Semi-local Liouville equivalence of complex Hamiltonian systems defined by rational Hamiltonian

Martynchuk, Nikolay

Published in:
Topology and its applications

DOI:
10.1016/j.topol.2015.05.090

Publication date:
2015

Topology and its Applications 191 (2015) 119–130

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Semi-local Liouville equivalence of complex Hamiltonian systems defined by rational Hamiltonian

N.N. Martynchuk [a,b,*]

[a] *Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia*
[b] *Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, PO Box 407, 9700 AK, Groningen, The Netherlands*

ARTICLE INFO

Article history:
Received 25 June 2014
Received in revised form 17 December 2014
Accepted 31 May 2015
Available online 11 June 2015

Keywords:
Hamiltonian system
Liouville theorem
Topological equivalence

ABSTRACT

Consider the holomorphic function $f = az^2 + R(w)$, where z and w are complex variables and R is a rational function. Let $D_{\xi_0,\varepsilon}$ be a small disc around $\xi_0 \in \mathbb{C}$. Function f defines the foliation in the neighborhood $f^{-1}(D_{\xi_0,\varepsilon})$ of the (singular) fiber $f^{-1}(\xi_0)$. We give a complete topological classification of such foliations.

1. Introduction

The aim of this paper is to give a *semi-local topological classification* (i.e. the classification up to *semi-local TR-equivalence*, see Definition 2.3) of foliations generated by the holomorphic function $f = az^2 + R(w)\colon M_{\mathbb{C}} \to \mathbb{C}$. Here $a \in \mathbb{C} \setminus \{0\}$ and $M_{\mathbb{C}} = \mathbb{C} \times (\mathbb{C} \setminus \{d_1, \dots, d_s\})$, where $d_1, \dots, d_s$ are the poles of the rational function R. The problem of semi-local topological classification of such systems came from the theory of integrable Hamiltonian systems. Let us explain how this problem originated and give a brief overview of closely related questions.

Suppose we have an integrable Hamiltonian system (M, ω, H), $\dim_{\mathbb{R}} M = 2N$, with pairwise *involutive* first integrals $H = H_1, \dots, H_N$. Consider the *momentum map* $\Phi = (H_1, \dots, H_N)\colon M \to \mathbb{R}^N$ and the corresponding *Liouville foliation* of the phase space M, i.e. the decomposition of M into connected components of $\Phi^{-1}(c),\ c \in \mathbb{R}^N$. If the vector fields $\operatorname{sgrad} H_i = \omega^{-1}(dH_i)$ are *complete*, the system is called *completely integrable*. In this case we can use the Liouville theorem (see [1]), which describes the topology of each nonsingular fiber $\Phi^{-1}(c)$, the topology of the foliation in a neighborhood of each connected compact nonsingular fiber and also the action-angle coordinates in this neighborhood.

* Correspondence to: Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia.
E-mail address: mnick45@bk.ru.

http://dx.doi.org/10.1016/j.topol.2015.05.090
0166-8641/

In particular, the *Liouville theorem* states that for each connected compact nonsingular fiber of a completely integrable Hamiltonian system with N degrees of freedom there exists a small neighborhood of this fiber that is fiberwise homeomorphic to a direct product $D^N \times T^N$ of an N-dimensional disk and an N-dimensional torus. It is natural to ask what happens in the general case: how to classify *Liouville foliations* in small neighborhoods of (singular) fibers of integrable systems up to *Liouville equivalence*, i.e. how to give a *semi-local Liouville classification*?

The semi-local Liouville classification of completely integrable Hamiltonian systems with hyperbolic singularities was given in the work [22], with focus–focus singularities — in [24]. The global topological structure of a regular isoenergy 3-surface of a completely integrable nondegenerate system with 2 degrees of freedom is described by the *Fomenko–Zieschang invariant* (the *marked molecule*), see [1] and [7].

Primarily, the works [1,5–11] and also [22,24] dealt with completely integrable Hamiltonian systems with compact fibers and nondegenerate singularities. Because of this A. Fomenko suggested to extend the developed theory onto integrable systems that do not satisfy the above conditions. In particular, A. Fomenko stated a problem of generalizing the Liouville theorem for integrable systems with *incomplete flows*, namely, for every integrable system from some "natural" class describe the topology of each (singular) fiber $\Phi^{-1}(c)$, the topology of the foliation in a neighborhood of each (nonsingular) fiber and construct the analogue of the action-angle coordinates; here Φ is a momentum map of the corresponding system with N degrees of freedom and $c \in \mathbb{R}^N$. For these reasons A. Fomenko suggested a special class of integrable systems: $(M_{\mathbb{C}}, \operatorname{Re}(dz \wedge dw), \operatorname{Re} f)$, where f is a holomorphic function on a complex manifold $M_{\mathbb{C}} \subset \mathbb{C}^2$, see [15–20]. Because of the Cauchy–Riemann equations the Poisson bracket $\{\operatorname{Re}(f), \operatorname{Im}(f)\} = 0$. Therefore, the Liouville foliation corresponding to a system $(M_{\mathbb{C}}, \operatorname{Re}(dz \wedge dw), \operatorname{Re} f)$ is generated by the momentum map $\Phi = (\operatorname{Re} f, \operatorname{Im} f)$, which is just f. The first one who pointed this out was H. Flashka (see [4]).

Deformations of level surfaces of (Laurent) polynomials, their topology and homotopy type were studied in [2], [3] and [12]. Topological properties of *elliptic foliations* on non-singular compact complex manifolds can be found in works [13] and [14]. The semi-local topological classification as well as the analogue of the Liouville theorem for Hamiltonian systems defined by a complex *hyperelliptic Hamiltonian* are described in [16] and [17].

In this paper we consider integrable Hamiltonian systems $(M_{\mathbb{C}}, \operatorname{Re}(dz \wedge dw), \operatorname{Re} f)$ defined by a *hyperelliptic rational Hamiltonian* $f(z,w) = az^2 + R(w)$. Recall that $a \in \mathbb{C} \setminus \{0\}$, R is a rational function and $M_{\mathbb{C}} \subset \mathbb{C}^2$ is the domain of the holomorphic function f, i.e. $M_{\mathbb{C}} = \mathbb{C} \times (\mathbb{C} \setminus \{d_1, \ldots, d_s\})$, where $d_1, \ldots, d_s$ are the poles of the rational function R. The main result is a complete classification of such systems (we also say rational Hamiltonians) up to semi-local *TR*-equivalence (and up to semi-local Liouville equivalence, see Definition 2.4). Namely, the following theorem holds:

Theorem 1.1. *Suppose $\xi_0 \neq f_1(0,\infty)$ and $\xi_0 \neq f_2(0,\infty)$ or $\xi_0 = f_1(0,\infty) = f_2(0,\infty)$. Then rational Hamiltonians f_1 and f_2 are semi-locally TR-equivalent (Liouville equivalent) with respect to ξ_0 iff fibers $\mathrm{T}^1_{\xi_0}$ and $\mathrm{T}^2_{\xi_0}$ are homeomorphic and have the same sets of multiplicities of singular points including the multiplicity of the point $(0,\infty)$ when $\xi_0 = f_j(0,\infty)$, $j = 1, 2$.*

In the case $\xi_0 = f_1(0,\infty) \neq f_2(0,\infty)$ Hamiltonians f_1 and f_2 are not semi-locally Liouville equivalent with respect to ξ_0.

The condition that fibers $\mathrm{T}^1_{\xi_0}$ and $\mathrm{T}^2_{\xi_0}$ are homeomorphic is obviously necessary for semi-local *TR*-equivalence. The fact that multiplicities of singular points of a rational Hamiltonian are invariants of semi-local Liouville equivalence (and hence of semi-local *TR*-equivalence) can be easily obtained using methods developed in [21] (see the beginning of the proof of Theorem 4.2). In order to prove other statements of the theorem we will

1) compute the difference $\chi(\mathrm{T}^j_\xi) - \chi(\mathrm{T}^j_{\xi_0})$, where ξ is close but not equal to ξ_0 and χ is an Eulerian characteristic (see Lemma 3.4),
2) switch to "semi-local" normal forms of rational Hamiltonians f_j (see Theorem 4.1).

Note that for hyperelliptic Hamiltonians $f = az^2 + P_n(w)$ we always have $f(0, \infty) = \infty$. Since the set of multiplicities of singular points of the layer $f^{-1}(\xi_0)$ completely determines its topology (whenever the degree n of the polynomial $P_n = P_n(w)$ is fixed), our classification theorem reduces to the one, obtained in [16] and [17]. Obviously, it is not true for rational Hamiltonians, i.e. if $f = az^2 + \dfrac{A_n(w)}{B_m(w)}$, where $A = A_n(w)$ and $B = B_m(w)$ are relatively prime polynomials of degrees n and m respectively, then the set of multiplicities of singular points of the layer $f^{-1}(\xi_0)$ does not uniquely determine its topology (for fixed n and m).

The paper is organized as follows. Preparatory work is done in the section "Required statements". The construction of the "semi-local" normal form of a rational Hamiltonian and a complete proof of the classification theorem is given in the section "Main results".

2. Definitions

In this section we give necessary definitions and introduce notation to make our results precise.

Definition 2.1. Consider two holomorphic (continuous) functions $f_1 \colon M_1 \to \mathbb{C}$ and $f_2 \colon M_2 \to \mathbb{C}$, where M_1 and M_2 are complex manifolds (topological spaces). Suppose there exists a biholomorphism (homeomorphism) $h \colon M_1 \to M_2$ such that $f_1 = f_2 \circ h$. We say that f_1 and f_2 are (*topologically*) *right equivalent* or simply *(T)R-equivalent*.

Let $f = az^2 + R(w)$ be a function of complex variables $(z, w) \in \mathbb{C} \times (\mathbb{C} \setminus \{d_1, \dots, d_s\})$ such that $a \neq 0$, $\dfrac{d}{dw}R(w) \not\equiv 0$, where d_j, $j = 1, \dots, s$, are the poles of the rational function R. We say that f is a *(hyperelliptic) rational Hamiltonian* of the corresponding Hamiltonian system $(\mathbb{C} \times (\mathbb{C} \setminus \{d_1, \dots, d_s\}), \mathrm{Re}(dz \wedge dw), \mathrm{Re}\, f)$.

Let $f \colon M \to \mathbb{C}$ be a function on a space M. By the *foliation* of M generated by f we mean the decomposition of M into the *fibers* (level surfaces) $\mathrm{T}_\xi = f^{-1}(\xi)$, $\xi \in \mathbb{C}$.

Suppose M is a manifold and f is a smooth function on it. Then a fiber T_ξ, $\xi \in \mathbb{C}$, is called *nonsingular* if for every point $P \in \mathrm{T}_\xi$ we have $df|_P \neq 0$.

Definition 2.2. Consider a rational Hamiltonian $f = az^2 + R(w)$ and a point $P = (0, w_0)$. Suppose $(R(w) - f(P))^{(j)}|_{w_0} = 0$ for $j = 0, \dots, k-1$, $R^{(k)}(w_0) \neq 0$. We say that k is the *multiplicity* (and $k-1$ is *Milnor number*, see [21]) of the point P.

If $k = 1$ we say that P is a *simple* point of the fiber $\mathrm{T}_{f(P)}$, otherwise (if $k \geq 2$) we say that P is a *singular* point of this fiber.

Note that P is a singular point of the rational Hamiltonian f, i.e. $df|_P = 0$, iff P is a singular point of the fiber $\mathrm{T}_{f(P)}$. Thus, each fiber T_ξ, $\xi \in \mathbb{C}$, is nonsingular iff it has only simple points.

We will be interested in the topology of the foliation of $f^{-1}(D_{\xi_0,\varepsilon})$ generated by a rational Hamiltonian f, where $D_{\xi_0,\varepsilon}$ is a small disc in $\mathbb{C}$ around ξ_0.

Definition 2.3. Consider two rational Hamiltonians f_1, f_2 and a point $\xi_0 \in \mathbb{C}$. Suppose $f_1|_{f_1^{-1}(D_{\xi_0,\varepsilon})}$ and $f_2|_{f_2^{-1}(D_{\xi_0,\varepsilon})}$ are *TR*-equivalent for some $\varepsilon > 0$. Then we say that f_1 and f_2 (or the corresponding Hamiltonian systems) are *semi-locally TR-equivalent* with respect to ξ_0.

Definition 2.4. Consider two rational Hamiltonians f_1, f_2 and a point $\xi_0 \in \mathbb{C}$. We say that f_1 and f_2 (or the corresponding Hamiltonian systems) are *semi-locally Liouville equivalent* (or *semi-locally TRL-equivalent*) with respect to ξ_0 if the corresponding foliations of $f_1^{-1}(D_{\xi_0,\varepsilon})$ and $f_2^{-1}(D_{\xi_0,\varepsilon})$ are fiberwise homeomorphic for any small $\varepsilon > 0$.

We will also need the notion of *local TR-equivalence* and *local Liouville equivalence*.

Definition 2.5. Consider two rational Hamiltonians f_1, f_2 and points P_1, P_2. Let U_1 and U_2 be neighborhoods of P_1 and P_2 resp. Suppose $f_1|_{U_1}$ and $f_2|_{U_2}$ are *TR*-equivalent via the map h such that $h(P_1) = P_2$. Then we say that f_1 and f_2 are *locally TR-equivalent* with respect to P_1 and P_2.

Definition 2.6. Consider two rational Hamiltonians f_1, f_2 and points P_1, P_2. We say that f_1 and f_2 are *locally Liouville equivalent* (or *locally TRL-equivalent*) with respect to P_1 and P_2 if there exist arbitrarily small neighborhoods U_1 and U_2 of P_1 and P_2 resp. such that the corresponding foliations of U_1 and U_2 are fiberwise homeomorphic via the map h such that $h(P_1) = P_2$.

From the definitions it is easily seen that semi-local (local) *TR*-equivalence of rational Hamiltonians implies their semi-local (local) Liouville equivalence. It is known (see [23]) that the converse holds for local equivalence. We will show that the same is true in the semi-local case.

3. Required statements

Consider a hyperelliptic rational Hamiltonian $f(z,w) = az^2 + R(w)$, i.e. $R(w) = \dfrac{A_n(w)}{B_m(w)}$, where $A = A_n(w)$ and $B = B_m(w)$ are relatively prime polynomials of degrees $n \geq 0$ and $m \geq 0$, $R(w) \not\equiv const$ (without loss of generality we will assume that $a = 1$ and $n \neq m$). On the fiber T_{ξ_0} we have finite number $s^P \geq 0$ of singular points $P_1, \ldots, P_{s^P}$ and finite number $s^Q \geq 0$ of simple points Q_k, $k = 1, \ldots, s^Q$. Let $l_1, \ldots, l_{s^P}$ denote multiplicities of singular points $P_1, \ldots, P_{s^P}$. Put $V_{\varepsilon,l} = \{(z', w') \in \mathbb{C}^2 \mid |z'^2 + w'^l| < \varepsilon,\ |w'| < (2\varepsilon)^{1/l}\}$, where $l \in \mathbb{N}$ and $\varepsilon > 0$ is arbitrarily small.

Next lemma is well known (see, e.g. [19, §2, Lemma 4]). It shows that singularities of Hamiltonian systems defined by hyperelliptic rational Hamiltonians are of the type A_k, $k \in \mathbb{N}$.

Lemma 3.1. *For each P_j, $j = 1, \ldots, s^P$, there exists a 4-dimensional neighborhood U_j^P of P_j such that $f|_{U_j^P}$ is R-equivalent to $g_j^P: V_{\varepsilon,l_j} \to \mathbb{C}$, where $g_j^P(z', w') = z'^2 + w'^{l_j} + \xi_0$.*

For each Q_k, $k = 1, \ldots, s^Q$, there exists a 4-dimensional neighborhood U_k^Q of Q_k such that $f|_{U_k^Q}$ is R-equivalent to $g_k^Q: V_{\varepsilon,1} \to \mathbb{C}$, where $g_k^Q(z', w') = z'^2 + w' + \xi_0$.

Moreover, we can assume that neighborhoods $\overline{U_j^P}$, $\overline{U_k^Q}$ are pairwise disjoint.

Proof. Consider a singular point $P_j = (0, w_j)$ of the rational Hamiltonian f. There exists a neighborhood of this point in which $f(z,w) = z^2 + g(w)(w - w_j)^{l_j} + \xi_0$, where $g = g(w)$ is a holomorphic function such that $g(w_j) \neq 0$. Let U^w be a small neighborhood of the point w_j such that (for some branch of the root $\sqrt[l_j]{\ }$) the map

$$\phi_{P,j}: w \mapsto w' = (w - w_j)\sqrt[l_j]{g(w)} \tag{1}$$

is a diffeomorphism between U^w and $\phi_{P,j}(U^w)$. Let $h_{P,j} = \mathrm{id}_{\mathbb{C}} \times \phi_{P,j}$. Take $\varepsilon > 0$ such that $V_{\varepsilon,l_j} \subset h_{P,j}(\mathbb{C} \times U^w) = \mathbb{C} \times \phi(U^w)$ and put $U_j^P = h_{P,j}^{-1}(V_{\varepsilon,l_j})$. It is easily seen that $f|_{U_j^P}$ and g_j^P are *R*-equivalent via the map $h_{P,j}: U_j^P \to V_{\varepsilon,l_j}$. Similarly, we can deal with simple points Q_k. □

Consider an open 2-dimensional disk $D_{\xi_0,\varepsilon}$ around a (singular) value ξ_0 of the rational Hamiltonian $f = z^2 + R(w)$. As above, let $P_1, \dots, P_{s^P}$ be singular points of the fiber T_{ξ_0} with multiplicities $l_j,\ j = 1, \dots, s^P$, and $Q_1, \dots, Q_{s^Q}$ be simple points of this fiber. In what follows by a neighborhood of a finite set of points we will mean a union of connected neighborhoods of these points with pairwise disjoint closures.

Lemma 3.2. *Suppose $\xi_0 \neq f(0,\infty) := \lim_{w\to\infty} R(w) \in \overline{\mathbb{C}}$. Then for every 4-dimensional neighborhood U' of the set of points P_j and Q_k there exist $\varepsilon > 0$ and a 4-dimensional neighborhood $U \subset U'$ of the set of points P_j and Q_k such that $f|_{f^{-1}(D_{\xi_0,\varepsilon})\setminus\overline{U}}$ is R-equivalent to $\mathrm{Pr}_1\colon D_{\xi_0,\varepsilon} \times \left(\mathrm{T}_{\xi_0} \setminus \overline{U}\right) \to D_{\xi_0,\varepsilon}$, where $\mathrm{Pr}_1(\xi,\eta) = \xi$.*

Proof. Let U^w be a circular neighborhood of the set of points $\mathrm{Pr}_w(P_j)$ and $\mathrm{Pr}_w(Q_k)$, where Pr_w is a projection on the plane $\mathbb{C}^w$ of the complex variable w. Consider the holomorphic function $u(\xi,w) = \xi - R(w)$ of complex variables ξ and w. Since $\xi_0 \neq R(\infty) = f(0,\infty)$, function $u(\xi_0,w)$ extends to a holomorphic map from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$, which zeros are precisely P_j and Q_k. Therefore, there exists $\varepsilon > 0$ such that $\inf_{w\notin U^w} |u(\xi_0,w)| \geq \varepsilon$. Put

$$U = \{(\pm\sqrt{u(\xi,w)}, w) \in \mathbb{C}^2 \mid \xi \in D_{\xi_0,\varepsilon},\ w \in U^w\}. \tag{2}$$

Reducing, if necessary, the neighborhood U^w and $\varepsilon > 0$, we obtain the inclusion $U \subset U'$. Let us now prove that U is as required.

Consider the case when the manifold $\mathrm{T}_{\xi_0} \setminus \overline{U}$ is connected. Take a point $(z_0,w_0) \in \mathrm{T}_{\xi_0} \setminus \overline{U}$. We choose the branch of the root so that $z_0 = \sqrt{u(\xi_0,w_0)}$. Obviously, for every $\xi \in D_{\xi_0,\varepsilon}$ and for every w in a small neighborhood of w_0 we have a uniquely defined value of the root $\sqrt{u(\xi,w)}$, which smoothly depends on ξ and w.

Note that $(\sqrt{u(\xi,w)}, w) \in \mathrm{T}_\xi \setminus U$ for all $w \notin U^w$, $\xi \in D_{\xi_0,\varepsilon}$. Therefore, for every $\xi \in D_{\xi_0,\varepsilon}$ the manifold $\mathrm{T}_\xi \setminus U$ is a Riemann surface of a multivalued analytic function $u_\xi = u(\xi,w)$, $w \notin U^w$.

Take a curve $\gamma = \gamma(t) \subset \mathbb{C}^w,\ t \in [0,1]$, that starts at w_0 and doesn't pass through the poles of the rational function R and U^w. It induces a curve $u(\xi,\gamma(t)) \subset \mathbb{C}^u$. The value of the root $\sqrt{u(\xi,\gamma(1))}$, depends on the parity of the "number of revolutions" of the curve $u(\xi,\gamma(t))$ around zero in $\mathbb{C}^u$. By the number of revolutions of the curve $u(\xi,\gamma(t))$ we mean an integer $r = r(u(\xi,\gamma(t)))$ such that $2\pi r \leq \mathrm{Arg}(u(\xi,\gamma(t))) < 2\pi(r+1)$, where $\mathrm{Arg}(u(\xi,\gamma(t)))$ is the increment of argument of u along the curve $u(\xi,\gamma(t))$. Since $\inf_{w\notin U^w} |u(\xi_0,w)| \geq \varepsilon$, this number of revolutions coincides with the number of revolutions of the curve $u(\xi,\gamma(t)) + \xi_0 - \xi$. The latter is $u(\xi_0,\gamma(t))$.

Consider the map $\mu\colon f^{-1}(D_{\xi_0,\varepsilon}) \setminus \overline{U} \to \mathrm{T}_{\xi_0}$, defined by the formula $\mu(z,w) = (v(z,w),w)$, where $v = v(z,w)$ is a function that can be constructed as follows. Take $(z,w) \in \mathrm{T}_\xi \setminus \overline{U}$. Let $\gamma = \gamma(t),\ t \in [0,1]$, be a curve such that $\gamma(0) = w_0, \gamma(1) = w$ and $z = \sqrt{u(\xi,w)}$. Put $v(z,w) = \sqrt{u(\xi_0,w)}$, where the value of the root $\sqrt{u(\xi_0,w)}$ is determined by the same curve γ. Since $\mathrm{T}_{\xi_0} \setminus \overline{U}$ is connected, we see that $v = v(z,w)$ is well defined on $f^{-1}(D_{\xi_0,\varepsilon}) \setminus \overline{U}$. Moreover, $v^2(z,w) + R(w) = \xi_0$. Thus, the map $\mu = \mu(z,w)$ is well defined on $f^{-1}(D_{\xi_0,\varepsilon}) \setminus \overline{U}$, the restriction of μ to the fiber T_{ξ_0} is the identity map: $\mu(z,w)|_{\mathrm{T}_{\xi_0}} = \mathrm{id}$ and $\mu(z,w)|_{\mathrm{T}_\xi\setminus\overline{U}}$ is a biholomorphism between $\mathrm{T}_\xi \setminus \overline{U}$ and $\mathrm{T}_{\xi_0} \setminus \overline{U}$ for every $\xi \in D_{\xi_0,\varepsilon}$

Define $h_1\colon f^{-1}(D_{\xi_0,\varepsilon}) \setminus \overline{U} \to D_{\xi_0,\varepsilon} \times \left(\mathrm{T}_{\xi_0} \setminus \overline{U}\right)$ as follows:

$$h_1(z,w) = (f(z,w), \mu(z,w))\ (\mu(z,w) \text{ is a point on } \mathrm{T}_{\xi_0}). \tag{3}$$

Let $H_1 = i \circ h_1$, where $i\colon D_{\xi_0,\varepsilon} \times \left(\mathrm{T}_{\xi_0} \setminus \overline{U}\right) \hookrightarrow D_{\xi_0,\varepsilon} \times \mathbb{C}^2$ is the inclusion map. It is easily seen that the Jacobian matrix of H_1 has the maximum rank at every point. Indeed, the Jacobian matrix has the form:

$$\begin{pmatrix} f_z & v_z & 0 \\ f_w & v_w & 1 \end{pmatrix},$$

and on the set $f^{-1}(D_{\xi_0,\varepsilon})\setminus\overline{U}$ we have the inequality $f_z \neq 0$. Since h_1 is a complex-differentiable bijection and the differential dh_1 is an isomorphism at each point, we conclude that h_1 is a biholomorphism. Obviously, $f|_{f^{-1}(D_{\xi_0,\varepsilon})\setminus\overline{U}} = \mathrm{Pr}_1 \circ h_1$. Therefore, we are done with the case when the manifold $\mathrm{T}_{\xi_0}\setminus\overline{U}$ is connected.

Suppose $\mathrm{T}_{\xi_0}\setminus\overline{U}$ is the union of two connected components. Then, for sufficiently small $\varepsilon > 0$, we have that each $\mathrm{T}_{\xi}\setminus\overline{U}$, $\xi \in D_{\xi_0,\varepsilon}$, is also the union of two connected components. Thus, we can repeat the above reasoning for each of the two components of $\mathrm{T}_{\xi_0}\setminus\overline{U}$ and get the required statement. □

Lemma 3.3. *Suppose $\xi_0 \neq f(0,\infty)$. Then for every 4-dimensional neighborhood V' of the set of points P_j there exist $\varepsilon > 0$ and a 4-dimensional neighborhood $V \subset V'$ of the set of points P_j such that $f|_{f^{-1}(D_{\xi_0,\varepsilon})\setminus V}$ is R-equivalent to $\mathrm{Pr}_1\colon D_{\xi_0,\varepsilon}\times\overline{L} \to D_{\xi_0,\varepsilon}$, where $\overline{L} = \mathrm{T}_{\xi_0}\setminus V$, $\mathrm{Pr}_1(\xi,\eta) = \xi$.*

Proof. Keep in mind the proof of the previous lemma. If T_{ξ_0} doesn't have simple points, then everything is done. Assume that there exist simple points Q_k. Take one of them, let it be a point $Q = (0, w_{\xi_0})$. For every $\xi \in D_{\xi_0,\varepsilon}$ we have a simple point $Q^{\xi} = (0, w_{\xi}) \in \mathrm{T}_{\xi}$ near Q. According to Lemma 3.1, in a neighborhood O_Q of Q there exist coordinates (z, w') in which Hamiltonian f can be written as $f(z,w') = z^2 + w' + \xi_0$. Note that in coordinates (z, w') we have $(z,w')(Q^{\xi}) = (0, \xi - \xi_0)$, and $Q^{\xi_0} = Q$. Let U_Q be the connected component of the neighborhood U from (2) such that $Q \in U_Q$. We can assume that $U_Q \subset O_Q$. We can also assume that for every $\xi \in D_{\xi_0,\varepsilon}$ the point $Q^{\xi} \in U_Q$ and there are no more simple or singular points in U_Q.

Consider some $\xi \in D_{\xi_0,\varepsilon}$ and a neighborhood $\mathrm{Pr}_w(U_Q) \subset U^w$ of the point w_{ξ_0}. Let φ_{ξ} be a homeomorphism of the plain $\mathbb{C}^w$ that coincides with the identity map in a small neighborhood of $\mathbb{C}^w \setminus \mathrm{Pr}_w(U_Q)$ and also coincides with the map (written in a new coordinate w') $w' \mapsto w' + \xi_0 - \xi$ in a small neighborhood of w_{ξ}. Such a homeomorphism can be obtained by a shift along the integral curves of the vector field X_{ξ} that is constructed as follows. Let W_1 and W_2 be neighborhoods of a point w_{ξ_0} such that $\overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset \mathrm{Pr}_w(U_Q)$ and $w_{\xi} \in W_1$ for every $\xi \in D_{\xi_0,\varepsilon}$. There exists a smooth function $\alpha\colon \mathbb{C}^w \to \mathbb{R}$ (α doesn't depend on ξ) such that $\alpha|_{\mathbb{C}^w\setminus W_2} \equiv 0$ and $\alpha|_{W_1} \equiv 1$. We set $X_{\xi} = \alpha(w)(\xi_0 - \xi)\dfrac{d}{dw'}$, then corresponding maps φ_{ξ} are as required. Moreover, they smoothly depend on $\xi \in D_{\xi_0,\varepsilon}$.

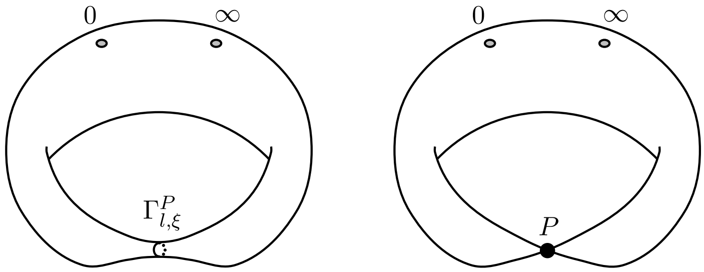
Let us define a map $h_2\colon O_Q \to D_{\xi_0,\varepsilon}\times \mathrm{T}_{\xi_0}$, where O_Q is the above neighborhood of Q in which we have new coordinates (z, w') from Lemma 3.1 (generally speaking, with $\varepsilon' \neq \varepsilon$ in its statement). To construct the map h_2 we introduce a family of maps $g_{\xi}\colon \mathrm{T}_{\xi}\cap O(Q) \to \mathrm{T}_{\xi_0}$, $\xi \in D_{\xi_0,\varepsilon}$, defined by the following formula $(z,w')(g_{\xi}(z,w)) = \left(\sqrt{-w'(\gamma_{\xi}(w))}, w'(\gamma_{\xi}(w))\right)$. We choose the branch of the root so that g_{ξ_0} is the inclusion map $\mathrm{T}_{\xi_0}\cap O(Q) \hookrightarrow \mathrm{T}_{\xi_0}$. Put $h_2(z,w) = (f(z,w), g_{f(z,w)}(z,w))$. Obviously, this map can be extended to a neighborhood of the set of simple points Q_k.

Now we "combine" the map h_1 from (3) with h_2. The resulting map $f^{-1}(D_{\xi_0,\varepsilon})\setminus\overline{V} \to D_{\xi_0,\varepsilon}\times\left(\mathrm{T}_{\xi_0}\setminus\overline{V}\right)$, where $V \subset U$ is a union of connected components of the neighborhood U containing singular points P_j, is well defined. Moreover, it extends to a homeomorphism

$$h_3\colon f^{-1}(D_{\xi_0,\varepsilon})\setminus V \to D_{\xi_0,\varepsilon}\times(\mathrm{T}_{\xi_0}\setminus V)\,. \tag{4}$$

We see that functions f and Pr_1 (considered on the sets $f^{-1}(D_{\xi_0,\varepsilon})\setminus V$ and $D_{\xi_0,\varepsilon}\times\overline{L}$ resp.) are *TR*-equivalent via the map h_3. □

Corollary 3.1. *Suppose T_{ξ_0} is nonsingular. If $\xi_0 \neq f(0,\infty)$, then for sufficiently small $\varepsilon > 0$ the foliation of $f^{-1}(D_{\xi_0,\varepsilon})$ generated by the rational Hamiltonian f is a trivial bundle $\mathrm{T}_{\xi_0}\times D_{\xi_0,\varepsilon}$. If $\xi_0 = f(0,\infty)$, then the corresponding foliation is not a trivial bundle, since $\chi(\mathrm{T}_{\xi}) - \chi(\mathrm{T}_{\xi_0}) < 0$, where $\xi \neq \xi_0$ is close to ξ_0 (see Lemma 3.4).*



Picture 1. Layers T_ξ and T_{ξ_0}.

Since we assumed that $n \neq m$ (see the beginning of Section 3), condition $\xi_0 = f(0, \infty)$ implies $n < m$ and $\xi_0 = 0$. In this case put $l_0 = m - n$ (multiplicity of the point $(0, \infty)$). Let l_j denote multiplicities of singular points P_j on T_{ξ_0}. By the direct computation of the topology of a given fiber we get

Lemma 3.4. $\chi(\mathrm{T}_\xi) - \chi(\mathrm{T}_{\xi_0}) = \sum_{j=1}^{s^P}(1 - l_j)$ *when* $\xi_0 \neq f(0, \infty)$ *and* $\chi(\mathrm{T}_\xi) - \chi(\mathrm{T}_{\xi_0}) = -l_0 + \sum_{j=1}^{s^P}(1 - l_j)$ *when* $\xi_0 = f(0, \infty)$.

Remark 3.1. There is a "geometrical" way to prove Lemma 3.4. Namely, if $P \in \mathrm{T}_{\xi_0}$ is a singular point, then there exist a neighborhood V_P of this point and coordinates (z, w') such that Hamiltonian $f = z^2 + w'^{\,l} + \xi_0$ for some $l \geq 2$. If $\varepsilon > 0$ is small enough, then for each $\xi \in D_{\xi_0,\varepsilon}$, $\xi \neq \xi_0$ we have a "vanishing graph" $\Gamma^P_{l,\xi} \subset V_P \cap \mathrm{T}_\xi$ on two vertices with l edges connecting these vertices. This graph $\Gamma^P_{l,\xi}$ shrinks to a point P as $\xi \to \xi_0$ (see [16, §3, propositions 1 and 2]). Therefore, to calculate $\chi(\mathrm{T}_\xi) - \chi(\mathrm{T}_{\xi_0})$ we can use the additivity of Eulerian characteristic χ and the fact that $\chi(\Gamma^P_{l,\xi}) = 2 - l$.

Example 3.1. Let $f(z, w) = z^2 + w + 1/w$, $\xi_0 = 2$ and $\xi \in D_{\xi_0,\varepsilon}$, $\xi \neq \xi_0$, where $\varepsilon > 0$ is small enough. It is easily seen that $P = (0, 1)$ is the only singular point of the fiber T_{ξ_0}. Since multiplicity l of this singular point equals 2, the corresponding "vanishing graph" $\Gamma^P_{l,\xi} \subset \mathrm{T}_\xi$ is a "vanishing circle", and $\chi(\mathrm{T}_{\xi_0}) - \chi(\mathrm{T}_\xi) = 1$. Moreover, T_ξ is a torus with 2 punctures and T_{ξ_0} is a sphere with two punctures and a pair of identified points (see Picture 1).

Suppose $\xi_0 \neq f(0, \infty)$. Take a singular point P_j and a map $\phi_{P,j}$ from (1). It is easily seen that if we define a neighborhood V of the set of singular points P_j so that each connected component of $\mathrm{Pr}_w(V)$ is $\phi^{-1}_{P,j}(D_{0,(2\varepsilon)^{1/l_j}})$ for some j, $j = 1, \ldots, s^P$, it will satisfy the assertion of Lemma 3.3. From now on we will assume that V is chosen in exactly this way.

Remark 3.2. Note that in the case when $R(w) = w^{l_j}$ and $\xi_0 = 0$, our neighborhood V is just V_{ε,l_j} (see Lemma 3.1).

As above, let l_j, $j = 1, \ldots, s^P$, denote multiplicities of singular points P_j on the fiber T_{ξ_0}. Put $\overline{V}^4_\varepsilon = \bigsqcup_{j=1}^{s^P} \overline{V}_{\varepsilon,l_j}$. Define $g\colon \overline{V}^4_\varepsilon \to \mathbb{C}$ by the formula $g|_{\overline{V}_{\varepsilon,l_j}}(z', w') = z'^{\,2} + w'^{\,l_j} + \xi_0$. Let V_P be the connected component of V, such that $P \in V_P$. By definition of V we see, that in coordinates (z, w') from Lemma 3.1 neighborhood V_P coincides with V_{ε,l_j}. Therefore, there exists a biholomorphism $h_4\colon V \to N^4_\varepsilon$ such that $g|_{V^4_\varepsilon} = f|_V \circ h_4$. Thus, we get

Corollary 3.2. *Suppose* $\xi_0 \neq f(0, \infty)$. *Then there exists* $\varepsilon > 0$ *such that* $f|_{\overline{V}}$ *and* $g|_{\overline{V}^4_\varepsilon}$ *are TR-equivalent.*

We will also need the following lemma

Lemma 3.5. *Let X, X'_1, X'_2 be topological spaces. Suppose X_1 and X_2 are closed subsets of X such that $X_1 \cup X_2 = X$. Let $H_j : X_j \to X'_j$, $j = 1, 2$ be homeomorphisms. Glue X'_1 and X'_2 via the map $H_{12} : H_1(X_1 \cap X_2) \to H_2(X_1 \cap X_2)$, where $H_{12} = H_2 \circ H_1^{-1}|_{H_1(X_1 \cap X_2)}$. Denote this gluing by $\sim_{12}$. Then, the map $H : X \to (X'_1 \sqcup X'_2)/\sim_{12}$, defined by the rule $H|_{X_j} = H_j$, $j = 1, 2$, is a homeomorphism.*

Proof. By definition, H is an open bijection. Therefore, we only need to check if this map is continuous. Let O be an open set in $(X'_1 \sqcup X'_2)/\sim_{12}$. We now prove that the set $H^{-1}(O) = H_1^{-1}(O \cap X'_1) \cup H_2^{-1}(O \cap X'_2)$ is open in X. Take a point $x \in H_1^{-1}(O \cap X'_1) \cup H_2^{-1}(O \cap X'_2)$ and suppose $x \in X_1 \cap X_2$. Then there exist a neighborhood $V_1 \subset X_1$ of x and a neighborhood $V_2 \subset X_2$ of x such that $H_1(V_1) \subset O \cap X'_1$ and $H_2(V_2) \subset O \cap X'_2$. Consider open sets $U_1 \subset X$ and $U_2 \subset X$ such that $U_1 \cap X_1 = V_1$, $U_2 \cap X_2 = V_2$. Put $U = U_1 \cap U_2$. It is easily seen that $x \in U \subset H^{-1}(O)$. Suppose $x \notin X_1 \cap X_2$. Then there also exists a neighborhood $U \subset X$ of x such that $U \subset H^{-1}(O)$, because H_1 and H_2 are homeomorphisms, and also because X_1 and X_2 are closed subsets of X. □

4. Main results

Now we are ready to introduce a "semi-local" normal form of a rational Hamiltonian f. Consider the following set

$$\partial^+ V_{\varepsilon,l_j} = \{(z, w) \in \mathbb{C}^2 \mid |z^2 + w^{l_j}| < \varepsilon, |w| = (2\varepsilon)^{1/l_j}\}.$$

Let (z', w') be our new coordinates, i.e. $w' = \phi_{P,j}(w)$ and $z' = z$, see (1). Let $\mu(z, w)$ be a map as in formula (3) and $\overline{L} = \mathrm{T}_{\xi_0} \setminus V$ be as in Lemma 3.3. We can assume that for each $(z', w') \in \partial^+ V_{\varepsilon,l_j}$ we have a well-defined value $\mu(z', \phi_{P,j}^{-1}(w')) = \left(\sqrt{-w'^{\,l_j}}, \phi_{P,j}^{-1}(w')\right)$. Consider functions $\nu_j : \partial^+ V_{\varepsilon,l_j} \to D_{\xi_0,\varepsilon} \times \partial\overline{L}$, where

$$\nu_j(z', w') = \left(z'^{\,2} + w'^{\,l_j} + \xi_0,\ \mu(z', \phi_{P,j}^{-1}(w'))\right) = \left(z'^{\,2} + w'^{\,l_j} + \xi_0, \sqrt{-w'^{\,l_j}}, \phi_{P,j}^{-1}(w')\right).$$

Note that each ν_j is a homeomorphism. Therefore, we can glue spaces ${}^+N^4_\varepsilon = \bigsqcup_{j=1}^{s^P} (V_{\varepsilon,l_j} \cup \partial^+ V_{\varepsilon,l_j})$ and $D_{\xi_0,\varepsilon} \times \overline{L}$ via the maps ν_j. Denote this gluing by $\sim$. Consider the following space

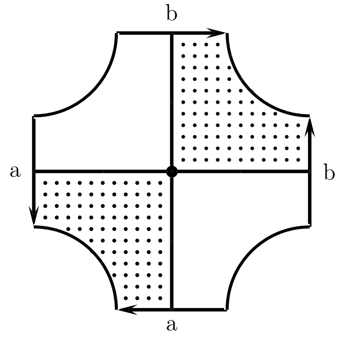
$$M^4 = {}^+N^4_\varepsilon \sqcup \left(D_{\xi_0,\varepsilon} \times \overline{L}\right) / \sim$$

and a function $G : M^4 \to \mathbb{C}$ that is defined as follows. On the set $D_{\xi_0,\varepsilon} \times \overline{L}$ let $G = \mathrm{Pr}_1$, and on each set $V_{\varepsilon,l_j} \cup \partial^+ V_{\varepsilon,l_j}$ let $G(z', w') = z'^{\,2} + w'^{\,l_j} + \xi_0$. Note that $(z', w') \in \partial^+ V_{\varepsilon,l_j}$ implies $G(z', w') = \mathrm{Pr}_1 \circ \nu_j(z', w')$, so G is well defined. We have the following

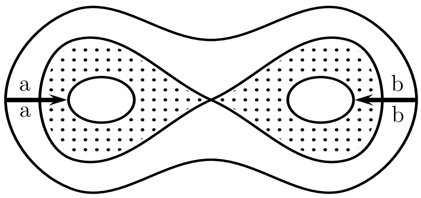
Theorem on the normal form 4.1. *Suppose $\xi_0 \neq f(0, \infty)$. Then there exists $\varepsilon > 0$ such that functions $f|_{f^{-1}(D_{\xi_0,\varepsilon})}$ and G are TR-equivalent.*

Proof. We will construct a homeomorphism $h : f^{-1}(D_{\xi_0,\varepsilon}) \to M^4$ such that $f|_{f^{-1}(D_{\xi_0,\varepsilon})} = G \circ h$. Let

$$X = f^{-1}(D_{\xi_0,\varepsilon}),\ X_1 = f^{-1}(D_{\xi_0,\varepsilon}) \setminus V,\ X_2 = \overline{V} \cap f^{-1}(D_{\xi_0,\varepsilon}),$$
$$X'_1 = D_{\xi_0,\varepsilon} \times \overline{L} \text{ and } X'_2 = {}^+N^4_\varepsilon.$$



Picture 2. Cross.



Picture 3. Atom.

We see that $X = X_1 \cup X_2$, where X_1 and X_2 are closed subsets of X. According to the proofs of Lemma 3.3 and Corollary 3.2, we let $H_1 = h_3$ and $H_2 = h_4$. Take the corresponding homeomorphism H from Lemma 3.5 and let $h = H$, i.e. we define h as follows:

$$h(z,w) = \begin{cases} h_3(z,w) & \text{when } (z,w) \in f^{-1}(D_{\xi_0,\varepsilon}) \setminus V, \\ h_4(z,w) = (z',w') & \text{when } (z,w) \in \overline{V} \cap f^{-1}(D_{\xi_0,\varepsilon}). \end{cases}$$

The map h is well defined. Indeed, consider $(z,w) \in X_1 \cap X_2 = \partial\overline{V} \cap f^{-1}(D_{\xi_0,\varepsilon})$. Then $\nu_j \circ h_4(z,w) = \nu_j(z',w') = (z'^2 + w'^{l_j} + \xi_0, \mu(z,w)) = (f(z,w), \mu(z,w)) = h_3(z,w)$. From Lemma 3.5 we get that h is a homeomorphism. Moreover, it is easily seen that $f|_{f^{-1}(D_{\xi_0,\varepsilon})} = G \circ h$. Indeed, let $(z,w) \in X_1 = f^{-1}(D_{\xi_0,\varepsilon}) \setminus V$. Then the composition $G \circ h(z,w) = f(z,w)$ by definition. Let $(z,w) \in X_2 = \overline{V} \cap f^{-1}(D_{\xi_0,\varepsilon})$. Then $G \circ h(z,w) = z'^2 + w'^{l_j} + \xi_0 = f(z,w)$, and theorem is proved. □

Remark 4.1. The gluing $\sim$ from above is not uniquely determined. In fact, it has one degree of freedom. Because of this, we can assume that for each rational Hamiltonian f such that $\xi_0 \neq f(0,\infty)$, with specified topological type of the fiber T_{ξ_0} and set of multiplicities of singular points on this fiber, the corresponding normal form is the same (see the proof of Theorem 4.2).

Remark 4.2. Suppose P is the only singular point on the fiber T_{ξ_0} and $\xi_0 \neq f(0,\infty)$. One may notice that the foliation of $f^{-1}(D_{\xi_0,\varepsilon})$ (considered up to fiberwise homeomorphism) generated by f is the 4-dimensional analogue of the *atom* (see [1] and [9]) of a singularity and the set ${}^+N^4_\varepsilon$ is the analogue of the so-called *cross* (see Picture 2 and Picture 3). As in 2-dimensional case “atom” $f^{-1}(D_{\xi_0,\varepsilon})$ of the singularity P can be obtained from the “cross” ${}^+N^4_\varepsilon$ (and a “ribbon” $D_{\xi_0,\varepsilon} \times \overline{L}$) via the appropriate gluing (see Theorem 4.1).

In the above construction we assumed that $\xi_0 \neq f(0,\infty)$. Similarly, we can construct a normal form of a rational Hamiltonian f in the case $\xi_0 = f(0,\infty)$. Indeed, condition $\xi_0 = f(0,\infty) = R(\infty)$ implies that the rational function $R(w)$ has at least one pole w_0 in $\mathbb{C}^w$. Make the change of variable $w \to w' = \frac{1}{w - w_0}$. Obviously, functions $f = f(z,w)$ and $g = g(z,w') = f(z, w_0 + \frac{1}{w'})$ are R-equivalent (we assume that points (z,w') with $w' = 0$ do not belong to the domain of g). Note that $w' = 0$ is not a pole of the function $g(0,w')$ since $g(0,0) = \xi_0 = f(0,\infty) \neq \infty$. In fact, this “punctured” point is a removable singularity of the function $g(0, w')$. Therefore, we can define multiplicity l_0 of the point $(0, \infty)$ of the fiber T_{ξ_0} as multiplicity of zero $w' = 0$ of the function $g(0, w') - \xi_0$. It is easily seen that l_0 is well defined, i.e. it doesn't depend on the choice of a pole of R. Indeed, condition $\xi_0 = R(\infty) = \dfrac{A_n(\infty)}{B_m(\infty)}$ implies $\xi_0 = 0$ (since $n < m$), and $l_0 = m - n$. Thus, since $g(0, \infty) \neq \xi_0$, we can construct the normal form (M^4, G) as in Theorem 4.1, where $M^4 \approx g^{-1}(D_{\xi_0,\varepsilon})$ has an additional puncture in the plain $\mathbb{C}^{w'}$ at zero. Note that we may have $l_0 = 1$, but it will not impede the construction of the normal form because of Lemma 3.1.

Consider two (hyperelliptic) rational Hamiltonians f_1 and f_2. We have the following

Theorem 4.2. *Suppose $\xi_0 \neq f_1(0, \infty)$ and $\xi_0 \neq f_2(0, \infty)$ or $\xi_0 = f_1(0, \infty) = f_2(0, \infty)$. Then rational Hamiltonians f_1 and f_2 are semi-locally Liouville equivalent with respect to ξ_0 iff fibers $\mathrm{T}^1_{\xi_0}$ and $\mathrm{T}^2_{\xi_0}$ are homeomorphic and have the same sets of multiplicities of singular points (including the multiplicity of the (singular) point $(0, \infty)$ when $\xi_0 = f_j(0, \infty)$, $j = 1, 2$).*

In the case $\xi_0 = f_1(0, \infty) \neq f_2(0, \infty)$ Hamiltonians f_1 and f_2 are not semi-locally Liouville equivalent with respect to ξ_0.

Proof. Suppose f_1 and f_2 are semi-locally Liouville equivalent with respect to ξ_0. At first, we are going to show that fibers $\mathrm{T}^1_{\xi_0}$ and $h(\mathrm{T}^1_{\xi_0})$ must have the same sets of multiplicities of singular points. Because of Lemma 3.1 it is sufficient to show that local Liouville equivalence of $g_k = z^2 + w^k$ and $g_l = z^2 + w^l$ (with respect to P_k and P_l, $P_k = P_l = (0, 0)$) implies $k = l$.

Suppose g_k and g_l are locally Liouville equivalent via the map h^{loc}. Let $B_\delta \in \mathbb{C}^2$ a closed ball of radius δ around $(0, 0)$. It is well known that for sufficiently small $\delta > 0$ first homology groups $\mathrm{H}_1(B_\delta \cap g_k^{-1}(\xi)) = \mathbb{Z}^{k-1}$ for every small $\xi \in \mathbb{C}$. We choose δ, δ_1 and δ_2, $0 < \delta < \delta_1 < \delta_2$, so that $B_\delta \subset h^{loc}(B_{\delta_1}) \subset B_{\delta_2}$, and the inclusion map $i_1 \colon B_\delta \to B_{\delta_2}$ induces a homotopy equivalence between $B_\delta \cap g_k^{-1}(\xi)$ and $B_{\delta_2} \cap g_k^{-1}(\xi)$ for all sufficiently small ξ. Then it is easily seen that the inclusion map $i_2 \colon B_\delta \to h^{loc}(B_{\delta_1})$ induces an injective homomorphism $\mathrm{H}_1(B_\delta \cap g_k^{-1}(\xi)) \hookrightarrow \mathrm{H}_1(h(B_{\delta_1}) \cap g_k^{-1}(\xi))$ of abelian groups. Thus, if $\delta_2 > 0$ was small enough, for some ξ we get an injective homomorphism $\mathbb{Z}^{k-1} \hookrightarrow \mathbb{Z}^{l-1}$, so $k \leq l$. Similarly, $k \geq l$.

Now let us prove that the foliations of $f_1^{-1}(D_{\xi_0,\varepsilon})$ and $f_1^{-1}(D_{\xi_0,\varepsilon})$ are fiberwise homeomorphic via a map h such that $h(\mathrm{T}^1_{\xi_0}) = \mathrm{T}^2_{\xi_0}$. Note that since $\varepsilon > 0$ is arbitrary small, fibers T^j_ξ, $j = 1, 2$, where $\xi_0 \neq \xi \in D_{\xi_0,\varepsilon}$, are nonsingular. Therefore, if $\mathrm{T}^1_{\xi_0}$ is a singular fiber, then $h(\mathrm{T}^1_{\xi_0})$ is also a singular fiber. The case of nonsingular fibers $\mathrm{T}^1_{\xi_0}$ and $\mathrm{T}^2_{\xi_0}$ is clear because of Corollary 3.1.

Thus, if f_1 and f_2 are semi-locally Liouville equivalent with respect to ξ_0, then fibers $\mathrm{T}^1_{\xi_0}$ and $\mathrm{T}^2_{\xi_0}$ are homeomorphic and have the same sets of multiplicities of singular points. Therefore, using Lemma 3.4 we immediately get the second assertion of the theorem.

Let us now prove the first assertion of the theorem. Consider the case when $\xi_0 \neq f_1(0, \infty)$ and $\xi_0 \neq f_2(0, \infty)$. *Necessity* is already proved.

Sufficiency. Let P^1_j and P^2_j, $j = 1, \ldots, s^{P^1} = s^{P^2}$ be singular points on fibers $\mathrm{T}^1_{\xi_0}$ and $\mathrm{T}^2_{\xi_0}$ such that for each j multiplicities $l^1_j = l^2_j$ coincide. Let $\varepsilon > 0$ be sufficiently small. Denote by V^1 and V^2 the neighborhoods of singular points P^1_j and P^2_j as in Theorem 4.1. Let $\overline{L}_1 = \mathrm{T}^1_{\xi_0} \setminus V^1$ and $\overline{L}_2 = \mathrm{T}^2_{\xi_0} \setminus V^2$. We have the normal forms (M^4_1, G_1) and (M^4_2, G_2) of the Hamiltonians f_1 and f_2, where

$$M^4_1 = \left(D_{\xi_0,\varepsilon} \times \overline{L}_1\right) \sqcup \left(\bigsqcup_{j=1}^{s^{P1}} (V_{\varepsilon,l_j} \cup \partial^+ V_{\varepsilon,l_j})\right) / \sim_1 \quad \text{and}$$

$$M^4_2 = \left(D_{\xi_0,\varepsilon} \times \overline{L}_2\right) \sqcup \left(\bigsqcup_{j=1}^{s^{P2}} (V_{\varepsilon,l_j} \cup \partial^+ V_{\varepsilon,l_j})\right) / \sim_2 .$$

We know that $G_1\colon M_1^4 \to \mathbb{C}$ and $G_2\colon M_2^4 \to \mathbb{C}$ are *TR*-equivalent to $f_1|_{f_1^{-1}(D_{\xi_0,\varepsilon})}$ and $f_2|_{f_2^{-1}(D_{\xi_0,\varepsilon})}$ resp. Therefore, it is sufficient to show that functions G_1 and G_2 are *TR*-equivalent.

Since fibers $\mathrm{T}^1_{\xi_0}$ and $\mathrm{T}^2_{\xi_0}$ are homeomorphic and have the same sets of multiplicities of singular points, there exists an orientation-preserving homeomorphism $\mathrm{T}^1_{\xi_0} \to \mathrm{T}^2_{\xi_0}$ such that for each $j = 1, \ldots, s^{P^1}$ the singular point P^1_j goes to P^2_j. In particular, manifolds with boundary $\mathrm{T}^1_{\xi_0} \setminus V^1$ and $\mathrm{T}^2_{\xi_0} \setminus V^2$ are homeomorphic. For every connected component of $D_{\xi_0,\varepsilon} \times \partial(\mathrm{T}^1_{\xi_0} \setminus V^1)$ consider its small blowing in $D_{\xi_0,\varepsilon} \times (\mathrm{T}^1_{\xi_0} \setminus V^1)$ which is homeomorphic to a direct product $D_{\xi_0,\varepsilon} \times [0,1] \times S^1$ and add this blowings to the set $\bigsqcup_j V_{\varepsilon,l_j} \approx V^1$. Denote the resulting set by $\widetilde{V}^1$. Note that $\widetilde{V}^1$ is a closed subset of M_1^4. Similarly, we can construct a closed subset $\widetilde{V}^2 \subset M_2^4$. It is easily seen that there exists an orientation-preserving homeomorphism $\mathrm{id} \times h_L\colon D_{\xi_0,\varepsilon} \times (\mathrm{T}^1_{\xi_0} \setminus \widetilde{V}^1) \to D_{\xi_0,\varepsilon} \times (\mathrm{T}^2_{\xi_0} \setminus \widetilde{V}^2)$.

On each set $V_{\varepsilon,l_j} \cup \partial^+ V_{\varepsilon,l_j}$ consider a homeomorphism h_V, defined by the formula $(z,w) \mapsto (z,w)$. Consider a connected component of $\widetilde{V}^1 \setminus \bigsqcup_j V_{\varepsilon,l_j}$, which is homeomorphic to $D_{\xi_0,\varepsilon} \times [0,1] \times S^1$, and the restrictions of the maps h_L and h_V to the corresponding parts of the boundary $\partial(\widetilde{V}^1 \setminus \bigsqcup_j V_{\varepsilon,l_j})$ (which are homeomorphic to a solid torus $D_{\xi_0,\varepsilon} \times S^1$). By the construction of $\sim_1$ and $\sim_2$ induced orientations on this solid tori are compatible. Therefore, there exists a homeomorphism $\widetilde{V}^1 \setminus \bigsqcup_j V_{\varepsilon,l_j} \to \widetilde{V}^2 \setminus \bigsqcup_j V_{\varepsilon,l_j}$ that “glue” h_L and h_V into a homeomorphism $h_M\colon M_1^4 \to M_2^4$ (use the following fact: the space of orientation-preserving homeomorphisms of a circle is arcwise connected). It is easily seen that $G_1 = G_2 \circ h_M$, so we are done with the case $\xi_0 \neq f_1(0,\infty)$ and $\xi_0 \neq f_2(0,\infty)$.

Now consider the case $\xi_0 = f_1(0,\infty)$ and $\xi_0 = f_2(0,\infty)$. *Necessity* follows from the above reasoning and Lemma 3.4.

Sufficiency. Consider functions $g_1 = g_1(z,w')$ and $g_1 = g_2(z,w')$ that are *R*-equivalent to f_1 and f_2 and have additional punctures in $\mathbb{C}^{w'}$ at zero. We know that fibers $g_1^{-1}(\xi_0)$ and $g_2^{-1}(\xi_0)$ are homeomorphic and have the same sets of multiplicities of singular points and the same multiplicity of the point $(z,w')(0,\infty) = (0,0))$. Since $g_1(0,\infty) \neq \xi_0$ and $g_2(0,\infty) \neq \xi_0$, functions g_1 and g_2 (considered as Hamiltonians) are semi-locally *TR*-equivalent with respect to ξ_0. Thus, theorem is proved. □

Remark 4.3. From the proof of Theorem 4.2 we easily get that its assertion holds not only for semi-local Liouville equivalence, but also for semi-local *TR*-equivalence.

Note that order of a pole of a rational Hamiltonian f is not an invariant even of smooth semi-local Liouville equivalence, as the following example shows

Example 4.1. Consider rational Hamiltonians

$$f_1(z,w) = z^2 + \frac{(w-3)(w-4)}{(w-1)(w-2)} \quad \text{and}$$

$$f_2(z,w) = z^2 + \frac{(w-1)(w-2)(w-3)(w-4)}{(w-5)^2}.$$

Parity of the poles of the functions $R_1 = f_1(0,w)$ and $R_2 = f_2(0,w)$ are different. Despite this, f_1 and f_2 are semi-locally *TR*-equivalent with respect to $\xi_0 = 0$, since fibers T^1_0 and T^2_0 are homeomorphic and since the corresponding foliations near them are trivial bundles.

Acknowledgements

The author is indebted to A. Fomenko and E. Kudryavtseva for their help and valuable advice.

References

[1] A. Bolsinov, A. Fomenko, Integrable Hamiltonian Systems. Geometry, Topology, Classification, Vol. 1, Chapman and Hall, Boca Raton, FL, 2004.
[2] S. Broughton, On the topology of polynomial hypersurfaces, I, Proc. AMS Symp. Pure. Math. 40 (1983) 165–178.
[3] S. Dirk, M. Tibar, Deformations of polynomials, boundary singularities and monodromy, Mosc. Math. J. 3 (2) (2003) 661–679.
[4] H. Flaschka, A remark on integrable Hamiltonian systems, Phys. Lett. A 131 (9) (1988) 505–508.
[5] A. Fomenko, The symplectic topology of completely integrable Hamiltonian systems, Russ. Math. Surv. 44 (1) (1989) 181–219.
[6] A. Fomenko, V. Sharko, Exact round Morse functions, inequalities of Morse type and integrals of Hamiltonian systems, Ukr. Math. J. 41 (6) (1989) 621–629.
[7] A. Fomenko, H. Zieschang, A topological invariant and a criterion for the equivalence of integrable Hamiltonian systems with two degrees of freedom, Math. USSR, Izv. 36 (3) (1991) 567–596.
[8] A. Fomenko, Symplectic Geometry, second revised edition, Gordon and Breach, 1995.
[9] A. Fomenko, E. Kudryavtseva, I. Nikonov, Maximum symmetric cellular partitions of surfaces and their coverings, Sb. Math. 199 (9) (2008) 3–96.
[10] A. Fomenko, A. Konyaev, New approach to symmetries and singularities in integrable Hamiltonian systems, Topol. Appl. 159 (2012) 1964–1975.
[11] A. Fomenko, A. Konyaev, Algebra and Geometry Through Hamiltonian Systems, in: Continuous and Distributed Systems, in: Theory and Applications, Solid Mechanics and Its Applications, vol. 211, 2014, pp. 3–21.
[12] A. Khovanskii, Newton polyhedra, and the genus of complete intersections, Funct. Anal. Appl. 12 (1) (1978) 38–46.
[13] K. Kodaira, On the structure of compact complex analytic surfaces, I, Am. J. Math. 50 (1963) 218–221.
[14] K. Kodaira, On the structure of compact complex analytic surfaces, II, Am. J. Math. 51 (1964) 1100–1104.
[15] E. Kudryavtseva, The Liouville-type theorem for integrable Hamiltonian systems with incomplete flows, Rep. RAS 445 (4) (2012) 383–385.
[16] E. Kudryavtseva, T. Lepski, The topology of Lagrangian foliations of integrable systems with hyperelliptic Hamiltonian, Transl. Sb. Math. 202 (3) (2011) 373–411.
[17] E. Kudryavtseva, T. Lepski, The topology of foliation and the Liouville theorem for integrable systems with incomplete flows, Sb. Tr. Semin. Vektorn. Tenzorn. Anal. 28 (2011) 106–149.
[18] E. Kudryavtseva, T. Lepski, Integrable Hamiltonian systems with incomplete flows and Newton's polygons, Contemp. Probl. Math., Mech. 6 (3) (2011) 42–55.
[19] T. Lepski, Incomplete integrable Hamiltonian systems with complex polynomial Hamiltonian of small degree, Sb. Math. 201 (10) (2010) 1511–1538.
[20] N. Martynchuk, Complex Hamiltonian systems on $\mathbb{C}^2$ with Hamiltonian function of low Laurent degree, Mosc. Univ. Math. Bull. 70 (2) (2015) 53–59.
[21] J. Milnor, Singular Points of Complex Hypersurfaces, Princeton Univ. Press, NJ, 1968.
[22] A. Oshemkov, Classification of hyperbolic singularities of rank 0 of integrable Hamiltonian systems, Math. Sb. 201 (8) (2010) 63–102.
[23] O. Saeki, Topological types of complex isolated hypersurface singularities, Kodai Math. J. 12 (1989) 23–29.
[24] N. Zung, A note on focus–focus singularities, Differ. Geom. Appl. 7 (1997) 123–130.