University of Groningen

On the controllability and stabilizability of linear complementarity systems

Camlibel, M. K.

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2019

[Link to publication in University of Groningen/UMCG research database](#)

On the Controllability and Stabilizability of Linear Complementarity Systems

M.K. Camlibel *

* *Bernoulli Institute of Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, The Netherlands (e-mail: m.k.camlibel@rug.nl).*

Abstract: This paper studies controllability and stabilizability of linear complementarity systems that can be cast as continuous piecewise affine dynamical systems. Under a certain right-invertibility assumption, we present a la Hautus necessary and sufficient conditions for both controllability and stabilizability.

Keywords: Controllability, stabilizability, linear complementarity systems, continuous piecewise affine dynamical systems

1. INTRODUCTION

Controllability is not only a fundamental system-theoretic property but also a prerequisite virtually for all control design problems. Kalman's and Hautus' tests for controllability are among the classical results of linear system theory. Apart from finite-dimensional linear time-invariant systems, there are only a few cases for which easily verifiable controllability conditions exist. For instance, necessary and sufficient conditions that are linear in nature exist for linear switched systems with state-independent switching Sun et al. (2002). This result relies on the fact that the switching signal can be considered as an external input. Yet another instance for which controllability admits a simple characterization is the case of continuous piecewise affine dynamical systems Thuan and Camlibel (2014). Contrary to linear switched systems, piecewise affine dynamical systems exhibit state-dependent switching phenomenon that makes the analysis much harder. Nevertheless, Hautus-like tests can be devised by exploiting the structure imposed by continuity of the vector field. Moreover, this Hautus-like controllability test also extends to stabilizability.

In this paper, we study a particular class of continuous piecewise affine dynamical systems, namely linear complementarity systems. Complementarity systems are encountered in a multitude of applications in various disciplines of science and engineering including electrical/mechanical engineering and operations research van der Schaft and Schumacher (1998, 2000); Brogliato (2003); Heemels and Brogliato (2003); Camlibel et al. (2004); Schumacher (2004). We refer to van der Schaft and Schumacher (1996); Heemels et al. (2000); Camlibel (2001); Camlibel and Schumacher (2002); Camlibel et al. (2003); Heemels et al. (2002); Camlibel et al. (2002); Camlibel (2007); Camlibel et al. (2008b, 2014) for the work on the analysis of general LCSs.

Under certain conditions, a linear complementarity system can be cast as a continuous piecewise affine dynamical system. We will exploit the structure of such piecewise affine systems and the results in Thuan and Camlibel (2014) in order to present tailor-made necessary and sufficient conditions for controllability and stabilizability of linear complementarity systems.

The organization of the paper is as follows. In Section 2, we will introduce the class of linear complementarity systems. Section 3 is devoted to continuous piecewise affine dynamical systems and Hautus-like controllability/stabilizability tests for these systems. Subsequently, Section 4 will discuss the relationships between linear complementarity systems and continuous piecewise affine systems. This will be followed by the main results and their proofs in Section 5. Finally, the paper closes with the conclusions in Section 6.

2. LINEAR COMPLEMENTARITY SYSTEMS

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ez(t) \tag{1a}$$
$$w(t) = Cx(t) + Du(t) + Fz(t) \tag{1b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $(z, w) \in \mathbb{R}^{p+p}$. When the external variables (z, w) satisfy the so-called complementarity relations

$$0 \leqslant z(t) \perp w(t) \geqslant 0 \tag{1c}$$

where the inequalities are componentwise and $\perp$ denotes orthogonality, that is $z \perp w$ if and only if $z^T w = 0$. We call the overall system (1) a *linear complementarity system* (LCS).

3. CONTINUOUS PIECEWISE AFFINE SYSTEMS

In order to introduce continuous piecewise affine dynamical systems, we first begin with piecewise affine functions and their properties.

A function $\psi : \mathbb{R}^\nu \to \mathbb{R}^\ell$ is said to be *affine* if there exist a matrix $H \in \mathbb{R}^{\ell \times \nu}$ and a vector $g \in \mathbb{R}^\ell$ such that $\psi(x) = Hx + g$ for all $x \in \mathbb{R}^\nu$. A function ϕ is called *piecewise affine* if there exists a finite family of affine

functions $\psi_i : \mathbb{R}^\nu \to \mathbb{R}^\ell$ with $i = 1, 2, \ldots, r$ such that $\phi(x) \in \{\psi_1(x), \psi_2(x), \ldots, \psi_r(x)\}$ for all $x \in \mathbb{R}^\nu$.

Continuous piecewise affine functions admit useful geometrical representations that will be used later on this paper. To elaborate, we first need to introduce some terminology. A set of $\mathbb{R}^\nu$ is called a *polyhedron* if it is the intersection of a finite family of closed half-spaces or hyperplanes in $\mathbb{R}^\nu$. If $\mathcal{P} \subseteq \mathbb{R}^\nu$ is a polyhedron, then there exist a matrix $P \in \mathbb{R}^{k \times \nu}$ and a vector $q \in \mathbb{R}^k$ such that $\mathcal{P} = \{x \in \mathbb{R}^\nu \mid Px \geqslant q\}$. A finite collection Ξ of polyhedra of $\mathbb{R}^\nu$ is said to be a *polyhedral subdivision* of $\mathbb{R}^\nu$ if (see Facchinei and Pang (2002) for more details)

  i. the union of all polyhedra in Ξ is equal to $\mathbb{R}^\nu$,
 ii. each polyhedron in Ξ is of dimension ν, and
iii. the intersection of any two polyhedra in Ξ is either empty or a common proper face of both polyhedra

As stated next, polyhedral subdivisions appear in the description of continuous piecewise affine functions.

Proposition 1. (Facchinei and Pang (2002), Prop. 4.2.1). *A continuous function $\psi : \mathbb{R}^\nu \to \mathbb{R}^\ell$ is piecewise affine if and only if there exist a polyhedral subdivision $\Xi = \{\Xi_1, \Xi_2, \ldots, \Xi_r\}$ of $\mathbb{R}^\nu$ and a finite family of affine functions $\psi_i : \mathbb{R}^\nu \to \mathbb{R}^\ell$ with $i = 1, 2, \ldots, r$ such that ψ coincides ψ_i on Ξ_i for every $i = 1, 2, \ldots, r$.*

By a *continuous piecewise affine system* (CPAS), we mean a dynamical system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + \Phi(y(t)) \tag{2a}$$

$$y(t) = Cx(t) + Du(t) \tag{2b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, all involved matrices are of appropriate sizes, and $\Phi : \mathbb{R}^p \to \mathbb{R}^n$ is a continuous piecewise affine function.

Since the piecewise affine function Φ is continuous, the right-hand side of (2a) is globally Lipschitz (see e.g. (Facchinei and Pang, 2002, Proposition 4.2.2)). Thus, it follows from the theory of ordinary differential equations that the system (2) must admit a unique solution for each initial state x^0 and locally integrable input u. We denote this solution by $x^u(t; x^0)$, and denote the corresponding output by $y^u(t; x^0)$.

Next, we define controllability and stabilizability for CPASs.

Definition 2. We say that the system (2) is

 • *controllable* if for any two states x^0, x^{f} there exist $T > 0$ and a locally integrable input u such that $x^u(T; x^0) = x^{\mathrm{f}}$.
 • *stabilizable* if for any initial state x^0 there exists a locally integrable input u such that $\lim_{t \to \infty} x^u(t; x^0) = 0$.

In this paper, the following blanket assumption will be in force.

Assumption 3. The transfer matrix $D + C(sI - A)^{-1}B$ is right invertible as a rational matrix.

Similar invertibility assumptions are common in the analysis of linear systems with output constraints (see e.g. Saberi et al. (2002); Shi et al. (2003); Saberi et al. (2003); Heemels and Camlibel (2008)). The main restriction of the

assumption is that the number of inputs should be at least as the number of outputs. Whenever this is the case, right invertibility is generically satisfied.

A complete characterization of controllability and stabilizability of CPASs are established in Thuan and Camlibel (2014). In order to quote the results of Thuan and Camlibel (2014), we need to introduce the following terminology. For a nonempty convex set $\mathcal{S} \subseteq \mathbb{R}^n$, $\mathcal{S}^-$ denotes the dual cone

$$\mathcal{S}^- := \{\xi \in \mathbb{R}^n \mid \xi^T x \leqslant 0 \text{ for all } x \in \mathcal{S}\},$$

$\mathcal{S}^\perp$ denotes orthogonal complement in $\mathbb{C}^n$, and $C_r(\mathcal{S})$ denotes the recession cone

$$C_r(\mathcal{S}) := \{\xi \in \mathbb{R}^n \mid z + \lambda x \in \Lambda \text{ for all } \lambda \geqslant 0 \text{ and } z \in \mathcal{S}\}.$$

Necessary and sufficient conditions for controllability of CPASs are stated next.

Proposition 4. (Thuan and Camlibel (2014),Thm. 3.4]). *Consider a CPAS (2) satisfying Assumption 3. Suppose that the origin is contained in the closure of the convex hull of the image of the function Φ. Let $\Xi = \{\mathcal{Y}^1, \mathcal{Y}^2, \ldots, \mathcal{Y}^h\}$ be a polyhedral subdivision of $\mathbb{R}^p$ induced by Φ. Also let*

$$\Phi(y) = H^i y - g^i \text{ whenever } y \in \mathcal{Y}^i. \tag{3}$$

Then, the system (2) is controllable if and only if the following four implications hold:

  i. $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$, $z^* g^i = 0$, and
$$z^* \begin{bmatrix} A + H^i C - \lambda I & B + H^i D \end{bmatrix} = 0 \; \forall i \implies z = 0.$$

 ii. $\lambda \in \mathbb{R}$, $z \in \mathbb{R}^n$, $(z, w^i) \in (\{g^i\} \times \mathcal{Y}^i)^-$, and
$$\begin{bmatrix} z \\ w^i \end{bmatrix}^T \begin{bmatrix} A + H^i C - \lambda I & B + H^i D \\ C & D \end{bmatrix} = 0 \; \forall i \implies z = 0.$$

iii. $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \neq 0$, $z \in \mathbb{C}^n$, $w^i \in C_r(\mathcal{Y}_i)^\perp$, and
$$\begin{bmatrix} z \\ w^i \end{bmatrix}^* \begin{bmatrix} A + H^i C - \lambda I & B + H^i D \\ C & D \end{bmatrix} = 0 \; \forall i \implies z = 0.$$

 iv. $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $z \in \mathbb{R}^n$, $w^i \in C_r(\mathcal{Y}_i)^-$, and
$$\begin{bmatrix} z \\ w^i \end{bmatrix}^T \begin{bmatrix} A + H^i C - \lambda I & B + H^i D \\ C & D \end{bmatrix} = 0 \; \forall i \implies z = 0.$$

Stabilizability of CPASs can also be characterized by similar spectral conditions.

Proposition 5. ((Thuan and Camlibel, 2014, Thm. 3.6)). *Consider a CPAS (2) satisfying Assumption 3. Suppose that $\Phi(0) = 0$. Let $\Xi = \{\mathcal{Y}^1, \mathcal{Y}^2, \ldots, \mathcal{Y}^h\}$ be a polyhedral subdivision of $\mathbb{R}^p$ induced by Φ. Also let*

$$\Phi(y) = H^i y - g^i \text{ whenever } y \in \mathcal{Y}^i. \tag{4}$$

Then, the system (2) is stabilizable if and only if the following four implications hold:

  i. $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geqslant 0$, $z \in \mathbb{C}^n$, $z^* g^i = 0$ and
$$z^* \begin{bmatrix} A + H^i C - \lambda I & B + H^i D \end{bmatrix} = 0 \; \forall i \implies z = 0.$$

 ii. $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$, $z \in \mathbb{R}^n$, $(z, w^i) \in (\{g^i\} \times \mathcal{Y}^i)^-$, and

$$\begin{bmatrix} z \\ w^i \end{bmatrix}^T \begin{bmatrix} A + H^i C - \lambda I & B + H^i D \\ C & D \end{bmatrix} = 0 \ \forall i \implies z = 0.$$

iii. $\lambda \in \mathbb{C}_+$, $z \in \mathbb{C}^n$, $w^i \in C_r(\mathcal{Y}_i)^\perp$, *and*

$$\begin{bmatrix} z \\ w^i \end{bmatrix}^* \begin{bmatrix} A + H^i C - \lambda I & B + H^i D \\ C & D \end{bmatrix} = 0 \ \forall i \implies z = 0.$$

iv. $\lambda \in \mathbb{R}$, $\lambda > 0$, $z \in \mathbb{R}^n$, $w^i \in C_r(\mathcal{Y}_i)^-$, *and*

$$\begin{bmatrix} z \\ w^i \end{bmatrix}^T \begin{bmatrix} A + H^i C - \lambda I & B + H^i D \\ C & D \end{bmatrix} = 0 \ \forall i \implies z = 0.$$

4. FROM LCS TO CPAS

A linear complementarity system of the form (1) can be cast as a CPAS of the form (2) when F is a P-matrix, that is a matrix whose principal minors are all positive. It is well-known that every positive definite matrix is a P-matrix (see for instance (Cottle et al., 1992, Thm. 3.1.6 and Thm. 3.3.7)). In particular, P-matrices play an important role in the context of linear complementarity problem, i.e. the problem of finding a p-vector z satisfying

$$0 \leqslant z \perp q + Fz \geqslant 0$$

for a given p-vector q and a $p \times p$ matrix F. We denote the linear complementarity problem for a given vector q and a given matrix F by $\mathrm{LCP}(q, F)$. If F is a P-matrix, $\mathrm{LCP}(q, F)$ admits a unique solution $z(q)$ for every $q \in \mathbb{R}^p$ (see (Cottle et al., 1992, Thm. 3.3.7)). In addition, one can verify that for every q there exists an index set $\alpha \subseteq \{1, 2, \ldots, p\}$ such that

$$\begin{bmatrix} -(F_{\alpha\alpha})^{-1} & 0_{|\alpha| \times |\alpha^c|} \\ -F_{\alpha^c \alpha}(F_{\alpha\alpha})^{-1} & I_{|\alpha^c|} \end{bmatrix} \begin{bmatrix} q_\alpha \\ q_{\alpha^c} \end{bmatrix} \geqslant 0 \tag{5a}$$

and the unique solution z of the $\mathrm{LCP}(q, F)$ is given by

$$z(q)_\alpha = -(F_{\alpha\alpha})^{-1} q_\alpha \quad \text{and} \quad z(q)_{\alpha^c} = 0. \tag{5b}$$

Here α^c denotes the set $\{1, 2, \ldots, p\} \setminus \alpha$. Furthermore, the map $q \mapsto z(q)$ is known to be Lipschitz continuous (Cottle et al., 1992, Lem. 7.3.10).

By using the Lipschitz continuity, it can be shown that Camlibel (2001) for every initial state $x_0 \in \mathbb{R}^n$ and locally integrable input u, there exists a unique locally absolutely continuous $x : \mathbb{R}_+ \to \mathbb{R}^n$ such that $x(0) = x_0$, (1a) is satisfied for almost all $t \geqslant 0$, and (1b)-(1c) are satisfied for all $t \geqslant 0$. Moreover, we can employ the parametrization given in (5) to conclude that there is one-to-one correspondence between the trajectories of the LCS (1) and those of the following *conewise linear system* (see Camlibel et al. (2008a) for more details)

$$\dot{x}(t) = Ax(t) + Bu(t) + \Psi\big(y(t)\big) \tag{6a}$$

$$y(t) = Cx(t) + Du(t) \tag{6b}$$

where Ψ is a Lipschitz continuous function given by

$$\Psi(y) = H^\alpha y_\alpha \text{ if } y \in \mathcal{Y}^\alpha \tag{7}$$

where $\alpha \subseteq \{1, 2, \ldots, p\}$,

$$H^\alpha = -E_{\bullet\alpha} F_{\alpha\alpha}^{-1} y_\alpha, \tag{8}$$

and

$$\mathcal{Y}^\alpha = \{ y \in \mathbb{R}^p \mid \begin{bmatrix} -(F_{\alpha\alpha})^{-1} & 0_{|\alpha| \times |\alpha^c|} \\ -F_{\alpha^c\alpha}(F_{\alpha\alpha})^{-1} & I_{|\alpha^c|} \end{bmatrix} \begin{bmatrix} y_\alpha \\ y_{\alpha^c} \end{bmatrix} \geqslant 0 \}. \tag{9}$$

5. MAIN RESULTS

In this section, our aim is first to streamline the conditions in Proposition 4 and Proposition 5.

Proposition 6. Consider an LCS of the form (1) *satisfying Assumption 3. Suppose that F is a P-matrix. Then, the following statements hold:*

(a) The system (2) *is controllable if and only if the following two implications hold:*

    i. $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$, and

$$z^* \begin{bmatrix} A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I & B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet} \end{bmatrix} = 0$$

    for all $\alpha \subseteq \{1, 2, \ldots, p\}$ implies $z = 0$.

    ii. $\lambda \in \mathbb{R}$, $z \in \mathbb{R}^n$, $w^\alpha \in (\mathcal{Y}^\alpha)^-$, and

$$\begin{bmatrix} z \\ w^\alpha \end{bmatrix}^T \begin{bmatrix} A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I & B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet} \\ C & D \end{bmatrix} = 0$$

    for all $\alpha \subseteq \{1, 2, \ldots, p\}$ implies $z = 0$.

(b) The system (2) *is stabilizable if and only if the following two implications hold:*

    i. $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geqslant 0$, $z \in \mathbb{C}^n$, and

$$z^* \begin{bmatrix} A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I & B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet} \end{bmatrix} = 0$$

    for all $\alpha \subseteq \{1, 2, \ldots, p\}$ implies $z = 0$.

    ii. $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$, $z \in \mathbb{R}^n$, $w^\alpha \in (\mathcal{Y}^\alpha)^-$, and

$$\begin{bmatrix} z \\ w^\alpha \end{bmatrix}^T \begin{bmatrix} A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I & B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet} \\ C & D \end{bmatrix} = 0$$

    for all $\alpha \subseteq \{1, 2, \ldots, p\}$ implies $z = 0$.

To prove this proposition, we need the following auxiliary result.

Lemma 7. Let $\mathcal{C} \subseteq \mathbb{R}^p$ be a polyhedral cone given by $\mathcal{C} = \{\xi \in \mathbb{R}^p \mid M\xi \geqslant 0\}$ for some matrix $M \in \mathbb{R}^{q \times p}$. Then, $\mathcal{C}^- = \{\eta \in \mathbb{R}^p \mid \eta = M^T \zeta \text{ for some } 0 \geqslant \zeta \in \mathbb{R}^q\}$. Moreover, if $\ker M^T = \{0\}$, then $\mathcal{C}^\perp = \{0\}$.

Proof. The first statement readily follows from Farkas' Lemma (see e.g. (Bertsekas, 2009, Prop. 2.3.1)). To prove the second statement, note that $\mathcal{C}^\perp = -\mathcal{C}^- \cap \mathcal{C}^-$. Therefore, $\eta \in \mathcal{C}^\perp$ if and only if there exist $\zeta_1 \geqslant 0$ and $\zeta_2 \leqslant 0$ such that $\eta = M^T \zeta_1 = M^T \zeta_2$. This means that $\zeta_1 - \zeta_2 \in \ker M^T$ and hence $\zeta_1 = \zeta_2 = 0$. Consequently, $\eta = 0$. ∎

Now, we are in a position to prove Proposition 6.

Proof of Proposition 6: The idea of the proof is to apply Proposition 4 for controllability and Proposition 5

for stabilizability. First note that, we have $\Psi(0) = 0$ due to (7)-(9). Therefore, hypotheses of both propositions are satisfied.

It follows from (9) that $\mathcal{Y}^\alpha$ is a polyhedral cone for all $\alpha \subseteq \{1, 2, \ldots, p\}$. As such, we have $C_r(\mathcal{Y}^\alpha) = \mathcal{Y}^\alpha$. Moreover, it follows from Lemma 7 that $\mathcal{Y}^\perp = \{0\}$ for all $\alpha \subseteq \{1, 2, \ldots, p\}$ since the matrix

$$\begin{bmatrix} -(F_{\alpha\alpha})^{-1} & 0_{|\alpha| \times |\alpha^c|} \\ -F_{\alpha^c\alpha}(F_{\alpha\alpha})^{-1} & I_{|\alpha^c|} \end{bmatrix}$$

is nonsingular.

As (8) does not contain translation term, that is $g^\alpha = 0$, we see that the condition (iv) of Proposition 4 is implied by the condition (ii) of Proposition 4 whereas the condition (iii) of Proposition 4 is implied by the condition (i) of Proposition 4. Moreover, the same reasoning holds if we replace Proposition 4 by Proposition 5 above.

Note that

$$H^\alpha C = -E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} \quad \text{and} \quad H^\alpha D = -E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet}.$$

Therefore, we can conclude that (a) is implied by Proposition 4 and (b) by Proposition 5.   ∎

Proposition 6 requires checking the involved conditions for all index sets $\alpha \subseteq \{1, 2, \ldots, p\}$. It turns out that the structure of LCSs can be exploited to obtain more streamlined conditions that can be checked much more easily as stated next.

Theorem 8. Consider an LCS of the form (1) satisfying Assumption 3. Suppose that F is a P-matrix. Then, the following statements hold:

(a) *The system (2) is controllable if and only if the following two implications hold:*

    i. $(A, [B\ E])$ *is controllable.*

    ii. $\lambda \in \mathbb{R}$, $z \in \mathbb{R}^n$, $w \leqslant 0$, *and*

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0 \quad and \quad \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} E \\ F \end{bmatrix} \geqslant 0$$

    implies $z = 0$ and $w = 0$.

(b) *The system (2) is stabilizable if and only if the following two implications hold:*

    i. $(A, [B\ E])$ *is stabilizable.*

    ii. $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$, $z \in \mathbb{R}^n$, $w \leqslant 0$, *and*

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0 \quad and \quad \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} E \\ F \end{bmatrix} \geqslant 0$$

    implies $z = 0$ and $w = 0$.

The proof of this theorem will rely on the following intermediate result.

Lemma 9. Suppose that (A, B, C, D) satisfies Assumption 3 and F is a P-matrix. Then, the following statements hold:

(i) *Let $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Then, for all $\alpha \subseteq \{1, 2, \ldots, p\}$*

$$z^* \begin{bmatrix} A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I & B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet} \end{bmatrix} = 0$$

    if and only if

$$z^* \begin{bmatrix} A - \lambda I & B & E \end{bmatrix} = 0.$$

(ii) *Let $\lambda \in \mathbb{R}$ and $z \in \mathbb{R}^n$. Then, for all $\alpha \subseteq \{1, 2, \ldots, p\}$ there exists $w^\alpha \in (\mathcal{Y}^\alpha)^-$ such that*

$$\begin{bmatrix} z \\ w^\alpha \end{bmatrix}^T \begin{bmatrix} A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I & B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet} \\ C & D \end{bmatrix} = 0$$

    if and only if there exists $w \leqslant 0$ such that

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0 \quad and \quad \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} E \\ F \end{bmatrix} \geqslant 0.$$

Proof. For the 'only if' part of (i), note that the choice $\alpha = \varnothing$ results in

$$z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0. \tag{10}$$

Therefore, it remains to show that $z^*E = 0$. Let $\alpha = \{1, 2, \ldots, p\}$. Then, we have

$$z^* \begin{bmatrix} A - EF^{-1}C - \lambda I & B - EF^{-1}D \end{bmatrix} = 0.$$

By using (10), we further obtain

$$0 = z^* \begin{bmatrix} EF^{-1}C & EF^{-1}D \end{bmatrix} = z^* EF^{-1} \begin{bmatrix} C & D \end{bmatrix}. \tag{11}$$

Note that $[C\ D]$ is full row rank due to Assumption 3. Therefore, it follows from (11) that $z^*E = 0$.

For the 'if' part of (i), note that $z^*E = 0$ implies that $z^*E_{\bullet\alpha} = 0$ for all $\alpha \subseteq \{1, 2, \ldots, p\}$. Therefore,

$$z^* \begin{bmatrix} A - \lambda I & B & E \end{bmatrix} = 0$$

implies

$$z^* \begin{bmatrix} A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I & B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet} \end{bmatrix} = 0$$

for all $\alpha \subseteq \{1, 2, \ldots, p\}$.

For the 'only if' part of (ii), we claim that $w^\varnothing \leqslant 0$,

$$\begin{bmatrix} z \\ w^\varnothing \end{bmatrix}^T \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0, \tag{12}$$

and

$$\begin{bmatrix} z \\ w^\varnothing \end{bmatrix}^T \begin{bmatrix} E \\ F \end{bmatrix} \geqslant 0. \tag{13}$$

Since (12) readily follows from the choice of $\alpha = \varnothing$, it remains to show that $w^\varnothing \leqslant 0$ and (13) holds. To do so, we first note that $\mathcal{Y}^\varnothing = \mathbb{R}_+^p$ due to (9). Since $w^\varnothing \in (\mathcal{Y}^\varnothing)^-$, we see that $w^\varnothing \leqslant 0$. For the rest, let $\bar\alpha = \{1, 2, \ldots, p\}$ and $\bar w = w^\alpha$. Then, we have

$$\begin{bmatrix} z \\ \bar w \end{bmatrix}^T \begin{bmatrix} A - EF^{-1}C - \lambda I & B - EF^{-1}D \\ C & D \end{bmatrix} = 0.$$

By using (12), we obtain

$$\begin{bmatrix} z \\ \bar w - w^\varnothing \end{bmatrix}^T \begin{bmatrix} -EF^{-1}C & -EF^{-1}D \\ C & D \end{bmatrix} = 0.$$

Hence, we see that

$$\left(\bar{w}^T - (w^\varnothing)^T - z^T EF^{-1}\right) [C\ D] = 0.$$

Since $[C\ D]$ is full row rank due to Assumption 3, we obtain

$$\begin{bmatrix} z \\ w^\varnothing \end{bmatrix}^T \begin{bmatrix} E \\ F \end{bmatrix} = \bar{w}^T F.$$

As such, it is enough to show that $\bar{w}^T F \geqslant 0$. Note that $\bar{w} \in (\mathcal{Y}^{\bar{\alpha}})^-$. Since

$$\mathcal{Y}^{\bar{\alpha}} = \{y \mid -F^{-1}y \geqslant 0\}$$

due to (9), it follows from Lemma 7 that

$$(\mathcal{Y}^{\bar{\alpha}})^- = \{\eta \mid \eta = (F^T)^{-1}\zeta \text{ for some } 0 \leqslant \zeta\}.$$

Therefore, $\bar{w} = (F^T)^{-1}\bar{\zeta}$ for some $\bar{\zeta} \geqslant 0$. Note that

$$\bar{w}^T F = \bar{\zeta}^T \geqslant 0.$$

For the 'if' part of (ii), w be such that $w \leqslant 0$,

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0, \tag{14}$$

and

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} E \\ F \end{bmatrix} \geqslant 0. \tag{15}$$

For all $\alpha \subseteq \{1, 2, \ldots, p\}$, define w^α by

$$w^\alpha_{\alpha^c} = w_{\alpha^c} \quad \text{and} \quad (w^\alpha_\alpha)^T = (w_\alpha)^T + z^T E_{\bullet\alpha} F_{\alpha\alpha}^{-1} \tag{16}$$

With this definition, we have

$$\begin{bmatrix} z \\ w^\alpha \end{bmatrix}^T \begin{bmatrix} A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I & B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet} \\ C & D \end{bmatrix}$$

$$= \begin{bmatrix} z \\ w^\alpha_\alpha \\ w^\alpha_{\alpha^c} \end{bmatrix}^T \begin{bmatrix} A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I & B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\alpha\bullet} \\ C_{\alpha\bullet} & D_{\alpha\bullet} \\ C_{\alpha^c\bullet} & D_{\alpha^c\bullet} \end{bmatrix}$$

$$= \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0.$$

It remains to show that $w^\alpha \in (\mathcal{Y}^\alpha)^-$. From (15), we see that

$$z^T E_{\bullet\alpha} + w^T_\alpha F_{\alpha\alpha} + w^T_{\alpha^c} F_{\alpha^c\alpha} \geqslant 0.$$

By using (16), we get

$$z^T E_{\bullet\alpha} + \left((w^\alpha_\alpha)^T - z^T E_{\bullet\alpha} F_{\alpha\alpha}^{-1}\right) F_{\alpha\alpha} + (w^\alpha_{\alpha^c})^T F_{\alpha^c\alpha} \geqslant 0.$$

This leads to

$$(w^\alpha_\alpha)^T F_{\alpha\alpha} + (w^\alpha_{\alpha^c})^T F_{\alpha^c\alpha} \geqslant 0. \tag{17}$$

In addition, we have

$$w^\alpha_{\alpha^c} \leqslant 0 \tag{18}$$

since $w \leqslant 0$. By combining (17) and (18), we can write

$$\begin{bmatrix} (F_{\alpha\alpha})^T & (F_{\alpha^c\alpha})^T \\ 0 & -I \end{bmatrix} \begin{bmatrix} w^\alpha_\alpha \\ w^\alpha_{\alpha^c} \end{bmatrix} \geqslant 0. \tag{19}$$

It follows from (9) and Lemma 7 that

$$\eta \in (\mathcal{Y}^\alpha)^- \iff \eta = \begin{bmatrix} (F_{\alpha\alpha})^{-T} & (F_{\alpha\alpha})^{-T}(F_{\alpha^c\alpha})^T \\ 0 & -I \end{bmatrix} \zeta.$$

for some $\zeta \geqslant 0$. Note that

$$\begin{bmatrix} (F_{\alpha\alpha})^{-T} & (F_{\alpha\alpha})^{-T}(F_{\alpha^c\alpha})^T \\ 0 & -I \end{bmatrix} \begin{bmatrix} (F_{\alpha\alpha})^T & (F_{\alpha^c\alpha})^T \\ 0 & -I \end{bmatrix} = I.$$

Therefore, (19) implies that $w^\alpha \in (\mathcal{Y}^\alpha)^-$. ∎

Now, we are in a position to prove Theorem 8.

Proof of Theorem 8: In view of Lemma 9, the conditions in the statement (a) of Proposition 6 are equivalent with the conditions in the statement of Theorem 8. This proves the controllability claims of Theorem 8. The claims about stabilizability also follows from Lemma 9 in a similar fashion.

6. CONCLUSIONS

In this paper, we studied controllability and stabilizability of linear complementarity systems. The main results are Hautus-like necessary and sufficient conditions for both controllability and stabilizability. By working under the assumption that the transfer matrix of the underlying linear system from inputs to one of the complementarity variable is right invertible, we first showed that the existing tests for controllability and stabilizability of continuous piecewise affine systems can be streamlined. Based on these streamlined tests, we presented tailor-made tests for linear complementarity systems.

Future research directions include relaxing the right-invertibility assumption on the one hand and studying linear complementarity systems that cannot be cast as continuous piecewise affine systems on the other hand.

REFERENCES

Bertsekas, D.P. (2009). *Convex Optimization Theory.* Athena Scientific.

Brogliato, B. (2003). Some perspectives on the analysis and control of complementarity systems. *IEEE Transactions on Automatic Control*, 48(6), 918–935.

Camlibel, M.K. (2001). *Complementarity Methods in the Analysis of Piecewise Linear Dynamical Systems.* Dissertation Series of Center for Economic Research. Tilburg University, The Netherlands. (ISBN: 90 5668 079 X).

Camlibel, M.K. (2007). Popov-Belevitch-Hautus type controllability tests for linear complementarity systems. *Systems and Control Letters*, 56, 381–387.

Camlibel, M.K., Heemels, W.P.M.H., and Schumacher, J.M. (2002). On linear passive complementarity systems. *European Journal of Control*, 8(3).

Camlibel, M.K., Heemels, W.P.M.H., and Schumacher, J.M. (2008a). Algebraic necessary and sufficient conditions for the controllability of conewise linear systems. *IEEE Transactions on Automatic Control*, 53(3), 762–774.

Camlibel, M.K., Heemels, W.P.M.H., and Schumacher, J.M. (2008b). A full characterization of stabilizability

of bimodal piecewise linear systems with scalar inputs. *Automatica*, 44(5), 1261–1267.

Camlibel, M.K., Heemels, W.P.M.H., van der Schaft, A.J., and Schumacher, J.M. (2003). Switched networks and complementarity. *IEEE Transactions on Circuits and Systems I*, 50(8), 1036–1046.

Camlibel, M.K., Iannelli, L., and Vasca, F. (2004). Modelling switching power converters as complementarity systems. In *Proc. of the 43th IEEE Conference on Decision and Control*. Paradise Islands (Bahamas).

Camlibel, M.K., Iannelli, L., and Vasca, F. (2014). Passivity and complementarity. *Mathematical Programming*, 145(1-2), 531–563.

Camlibel, M.K. and Schumacher, J.M. (2002). Existence and uniqueness of solutions for a class of piecewise linear dynamical systems. *Linear Algebra and its Applications*, 351-352, 147–184.

Cottle, R.W., Pang, J.S., and Stone, R.E. (1992). *The Linear Complementarity Problem*. Academic Press, Inc., Boston.

Facchinei, F. and Pang, J.S. (2002). *Finite Dimensional Variational Inequalities and Complementarity Problems*. Springer.

Heemels, W.P.M.H. and Brogliato, B. (2003). The complementarity class of hybrid dynamical systems. *European Journal of Control*, 26(4), 651–677.

Heemels, W.P.M.H. and Camlibel, M.K. (2008). Null controllability of discrete-time linear systems with input and state constraints. In *47th IEEE Conference on Decision and Control*. Cancun (Mexico).

Heemels, W.P.M.H., Camlibel, M.K., and Schumacher, J.M. (2002). On the dynamic analysis of piecewise-linear networks. *IEEE Trans. on Circuits and Systems–I: Fundamental Theory and Applications*, 49(3), 315–327.

Heemels, W.P.M.H., Schumacher, J.M., and Weiland, S. (2000). Linear complementarity systems. *SIAM J. Appl. Math.*, 60(4), 1234–1269.

Saberi, A., Han, J., and Stoorvogel, A.A. (2002). Constrained stabilization problems for linear plants. *Automatica*, 38, 639–654.

Saberi, A., Stoorvogel, A.A., Shi, G., and Sannuti, P. (2003). Output regulation of linear plants subject to constraints. *International Journal of Control*, 76(2), 149–164.

Schumacher, J.M. (2004). Complementarity systems in optimization. *Mathematical Programming Series B*, 101, 263–295.

Shi, G., Saberi, A., Stoorvogel, A.A., and Sannuti, P. (2003). Output regulation of discrete-time linear plants subject to state and input constraints. *International Journal of Robust and Nonlinear Control*, 13, 691–713.

Sun, Z., Ge, S., and Lee, T. (2002). Controllability and reachability criteria for switched linear systems. *Automatica*, 38(5), 775–786.

Thuan, L.Q. and Camlibel, M.K. (2014). Controllability and stabilizability of piecewise affine dynamical systems. *SIAM Journal on Control and Optimization*, 52(3), 1914–1934.

van der Schaft, A.J. and Schumacher, J.M. (1996). The complementary-slackness class of hybrid systems. *Mathematics of Control, Signals and Systems*, 9, 266–301.

van der Schaft, A.J. and Schumacher, J.M. (1998). Complementarity modelling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4), 483–490.

van der Schaft, A.J. and Schumacher, J.M. (2000). *An Introduction to Hybrid Dynamical Systems*. Springer-Verlag, London.