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Convexities related to path properties on graphs

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Abstract

A feasible family of paths in a connected graph G is a family that contains at least one path between any pair of vertices in G . Any feasible path family defines a convexity on G . Well-known instances are: the geodesics, the induced paths, and all paths. We propose a more general approach for such ‘path properties’. We survey a number of results from this perspective, and present a number of new results. We focus on the behaviour of such convexities on the Cartesian product of graphs and on the classical convexity invariants, such as the Carathéodory, Helly and Radon numbers in relation with graph invariants, such as the clique number and other graph properties.

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1. Introduction

In [13], the notion of transit function is introduced as a means to study how to move around in discrete structures. Basically, it is a function satisfying three simple axioms on a

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set V , which is provided with a structure σ . Prime examples of such a structure are: a set of edges E , so that we are considering a graph $G = (V, E)$, or a partial ordering \leq , so that we are considering a partially ordered set (V, \leq) . Then the idea is to study transit functions that have additional properties defined in terms of the structure σ . For instance, the transit function may be defined in terms of paths in the graph $G = (V, E)$. Such transit functions are called *path transit functions* on G in [13]. A prime example is the interval function $I: V \times V \rightarrow 2^V$ of a connected graph G , where $I(u, v)$ is the set of the vertices lying on shortest paths between u and v . Other examples are the induced-path transit function, and the all-paths transit function. Any transit function on (V, σ) defines a natural convexity on V . The convexities associated with the three mentioned path transit functions have already been studied extensively. Some relevant references are: for the geodesic convexity [6,10,12,15,21], for the induced-path convexity [5,14], and for the all-paths convexity (or the coarse convexity) [2,4,17]. In [13], a wide variety of prototype problems to be studied for transit functions and their convexities is presented. In this paper, we focus on one such type of problems.

Paths transit functions are the topic of this paper, in particular the above-mentioned three examples and transit functions constructed from these. By choosing the perspective of transit functions, we propose a unifying approach for the study of such path properties. This approach suggests also various new questions for future research. We study the behaviour of these functions under Cartesian products of graphs, and we study the various invariants of the associated convexities, such as the Carathéodory, Helly, and Radon numbers. Along the way, we survey some results in the literature related to these topics.

2. Preliminaries

In this section, we present some of the basic ideas from [13] on transit functions. Throughout the paper $G = (V, E)$ is a connected, simple, loopless graph. A *transit function* on G is a function $R: V \times V \rightarrow 2^V$ satisfying the following three axioms:

- (t1) $u \in R(u, v)$ for all u and v in V ,
- (t2) $R(u, v) = R(v, u)$ for all u and v in V ,
- (t3) $R(u, u) = \{u\}$.

Axioms of the type (t1)–(t3), which are in terms of R only, are called *transit axioms*. Let R and S be transit functions on the graph G . The *join* of R and S is the transit function $R \vee S$ defined by $(R \vee S)(u, v) = R(u, v) \cup S(u, v)$. The *meet* of R and S is the transit function $R \wedge S$ defined by $(R \wedge S)(u, v) = R(u, v) \cap S(u, v)$. With this join and meet the family of all transit functions on G is a lattice. We denote this lattice by L_G . Note that the structure of G is not relevant for L_G . But our interest is in substructures of L_G that reflect the structure of G . The partial ordering \leq of this lattice can simply be described by $R \leq S$ if and only if $R(u, v) \subseteq S(u, v)$, for all u, v in V . The universal lower bound of this lattice is the *discrete transit function* 0 defined by $0(u, u) = \{u\}$, for all u in V , and $0(u, v) = \{u, v\}$, for all distinct u and v in V . The universal upper bound is the *trivial transit function* 1 defined by $1(u, u) = \{u\}$, for all u in V , and $1(u, v) = V$, for all distinct u and v in V .

A subset W of V is R -convex if $R(u, v) \subseteq W$, for all u, v in W . The family \mathcal{C}_R of all R -convex sets in V is an abstract convexity: it is closed under intersections and nested unions, and both \emptyset and V are R -convex. Note that, in the finite case, the condition on nested unions can be deleted. The convexity \mathcal{C}_0 of the discrete transit function 0 is the *discrete convexity*: every subset is convex. The convexity \mathcal{C}_1 of the trivial transit function 1 is the *trivial convexity*. Note that we assume that singletons are always convex. This is no real restriction of the notion of a convexity, because if we add all missing singletons to a convexity, then it remains a convexity. Thus the empty set \emptyset , the singletons $\{u\}$ and the whole set V are the *trivial convex sets* of a convexity. The smallest R -convex subset containing a subset W of V is denoted by $\langle W \rangle_R$ and is called the R -convex hull of W . Note that two different transit functions R and S may give rise to the same convexity, that is, $\mathcal{C}_R = \mathcal{C}_S$. An R -convex subgraph H of a graph G is a subgraph induced by an R -convex set in G . Since no confusion can arise, we will not always distinguish between a convex subset and the convex subgraph induced by this set. Convexities defined by a transit function are called interval convexities, or interval spaces in e.g. [1,21]. For a detailed account on abstract convexities, see, for example [21].

Let Φ be a property of paths in G , for instance the property of being a *geodesic*, i.e. a shortest path. A Φ -path is a path having property Φ . Formally, we take a *path property* Φ to be a subset of the set of all paths in G . Thus, if P is a Φ -path, then we may denote that also as $P \in \Phi$. Let u and v be vertices of G . Then $\Phi(u, v)$ denotes the subset of all u, v -paths in Φ . We will only consider *feasible* path properties, that is, path properties Φ such that $\Phi(u, v) \neq \emptyset$, for all u, v in V . So all path properties in the sequel are presumed to be feasible without mention. The Φ -path transit function, or Φ -path function for short, on G is the transit function R_Φ defined by

$$R_\Phi(u, v) = \{x \in V \mid x \text{ is on some } \Phi\text{-path in } G\}.$$

Note that the subgraph induced by $R_\Phi(u, v)$ is a connected subgraph of G . If no confusion arises, we call a Φ -path transit function a path transit function. The convexity \mathcal{C}_{R_Φ} will also be denoted as \mathcal{C}_Φ . If R_{Φ_1} and R_{Φ_2} are two path transit functions, then $R_{\Phi_1} \wedge R_{\Phi_2}$ need not be a path transit function. For example, if $\Phi_1 = \text{'shortest'}$ and $\Phi_2 = \text{'longest'}$, then $R_{\Phi_1} \wedge R_{\Phi_2}$ usually will not be a path transit function. However, $R_{\Phi_1} \vee R_{\Phi_2}$ is always a path transit function, namely of the path property $\Phi = \Phi_1 \cup \Phi_2$. Hence, the family of the path transit functions on G is a join semi-lattice of L_G , denoted by $L_{p(G)}$. Clearly, the *all-paths transit function* on G defined by

$$A_G(u, v) = \{x \in V \mid x \text{ lies in some } u, v\text{-path in } G\},$$

is a universal upper bound of $L_{p(G)}$.

3. The lattice of convexities

In this section, we study the relation between the lattice of transit functions on a connected graph $G = (V, E)$ and the lattice of associated convexities.

Let \mathcal{L}_G be the family of R -convexities \mathcal{C}_R on V with R in L_G . For R and S in L_G , we define $\mathcal{C}_R \wedge \mathcal{C}_S = \mathcal{C}_R \cap \mathcal{C}_S$ and $\mathcal{C}_R \vee \mathcal{C}_S = \{U \cap W \mid U \in \mathcal{C}_R, W \in \mathcal{C}_S\}$. Then \mathcal{L}_G is a

lattice with join $\mathcal{C}_R \vee \mathcal{C}_S$ and meet $\mathcal{C}_R \wedge \mathcal{C}_S$. The partial order \leq of this lattice is defined by $\mathcal{C}_1 \leq \mathcal{C}_2$ if $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Note that, for any two transit functions R and S on G , we have

$$\text{if } R \leq S \text{ then } \mathcal{C}_R \leq \mathcal{C}_S.$$

The relation between meets and joins in the lattices L_G and \mathcal{L}_G is given in Theorem 1. Note that the structure of G does not play a role in this result. But it may when we consider subsets of the lattice.

Theorem 1. *Let R and S be transit functions on a connected graph G . Then $\mathcal{C}_R \wedge \mathcal{C}_S = \mathcal{C}_{R \vee S}$ and $\mathcal{C}_R \vee \mathcal{C}_S \subseteq \mathcal{C}_{R \wedge S}$.*

Proof. First we prove the formula for the meet $\mathcal{C}_R \wedge \mathcal{C}_S$:

$$\begin{aligned} W \in \mathcal{C}_{R \vee S} &\Leftrightarrow (R \vee S)(u, v) \subseteq W \text{ for all } u, v \in W \\ &\Leftrightarrow R(u, v) \cup S(u, v) \subseteq W \text{ for all } u, v \in W \\ &\Leftrightarrow R(u, v) \subseteq W \text{ and } S(u, v) \subseteq W \text{ for all } u, v \in W \\ &\Leftrightarrow W \in \mathcal{C}_R \text{ and } W \in \mathcal{C}_S \\ &\Leftrightarrow W \in \mathcal{C}_R \cap \mathcal{C}_S = \mathcal{C}_R \wedge \mathcal{C}_S. \end{aligned}$$

Next, we prove the formula for the join $\mathcal{C}_R \vee \mathcal{C}_S$. Choose any subset $W \in \mathcal{C}_R \vee \mathcal{C}_S$. By definition, there exist subsets $X \in \mathcal{C}_R$ and $Y \in \mathcal{C}_S$ such that $W = X \cap Y$. Now, for any $u, v \in W = X \cap Y$, we have $u, v \in X$ as well as $u, v \in Y$, so that $R(u, v) \subseteq X$ as well as $S(u, v) \subseteq Y$. This implies that $R(u, v) \wedge S(u, v) = R(u, v) \cap S(u, v) \subseteq X \cap Y = W$, whence $W \in \mathcal{C}_{R \wedge S}$. \square

The following example shows that we may have proper inclusion in the case of the join in Theorem 1. We take the complete graph K_5 on the vertex set $\{1, 2, 3, 4, 5\}$. We define the transit functions R and S as follows: $R(u, u) = S(u, u) = \{u\}$, for all vertices u ; $R(1, 2) = \{1, 2, 3\}$, $R(2, 3) = \{2, 3, 4\}$; $S(1, 2) = \{1, 2, 5\}$, $S(2, 5) = \{2, 4, 5\}$; and $R(u, v) = S(u, v) = \{u, v\}$ for any other pair of distinct vertices u and v . Then, we have $\langle \{1, 2\} \rangle_R = \{1, 2, 3, 4\}$, $\langle \{1, 2\} \rangle_S = \{1, 2, 4, 5\}$. Hence, we have $\langle \{1, 2\} \rangle_{\mathcal{C}_R \vee \mathcal{C}_S} = \langle \{1, 2\} \rangle_R \cap \langle \{1, 2\} \rangle_S = \{1, 2, 3, 4\} \cap \{1, 2, 4, 5\} = \{1, 2, 4\}$. On the other hand, we have $(R \wedge S)(1, 2) = R(1, 2) \cap S(1, 2) = \{1, 2, 3\} \cap \{1, 2, 5\} = \{1, 2\}$, so that $\langle \{1, 2\} \rangle_{R \wedge S} = \{1, 2\}$. Note that, since the graph was complete, the transit functions are trivially path transit functions.

4. Examples of path properties

In this section, we collect a number of specific path transit functions and list some basic facts. Let $G = (V, E)$ be a connected graph. If no confusion arises, then we may write F instead of F_G , for any function F_G on G .

4.1. The geodesic transit function

Let Γ be the family of all geodesics in G , and let d be the distance function of G . Then the geodesic transit function R_Γ of G is the well-known interval function I_G of G (see [12]),

which is defined as follows:

$$I_G(u, v) = \{x \in V \mid d(u, x) + d(x, v) = d(u, v)\}.$$

for u, v in V . The function I and the geodesic convexity of a connected graph G are important tools for the study of metric properties of G , see e.g. [4,12]. An example of a class of graphs where these tools are indispensable, is that of median graphs. These are defined by the property that, for any triple of vertices, the intervals between the pairs of the triple intersect in exactly one vertex. Prime examples are trees, hypercubes and grid graphs. There is by now a rich structure theory available for median graphs, see e.g. [10,12]. The definition of I is in terms of the distance function of G . In [15,16], Nebeský gave an interesting characterization of the interval function using transit axioms only. Thus, I is characterized without any reference to metric notions. It may be noted that no simple characterizations are available for the geodesically convex sets in a graph.

4.2. The induced-path transit function

The *induced-path transit function* J_G of G is defined as follows:

$$J_G(u, v) = \{z \in V \mid z \text{ lies on some induced } u, v\text{-path in } G\}$$

for each $u, v \in V$. The convexity of J is also known as the *minimal path convexity*, see e.g. [5,8]. The analogue of median graphs in the case of the function J is studied in [14]. The characterization of this transit function in terms of transit axioms alone seems to be difficult, but its convex sets are nicely characterized. Recall that a *clique* of G is a subset of V of pairwise adjacent vertices. We say that a clique S *separates* a vertex v from a subset W of V if every path between v and W passes through S . Note that, if W is a clique in itself, then, by definition, W is a clique separating v from W . The following characterization of the J -convex hull is due to Duchet [5]: *in a connected graph G a vertex v belongs to the J -convex hull of a subset W of V if and only if no clique of $G - v$ separates v and W .*

4.3. The all-paths transit function

The *all-paths transit function* A_G of G was already defined above in Section 2:

$$A_G(u, v) = \{x \in V \mid x \text{ lies on some } u, v\text{-path in } G\}$$

for u, v in V . It is the universal upper bound in the join-semilattice of all path transit functions. The convexity generated by A was studied in [5,17], where it is called the *coarsest convexity*. A characterization of A in terms of transit axioms only was recently established in [2]. The all-paths function has a nice structure reflecting the block-cut-vertex structure of the graph. Recall that a *block* of a graph is a maximal 2-connected subgraph. If G is 2-connected or if G is K_1 or K_2 , then A is the trivial transit function 1_G of G . If G contains a cut-vertex, then A is a non-trivial transit function. In this case G can be considered to be a tree of blocks. A ‘subtree of blocks’ is a non-trivial connected subgraph such that if it contains two vertices of a block then it contains the whole block. The non-trivial A -convex subgraphs are the proper subtrees of blocks.

4.4. The I_j -path transit function

For $j \geq 0$, the path transit function I_j is defined by

$$I_j(u, v) = \{z \in V \mid z \text{ lies on a } u, v\text{-path of length } \leq d(u, v) + j \text{ in } G\}$$

for any $u, v \in V$. Clearly, we have $I_k \leq I_{k+1}$, for every $k \geq 0$. Not much is known about this path transit function or its associated convexity. We present it here mainly because it seems to be a natural transit function for further study.

4.5. The triangle-path transit functions

Let $P = u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_k$ be a path in G . Let z_i be a vertex not on P but adjacent to two consecutive vertices u_i, u_{i+1} of P . Then we say that the path $Q = u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_i \rightarrow z_i \rightarrow u_{i+1} \rightarrow \dots \rightarrow u_k$ is obtained from P by replacing the edge $u_i \rightarrow u_{i+1}$ by a triangle. A *triangular extension* of a path P is a path Q obtained from P by replacing some of the edges of P by triangles. We will call P a triangular extension of itself as well. Let Φ be a (feasible) path property on G . Then Φ^Δ is the path property defined by

$$\Phi^\Delta = \{Q \mid Q \text{ is a triangular extension of some path in } \Phi\}.$$

Note that we have $\Phi \subseteq \Phi^\Delta$, with equality if and only if no path in Φ is involved in a triangle. In particular, we have equality in the case of a triangle-free graph. The path property $\Phi^{k\Delta}$ is defined recursively by $\Phi^{0\Delta} = \Phi$ and $\Phi^{k\Delta} = (\Phi^{(k-1)\Delta})^\Delta$, for $k \geq 1$.

Let R be a Φ -path transit function on G . Then R^Δ is the path transit function on G defined by

$$R^\Delta = R_{\Phi^\Delta}.$$

The transit function R^Δ is a *triangle-path transit function*. Note that $R^\Delta = R$ if no path in Φ is involved in a triangle. Recursively, we define $R^{k\Delta}$ by $R^{0\Delta} = R$ and $R^{k\Delta} = (R^{(k-1)\Delta})^\Delta$, for $k \geq 1$, see [13]. Clearly, $R^{k\Delta}$ is a path transit function as well. The following lemma follows immediately from the definitions.

Lemma 2. $R_\Phi^{k\Delta} = R_{\Phi^{k\Delta}}$, and $R^{k\Delta} \leq R^{(k+1)\Delta}$ for $k \geq 0$.

Note that we trivially have $A^\Delta = A$. But in general we will have $R^{(k-1)\Delta} < R^{k\Delta}$ if G contains triangles and k is not too large. For the associated convexities however, the situation can be quite different as is shown by the transit functions I_j , with $j \geq 0$, and the transit function J .

Proposition 3. Let G be a graph, and let Φ be a path property such that the path $u \rightarrow v$ is in Φ , for any edge uv of G . Then, for $k \geq 1$, $\mathcal{C}_{R^{k\Delta}} = \mathcal{C}_{R^{(k+1)\Delta}}$.

Proof. Using Lemma 2, we deduce that $\mathcal{C}_{R^{(k+1)\Delta}} \subseteq \mathcal{C}_{R^{k\Delta}}$. To prove that $\mathcal{C}_{R^{k\Delta}} \subseteq \mathcal{C}_{R^{(k+1)\Delta}}$, let W be a set in $\mathcal{C}_{R^{k\Delta}}$. Take any two vertices u and v in W . Let z be a vertex in $R^{(k+1)\Delta}(u, v)$ —

$R^{k\Delta}(u, v)$. Then, there exists a u, v -path P in $\Phi^{k\Delta}$ such that z is adjacent to two consecutive vertices x and y on P . Note that x and y are in W . Now $x \rightarrow y$ is a path in Φ . Since $k \geq 1$, we know that $R^\Delta(x, y) \subseteq R^{k\Delta}(x, y) \subseteq W$. Clearly, we have $z \in R^\Delta(x, y)$, whence $z \in W$. Thus, it follows that $R^{(k+1)\Delta}(u, v) \subseteq W$, by which we have shown that $W \in \mathcal{C}_{R^{(k+1)\Delta}}$. \square

All path properties in this section satisfy the condition in the Proposition. But it leaves open the question whether it holds for any other path property:

For which other path transit functions R on G do the triangular path functions $R^{k\Delta}$ define the same convexity on G , for all $k \geq 1$?

Since the I -convex sets are difficult to characterize, we may expect that the I^Δ -convex sets are also difficult to characterize. For J^Δ we have the following characterization, see [3]: let $G = (V, E)$ be a connected graph, and let $W \subseteq V$; then a vertex v does not belong to the J^Δ -convex hull of W if and only if there exists a clique M separating v and W in such a way that any two paths connecting v to two distinct vertices of M contain vertices that induce a cycle of length at least 4 in G .

Clearly, we have $I^{k\Delta} \leq I_k$, for every $k \geq 0$.

5. Path transit functions of Cartesian products of graphs

In this section, we discuss path transit functions on Cartesian products of graphs. First, we recall the definition of Cartesian product. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The *Cartesian product* $G_1 \square G_2$ of G_1 and G_2 is the graph with vertex set $V_1 \times V_2$, where two vertices $(u_1, u_2), (v_1, v_2)$ in $V_1 \times V_2$ are joined by an edge if and only if either $u_1 = v_1$ and $u_2 v_2 \in E_2$ or $u_2 = v_2$ and $u_1 v_1 \in E_1$. The *i -th projection* of $G_1 \square G_2$ is the mapping π_i defined by $\pi_i(u_1, u_2) = u_i$, for $i = 1, 2$. Note that these projections are graph homomorphisms. Also note that paths need not be projected on paths. If, say, G_2 is the *trivial graph* K_1 , then π_1 is an isomorphism between G_1 and $G_1 \square G_2$.

Let R_1 be a transit function on G_1 , and let R_2 be a transit function on G_2 . Then the function $R_1 \square R_2: V(G_1 \square G_2) \times V(G_1 \square G_2) \rightarrow V(G_1 \square G_2)$ defined by

$$R_1 \square R_2((u_1, u_2), (v_1, v_2)) = R_1(u_1, v_1) \times R_2(u_2, v_2)$$

is a transit function on $G_1 \square G_2$.

Let \mathcal{C}_1 and \mathcal{C}_2 be two convexities on the sets V_1 and V_2 , respectively. Then

$$\mathcal{C}_1 \square \mathcal{C}_2 = \{X \times Y \mid X \in \mathcal{C}_1, Y \in \mathcal{C}_2\},$$

is a convexity as well, see e.g. [18,19,21]. Moreover, also the following equality holds (see e.g. [21]):

$$\mathcal{C}_{R_1 \square R_2} = \mathcal{C}_{R_1} \square \mathcal{C}_{R_2}.$$

We want to specialize these equalities for the case of path properties. Let Φ_1 be a path property on G_1 , and let Φ_2 be a path property on G_2 . Now the question is how to construct a path property on $G_1 \square G_2$ starting from Φ_1 and Φ_2 . Let u_i and v_i be vertices in G_i and

let P_i be a path in $\Phi_i(u_i, v_i)$, for $i = 1, 2$. Then, intuitively, a $(u_1, u_2), (v_1, v_2)$ -path in $(\Phi_1 \square \Phi_2)((u_1, u_2), (v_1, v_2))$ should be constructed in some way from paths P_1 and P_2 , or otherwise stated, should be some path between (u_1, u_2) and (v_1, v_2) in $P_1 \square P_2$. Note that, for any such path Q , we have $\pi_i(Q) = P_i$, for $i = 1, 2$. There are many possible choices. Some choices make more sense than others. A choice that would certainly make sense is all paths Q such that the length $l(Q)$ equals $l(P_1) + l(P_2)$. But when we look from the perspective of the associated transit functions, it turns out that, loosely speaking, it does not matter what choice we make. This is made precise in the following way. Let $(\Phi_1 \square \Phi_2)((u_1, u_2), (v_1, v_2))$ be the set of paths Q in $G_1 \square G_2$ such that the projection $\pi_i(Q)$ of Q is a Φ_i -path in G_i between u_i and v_i , for $i = 1, 2$. Note that, for any P_1 in $\Phi_1(u_1, v_1)$ and P_2 in $\Phi_2(u_2, v_2)$, all paths in $P_1 \square P_2$ between (u_1, u_2) and (v_1, v_2) are in $(\Phi_1 \square \Phi_2)((u_1, u_2), (v_1, v_2))$. It is obvious that, Φ_1 and Φ_2 being feasible, $\Phi_1 \square \Phi_2$ is a feasible path property on $G_1 \square G_2$. The following proposition tells us that our choice of $\Phi_1 \square \Phi_2$ does not contain ‘too many’ paths.

Proposition 4. *Let G_1 and G_2 be two connected graphs, and let Φ_1 be a path property on G_1 and Φ_2 be a path property on G_2 . Then*

$$R_{\Phi_1 \square \Phi_2} = R_{\Phi_1} \square R_{\Phi_2}.$$

Proof. Take two vertices (u_1, u_2) and (v_1, v_2) in $G_1 \square G_2$. Let (z_1, z_2) be a vertex in $R_{\Phi_1 \square \Phi_2}((u_1, u_2), (v_1, v_2))$. Let Q be a path in $\Phi_1 \square \Phi_2((u_1, u_2), (v_1, v_2))$ containing (z_1, z_2) such that $P_1 = \pi_1(Q)$ is a Φ_1 -path in G_1 and $P_2 = \pi_2(Q)$ is a Φ_2 -path in G_2 . Then P_i contains z_i , so that z_i lies in $R_{\Phi_i}(u_i, v_i)$, for $i = 1, 2$. Hence (z_1, z_2) lies in $R_{\Phi_1} \square R_{\Phi_2}((u_1, u_2), (v_1, v_2))$.

Conversely, let (z_1, z_2) be a vertex in $R_{\Phi_1} \square R_{\Phi_2}((u_1, u_2), (v_1, v_2))$. Then there exists a Φ_i -path P_i between u_i and v_i in G_i containing z_i , for $i = 1, 2$. Let Q be the path in $G_1 \square G_2$ constructed as follows: loosely speaking, we start in (u_1, u_2) . Now, we walk along the copy of P_1 fixing u_2 until we arrive at (z_1, u_2) . Then we continue along the copy of P_2 fixing z_1 until we arrive at (z_1, v_2) . Along the way we passed through (z_1, z_2) . Finally, we continue along P_1 fixing v_2 until we arrive at (u_2, v_2) . Clearly, we have $\pi_1(Q) = P_1$ and $\pi_2(Q) = P_2$. This implies that (z_1, z_2) lies in $R_{\Phi_1 \square \Phi_2}((u_1, u_2), (v_1, v_2))$, and we are done. \square

Corollary 5. *Let G_1 and G_2 be two connected graphs, and let Φ_1 be a path property on G_1 and Φ_2 be a path property on G_2 . Then*

$$\mathcal{C}_{\Phi_1 \square \Phi_2} = \mathcal{C}_{\Phi_1} \square \mathcal{C}_{\Phi_2}.$$

Proof. By definition, $R_{\Phi_1 \square \Phi_2}$ is the transit function of $\mathcal{C}_{\Phi_1 \square \Phi_2}$. It is straightforward to check that $R_{\Phi_1} \square R_{\Phi_2}$ is the transit function of $\mathcal{C}_{\Phi_1} \square \mathcal{C}_{\Phi_2}$. \square

Let G_1 and G_2 be two non-trivial connected graphs. The following equality is part of folklore and follows immediately from the definitions:

$$I_{G_1 \square G_2} = I_{G_1} \square I_{G_2}. \quad (1)$$

We could formulate this feature as follows. As before let Γ be the path property ‘shortest’, by which we formally mean that Γ is the set of all geodesics in a graph G . In this terminology

we can write $I_G = R_\Gamma(G)$. Then (1) could be written as $R_\Gamma(G_1 \square G_2) = R_\Gamma(G_1) \square R_\Gamma(G_2)$. Hence, by Proposition 4, we have $R_\Gamma(G_1 \square G_2) = R_{\Gamma \square \Gamma}(G_1 \square G_2)$. Loosely speaking, for the property $\Gamma =$ ‘shortest’ we have $R_\Gamma = R_{\Gamma \square \Gamma}$. This gives rise to the following question. Let Ψ be a graph property that can be defined on any graph similar to ‘shortest’. Instances are the path functions given in Section 4. Which graph properties in this sense are ‘product stable’? By this we mean

For which such path properties Ψ does the following equality hold:

$$R_\Psi(G_1) \square R_\Psi(G_2) = R_\Psi(G_1 \square G_2)?$$

Equality for the transit functions yields trivially equality for the associated convexities. But, inequality for the transit functions does not necessarily imply inequality for the associated convexities. So, we have also the following question:

For which such path properties Ψ does the following equality hold:

$$\mathcal{C}_\Psi(G_1 \square G_2) = \mathcal{C}_\Psi(G_1) \square \mathcal{C}_\Psi(G_2)?$$

First let Ψ be the property ‘all-paths’, that is, $R_\Psi = A_G$. Note that, for any two non-trivial connected graphs G_1 and G_2 , their Cartesian product $G_1 \square G_2$ is 2-connected. Hence, $A_{G_1 \square G_2}$ is the trivial transit function on $G_1 \square G_2$. On the other hand, if at least one of G_1 and G_2 contains a cut-vertex, say G_1 contains a cut-vertex, then A_{G_1} is not the trivial transit function, whence also $A_{G_1} \square A_{G_2}$ is not the trivial transit function. So, as soon as at least one of the factors of a Cartesian product contains a cut-vertex, then we have $A_{G_1} \square A_{G_2} < A_{G_1 \square G_2}$. But if each of the factors is either K_2 or 2-connected, then we have $A_{G_1} \square A_{G_2} = A_{G_1 \square G_2} = 1_{G_1 \square G_2}$. Summarizing, for non-trivial connected graphs G_1 and G_2 we have

$$A_{G_1} \square A_{G_2} \leq A_{G_1 \square G_2} = 1_{G_1 \square G_2}.$$

Note that any block of a graph G is A_G -convex. Assume that both G_1 and G_2 contain a cut-vertex, and let B_1 be a block in G_1 and B_2 in G_2 . Then B_i is A_{G_i} -convex, for $i = 1, 2$, so $B_1 \square B_2$ is $A_{G_1} \square A_{G_2}$ -convex, but it is not $A_{G_1 \square G_2}$ -convex, since $A_{G_1 \square G_2}$ is the trivial transit function.

For the induced-path function J the answer on the above questions is also negative. Let Ω be the property ‘induced’, so that $J_G = R_\Omega(G)$. Take an edge uv in G_1 and an edge xy in G_2 . Then $J_{G_1}(u, v) \square J_{G_2}(x, y) = \{u, v\} \times \{x, y\}$. But in general $J_{G_1 \square G_2}((u, x), (v, y))$ is a much larger set, because we may find many induced paths going out of $\{u, v\} \times \{x, y\}$. If we take the edges uv and xy to be such that they are not in a triangle, then we have $J_{G_1}^{k\Delta}(u, v) = \{u, v\}$ and $J_{G_2}^{k\Delta}(x, y) = \{x, y\}$. And again we will have that, in general, $J_{G_1 \square G_2}^{k\Delta}((u, x), (v, y))$ is a much larger set. To show that we have inequality for the convexities, just note that any edge is J -convex, and any edge not on a triangle is $J^{k\Delta}$ -convex.

On the other hand, take any vertex (z_1, z_2) in $J_{G_1}^{k\Delta} \square J_{G_2}^{k\Delta}((u_1, u_2), (v_1, v_2))$. Then z_1 is on an $\Omega_{G_1}^{k\Delta}$ -path P_1 between u_1 and v_1 in G_1 and z_2 is on an $\Omega_{G_2}^{k\Delta}$ -path P_2 between u_2 and v_2 in G_2 . From these two paths we easily construct an $\Omega_{G_1 \square G_2}^{k\Delta}$ -path between (u_1, u_2) and

(v_1, v_2) in $G_1 \square G_2$ containing (z_1, z_2) . Hence we have

$$J_{G_1}^{k\Delta} \square J_{G_2}^{k\Delta} \leq J_{G_1 \square G_2}^{k\Delta}$$

for $k \geq 0$.

It is obvious that I_j is not product stable, for any $j \geq 1$. But in this case we can say even more. Let d, j, k be positive integers, let G_1 be the path P_{d+1} of length d , and let G_2 be the cycle C_n on $n = d + j + k$ vertices. Note that on G_1 we have $I_j = I$. Let u, v be the end points of G_1 , and let x, y be two vertices on G_2 at distance d . Then we have $I_k(x, y) = I(x, y)$. So the Cartesian product of these two intervals is a proper subset of the vertex set of $G_1 \square G_2$. On the other hand, $I_{j+k}((u, x), (v, y))$ in $G_1 \square G_2$ is the whole vertex set. It is easy to see that we have

$$I_j \square I_k(G_1 \square G_2) \leq I_{j+k}(G_1 \square G_2).$$

Let G_1 and G_2 both be the triangle graph K_3 on the vertices u, v, w . Then $I_{G_1}^\Delta(u, v) \times I_{G_2}^\Delta(u, v)$ is the whole vertex set of $G_1 \square G_2$. On the other hand, the vertex (w, w) is not in $I_{G_1 \square G_2}^\Delta((u, u), (v, v))$. So also I^Δ is not product stable. Similar examples can be used to show that $I^{k\Delta}$ is not product stable, for any $k \geq 1$. However, in this case, the situation for the convexities is different.

Theorem 6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected graphs, and let $k \geq 0$. Then $\mathcal{C}_{I^{k\Delta}}(G_1 \square G_2) = \mathcal{C}_{I^{k\Delta}}(G_1) \square \mathcal{C}_{I^{k\Delta}}(G_2)$.

Proof. The case $k = 0$ is a special case: it follows immediately from $I(G_1 \square G_2) = I(G_1) \square I(G_2)$, which is part of folklore. The proof for $k \geq 1$ is by induction on k . First we prove that $\mathcal{C}_{I^\Delta}(G_1 \square G_2) = \mathcal{C}_{I^\Delta}(G_1) \square \mathcal{C}_{I^\Delta}(G_2)$.

Choose any set W in $\mathcal{C}_{I^\Delta}(G_1) \square \mathcal{C}_{I^\Delta}(G_2)$. Then $W = X \times Y$ with X in $\mathcal{C}_{I^\Delta}(G_1)$ and Y in $\mathcal{C}_{I^\Delta}(G_2)$. Take any two vertices (u_1, u_2) and (v_1, v_2) in W , and let (z_1, z_2) be a vertex in $I_{G_1 \square G_2}^\Delta((u_1, u_2), (v_1, v_2))$. If (z_1, z_2) is on some geodesic between (u_1, u_2) and (v_1, v_2) in $G_1 \square G_2$, then (z_1, z_2) is in $I_{G_1}(u_1, v_1) \times I_{G_2}(u_2, v_2)$, whence in W , and we are done. Otherwise, there exists a geodesic P between (u_1, u_2) and (v_1, v_2) in $G_1 \square G_2$, such that (z_1, z_2) is adjacent to two consecutive vertices (x_1, x_2) and (y_1, y_2) on P . Now (z_1, z_2) , (x_1, x_2) and (y_1, y_2) form a triangle in $G_1 \square G_2$. This is only possible if either x_1, y_1, z_1 form a triangle in G_1 and $x_2 = y_2 = z_2$ in G_2 , or x_2, y_2, z_2 form a triangle in G_2 and $x_1 = y_1 = z_1$ in G_1 . In the first case we have z_1 in X , since X is $I_{G_1}^\Delta$ -convex, and trivially z_2 in Y . In the second case we have z_2 in Y , since Y is $I_{G_2}^\Delta$ -convex, and trivially z_1 in X . So we conclude that in both cases that (z_1, z_2) lies in W . This shows that W is in $\mathcal{C}_{I^\Delta}(G_1 \square G_2)$.

Conversely, choose any set W in $\mathcal{C}_{I^\Delta}(G_1 \square G_2)$. We will prove that $\pi_i(W)$ is $I_{G_i}^\Delta$ -convex in G_i , for $i = 1, 2$, and that $W = \pi_1(W) \times \pi_2(W)$.

Choose any two vertices u_1 and v_1 in $\pi_1(W)$. Then, by the definition of projections, there exist vertices u_2, v_2 in $\pi_2(W)$ such that (u_1, u_2) and (v_1, v_2) are vertices in W . Let P_2 be a geodesic between u_2 and v_2 in G_2 . Take any vertex z_1 in $I_{G_1}^\Delta(u_1, v_1)$. Then there is a geodesic P_1 between u_1 and v_1 in G_1 such that either z_1 is on P_1 or z_1 is adjacent to two consecutive vertices x_1, y_1 on P_1 . Now $Q = (P_1 \square \{u_2\}) \cup (\{v_1\} \square P_2)$ is a subgraph of

$G_1 \square G_2$, which is a geodesic between (u_1, v_1) and (u_2, v_2) . Then either (z_1, u_2) lies on Q or (z_1, u_2) is adjacent to the two consecutive vertices $(x_1, u_2), (y_1, u_2)$ on Q . Since W is $I_{G_1 \square G_2}^\Delta$ -convex, it follows in both cases that (z_1, u_2) lies in W . But this implies that z_1 lies in $\pi_1(W)$. Hence $\pi_1(W)$ is $I_{G_1}^\Delta$ -convex. Similarly, we deduce that $\pi_2(W)$ is $I_{G_2}^\Delta$ -convex.

Clearly, we have $W \subseteq \pi_1(W) \times \pi_2(W)$. So let (z_1, z_2) be a vertex in $\pi_1(W) \times \pi_2(W)$. By the definition of projections, there exists a vertex x_2 in $\pi_2(W)$ such that (z_1, x_2) lies in W , and there exists a vertex x_1 in $\pi_1(W)$ such that (x_1, z_2) lies in W . Let P_1 be a geodesic between x_1 and z_1 in G_1 , and let P_2 be a geodesic between z_2 and x_2 in G_2 . Then $Q = (P_1 \square \{z_2\}) \cup (\{z_1\} \square P_2)$ is a geodesic between (x_1, z_2) and (z_1, x_2) in $G_1 \square G_2$ containing (z_1, z_2) . Since W is $I_{G_1 \square G_2}^\Delta$ -convex, it follows that (z_1, z_2) lies in W . This shows that $W = \pi_1(W) \times \pi_2(W)$, which concludes the proof in the case $k = 1$.

Finally, let $k > 1$. Then we have

$$\begin{aligned} \mathcal{C}_{I^{k\Delta}}(G_1 \square G_2) &= \mathcal{C}_{I^{(k-1)\Delta}}(G_1 \square G_2) && \text{(by Proposition 3)} \\ &= \mathcal{C}_{I^{(k-1)\Delta}}(G_1) \square \mathcal{C}_{I^{(k-1)\Delta}}(G_2) && \text{(by induction)} \\ &= \mathcal{C}_{I^{k\Delta}}(G_1) \square \mathcal{C}_{I^{k\Delta}}(G_2) && \text{(by Proposition 3).} \end{aligned}$$

This concludes the proof. \square

6. Convexity invariants

In this section, we survey the classical convexity invariants such as the Helly, Carathéodory, and Radon numbers and the exchange number (see [7,11,20]) for the path properties of Section 4, except for I_j , of which not much is known as yet. Along the way we give improvements of some of the known bounds. We start with shortly recalling the various definitions. Let Φ be a path property. A Φ -copoint of a point p of V is a maximal Φ -convex subset of V not containing p . The *Carathéodory number* c of the convexity space \mathcal{C} is the smallest integer (if it exists) such that for any finite subset F of V , $\langle F \rangle_{\mathcal{C}} = \bigcup \{ \langle S \rangle_{\mathcal{C}} \mid S \subseteq F, |S| \leq c \}$. The *exchange number* e of \mathcal{C} is the smallest integer (if it exists) such that for any subset F of V with $|F| \geq e$ and any point p in F , we have $\langle F - p \rangle_{\mathcal{C}} \subseteq \bigcup \{ \langle F - a \rangle_{\mathcal{C}} \mid a \in F - p \}$. The *Helly number* h of \mathcal{C} is the smallest integer (if it exists) such that every family of convex sets with an empty intersection contains a subfamily of at most h members with an empty intersection. Equivalently, h is the smallest natural number such that $\bigcap_{s \in F} \langle F - s \rangle_{\mathcal{C}} \neq \emptyset$ for every $(h + 1)$ -element subset F of V . The *Radon number* r of \mathcal{C} is the smallest integer (if it exists) such that every r -element set $S \subseteq V$ admits a Radon partition, that is, a partition $S = S_1 \cup S_2, (S_1 \cap S_2 = \emptyset)$ with $\langle S_1 \rangle_{\mathcal{C}} \cap \langle S_2 \rangle_{\mathcal{C}} \neq \emptyset$. The m -th *Radon number*, denoted by r_m , is the smallest number (if it exists) such that every r_m -element set $W \subseteq V$ admits a Radon m -partition, that is a partition of S into m pairwise disjoint subsets W_1, W_2, \dots, W_m such that $\langle W_1 \rangle_{\mathcal{C}} \cap \langle W_2 \rangle_{\mathcal{C}} \cap \dots \cap \langle W_m \rangle_{\mathcal{C}} \neq \emptyset$.

The *clique number* ω is the cardinality of the largest clique in G . A subset $S \subseteq V$ is called a *convex-independent set* if $x \notin \langle S - x \rangle_{\mathcal{C}}$ for every $x \in S$. The *rank* of \mathcal{C} is the supremum of the cardinalities of the independent subsets of V . The *hull number* u of \mathcal{C} is the infimum of the cardinalities of subsets S of V such that $\langle S \rangle_{\mathcal{C}} = V$.

6.1. The geodesic convexity

The geodesic convexity is in some sense “universal” with respect to the above mentioned invariants, namely in [6] it is observed that for every convexity on a finite set V , with Helly, Radon and m -th Radon numbers h , r and r_m , respectively, there is a finite connected graph G whose geodesic convexity has Helly number h , Radon number r and m -th Radon number at least r_m . So far no relationships between the invariants Carathéodory, Helly and Radon numbers and any known graph parameter are known. Note that the n -cube Q_n has $h = 2$, $c = n$ and $r = \lceil \log_2(n + 1) \rceil + 2$.

6.2. The induced-path convexity

For the induced-path convexity, Duchet determined in [5] the relationships between the Helly and Radon numbers and the clique number. It is also shown there that the Carathéodory number c satisfies $c \leq 2$. Using the inequality $e \leq c + 1$ [18], it follows that the exchange number satisfies $e \leq 3$. Duchet’s result is as follows.

Theorem (Duchet, 1988). *For the J -convexity, the Carathéodory number satisfies $c \leq 2$, the Helly number satisfies $h = \omega$ and the Radon number satisfies $r = \omega + 1$ if $\omega \geq 3$ and $r \leq 4$ if $\omega \leq 2$.*

In Theorem 8, we will characterize the cases $r = 3$ and 4 for triangle-free graphs, i.e. graphs with $\omega \leq 2$. First we need some preliminaries. A *cut-edge* is an edge in G such that the removal of its end-vertices disconnects G .

Lemma 7. *Let $G = (V, E)$ be a 2-connected, triangle-free graph without cut-edges. Then the J -convex hull of any two non-adjacent vertices in G equals V .*

Proof. Note that, G being triangle-free and 2-connected, there are non-adjacent vertices. Let u and v be any pair of non-adjacent vertices in G , and let S be the convex hull of $\{u, v\}$. Assume that $S \neq V$. Choose any vertex w in $V - S$. By Menger’s theorem we can find two internally disjoint paths P_x and P_y starting in w and ending in distinct vertices x and y in S . We may take x to be the first vertex of P_x in S and y to be the first vertex of P_y in S . Now we deduce that xy is an edge. For, otherwise, we can find an induced path within $P_x \cup P_y$ between x and y going out of S . And this would contradict the fact that S is J -convex.

Since G does not contain cut-edges, there must be a path P_z from w to a vertex z in S distinct from x and y such that z is the first vertex of P_z in S . As in the case of xy , we deduce that xz and yz are edges as well. But now we have created a triangle on x , y , and z , which is impossible. This implies that $S = V$, so that the convex hull of u and v is the whole graph. \square

Let G be a 2-connected, triangle-free graph. An *atom* of G is a maximal 2-connected subgraph of G not having a cut-edge. The *atom-cut-edge tree* $T(G)$ of G is the graph with the atoms and the cut-edges of G as its vertices, and two vertices in $T(G)$ are

adjacent whenever one of them is a cut-edge and the other is an atom containing that edge. Note that, within that atom, the edge is *not* a cut-edge. It is straightforward to verify that $T(G)$ is indeed a tree.

Theorem 8. *Let G be a connected triangle-free graph with at least three vertices. The Radon number r of the J -convexity of G is 3 if and only if either G is a path or G is 2-connected and the atom-cut-edge tree of G is a path. In all other cases $r = 4$.*

Proof. If G is a path of length at least 3, then clearly $r = 3$. If G is not a path and not 2-connected, then let v be a cut-vertex of G of degree at least three. Any set of three neighbors of v with two neighbors belonging to distinct components of $G - v$ has no Radon partition. So we have $r = 4$.

So let G be 2-connected. First, we determine the J -convex hull of two non-adjacent vertices u and v . If they lie in the same atom H of G , then, by Lemma 7, their convex hull is H . So suppose that they lie in different atoms H_u and H_v . In the case that u is on a cut-edge uu' , then we choose H_u to be the atom such that any induced u, v -path contains vertices of H_u different from u and u' . We make a similar choice in case v is on a cut-edge. Let P be the path in $T(G)$ between H_u and H_v , and let $H_u, H_1, \dots, H_k, H_v$ be the atoms on P in the order that we encounter them while going from H_u to H_v along P . Let H be the subgraph consisting of the union of these atoms. We will show that H is the J -convex hull of u and v . By Lemma 7, it suffices to show that the J -convex hull contains two non-adjacent vertices of every atom in H .

By the choice of H_u , the vertex u does not lie on the cut-edge xy between H_u and H_1 . There exists an induced path in H_u between u and x . This path can be extended to an induced u, v -path, so that x is in the J -convex hull of $\{u, v\}$. Similarly, the same holds for y . Since G is triangle-free, u cannot be adjacent to both x and y . So there are two non-adjacent vertices of H_u in the J -convex hull of $\{u, v\}$. Now, we replace H_u and u by H_1 and x (or y), respectively, and deduce that also H_1 is in the J -convex hull of $\{u, v\}$. Proceeding in this way, we deduce that H is precisely the J -convex hull of $\{u, v\}$.

From these observations we easily deduce that, if $T(G)$ is a path, then any three vertices of G admit a Radon partition, so that $r = 3$.

Finally, if $T(G)$ is not a path, then take three different end vertices of $T(G)$. In each of the corresponding atoms of G , choose a vertex that is not on a cut-edge. Then our observations above tell us that there is no Radon partition for these three vertices. Hence we have $r = 4$. This concludes the proof. \square

From the characterization of the J -convex hull in Section 4.2 we know that, for any connected graph G and any vertex p , any two distinct copoints of p are non-intersecting. For, consider two distinct copoints U_p and W_p of a vertex p of G . Since U_p and W_p are distinct J -convex sets, they are separated by a clique and hence have no vertex in common. Therefore U_p and W_p are non-intersecting. Let $m, k \geq 1$. A convexity \mathcal{C} on V has the \mathcal{C} -copoint intersection property $CIP(m, k)$ if and only if for each p in V , it holds that any set of m distinct \mathcal{C} -copoints at p contains a k -subset with an empty intersection. In [9], the following result was proved.

Theorem (Jamison, 1981). *Let the convexity \mathcal{C} on V satisfy $CIP(3, 2)$ with finite Helly number h . Then for each $m \geq 1$, $r_m \leq 2m$ if $h = 2$, and $r_m = (m - 1)h + 1$ if $h \geq 3$.*

By the above observations, the J -convexity satisfies $CIP(3, 2)$. Therefore, we have the following result.

Corollary 9. *The J -convexity on a connected graph satisfies $r_m \leq 2m$ if $\omega = 2$ and $r_m = (m - 1)\omega + 1$ if $\omega \geq 3$.*

6.3. Triangle-path convexities

By Proposition 3, we need to consider only I^Δ and J^Δ . As in the case of the geodesic convexity, no bound or relationship between the invariants of the I^Δ -convexity and any other known graph parameter is known. But, for the J^Δ -convexity, the bounds for the invariants are known. The following result can be found in [3]: the J^Δ -convexity has Carathéodory number $c = 2$, exchange number $e = 3$, Helly number $h = 2$ and Radon number r satisfying $3 \leq r \leq 4$.

From the characterization of the J^Δ -convex hull mentioned in Section 4.5, we get, similar to the J -convexity, the following result for the J^Δ -convexity in a connected graph G : *for the J^Δ -convexity, given any vertex p of G , any two distinct copoints of p are non-intersecting.*

The J^Δ -convexity satisfies $CIP(3, 2)$, by the previous discussion. Therefore as a corollary of the theorem of Jamison [9], we have the following theorem.

Theorem 10. *Let $m \geq 1$. The m th Radon number for the J^Δ -convexity satisfies $r_m \leq 2m$.*

6.4. The all-paths convexity

The Carathéodory, Helly and Radon numbers for the all-paths convexity were investigated in [17]. Recall that the *block-cut-vertex tree* $B(G)$ of a connected graph G has the blocks and cut-vertices of G as its vertices and two vertices of $B(G)$ are adjacent whenever one of them is a cut-vertex and the other a block such that the cut-vertex is a vertex of the block. The hull number u and the rank of the all-paths convexity can be phrased in terms of $B(G)$. We summarize these results in the following theorem.

Theorem 11. *For the all-paths convexity, the Carathéodory number satisfies $c = 2$, the exchange number satisfies $e = 3$, the Helly number satisfies $h = 2$, the Radon number satisfies $3 \leq r \leq 4$, and the m th Radon number satisfies $r_m \leq 2m$. The hull number and the rank are both equal to the number of end vertices in $B(G)$.*

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