoint follows from the Riesz representation theorem (see [28, Chapter 2]). The Krein space and Hilbert space concepts of the adjoint operator are related. If we view $A \in \mathrm{B}(\mathcal{K},\mathcal{H})$ as an element of $\mathrm{B}(|\mathcal{K}|,|\mathcal{H}|)$ and write $A^\times \in \mathrm{B}(|\mathcal{H}|,|\mathcal{K}|)$ for its Hilbert space adjoint, then

$$A^* = J_{\mathcal{H}}A^\times J_{\mathcal{K}}. \tag{2.2.2}$$

Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces. By a *linear relation* $\mathbf{R}$ in $\mathcal{H}\times\mathcal{K}$ we mean a subspace $\mathbf{R}$ of $\mathcal{H}\times\mathcal{K}$. The domain of $\mathbf{R}$ is defined by

$$\text{dom}\,\mathbf{R} = \{h \in \mathcal{H} \mid (h,k) \in \mathbf{R} \text{ for some } k \in \mathcal{K}\}$$

and the range of $\mathbf{R}$ by

$$\text{ran}\,\mathbf{R} = \{k \in \mathcal{K} \mid (h,k) \in \mathbf{R} \text{ for some } h \in \mathcal{H}\}.$$

If

$$\langle k,k\rangle_{\mathcal{K}} \le \langle h,h\rangle_{\mathcal{H}}, \quad (h,k) \in \mathbf{R}, \tag{2.2.3}$$

we say that the relation $\mathbf{R}$ is *contractive.* If equality holds in (2.2.3) we say that $\mathbf{R}$ is *isometric.*

The following theorem is due to Shmul'yan [35]; for a proof see [10].

Theorem 2.2.1 *Let $\mathbf{R}$ be a linear relation on $\mathcal{H}\times\mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are Pontryagin spaces having the same negative index.*

(i) *If $\mathbf{R}$ is a densely defined contraction, then its closure is the graph of a contraction in $\mathrm{B}(\mathcal{H},\mathcal{K})$.*

(ii) *If* $\mathbf{R}$ *is a densely defined isometry, then its closure is the graph of an isometry in* $\mathrm{B}(\mathcal{H},\mathcal{K})$.

(iii) *If* $\mathbf{R}$ *is a densely defined isometry which has dense range, then its closure is the graph of a unitary operator in* $\mathrm{B}(\mathcal{H},\mathcal{K})$.

2.3 Reproducing kernel Pontryagin spaces

2.3.1 General Facts

Let $\mathcal{P}$ be a Pontryagin space whose elements are $n-$vector functions defined on some open subset Ω of $\mathbb{C}$. The space $\mathcal{P}$ is said to be a *reproducing kernel Pontryagin space* if there exists an $n \times n$ matrix kernel $K(z,w)$ on $\Omega \times \Omega$ such that for every fixed choice of $w \in \Omega$ and $\boldsymbol{\xi} \in \mathbb{C}^n$,

(1) the function $z \mapsto K(z,w)\boldsymbol{\xi}$ belongs to $\mathcal{P}$ and

(2) $\langle f, K(\,\cdot\,,w)\boldsymbol{\xi}\rangle_{\mathcal{P}} = \langle f(w), \boldsymbol{\xi}\rangle_{\mathbb{C}^n}$, for every $f \in \mathcal{P}$.

We refer to the kernel $K(z,w)$ as the *reproducing kernel* for $\mathcal{P}$. In this case we write $\mathcal{P}(K)$ instead of $\mathcal{P}$. It turns out that the set of functions $K(\,\cdot\,,w)\boldsymbol{\xi}$, $w \in \Omega$ and $\boldsymbol{\xi} \in \mathbb{C}^n$, is total in $\mathcal{P}(K)$, that is,

$$\overline{\mathrm{span}}\ \{\ K(\,\cdot\,,w)\boldsymbol{\xi} \mid w \in \Omega,\ \boldsymbol{\xi} \in \mathbb{C}^n\ \} = \mathcal{P}(K).$$

Indeed, if $f \in \mathcal{P}(K)$ is orthogonal to all such elements, then (2) implies

$$\langle f(w), \boldsymbol{\xi}\rangle_{\mathbb{C}^n} = 0, \quad w \in \Omega,\ \boldsymbol{\xi} \in \mathbb{C}^n,$$

and hence $f = 0$.

If $\mathcal{P}(K)$ is a reproducing kernel Pontryagin space, then its reproducing kernel $K(z,w)$ is unique and is such that $\mathrm{sq}_- K(z,w) = \mathrm{ind}_- \mathcal{P}(K)$ (see [10, Theorem 1.1.2]). On the other hand, if $K(z,w)$ is a Hermitian kernel with κ negative squares then there exists a unique Pontryagin space $\mathcal{P}(K)$ of negative index κ with reproducing kernel $K(z,w)$ (see [10, Theorem 1.1.3]).

We shall frequently apply the following two theorems. For more details and more general statements we refer to [10, Section 1.5].

Theorem 2.3.1 *Let* $K(z,w)$, $K_1(z,w)$, *and* $K_2(z,w)$ *be* $n \times n$ *matrix kernels on* Ω *having a finite number of negative squares such that*

$$K(z,w) = K_1(z,w) + K_2(z,w).$$

Then

(i) $\mathrm{sq}_- K(z,w) \le \mathrm{sq}_- K_1(z,w) + \mathrm{sq}_- K_2(z,w)$.

(ii) *The following statements are equivalent:*

(a) $\mathcal{P}(K) = \mathcal{P}(K_1) \oplus \mathcal{P}(K_2)$.

(b) $\mathcal{P}(K_1) \subset \mathcal{P}(K)$ *and the inclusion mapping* $\mathcal{P}(K_1) \hookrightarrow \mathcal{P}(K)$ *is isometric.*

(c) $\mathcal{P}(K_1) \cap \mathcal{P}(K_2) = \{0\}$.

(iii) *If* (a), (b), *and* (c) *in* (ii) *hold then* $\mathrm{sq}_- K(z,w) = \mathrm{sq}_- K_1(z,w) + \mathrm{sq}_- K_2(z,w)$.

Theorem 2.3.2 *Let* $K(z,w)$ *and* $K_1(z,w)$ *be such that*

$$K(z,w) = A(z)K_1(z,w)A(w)^*,$$

where $K_1(z,w)$ *is an* $n \times n$ *matrix kernel on* Ω *having a finite number of negative squares and* $A(z)$ *is an* $m \times n$ *matrix function on* Ω*. Then*

(i) $\mathrm{sq}_- K(z,w) \leq \mathrm{sq}_- K_1(z,w)$,

(ii) $\mathcal{P}(K) = A\mathcal{P}(K_1)$,

(iii) *the operator of multiplication by* $A(z)$*:*

$$\mathcal{P}(K_1) \ni f(z) \mapsto A(z)f(z) \in \mathcal{P}(K)$$

is a unitary mapping if and only if the set

$$\{f(z) \in \mathcal{P}(K_1) \mid A(z)f(z) = 0, z \in \Omega\} = \{0\}.$$

In this case $\mathrm{sq}_- K(z,w) = \mathrm{sq}_- K_1(z,w)$.

We use a different notation for the reproducing kernel spaces associated with a function $S(z)$ from the class $\mathbf{S}_\kappa(\mathbb{C}^n, J)$. By definition the kernels $K_S(z,w)$, $K_{S^\#}(z,w)$, and $D_S(z,w)$ have κ negative squares and we write $\mathcal{P}(S)$, $\mathcal{P}(S^\#)$, and $\mathcal{D}(S)$ instead of $\mathcal{P}(K_S)$, $\mathcal{P}(K_{S^\#})$, and $\mathcal{P}(D_S)$, respectively. Which signature matrix J is considered here will be clear from the context. For generalized Schur functions $S(z)$ we have $J = I$, and when $n = 2$ and J is given by (2.1.6) we write $\Theta(z)$ instead of $S(z)$ and then the spaces are denoted by $\mathcal{P}(\Theta)$, $\mathcal{P}(\Theta^\#)$, and $\mathcal{D}(\Theta)$. In Section 2.4 we give examples of such Θ's for which the corresponding reproducing kernel Pontryagin spaces are finite dimensional. Recall that a rational $n \times n$ matrix function $S(z)$ is said to be *J-unitary* on the unit circle $\mathbb{T}$ if

$$S(z)JS(z)^* = J, \quad |z| = 1, \; z \in \mathrm{hol}(S).$$

Theorem 2.3.3 *Let* $S(z)$ *be a meromorphic* $n \times n$ *matrix function. The following three statements are equivalent:*

(1) $S(z)$ *is rational and* J*-unitary on the unit circle* $\mathbb{T}$.

(2) *The kernel $K_S(z,w)$ has a finite number of negative squares and the space $\mathcal{P}(S)$ is finite dimensional.*

(3) *The kernel $K_{S^\#}(z,w)$ has a finite number of negative squares and the space $\mathcal{P}(S^\#)$ is finite dimensional.*

(4) *The kernel $D_S(z,w)$ has a finite number of negative squares and the space $\mathcal{D}(S)$ is finite dimensional.*

The equivalence of statements (1) and (2) is proved in [11, Theorem 8.1]. In view of Theorem 2.1.1, statements (1) and (2) are equivalent to statements (3) and (4).

2.3.2 Kernels on $\mathbb{C}^n_J$

Theorem 2.1.1 can be proved along the lines of the proof of a similar theorem in [9], see [9, Theorem 0.2]. Instead we shall prove Theorem 2.1.1 using this theorem. Let J be an $n \times n$ signature matrix and denote by $\mathbb{C}^n_J$ the space $\mathbb{C}^n$ equipped with the indefinite inner product

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{\mathbb{C}^n_J} = \boldsymbol{\beta}^* J \boldsymbol{\alpha}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^n.$$

Let $S(z)$ be a meromorphic $n \times n$ matrix function on $\mathbb{D}$ and for each $z \in \operatorname{hol}(S)$ consider $S(z)$ as a mapping from $\mathbb{C}^n_J$ to $\mathbb{C}^n_J$. Then its adjoint $S(z)^*$ is given by

$$S(z)^* = JS(z)^\times J,$$

where $S(z)^\times$ is the adjoint of $S(z)$ considered as a mapping from $|\mathbb{C}^n_J| = \mathbb{C}^n$ to itself, see (2.2.2). We define the following kernels.

$$H_S(z,w) := \frac{I - S(z)S(w)^*}{1 - zw^*} : \mathbb{C}^n_J \to \mathbb{C}^n_J, \tag{2.3.1}$$

$$H_{S^\#}(z,w) := \frac{I - S^\#(z)S^\#(w)^*}{1 - zw^*} : \mathbb{C}^n_J \to \mathbb{C}^n_J,$$

and
(2.3.2)

$$B_S(z,w) := \begin{pmatrix} \dfrac{I - S(z)S(w)^*}{1 - zw^*} & \dfrac{S(z) - S(w^*)}{z - w^*} \\ \dfrac{S^\#(z) - S^\#(w^*)}{z - w^*} & \dfrac{I - S^\#(z)S^\#(w)^*}{1 - zw^*} \end{pmatrix} : \begin{pmatrix} \mathbb{C}^n_J \\ \mathbb{C}^n_J \end{pmatrix} \to \begin{pmatrix} \mathbb{C}^n_J \\ \mathbb{C}^n_J \end{pmatrix}.$$

We say the kernel $H_S(z,w)$ has κ negative squares if for any integer $\ell > 0$, points $w_1, w_2, \ldots, w_\ell \in \operatorname{hol}(S)$, and vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_\ell \in \mathbb{C}^n_J$, the Hermitian matrix

$$\left(\langle H(w_i, w_j) \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j \rangle_{\mathbb{C}^n_J} \right)_{i,j=1}^{\ell}$$

has at most κ and at least one matrix of this form has exactly κ negative eigenvalues, counting multiplicities. That the kernels $H_{S^\#}(z,w)$ and $B_S(z,w)$ have κ negative squares is defined in a similar way. Theorem 0.2 in [9] can now be stated as follows.

Theorem 2.3.4 *If one of the kernels $H_S(z,w)$, $H_{S^\#}(z,w)$, or $B_S(z,w)$ has κ negative squares so do the other two.*

We are now in a position to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. We use the following relations:

$$\begin{cases} K_S(z,w) = H_S(z,w)J, \\ K_{S^\#}(z,w) = JH_{S^\#}(z,w), \\ D_S(z,w) = \widehat{B}_S(z,w)\begin{pmatrix} J & 0 \\ 0 & J\end{pmatrix}, \end{cases} \tag{2.3.3}$$

where

$$\widehat{B}_S(z,w) = \begin{pmatrix} I & 0 \\ 0 & J\end{pmatrix} B_S(z,w)\begin{pmatrix} I & 0 \\ 0 & J\end{pmatrix}. \tag{2.3.4}$$

That $\mathrm{sq}_-K_S(z,w) = \mathrm{sq}_-H_S(z,w)$ follows from the first equality in (2.3.3), as it implies

$$\langle K_S(w_i,w_j)\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j\rangle_{\mathbb{C}^n} = \langle H_S(w_i,w_j)J\boldsymbol{\alpha}_i, \boldsymbol{J\alpha}_j\rangle_{\mathbb{C}^n_J}. \tag{2.3.5}$$

The other equalities in (2.3.3) imply in the same way that

$$\mathrm{sq}_-K_{S^\#}(z,w) = \mathrm{sq}_-H_{S^\#}(z,w), \quad \mathrm{sq}_-D_S(z,w) = \mathrm{sq}_-\widehat{B}_S(z,w).$$

Since $\begin{pmatrix} I & 0 \\ 0 & J\end{pmatrix}$ is invertible, Theorem 2.3.2 with $A(z) = \begin{pmatrix} I & 0 \\ 0 & J\end{pmatrix}$ implies $\mathrm{sq}_-B_S(z,w) = \mathrm{ind}_-\widehat{B}_S(z,w)$ and hence $\mathrm{sq}_-D_S(z,w) = \mathrm{sq}_-B_S(z,w)$. The theorem now follows from Theorem 2.3.4 above. ■

From the proof of Theorem 2.1.1 we see that if one of the kernels $H_S(z,w)$, $B_S(z,w)$, $\widehat{B}_S(z,w)$, $K_S(z,w)$, and $D_S(z,w)$, has κ negative squares then all the others have κ negative squares also. If $S(z) \in \mathbf{S}_\kappa(\mathbb{C}^n, J)$, we denote the reproducing kernel Pontryagin spaces corresponding to these kernels by $\mathcal{H}(S)$, $\mathcal{K}(S)$, $\widehat{\mathcal{K}}(S)$, $\mathcal{P}(S)$, and $\mathcal{D}(S)$ respectively.

Theorem 2.3.5 *For $S(z) \in \mathbf{S}_\kappa(\mathbb{C}^n, J)$ we have:*

(i) *$\mathcal{H}(S) = \mathcal{P}(S)$ and the inclusion mapping $\iota : \mathcal{H}(S) \hookrightarrow \mathcal{P}(S)$ is unitary. In particular, $\iota H_S(z,w)\boldsymbol{\alpha} = K_S(z,w)J\boldsymbol{\alpha}, \quad \boldsymbol{\alpha} \in \mathbb{C}^n$.*

(ii) *The mapping*

$$\omega \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} h \\ Jk \end{pmatrix}, \quad \begin{pmatrix} h \\ k \end{pmatrix} \in \mathcal{K}(S),$$

is a unitary mapping from $\mathcal{K}(S)$ *onto* $\mathcal{D}(S)$. *In particular,*

$$\omega B_S(z,w) \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} = D_S(z,w) \begin{pmatrix} J\boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \mathbb{C}^{2n}.$$

Proof We use the same notation as in the proof of Theorem 2.1.1.
(i) From the first equality in (2.3.3) we have that for $v, w \in \operatorname{hol}(S)$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^n$,

$$\begin{aligned} \langle H_S(\,\cdot\,,w)\boldsymbol{\alpha},\ H_S(\,\cdot\,,v)\boldsymbol{\beta}\rangle_{\mathcal{H}(S)} &= \langle H_S(v,w)\boldsymbol{\alpha},\ \boldsymbol{\beta}\rangle_{\mathbb{C}^n_J} = \langle JK_S(v,w)J\boldsymbol{\alpha},\ \boldsymbol{\beta}\rangle_{\mathbb{C}^n} \\ &= \langle K_S(v,w)J\boldsymbol{\alpha},\ J\boldsymbol{\beta}\rangle_{\mathbb{C}^n} = \langle K_S(\,\cdot\,,w)J\boldsymbol{\alpha},\ K_S(\,\cdot\,,v)J\boldsymbol{\beta}\rangle_{\mathcal{P}(S)}. \end{aligned}$$

It follows that the relation

$$\mathbf{R} = \operatorname{span}\{\{H_S(z,w)\boldsymbol{\alpha},\ \ K_S(z,w)J\boldsymbol{\alpha}\} \mid w \in \operatorname{hol}(S), \boldsymbol{\alpha} \in \mathbb{C}^n\}$$

in $\mathcal{H}(S) \times \mathcal{P}(S)$ is isometric. Since it is densely defined and has dense range, we conclude by Theorem (2.2.1) (iii) that its closure defines the graph of a unitary operator $\iota \in \mathrm{B}(\mathcal{H}(S), \mathcal{P}(S))$. Since the restriction $\iota|_{\operatorname{dom}\mathbf{R}}$ is the identity operator on $\operatorname{dom}\mathbf{R}$, ι itself is the identity operator.
(ii) The arguments in (i) can be repeated to show that the third equality in (2.3.3) implies that $\widehat{\mathcal{K}}(S) = \mathcal{D}(S)$ and that the inclusion mapping $\iota_2 : \widehat{\mathcal{K}}(S) \hookrightarrow \mathcal{D}(S)$ is unitary. Formula (2.3.4) and Theorem 2.3.2 imply that the operator of multiplication by $\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$ is a unitary mapping from $\mathcal{K}(S)$ onto $\widehat{\mathcal{K}}(S)$. Since ω is the composition of these two unitary mappings:

$$\omega = \iota_2 \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix},$$

ω is unitary also. The last equality in the theorem follows from (2.3.3) and (2.3.4). ∎

2.3.3 The projections from $\mathcal{D}(S)$ onto $\mathcal{P}(S)$ and $\mathcal{P}(S^\#)$

In this subsection we describe the projection mappings $\pi_S : \mathcal{D}(S) \to \mathcal{P}(S)$ and $\pi_{S^\#} : \mathcal{D}(S) \to \mathcal{P}(S^\#)$.

Theorem 2.3.6 *For* $S(z) \in \mathbf{S}_\kappa(\mathbb{C}^n, J)$, *the operators* π_S *and* $\pi_{S^\#}$ *defined by*

$$\pi_S \begin{pmatrix} h \\ k \end{pmatrix} = h \ \ \text{and} \ \ \pi_{S^\#} \begin{pmatrix} h \\ k \end{pmatrix} = k$$

are coisometries from $\mathcal{D}(S)$ onto $\mathcal{P}(S)$ and $\mathcal{P}(S^{\#})$ respectively. Moreover, if $\Lambda = \pi_{S^{\#}}\pi_S^*$ then we have for $\boldsymbol{\alpha} \in \mathbb{C}^n$, $h \in \mathcal{P}(S)$, and $k \in \mathcal{P}(S^{\#})$,
(2.3.6)

$$\Lambda K_S(z,w)\boldsymbol{\alpha} = \frac{S^{\#}(z) - S^{\#}(w)}{z - w^*}\boldsymbol{\alpha}, \quad \pi_S^* h = \begin{pmatrix} h \\ \Lambda h \end{pmatrix}, \quad \pi_{S^{\#}}^* k = \begin{pmatrix} \Lambda^* k \\ k \end{pmatrix}.$$

Proof By Theorem 2.1.1, the spaces $\mathcal{P}(S)$, $\mathcal{P}(S^{\#})$ and $\mathcal{D}(S)$ have the same negative index κ. Let $\mathbf{0}$ be the zero vector in $\mathbb{C}^n$. With $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \mathbb{C}^n$ we define the relations $\mathbf{R}$ in $\mathcal{P}(S) \times \mathcal{D}(S)$ and $\mathbf{T}$ in $\mathcal{P}(S^{\#}) \times \mathcal{D}(S)$ by

$$\mathbf{R} = \text{span} \left\{ \left\{ K_S(z,w)\boldsymbol{\alpha}, \ D_S(z,w)\begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{0} \end{pmatrix} \right\} \mid w \in \text{hol}\,(S) \cap \text{hol}\,(S^{\#}), \boldsymbol{\alpha} \in \mathbb{C}^n \right\}$$

and

$$\mathbf{T} = \text{span} \left\{ \left\{ K_{S^{\#}}(z,w)\boldsymbol{\gamma}, \ D_S(z,w)\begin{pmatrix} \mathbf{0} \\ \boldsymbol{\gamma} \end{pmatrix} \right\} \mid w \in \text{hol}\,(S) \cap \text{hol}\,(S^{\#}), \boldsymbol{\gamma} \in \mathbb{C}^n \right\}.$$

We show that the relation $\mathbf{R}$ is isometric and since it is densely defined we conclude by Theorem 2.2.1 (ii) that its closure defines a graph of an isometry in $\mathrm{B}\,(\mathcal{P}(S), \mathcal{D}(S))$. Let

$$\left(K_S(z,w)\boldsymbol{\alpha}, \ D_S(z,w)\begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{0} \end{pmatrix} \right), \quad \left(K_S(z,v)\boldsymbol{\beta}, \ D_S(z,v)\begin{pmatrix} \boldsymbol{\beta} \\ \mathbf{0} \end{pmatrix} \right)$$

be two pairs in the relation $\mathbf{R}$. Then

$$\begin{aligned}
\left\langle D_S(z,w)\begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{0} \end{pmatrix}, D_S(z,v)\begin{pmatrix} \boldsymbol{\beta} \\ \mathbf{0} \end{pmatrix} \right\rangle_{\mathcal{D}(S)} &= \left\langle D_S(v,w)\begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\beta} \\ \mathbf{0} \end{pmatrix} \right\rangle_{\mathbb{C}^n} \\
&= (\boldsymbol{\beta}^* \quad \mathbf{0}) \begin{pmatrix} \dfrac{J - S(v)JS(w)^*}{1 - vw^*} & \dfrac{S(v) - S(w^*)}{v - w^*} \\ \dfrac{S^{\#}(v) - S^{\#}(w^*)}{v - w^*} & \dfrac{J - S^{\#}(v)JS^{\#}(w)^*}{1 - vw^*} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{0} \end{pmatrix} \\
&= \boldsymbol{\beta}^* \frac{J - S(v)JS(w)^*}{1 - vw^*}\boldsymbol{\alpha} = \langle K_S(z,w)\boldsymbol{\alpha}, K_S(z,v)\boldsymbol{\beta} \rangle_{\mathcal{P}(S)}.
\end{aligned}$$

Hence the closure of $\mathbf{R}$ is the graph of an isometry V from $\mathcal{P}(S)$ into $\mathcal{D}(S)$. Setting $\pi_S = V^*$, we see that π_S is a coisometry from $\mathcal{D}(S)$ onto $\mathcal{P}(S)$, and from

$$\begin{aligned}
\left\langle \pi_S \begin{pmatrix} h \\ k \end{pmatrix}(w), \ \boldsymbol{\alpha} \right\rangle_{\mathbb{C}^n} &= \left\langle \pi_S \begin{pmatrix} h \\ k \end{pmatrix}, K_S(z,w)\boldsymbol{\alpha} \right\rangle_{\mathcal{P}(S)} \\
&= \left\langle \begin{pmatrix} h \\ k \end{pmatrix}, VK_S(z,w)\boldsymbol{\alpha} \right\rangle_{\mathcal{D}(S)} = \left\langle \begin{pmatrix} h \\ k \end{pmatrix}, D_S(z,w)\begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{0} \end{pmatrix} \right\rangle_{\mathcal{D}(S)} \\
&= \left\langle \begin{pmatrix} h(w) \\ k(w) \end{pmatrix}, \begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{0} \end{pmatrix} \right\rangle_{\mathbb{C}^{2n}} = \langle h(w), \boldsymbol{\alpha} \rangle_{\mathbb{C}^n},
\end{aligned}$$

it follows that $\pi_S \begin{pmatrix} h \\ k \end{pmatrix} = h$. Similar calculations show that the closure of $\mathbf{T}$ is the graph of an isometric operator from $\mathcal{D}(S)$ to $\mathcal{P}(\mathcal{S}^\#)$, whose adjoint coincides with the operator $\pi_{S^\#}$ defined in the theorem. From

$$\pi_S^* K_S(z,w)\boldsymbol{\alpha} = D_S(z,w)\begin{pmatrix} \boldsymbol{\alpha} \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{J - S(z)JS(w)^*}{1 - zw^*}\boldsymbol{\alpha} \\ \dfrac{S^\#(z) - S^\#(w^*)}{z - w^*}\boldsymbol{\alpha} \end{pmatrix},$$

it follows that the first equality in (2.3.6) holds and that

$$\pi_S^* K_S(z,w)\boldsymbol{\alpha} = \begin{pmatrix} K_S(z,w)\boldsymbol{\alpha} \\ \Lambda K_S(z,w)\boldsymbol{\alpha} \end{pmatrix}.$$

By continuity this last equality implies the second equality in (2.3.6). The third equality in (2.3.6) can be proved in a similar way. ∎

2.4 Examples of spaces $\mathcal{P}(\Theta)$ and $\mathcal{D}(\Theta)$

We now consider the special case when $n = 2$ and

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

mentioned in Section 2.1. As stated there, we write $\Theta(z)$ for the elements in $\mathbf{S}_\kappa(\mathbb{C}^2, J)$ instead of $S(z)$ and denote the corresponding kernels by $K_\Theta(z,w)$ and $D_\Theta(z,w)$. $\mathcal{P}(\Theta)$ and $\mathcal{D}(\Theta)$ stand for the corresponding reproducing kernel Pontryagin spaces. In the next three subsections we give a detailed description of some functions $\Theta(z)$ and the spaces $\mathcal{P}(\Theta)$ and $\mathcal{D}(\Theta)$ they generate. These are important in later chapters.

2.4.1 The J-unitary functions $\Theta_1(z)$, $\Theta_2(z)$, $\Theta_\infty(z)$, $\Theta_3(z)$, and $\Theta_4(z)$

Throughout the sequel we use the following notation. Here σ_0 is an arbitrary complex number and k and q are integers with $k \geq 1$ and $q \geq 0$.

(2.4.1) $$\Theta_1(z) = \frac{1}{\sqrt{1 - |\sigma_0|^2}} \begin{pmatrix} 1 & \sigma_0 \\ \sigma_0^* & 1 \end{pmatrix} \begin{pmatrix} z^k & 0 \\ 0 & 1 \end{pmatrix} \text{ if } |\sigma_0| < 1,$$

(2.4.2) $$\Theta_2(z) = \frac{1}{\sqrt{|\sigma_0|^2 - 1}} \begin{pmatrix} \sigma_0 & 1 \\ 1 & \sigma_0^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^k \end{pmatrix} \text{ if } |\sigma_0| > 1,$$

and

$$\Theta_\infty(z) = \begin{pmatrix} 1 & 0 \\ 0 & z^q \end{pmatrix}. \tag{2.4.3}$$

We note here that $\Theta_\infty(z)$ is a limiting case of $\Theta_2(z)$ when we let σ_0 go to infinity (and replace k by q). The definition of $\Theta_3(z)$ is more involved. Let $s_0 \neq 0, s_1, \dots, s_{k-1}$ be k given complex numbers and define the polynomial $Q(z) = Q(z; s_0, s_1, \cdots, s_{k-1})$ by

$$\begin{aligned} Q(z) &= Q(z; s_0, s_1, \cdots, s_{k-1}) \\ &= c_0 + c_1 z + \cdots + c_{k-1} z^{k-1} - (c_{k-1}^* z^{k+1} + \cdots + c_0^* z^{2k}) \end{aligned} \tag{2.4.4}$$

in which the complex numbers $c_0, \dots, c_{k-1}$ are determined by the relation

$$\begin{pmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ c_{k-1} & \cdots & c_1 & c_0 \end{pmatrix} \begin{pmatrix} s_0 & 0 & \cdots & 0 \\ s_1 & s_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ s_{k-1} & \cdots & s_1 & s_0 \end{pmatrix} = \sigma_0 I_k.$$

From the definition one can check that

$$z^{*k} Q(z) + z^k Q(z)^* = 0 \quad \text{for} \quad |z| = 1, \tag{2.4.5}$$

and if we set

$$p(z) = c_0 + c_1 z + \cdots + c_{k-1} z^{k-1} \tag{2.4.6}$$

then

$$Q(z) = p(z) - z^{2k} p(\frac{1}{z^*})^*. \tag{2.4.7}$$

We define $\Theta_3(z)$ and $\Theta_4(z)$ by

$$\Theta_3(z) = \begin{pmatrix} Q(z) + z^k & -\sigma_0 Q(z) \\ \sigma_0^* Q(z) & -Q(z) + z^k \end{pmatrix} \text{ if } |\sigma_0| = 1, \tag{2.4.8}$$

$$\Theta_4(z) = \Theta_3(z) \Theta_\infty(z). \tag{2.4.9}$$

Lemma 2.4.1 *The polynomial matrix functions $\Theta_1(z)$, $\Theta_2(z)$, $\Theta_\infty(z)$, $\Theta_3(z)$, and $\Theta_4(z)$ defined above are J-unitary on the unit circle $\mathbb{T}$.*

Proof For $|z| = 1$,

$$\begin{aligned}
\Theta_1(z)J\Theta_1(z)^* &= \frac{1}{1-|\sigma_0|^2}\begin{pmatrix} z^k & -\sigma_0 \\ \sigma_0^* z^k & -1 \end{pmatrix}\begin{pmatrix} z^{*k} & \sigma_0 z^{*k} \\ \sigma_0^* & 1 \end{pmatrix} \\
&= \frac{1}{1-|\sigma_0|^2}\begin{pmatrix} |z|^{2k}-|\sigma_0|^2 & \sigma_0(|z|^{2k}-1) \\ \sigma_0^*(|z|^{2k}-1) & |\sigma_0|^2|z|^{2k}-1 \end{pmatrix} = J, \\
\Theta_2(z)J\Theta_2(z)^* &= \frac{1}{|\sigma_0|^2-1}\begin{pmatrix} \sigma_0 & -z^k \\ 1 & -\sigma_0^* z^k \end{pmatrix}\begin{pmatrix} \sigma_0^* & 1 \\ z^{*k} & \sigma_0 z^{*k} \end{pmatrix} \\
&= \frac{1}{|\sigma_0|^2-1}\begin{pmatrix} |\sigma_0|^2-|z|^{2k} & \sigma_0(1-|z|^{2k}) \\ \sigma_0^*(1-|z|^{2k}) & 1-|\sigma_0|^2|z|^{2k} \end{pmatrix} = J, \\
\Theta_\infty(z)J\Theta_\infty(z)^* &= \begin{pmatrix} 1 & 0 \\ 0 & -z^q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{*q} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -|z|^{2q} \end{pmatrix} = J,
\end{aligned}$$

and by (2.4.5),

$$\begin{aligned}
\Theta_3(z)J\Theta_3(z)^* &= \begin{pmatrix} Q(z)+z^k & \sigma_0 Q(z) \\ \sigma_0^* Q(z) & Q(z)-z^k \end{pmatrix}\begin{pmatrix} Q(z)^*+z^{*k} & \sigma_0 Q(z)^* \\ -\sigma_0^* Q(z)^* & -Q(z)^*+z^{*k} \end{pmatrix} \\
&= \begin{pmatrix} z^kQ(z)^*+z^{*k}Q(z)+|z|^{2k} & \sigma_0(z^kQ(z)^*+z^{*k}Q(z)) \\ \sigma_0^*(z^kQ(z)^*+z^{*k}Q(z) & z^kQ(z)^*+z^{*k}Q(z)-|z|^{2k} \end{pmatrix} = J.
\end{aligned}$$

That $\Theta_4(z)$ is J-unitary follows from the fact that both $\Theta_\infty(z)$ and $\Theta_3(z)$ are J-unitary. ■

Lemma 2.4.1 and Theorem 2.3.3 imply that the spaces $\mathcal{P}(\Theta_j)$ and $\mathcal{D}(\Theta_j)$ for $j = 1, 2, 3, 4, \infty$ are finite dimensional.

2.4.2 The spaces $\mathcal{P}(\Theta_1)$, $\mathcal{P}(\Theta_2)$, $\mathcal{P}(\Theta_\infty)$, $\mathcal{P}(\Theta_3)$, and $\mathcal{P}(\Theta_4)$

In this subsection we describe the finite dimensional reproducing kernel Pontryagin spaces associated with the matrix functions $\Theta_1(z)$, $\Theta_2(z)$, Θ_∞, $\Theta_3(z)$, and Θ_4 defined by (2.4.1)–(2.4.8) and (2.4.9). The symbols $\mathbf{o}$, $\mathbf{u}$, $\mathbf{e_1}$, $\mathbf{e_2}$ will stand for the vectors

$$\mathbf{o} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ \sigma_0^* \end{pmatrix}, \quad \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.4.10}$$

and $\widehat{C}$ will be the $k \times k$ matrix

$$\widehat{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ c_{k-1} & 0 & 0 & 0 & \cdots & 0 \\ c_{k-2} & c_{k-1} & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ c_2 & \cdots & c_{k-2} & c_{k-1} & 0 & 0 \\ c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} & 0 \end{pmatrix}. \tag{2.4.11}$$

Theorem 2.4.2 *The space $\mathcal{P}(\Theta_1)$ is the Hilbert space spanned by the orthonormal basis*

$$\left\{rz^{n-1}\mathbf{u}\right\}_{n=1}^{k}, \; r = \frac{1}{\sqrt{1-|\sigma_0|^2}},$$

with Gram matrix $I_{k\times k}$.

Theorem 2.4.3 *The space $\mathcal{P}(\Theta_2)$ is the anti-Hilbert space spanned by the basis*

$$\left\{rz^{n-1}\mathbf{u}\right\}_{n=1}^{k}, \; r = \frac{1}{\sqrt{|\sigma_0|^2-1}},$$

with Gram matrix $-I_{k\times k}$.

Theorem 2.4.4 *The space $\mathcal{P}(\Theta_\infty)$ is the anti-Hilbert space with basis*

$$\left\{z^{n-1}\mathbf{e_2}\right\}_{n=1}^{q},$$

whose Gram matrix equals $-I_{q\times q}$.

Theorem 2.4.5 *The space $\mathcal{P}(\Theta_3)$ is the Pontryagin space spanned by the basis*

$$\left\{z^{n-1}\mathbf{u}\right\}_{n=1}^{k}, \; \left\{z^{n-1}(J\mathbf{u}-2z^kp(z^{-*})^*\mathbf{u})\right\}_{n=1}^{k}$$

with Gram matrix $2\begin{pmatrix} 0 & I_{k\times k} \\ I_{k\times k} & -2(\widehat{C}+\widehat{C}^*) \end{pmatrix}$, *where $\widehat{C}$ is given by* (2.4.11). *In particular, the elements of the space $\mathcal{P}(\Theta_3)$ are of the form*

$$t_1(z)\mathbf{u} + t_2(z)(J\mathbf{u} - 2z^kp(z^{-*})^*\mathbf{u}),$$

where $t_1(z)$ and $t_2(z)$ are polynomials of degree $\leq k-1$.

Theorem 2.4.6 *With $\Theta_4(z) = \Theta_3(z)\Theta_\infty(z)$, the space $\mathcal{P}(\Theta_4)$ can be decomposed as the orthogonal direct sum*

$$\mathcal{P}(\Theta_4) = \mathcal{P}(\Theta_3) \oplus \Theta_3\mathcal{P}(\Theta_\infty). \tag{2.4.12}$$

Moreover, the map

$$W : \mathcal{P}(\Theta_4) \ni f \mapsto \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \begin{pmatrix} \mathcal{P}(\Theta_3) \\ \mathcal{P}(\Theta_\infty) \end{pmatrix} \tag{2.4.13}$$

determined by the decomposition

$$f = f_1 + \Theta_3 f_2,$$

in accordance with (2.4.12)*, is unitary.*

Theorems 2.4.2 – 2.4.6 can be obtained from Theorems 2.4.8 – 2.4.12 below by use of the mapping π_Θ defined in Theorem 2.3.6, see Subsection 2.4.4.

Corollary 2.4.7 *We have*

$$\operatorname{ind}_- \mathcal{P}(\Theta_j) = \begin{cases} 0 & \text{if } j = 1, \\ k & \text{if } j = 2, 3, \\ k + q & \text{if } j = 4. \end{cases}$$

Proof Since $\mathcal{P}(\Theta_j)$ is a Hilbert space for $j = 1$ and an anti-Hilbert space for $j = 2$, the formula for its index in these cases is clear. For the case $j = 3$, the formula follows from the Gram matrix in Theorem 2.4.5: it has k negative (and k positive) eigenvalues. The result for the case $j = 4$ follows from the orthogonal decomposition in (2.4.12) and Theorems 2.4.4. ∎

2.4.3 The spaces $\mathcal{D}(\Theta_1)$, $\mathcal{D}(\Theta_2)$, $\mathcal{D}(\Theta_\infty)$, $\mathcal{D}(\Theta_3)$, and $\mathcal{D}(\Theta_4)$

In the following theorems the matrix functions $\Theta_1(z)$, $\Theta_2(z)$, $\Theta_\infty(z)$, $\Theta_3(z)$, and $\Theta_4(z)$ are as defined by (2.4.1)–(2.4.8) and (2.4.9).

Theorem 2.4.8 *The space $\mathcal{D}(\Theta_1)$ is the Hilbert space spanned by the orthonormal basis*

$$\left\{ \begin{pmatrix} r z^{n-1} \mathbf{u} \\ z^{k-n} \mathbf{e_1} \end{pmatrix} \right\}_{n=1}^{k}, \quad r = \frac{1}{\sqrt{1 - |\sigma_0|^2}},$$

with Gram matrix $I_{k \times k}$. In particular, the elements of the space $\mathcal{D}(\Theta_1)$ are of the form

$$\begin{pmatrix} r t(z) \mathbf{u} \\ z^{k-1} t(z^{-1}) \mathbf{e_1} \end{pmatrix},$$

where $t(z)$ is a polynomial of degree $\leq k - 1$.

Theorem 2.4.9 *The space $\mathcal{D}(\Theta_2)$ is the anti-Hilbert space spanned by the basis*

$$\left\{\begin{pmatrix} -rz^{n-1}\mathbf{u} \\ z^{k-n}\mathbf{e_2} \end{pmatrix}\right\}_{n=1}^{k}, \quad r=\frac{1}{\sqrt{|\sigma_0|^2-1}},$$

with Gram matrix $-I_{k\times k}$. In particular, the elements of the space $\mathcal{D}(\Theta_2)$ are of the form

$$\begin{pmatrix} -rt(z)\mathbf{u} \\ z^{k-1}t(z^{-1})\mathbf{e_2} \end{pmatrix},$$

where $t(z)$ is a polynomial of degree $\leq k-1$.

Theorem 2.4.10 *The space $\mathcal{D}(\Theta_\infty)$ is the anti-Hilbert space with basis*

$$\left\{\begin{pmatrix} -z^{n-1}\mathbf{e_2} \\ z^{q-n}\mathbf{e_2} \end{pmatrix}\right\}_{n=1}^{q},$$

whose Gram matrix equals $-I_{q\times q}$. In particular, the elements of the space $\mathcal{D}(\Theta_\infty)$ are of the form

$$\begin{pmatrix} -t(z)\mathbf{e_2} \\ z^{q-1}t(z^{-1})\mathbf{e_2} \end{pmatrix},$$

where $t(z)$ is a polynomial of degree $\leq q-1$.

For the next theorem we recall that $p(z)$ is the polynomial defined by (2.4.6).

Theorem 2.4.11 *The space $\mathcal{D}(\Theta_3)$ is the Pontryagin space spanned by the basis*

$$\left\{\begin{pmatrix} z^{n-1}\mathbf{u} \\ z^{k-n}J\mathbf{u} \end{pmatrix}\right\}_{n=1}^{k}, \quad \left\{\begin{pmatrix} z^{n-1}(J\mathbf{u}-2z^kp(z^{-*})^*\mathbf{u}) \\ z^{k-n}(\mathbf{u}-2z^kp(z^{-1})J\mathbf{u}) \end{pmatrix}\right\}_{n=1}^{k}$$

with Gram matrix $2\begin{pmatrix} 0 & I_{k\times k} \\ I_{k\times k} & -2(\widehat{C}+\widehat{C}^*) \end{pmatrix}$. In particular, the elements of the space $\mathcal{D}(\Theta_3)$ are of the form

$$\begin{pmatrix} t_1(z)\mathbf{u} \\ z^{k-1}t_1(z^{-1})J\mathbf{u} \end{pmatrix} + \begin{pmatrix} t_2(z)(J\mathbf{u}-2z^kp(z^{-*})^*\mathbf{u}) \\ z^{k-1}t_2(z^{-1})(\mathbf{u}-2z^kp(z^{-1})J\mathbf{u}) \end{pmatrix},$$

where $t_1(z)$ and $t_2(z)$ are polynomials of degree $\leq k-1$.

Theorem 2.4.12 *The space $\mathcal{D}(\Theta_4)$ can be decomposed as the orthogonal direct sum*

$$\mathcal{D}(\Theta_4)=\begin{pmatrix} 1 & 0 \\ 0 & \Theta_\infty \end{pmatrix}\mathcal{D}(\Theta_3)\oplus\begin{pmatrix} \Theta_3 & 0 \\ 0 & 1 \end{pmatrix}\mathcal{D}(\Theta_\infty). \tag{2.4.14}$$

Moreover, the map

$$W : \mathcal{D}(\Theta_4) \ni f \mapsto \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \begin{pmatrix} \mathcal{D}(\Theta_3) \\ \mathcal{D}(\Theta_\infty) \end{pmatrix} \tag{2.4.15}$$

determined by the decomposition

$$f = \begin{pmatrix} 1 & 0 \\ 0 & \Theta_\infty \end{pmatrix} f_1 + \begin{pmatrix} \Theta_3 & 0 \\ 0 & 1 \end{pmatrix} f_2,$$

in accordance with (2.4.14), *is unitary.*

2.4.4 Proofs of Theorems 2.4.2–2.4.6

Theorems 2.4.2–2.4.6 can be obtained from Theorems 2.4.8–2.4.12 by means of the projection mapping $\pi_\Theta : \mathcal{D}(\Theta) \to \mathcal{P}(\Theta)$ defined in Theorem 2.3.6. In all the cases it can be shown that π_Θ is injective and since it is a coisometry, we conclude that it is unitary. The unitarity of π_Θ implies that it maps a basis of $\mathcal{D}(\Theta)$ onto a basis of $\mathcal{P}(\Theta)$ and since the inner products are preserved, the Gram matrices associated with the two bases are the same. We show the injectivity of π_Θ for the case $\Theta(z) = \Theta_3(z)$. The remaining cases can be shown in a similar way. To do this we let

$$\mathbf{X}(z) = \begin{pmatrix} t_1(z)\mathbf{u} \\ z^{k-1}t_1(z^{-1})J\mathbf{u} \end{pmatrix} + \begin{pmatrix} t_2(z)(J\mathbf{u} - 2z^k p(z^{-*})^*\mathbf{u}) \\ z^{k-1}t_2(z^{-1})(\mathbf{u} - 2z^k p(z^{-1})J\mathbf{u}) \end{pmatrix}$$

be an element of $\mathcal{D}(\Theta_3)$ where $t_1(z)$ and $t_2(z)$ are polynomials of degree $\leq k-1$ and assume that $\pi_{\Theta_3}\mathbf{X}(z) = \mathbf{o}$. This implies that

$$[t_1(z) - 2z^k t_2(z)p(z^{-*})^*]\mathbf{u} + t_2(z)J\mathbf{u} = \mathbf{o}.$$

Since $\mathbf{u}$ and $J\mathbf{u}$ are linearly independent it follows that $t_1(z) = t_2(z) = 0$. Hence $\mathbf{X}(z) = \begin{pmatrix} \mathbf{o} \\ \mathbf{o} \end{pmatrix}$ and π_{Θ_3} is injective.

2.4.5 Proofs of Theorems 2.4.8–2.4.12

Since Theorems 2.4.8, 2.4.9, and 2.4.10 can be proved in a similar way , we give the proofs of Theorems 2.4.8, 2.4.11, and 2.4.12 only.

Proof of Theorem 2.4.8. We have that

$$D_{\Theta_1}(z,w) = \begin{pmatrix} r^2 \dfrac{1 - z^k w^{*k}}{1 - zw^*} \begin{pmatrix} 1 & \sigma_0 \\ \sigma_0^* & |\sigma_0|^2 \end{pmatrix} & r\dfrac{z^k - w^{*k}}{z - w^*} \begin{pmatrix} 1 & 0 \\ \sigma_0^* & 0 \end{pmatrix} \\ r\dfrac{z^k - w^{*k}}{z - w^*} \begin{pmatrix} 1 & \sigma_0 \\ 0 & 0 \end{pmatrix} & \dfrac{1 - z^k w^{*k}}{1 - zw^*} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

from which we derive the equality

$$D_{\Theta_1}(z,w)\begin{pmatrix} a\\ b\\ \mathbf{o}\end{pmatrix} = rw^{*(k-1)}D_{\Theta_1}(z,w^{-1})\begin{pmatrix}\mathbf{o}\\ a+\sigma_0 b\\ d\end{pmatrix},\ a,b,d\in\mathbb{C}.$$

Since $\mathcal{D}(\Theta_1)$ is spanned by the columns of $D_{\Theta_1}(z,w)$, this means that in fact it is spanned by

$$\frac{1}{r}D_{\Theta_1}(z,w)\begin{pmatrix}\mathbf{e_1}\\ \mathbf{o}\end{pmatrix} = \begin{pmatrix} r(1+zw^*+\cdots+z^{k-1}w^{*(k-1)})\mathbf{u}\\ (z^{k-1}+z^{k-2}w^*+\cdots+zw^{*(k-2)}+w^{*(k-1)})\mathbf{e_1}\end{pmatrix}.$$

We divide this element by $2\pi i w^{*n}$, integrate with respect to w^* over a circle around $w^*=0$ and, by Cauchy's theorem, obtain the basis elements described in part (i) of the theorem. By the reproducing property of the kernel we have

$$\left\langle \frac{1}{r}D_{\Theta_1}(z,w)\begin{pmatrix}\mathbf{e_1}\\ \mathbf{o}\end{pmatrix}, \frac{1}{r}D_{\Theta_1}(z,v)\begin{pmatrix}\mathbf{e_1}\\ \mathbf{o}\end{pmatrix}\right\rangle_{\mathcal{D}(\Theta_1)} = 1+vw^*+\cdots+v^{k-1}w^{*(k-1)}.$$

Dividing both sides by $-4\pi^2 v^m w^{*n}$, $1\le m,n\le k$, and integrating with respect to v and w^* over circles around the origin, we see that the Gram matrix associated with this basis is equal to $I_{k\times k}$. ■

Proof of Theorem 2.4.11. For this case we have that

$$D_{\Theta_3}(z,w) = \begin{pmatrix} \dfrac{1-z^k w^{*k}}{1-zw^*}J & \dfrac{z^k-w^{*k}}{z-w^*}I\\ \dfrac{z^k-w^{*k}}{z-w^*}I & \dfrac{1-z^k w^{*k}}{1-zw^*}J\end{pmatrix}$$

$$-\begin{pmatrix} \dfrac{z^kQ(w)^*+w^{*k}Q(z)}{1-zw^*}\begin{pmatrix}1 & \sigma_0\\ \sigma_0^* & 1\end{pmatrix} & \dfrac{Q(z)-Q(w^*)}{z-w^*}\begin{pmatrix}-1 & \sigma_0\\ -\sigma_0^* & 1\end{pmatrix}\\ \dfrac{Q(z^*)^*-Q(w)^*}{z-w^*}\begin{pmatrix}-1 & -\sigma_0\\ \sigma_0^* & 1\end{pmatrix} & \dfrac{z^kQ(w^*)+w^{*k}Q(z^*)^*}{1-zw^*}\begin{pmatrix}1 & -\sigma_0\\ -\sigma_0^* & 1\end{pmatrix}\end{pmatrix}.$$

From this it can be shown that

$$D_{\Theta_3}(z,w)\begin{pmatrix}J\mathbf{u}\\ \mathbf{o}\end{pmatrix} = w^{*(k-1)}D_{\Theta_3}(z,w^{-1})\begin{pmatrix}\mathbf{o}\\ \mathbf{u}\end{pmatrix},$$

and

$$w^{*(k-1)}D_{\Theta_3}(z,w^{-1})\begin{pmatrix}\mathbf{u}\\ \mathbf{o}\end{pmatrix} - D_{\Theta_3}(z,w)\begin{pmatrix}\mathbf{o}\\ J\mathbf{u}\end{pmatrix} = -2\frac{Q(w^*)}{w^{*k}}D_{\Theta_3}(z,w)\begin{pmatrix}\mathbf{o}\\ \mathbf{u}\end{pmatrix}.$$

These equalities imply that

$$\mathcal{D}(\Theta_3) = \text{span}\,\{D_{\Theta_3}(z,w)\begin{pmatrix}J\mathbf{u}\\ \mathbf{o}\end{pmatrix}, D_{\Theta_3}(z,w)\begin{pmatrix}\mathbf{u}\\ \mathbf{o}\end{pmatrix}\}. \tag{2.4.16}$$

Since

$$D_{\Theta_3}(z,w)\begin{pmatrix} J\mathbf{u} \\ \mathbf{o} \end{pmatrix} = \begin{pmatrix} \dfrac{1-z^k w^{*k}}{1-zw^*}\mathbf{u} \\ \dfrac{z^k - w^{*k}}{z-w^*}J\mathbf{u} \end{pmatrix},$$

we obtain, using integration as in the proof of Theorem 2.4.8, that

$$\text{span}\,\{D_{\Theta_3}(z,w)\begin{pmatrix} J\mathbf{u} \\ \mathbf{o} \end{pmatrix}\} = \text{span}\begin{pmatrix} z^{j-1}\mathbf{u} \\ z^{k-j}J\mathbf{u} \end{pmatrix}_{j=1}^{k} \tag{2.4.17}$$

is a neutral space which accounts for the 0 entry in the left upper corner of the Gram matrix. The elements on the right-hand side are linearly independent and their span coincides with the space of functions of the form

$$\begin{pmatrix} t(z)\mathbf{u} \\ z^{k-1}t(z^{-1})J\mathbf{u} \end{pmatrix},$$

where $t(z)$ is a polynomial of degree $\le k-1$. From

$$D_{\Theta_3}(z,w)\begin{pmatrix} \mathbf{u} \\ \mathbf{o} \end{pmatrix} = \begin{pmatrix} \dfrac{1-z^k w^{*k}}{1-zw^*}J\mathbf{u} - 2\dfrac{z^k Q(w)^* + w^{*k}Q(z)}{1-zw^*}\mathbf{u} \\ \dfrac{z^k - w^{*k}}{z-w^*}\mathbf{u} + 2\dfrac{Q(z^*)^* - Q(w)^*}{z-w^*}J\mathbf{u} \end{pmatrix}$$

and $Q(z) = p(z) - z^{2k}p(\frac{1}{z^*})^*$ (see (2.4.7)), we get

$$D_{\Theta_3}(z,w)\begin{pmatrix} \mathbf{u} \\ \mathbf{o} \end{pmatrix} =$$

$$\begin{pmatrix} \dfrac{1-z^k w^{*k}}{1-zw^*}J\mathbf{u} - 2z^k\dfrac{1-z^k w^{*k}}{1-zw^*}p(z^{-*})^*\mathbf{u} \\ \dfrac{z^k - w^{*k}}{z-w^*}\mathbf{u} - 2z^k\dfrac{z^k - w^{*k}}{z-w^*}p(\frac{1}{z})J\mathbf{u} \end{pmatrix} - \begin{pmatrix} t_w(z)\mathbf{u} \\ z^{k-1}t_w(z^{-1})J\mathbf{u} \end{pmatrix},$$

where

$$t_w(z) = 2\frac{z^k p(w)^* - z^k p(z^{-*})^* + w^{*k}p(z) - z^k w^{*2k}p(w^{-*})}{1-zw^*}$$

is a polynomial of degree $\le k-1$ in z. The span of the second summand is contained in the neutral subspace (2.4.17) and can be dropped from the formula when calculating the span on the right-hand side of (2.4.16). The remainder of the proof can be given by integration and using the reproducing property of the kernel as in the proof of Theorem 2.4.8 and is omitted. ■

Proof of Theorem 2.4.12. The orthogonal decomposition of $\mathcal{D}(\Theta_4)$ and the unitarity of the map follow from

(a) the equality

$$\begin{aligned} D_{\Theta_4}(z,w) &= \begin{pmatrix} 1 & 0 \\ 0 & \Theta_\infty(z) \end{pmatrix} D_{\Theta_3}(z,w) \begin{pmatrix} 1 & 0 \\ 0 & \Theta_\infty(w)^* \end{pmatrix} \\ &\quad + \begin{pmatrix} \Theta_3(z) & 0 \\ 0 & 1 \end{pmatrix} D_{\Theta_\infty}(z,w) \begin{pmatrix} \Theta_3(w)^* & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{2.4.18}$$

(b) the implication that if $f_1 \in \mathcal{D}(\Theta_3)$ and $f_2 \in \mathcal{D}(\Theta_\infty)$ then the identity

$$\begin{pmatrix} 1 & 0 \\ 0 & \Theta_\infty \end{pmatrix} f_1 + \begin{pmatrix} \Theta_3 & 0 \\ 0 & 1 \end{pmatrix} f_2 = 0$$

implies $f_1 = 0$ and $f_2 = 0$, and
(c) reproducing kernel methods as in Theorems 2.3.1 and 2.3.2.

The implication in (b) follows from

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^q \end{pmatrix} \mathcal{D}(\Theta_3) \cap \begin{pmatrix} Q(z) + z^k & -\sigma_0 & 0 & 0 \\ \sigma_0^* Q(z) & -Q(z) + z^k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathcal{D}(\Psi_q) = \{0\},$$

which can be verified by comparing the degrees of the elements in the two sets. ■