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Data Informativity for Analysis of Linear Systems With Convex Conic Constraints

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Abstract—This letter studies the informativity problem for reachability and null-controllability of constrained systems. To be precise, we will focus on an unknown linear systems with convex conic constraints from which we measure data consisting of exact state trajectories of finite length. We are interested in performing system analysis of such an unknown system on the basis of the measured data. However, from such measurements it is only possible to obtain a unique system explaining the data in very restrictive cases. This means that we can not approach this problem using system identification combined with model based analysis. As such, we will formulate conditions on the data under which any such system consistent with the measurements is guaranteed to be reachable or null-controllable. These conditions are stated in terms of spectral conditions and subspace inclusions, and therefore they are easy to verify.

Index Terms—Constrained control, linear systems, sampled-data control.

I. INTRODUCTION

THIS letter deals with the question: what can be inferred from an unknown constrained linear system on the basis of state measurements? A similar question, for unconstrained systems, has recently led to the development of the *informativity framework* in [1]. The observation at the center of this framework is that we can only conclude that the unknown system has a given property if *all* systems compatible with the measurements have this property. In the context of linear systems this has lead to, among others, results for analysis problems in [2] and control problems in [3], [4]. Parallel to the work performed within this framework, similar analysis problems are addressed in [5], while control problems are addressed in [6], [7].

In contrast to this earlier work, we will be focusing on conically constrained linear systems. Such conic constraints often arise naturally in modeling, taking the form of, e.g., nonnegativity constraints on the input or states. Specifically, we will

be looking at the class of difference inclusions of the form

$$x_{k+1} \in H(x_k)$$

where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex process, that is, a set-valued map whose graph is a convex cone. It is straightforward to show that any conically constrained linear system can be written as such a system and vice versa. Such difference inclusions arise naturally in many different contexts, including chemical reaction networks [8], von Neumann-Gale economic growth models [9] and cable-suspended robots [10], [11]. Lastly, as shown in, e.g., [12], [13], difference inclusions of convex processes can be used as meaningful approximations of more complex set-valued maps.

The many applications of convex processes have led to interest in the analysis of such systems. In particular, this letter will consider the system-theoretic properties of reachability and null-controllability. For a given convex process, tests for these properties have been developed in terms of spectral conditions. First among these were the characterizations of reachability and null-controllability in [14], [15]. However, the aforementioned characterizations only regard *strict* (nonempty everywhere) convex processes, which limits the applicability for our goals. In [16] both of these results are generalized to work for a class of nonstrict convex processes. These characterizations of [16] will be fundamental in our investigation of informativity. In this letter, we will be interested in analyzing whether these system-theoretic properties hold for *all* convex processes compatible with measured data. Specifically, the data we consider will consist of exact measurements of the state. This means that we leave studies involving measurement noise or partial state measurements as future extensions.

Apart from the aforementioned work, some results in data-driven analysis and control should be mentioned. With regard to unconstrained linear systems [17] analysis stability of an input/output system using time series data. The works [18]–[21] deal with data-based controllability and observability analysis. Lastly, many methods arising from Model Predictive Control (MPC) are well suited to constrained systems. For an overview of such methods, we refer to [22], [23]. More specifically, MPC has recently been brought into a data-based context in [24], [25].

The contribution of this letter is threefold:

- 1) We expand the informativity framework of [1] towards the class of convex processes. This framework will naturally lead to the formulation of a number of

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problems. In particular, we will illustrate the framework by resolving the problems of informativity for reachability and null-controllability.

- 2) We develop explicit tools to manipulate and perform analysis on convex processes with a polyhedral graph. Assuming polyhedrality will allow us to represent convex processes and the conditions required for reachability and null-controllability in a convenient way.
- 3) Lastly, we note the fact that polyhedral convex processes naturally arise from the aforementioned informativity problems with finite measurements. This allows us to combine the previous points to formulate tests *on measured state data* to conclude that all convex processes consistent with the data are reachable or null-controllable.

This letter is organized as follows: We begin in Section II with definitions of convex process and reachability and null-controllability. After this, Section III introduces informativity and formally states the problem we will consider in this letter. In Section IV, we will present some known results regarding the analysis of convex processes, which will be applied in Section V to our problem. We finalize this letter with conclusions in Section VI.

II. CONVEX PROCESSES

Given convex sets $\mathcal{S}, \mathcal{T} \subseteq \mathbb{R}^n$ and scalar $\rho \in \mathbb{R}$ we define the sum and scalar multiplication of sets as:

$$\mathcal{S} + \mathcal{T} := \{s + t \mid s \in \mathcal{S}, t \in \mathcal{T}\}, \quad \rho\mathcal{S} := \{\rho s \mid s \in \mathcal{S}\}.$$

We denote the closure of \mathcal{S} by $\text{cl}\mathcal{S}$. A *convex cone* is a nonempty convex set that is closed under nonnegative scalar multiplication.

A *set-valued map*, denoted $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a map taking elements of \mathbb{R}^n to subsets of \mathbb{R}^n . It is called a *convex process*, *closed convex process* or *linear process* if its graph

$$\text{gr } H := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y \in H(x)\}$$

is a convex cone, closed convex cone or subspace, respectively.

The *domain* and *image* of H are defined as $\text{dom } H = \{x \in \mathbb{R}^n \mid H(x) \neq \emptyset\}$ and $\text{im } H = \{y \in \mathbb{R}^n \mid \exists x \text{ s.t. } y \in H(x)\}$. If $\text{dom } H = \mathbb{R}^n$, we say that H is *strict*.

In this letter we consider systems described by a *difference inclusion* of the form

$$x_{k+1} \in H(x_k) \quad (1)$$

where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex process. Our main motivation for considering this class of systems is the fact that this class of systems captures the behavior of all linear systems with convex conic constraints. This will be made explicit in the following example.

Example 1: Consider states x_k in \mathbb{R}^n and inputs $u_k \in \mathbb{R}^m$. Let A and B be linear maps of appropriate dimensions and let $\mathcal{C} \subseteq \mathbb{R}^{n+m}$ be a convex cone. Consider the linear system with conic constraints given by:

$$x_{k+1} = Ax_k + Bu_k, \quad \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \mathcal{C}. \quad (2)$$

Note that this description can be applied to any combination of input, state and output constraints.

We can describe the dynamics of (2) by the difference inclusion (1) with the convex process H defined by:

$$H(x) := \left\{ Ax + Bu \mid \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{C} \right\}.$$

This reveals that we can study the properties of conically constrained linear systems by studying convex processes, without any loss of generality.

Next, we define a number of sets associated with the difference inclusion (1). A *q-step trajectory* is a (finite) sequence x_0, \dots, x_q such that (1) holds for all $k < q$. We define the *q-step behavior* as:

$$\mathfrak{B}_q(H) := \left\{ (x_k)_{k=0}^q \in (\mathbb{R}^n)^{q+1} \mid (x_k) \text{ satisfies (1)} \right\}.$$

Using this, we define the *reachable* and *null-controllable* sets by:

$$\begin{aligned} \mathcal{R}(H) &:= \{ \xi \mid \exists q, (x_k)_{k=0}^q \in \mathfrak{B}_q(H) \text{ s.t. } x_0 = 0, x_q = \xi \}, \\ \mathcal{N}(H) &:= \{ \xi \mid \exists q, (x_k)_{k=0}^q \in \mathfrak{B}_q(H) \text{ s.t. } x_0 = \xi, x_q = 0 \}. \end{aligned}$$

We say that a point $\xi \in \mathbb{R}^n$ is *reachable* if $\xi \in \mathcal{R}(H)$. That is, there exists a *q-step trajectory* from the origin to ξ . Similarly, we say a point $\xi \in \mathbb{R}^n$ is *null-controllable* if $\xi \in \mathcal{N}(H)$.

By a *trajectory* of (1), we mean a sequence $(x_k)_{k \in \mathbb{N}}$ such that (1) holds for all $k \geq 0$. The *behavior* is the set of all trajectories:

$$\mathfrak{B}(H) := \left\{ (x_k) \in (\mathbb{R}^n)^{\mathbb{N}} \mid (x_k) \text{ is a trajectory of (1)} \right\}.$$

The set of *feasible* states of the difference inclusion (1) is the set of states from which a trajectory emanates:

$$\mathcal{F}(H) := \{ \xi \mid \exists (x_k) \in \mathfrak{B}(H) \text{ with } x_0 = \xi \}.$$

Clearly, if H is a convex process, then $\mathcal{F}(H)$ is a convex cone.

It is important to stress that in general not every point in the state space is feasible: In Example 1, if we consider a point x_0 for which no u_0 satisfies the constraints, we have that $H(x_0) = \emptyset$. This means that x_0 is not a feasible point. As there is no need to reach or control states that violate the constraints, we say the system (1) is *reachable* or *null-controllable* if every feasible state is reachable or null-controllable respectively. In terms of the previously defined sets, these can be written as $\mathcal{F}(H) \subseteq \mathcal{R}(H)$ and $\mathcal{F}(H) \subseteq \mathcal{N}(H)$ respectively.

It is important to note that, as is the case for discrete-time linear systems, reachability and null-controllability are not equivalent notions.

III. PROBLEM FORMULATION

In this letter we are interested in analyzing the properties of an unknown system based on measurements performed on it. We will assume that the system under consideration is given by

$$x_{k+1} \in H_s(x_k)$$

where H_s is an unknown convex process. However, we do have access to a number of exact state measurements corresponding to (*q*-step) trajectories of H_s . It is clear to see that

we can view a single q -step trajectory as q separate 1-step trajectories. Therefore, without loss of generality, we assume that we measure single steps. That is, we are given a finite number of pairs $(x_k, y_k) \in \text{gr } H_s$, with $k = 0, \dots, T$

Suppose that we are interested in characterizing reachability of H_s . As H_s is unknown, it is indistinguishable from all other convex processes that could have generated the measurements. Therefore, we may only conclude that H_s is reachable if *all* convex processes that are compatible with the data are reachable. This motivates the following definition. Let Σ denote the set of all convex processes $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a finite set of measurements. Define the set of all convex processes compatible with these measurements by:

$$\Sigma_{\mathcal{D}} := \{H \in \Sigma \mid \mathcal{D} \subseteq \text{gr } H\}. \quad (3)$$

Recall that, in order to characterize whether H_s is reachable, we require all convex processes compatible with the measurements to be reachable. As such, we say that the data \mathcal{D} are *informative for reachability* if every $H \in \Sigma_{\mathcal{D}}$ is reachable. In a similar way we define *informativity for null-controllability*.

Note that informativity is fundamentally a property of the data and the system class, but *not* of the system H_s . This leads to the following problem formulation.

Problem 1: Provide necessary and sufficient conditions on the data \mathcal{D} under which the data are informative for reachability or null-controllability.

Remark 1: Following Example 1, it is clear that all convex processes consistent with the data are reachable if and only if all conically constrained linear systems consistent with the data are reachable. As these problems are equivalent we will only focus on formulations in terms of convex processes in the remainder of this letter.

It should be noted that, in certain cases, the informativity problem can be resolved trivially, as shown by the following example.

Example 2: Let $n = 1$, and assume that we measure the 2-step trajectory given by $x_0 = 0$, $x_1 = 1$, and $x_2 = -1$. Then we have $\mathcal{D} = \{(0, 1), (1, -1)\}$.

Note that nonnegative scalar multiples of these measurements are also (finite step) trajectories of any convex process in $\Sigma_{\mathcal{D}}$. As such, it is clear that for any $\alpha, \beta \geq 0$ we have 2-step trajectories $y_0 = 0$, $y_1 = \alpha$, $y_2 = -\alpha$ and $z_0 = 0$, $z_1 = 0$, $z_2 = \beta$. Furthermore, the sum of two such 2-step trajectories is one as well. Therefore $(0, \alpha, \beta - \alpha) \in \mathfrak{B}_2(H)$ for any H consistent with the data. As such, $\mathcal{R}(H) = \mathbb{R}$ for any $H \in \Sigma_{\mathcal{D}}$.

In general, however, resolving the problem is not this straightforward. To be precise, it is made difficult by two things. First of all, apart from trivial examples, the set $\Sigma_{\mathcal{D}}$ contains infinitely many convex processes. As such, it is usually not possible to take an approach based on identification. In addition, there may *not* exist q for a convex process H such that

$$\mathcal{R}(H) := \{\xi \mid \exists (x_k)_{k=0}^q \in \mathfrak{B}_q(H) \text{ s.t. } x_0 = 0, x_q = \xi\}.$$

Therefore, testing whether a given convex process is reachable or null-controllable is a nontrivial problem in itself (see, e.g., [14], [16]).

IV. ANALYSIS OF CONVEX PROCESSES

By definition a convex cone \mathcal{C} is closed under *conic combinations*: If $c_1, \dots, c_\ell \in \mathcal{C}$ then

$$\sum_{i=1}^{\ell} \alpha_i c_i \in \mathcal{C} \quad \forall \alpha_i \geq 0.$$

The set of all conic combinations of a set \mathcal{S} is called the *conic hull* and is denoted by $\text{cone } \mathcal{S}$. If there exists a finite set \mathcal{S} such that $\mathcal{C} = \text{cone } \mathcal{S}$ we say that \mathcal{C} is *finitely generated* or *polyhedral*. We denote the set of vectors of length ℓ with nonnegative and nonpositive elements by \mathbb{R}_+^ℓ and \mathbb{R}_-^ℓ respectively. Then, if $M \in \mathbb{R}^{k \times \ell}$ and \mathcal{S} is the set of columns of M , we have that

$$\text{cone } \mathcal{S} = M\mathbb{R}_+^\ell. \quad (4)$$

For a nonempty set $\mathcal{C} \subseteq \mathbb{R}^n$, we define the *negative* and *positive polar cone*, respectively,

$$\begin{aligned} \mathcal{C}^- &:= \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 0 \quad \forall x \in \mathcal{C}\}, \\ \mathcal{C}^+ &:= \{y \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0 \quad \forall x \in \mathcal{C}\}. \end{aligned}$$

Given sets \mathcal{C} and \mathcal{S} , we have that $(\mathcal{C}^-)^- = \text{cl}(\text{cone } \mathcal{C})$, and:

$$(\mathcal{C} + \mathcal{S})^- = \mathcal{C}^- \cap \mathcal{S}^-, \quad (\mathcal{C} \cap \mathcal{S})^- = \text{cl}(\mathcal{C}^- + \mathcal{S}^-). \quad (5)$$

Let A be a linear map and let \bullet^{-1} denotes the inverse image, that is, $A^{-1}\mathcal{C}^- = \{x \mid Ax \in \mathcal{C}^-\}$. Then if \mathcal{C} is a convex cone we have that (see, e.g., [26, Th. 2.4.3]):

$$(A^\top \mathcal{C})^- = A^{-1}\mathcal{C}^-. \quad (6)$$

The aforementioned properties also hold for the positive polar cone.

Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex process. We define *negative* and *positive dual* processes H^- and H^+ of H as follows:

$$p \in H^-(q) \iff \langle p, x \rangle \geq \langle q, y \rangle \quad \forall (x, y) \in \text{gr}(H), \quad (7a)$$

$$p \in H^+(q) \iff \langle p, x \rangle \leq \langle q, y \rangle \quad \forall (x, y) \in \text{gr}(H). \quad (7b)$$

Note that $H^+(q) = -H^-(-q)$ for all q . If H is a closed convex process, we know that $(H^+)^- = H$ and

$$H(0) = (\text{dom } H^+)^+ = (\text{dom } H^-)^-. \quad (8)$$

It is straightforward to check that

$$\text{gr}(H^-) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} (\text{gr } H)^-, \quad \text{gr}(H^+) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} (\text{gr } H)^+. \quad (9)$$

For a convex cone $\mathcal{C} \subseteq \mathbb{R}^n$, we define $\text{lin}(\mathcal{C}) = -\mathcal{C} \cap \mathcal{C}$ and $\text{Lin}(\mathcal{C}) = \mathcal{C} - \mathcal{C}$. We can now define two linear processes L_- and L_+ associated with H by

$$\text{gr}(L_-) := \text{lin}(\text{gr}(H)) \text{ and } \text{gr}(L_+) := \text{Lin}(\text{gr}(H)). \quad (10)$$

By definition, we therefore have

$$\text{gr}(L_-) \subseteq \text{gr}(H) \subseteq \text{gr}(L_+). \quad (11)$$

It is clear that L_- and L_+ are, respectively, the largest and the smallest (with respect to the graph inclusion) linear processes satisfying (11). We call L_- and L_+ , respectively, the minimal and maximal linear processes associated with H . If H is not

clear from context, we write $L_-(H)$ and $L_+(H)$ in order to avoid confusion.

If L is a linear process it is clear that the negative and positive dual processes are equal, which allows us to denote it by $L^\perp := L^- = L^+$. In fact, the minimal and maximal linear processes associated with a convex process enjoy the following additional properties:

$$L_-(H^-) = L_-(H^+) = L_+^\perp, \quad (12a)$$

$$L_+(H^-) = L_+(H^+) = L_-^\perp. \quad (12b)$$

For the reachable and null-controllable sets of L_- and L_+ we use the following shorthand notation:

$$\mathcal{R}_- := \mathcal{R}(L_-), \mathcal{R}_+ := \mathcal{R}(L_+).$$

$$\mathcal{N}_- := \mathcal{N}(L_-), \mathcal{N}_+ := \mathcal{N}(L_+).$$

We denote the image of a set \mathcal{S} under a convex process H by $H(\mathcal{S}) := \{y \in \mathbb{R}^n \mid \exists x \in \mathcal{S} \text{ s.t. } y \in H(x)\}$. A direct consequence of this definition is that

$$H(\mathcal{S}) = [0 \quad I_n](\text{gr}(H) \cap (\mathcal{S} \times \mathbb{R}^n)). \quad (13)$$

We can define powers of convex processes, by taking H^0 equal to the identity map, and letting for $q \geq 0$:

$$H^{q+1}(x) := H(H^q(x)) \quad \forall x \in \mathbb{R}^n.$$

We can define the inverse of a convex process by $H^{-1}(y) = \{x \mid y \in H(x)\}$. Note that this is always defined as a set-valued map. For higher negative powers of H we use the shorthand: $H^{-n}(x) = (H^{-1})^n(x)$.

Let $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a linear process, then we know that $\mathcal{F}(L) = L^-(\mathbb{R}^n)$ and $\mathcal{R}(L) = L^+(0)$. In addition

$$\mathcal{F}(L^\perp) = \mathcal{R}(L)^\perp, \quad (14a)$$

$$\mathcal{R}(L^\perp) = \mathcal{F}(L)^\perp. \quad (14b)$$

We will characterize reachability in terms of spectral conditions. For this we require one more definition: A real number λ and vector $\xi \in \mathbb{R}^n \setminus \{0\}$ form an *eigenpair* of H if $\lambda\xi \in H(\xi)$. In this case λ is called an *eigenvalue* and ξ is called an *eigenvector* of H .

In the following, we will need the assumption:

$$\text{dom } H + \mathcal{R}_- = \mathbb{R}^n. \quad (15)$$

As proven in [16, Th. 1, Lemma 7], we can characterize reachability in terms of eigenvalues of the dual process.

Theorem 1: Let H be a convex process such that (15) holds. Then, the following are equivalent:

- 1) H is reachable.
- 2) $\mathcal{R}(H) = \mathbb{R}^n$.
- 3) $\mathcal{R}_+ = \mathbb{R}^n$ and H^- has no nonnegative eigenvalues.

We now move towards null-controllability. It is tempting to think that null-controllability of H is equivalent to reachability of H^{-1} . However, while indeed it is true that $\mathcal{R}(H^{-1}) = \mathcal{N}(H)$, we do not necessarily have that $\mathcal{F}(H^{-1}) = \mathcal{F}(H)$.

As such, we require a characterization of null-controllability. This will be done under slightly more restrictive assumptions than Theorem 1. To be precise, we will assume both (15) and

$$\mathcal{R}_+ = \text{im } H + \mathcal{N}_- = \mathbb{R}^n. \quad (16)$$

The following was proven in [16, Th. 2, Lemma 9].

Theorem 2: Let H be a convex process such that (15) and (16) hold. Then, the following are equivalent:

- 1) H is null-controllable.
- 2) $\mathcal{N}(H) - \mathcal{R}(H) = \mathbb{R}^n$.
- 3) H^- has no positive eigenvalues.

The following shows why we require separate tests for these two properties.

Example 3: Recall that, as is the case for discrete time linear systems, a convex process can be null-controllable without being reachable. As a simple example consider the convex process given by:

$$\text{gr } H := \mathbb{R} \times \{0\}.$$

On the other hand, we know that reachability implies null-controllability for discrete time linear systems. For general convex processes this is not the case. As an example, let:

$$\text{gr } G := \{(x, y) \mid 0 \leq x \leq y\}.$$

Note that $\mathcal{R}(G) = \mathbb{R}_+ = \mathcal{F}(G)$, and therefore G is reachable. As any trajectory of G is a non-decreasing sequence, G is clearly not null-controllable. This means that in general tests for reachability can not be employed to obtain results for null-controllability.

These two theorems allow us to check for reachability and null-controllability without explicitly determining $\mathcal{R}(H)$ or $\mathcal{N}(H)$. This will be central in resolving Problem 1 in the next section.

V. INFORMATIVITY FOR CONVEX PROCESSES

We turn our attention to the context of informativity. Let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a finite set of measurements. We define the *most powerful unfalsified process*, $H_{\mathcal{D}}$, by:

$$\text{gr } H_{\mathcal{D}} := \text{cone } \mathcal{D}.$$

By definition we see that $H_{\mathcal{D}} \in \Sigma_{\mathcal{D}}$ and $\text{gr } H_{\mathcal{D}} \subseteq \text{gr } H$ if and only if $H \in \Sigma_{\mathcal{D}}$. Our goal is to find conditions on \mathcal{D} under which every $H \in \Sigma_{\mathcal{D}}$ is reachable or null-controllable. We start with the following theorem.

Theorem 3: Suppose that (15) holds for $H_{\mathcal{D}}$. Then $H_{\mathcal{D}}$ is reachable if and only if every $H \in \Sigma_{\mathcal{D}}$ is reachable.

Proof: Note that $H_{\mathcal{D}} \in \Sigma_{\mathcal{D}}$. Therefore the ‘if’ part is immediate. For the ‘only if’ part, assume that $H_{\mathcal{D}}$ is reachable. By Theorem 1, we have that $\mathcal{R}(H_{\mathcal{D}}) = \mathbb{R}^n$. Now let H be a convex process such that $\text{gr } H_{\mathcal{D}} \subseteq \text{gr } H$. As any q -step trajectory of $H_{\mathcal{D}}$ is one of H , it is immediate that $\mathcal{R}(H_{\mathcal{D}}) \subseteq \mathcal{R}(H)$. Therefore $\mathcal{R}(H) = \mathbb{R}^n$. This implies that H is reachable. ■

Remark 2: It is important to stress that a convex process H is defined to be reachable if $\mathcal{F}(H) \subseteq \mathcal{R}(H)$. Therefore a nonstrict convex process H can be reachable whilst $\mathcal{R}(H) \neq \mathbb{R}^n$. Now let $\text{gr } H \subseteq \text{gr } G$. Note that we may *not* conclude reachability of G from reachability of H in general. As an example, let $\text{gr } H = \{0\}$. This convex process is reachable, and its graph is contained in the graph of any other convex process, which are not necessarily reachable.

Next, we study null-controllability. It is clear that the reasoning of Remark 2 also applies to null-controllability. This leads to an important point of contrast between Theorem 1 and Theorem 2: Under the conditions of the latter the convex process H can be null-controllable even if $\mathcal{N}(H) \neq \mathbb{R}^n$.

Theorem 4: Suppose that (15) and (16) hold for $H_{\mathcal{D}}$. Then, $H_{\mathcal{D}}$ is null-controllable if and only if every $H \in \Sigma_{\mathcal{D}}$ is null-controllable.

Proof: Again the ‘if’ part is immediate. For the ‘only if’ part, assume that $H_{\mathcal{D}}$ is null-controllable. Let H be a convex process such that $\text{gr } H_{\mathcal{D}} \subseteq \text{gr } H$. As in the proof of Theorem 3, we see that $\mathcal{R}(H_{\mathcal{D}}) \subseteq \mathcal{R}(H)$ and $\mathcal{N}(H_{\mathcal{D}}) \subseteq \mathcal{N}(H)$. This implies that

$$\mathbb{R}^n = \mathcal{N}(H_{\mathcal{D}}) - \mathcal{R}(H_{\mathcal{D}}) \subseteq \mathcal{N}(H) - \mathcal{R}(H).$$

Note that we also have $\text{gr } L_{-}(H_{\mathcal{D}}) \subseteq \text{gr } L_{-}(H)$ and $\text{gr } L_{+}(H_{\mathcal{D}}) \subseteq \text{gr } L_{+}(H)$. Therefore, it is clear that (15) and (16) hold for H . This implies that H is null-controllable. ■

The question rests whether we can provide simple tests for reachability and null-controllability of $H_{\mathcal{D}}$ in terms of the data \mathcal{D} . In order to resolve this, we will begin by giving two equivalent representations of $H_{\mathcal{D}}$.

Denote $T = |\mathcal{D}|$ and $\mathcal{D} = \{(x_t, y_t) : t = 1, \dots, T\}$. We define the matrices $X, Y \in \mathbb{R}^{n \times T}$ by taking:

$$X := [x_1 \ x_2 \ \dots \ x_T], \quad Y := [y_1 \ y_2 \ \dots \ y_T].$$

Since cone \mathcal{D} is a convex cone, we have that $\mathcal{D}^+ = (\text{cone } \mathcal{D})^+$. As \mathcal{D} is a finite set, we have that cone \mathcal{D} and \mathcal{D}^+ are polyhedral cones. This means that there exists $\ell \in \mathbb{N}$ and $\eta_1, \dots, \eta_\ell \in \mathbb{R}^{2n}$, such that $\mathcal{D}^+ = \text{cone}\{\eta_1, \dots, \eta_\ell\}$. We can now define matrices $Z, W \in \mathbb{R}^{\ell \times n}$ by the following partition:

$$[Z \ -W] := [\eta_1 \ \dots \ \eta_\ell]^\top.$$

As cone \mathcal{D} is closed, it is equal to $(\mathcal{D}^+)^+$. Recall that $\text{gr } H_{\mathcal{D}} = \text{cone } \mathcal{D}$. Therefore, we can use (4) to represent $H_{\mathcal{D}}$ in the following ways:

$$\text{gr } H_{\mathcal{D}} = \begin{bmatrix} X \\ Y \end{bmatrix} \mathbb{R}_+^T = \left\{ (x, y) \mid [Z \ -W] \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}_+^\ell \right\}. \quad (17)$$

Immediately, we see that

$$\text{dom } H_{\mathcal{D}} = X\mathbb{R}_+^T \text{ and } \text{im } H_{\mathcal{D}} = Y\mathbb{R}_+^T.$$

Using (17) we can express the minimal and maximal linear processes of $H_{\mathcal{D}}$ as follows:

$$\begin{aligned} \text{gr } L_{-}(H_{\mathcal{D}}) &= \ker [Z \ -W], \\ \text{gr } L_{+}(H_{\mathcal{D}}) &= \text{im} \begin{bmatrix} X \\ Y \end{bmatrix}. \end{aligned}$$

For the characterizations of reachability and null-controllability in Theorem 1 and Theorem 2 respectively, we need the reachable and null-controllable sets of L_{+} and L_{-} . In order to characterize these in terms of the data \mathcal{D} , we first look at the image of a set under these linear processes. For a given set $\mathcal{S} \subseteq \mathbb{R}^n$ we can apply (13) to verify that:

$$\begin{aligned} L_{-}(H_{\mathcal{D}})(\mathcal{S}) &= W^{-1}Z\mathcal{S}, \\ L_{+}(H_{\mathcal{D}})(\mathcal{S}) &= YX^{-1}\mathcal{S}. \end{aligned}$$

Recall that for a linear process L the reachable set is *finitely determined* and $\mathcal{R}(L) = L^n(0)$. Combining the above with some slight abuse of notation, we can write:

$$\begin{aligned} \mathcal{R}(L_{-}(H_{\mathcal{D}})) &= (W^{-1}Z)^n\{0\}, \\ \mathcal{R}(L_{+}(H_{\mathcal{D}})) &= (YX^{-1})^n\{0\}. \end{aligned}$$

This characterizes the reachable sets of $L_{-}(H_{\mathcal{D}})$ and $L_{+}(H_{\mathcal{D}})$ using subspace algorithms with at most n steps. Following the same reasoning with negative powers, we obtain that:

$$\begin{aligned} \mathcal{N}(L_{-}(H_{\mathcal{D}})) &= (Z^{-1}W)^n\{0\}, \\ \mathcal{N}(L_{+}(H_{\mathcal{D}})) &= (XY^{-1})^n\{0\}. \end{aligned}$$

We now shift our focus to the negative dual of $H_{\mathcal{D}}$, and show that it can be represented in terms of X and Y or Z and W as well.

By (9) and the first representation of (17) we have that:

$$\text{gr}(H_{\mathcal{D}}^-) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \left(\begin{bmatrix} X \\ Y \end{bmatrix} \mathbb{R}_+^T \right)^-.$$

By (6) this implies that:

$$\text{gr}(H_{\mathcal{D}}^-) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} [X^\top \ Y^\top]^{-1} \mathbb{R}_-^T = [Y^\top \ -X^\top]^{-1} \mathbb{R}_-^T.$$

Similarly, we can begin from (9) and the second representation in (17) instead. As such, we can conclude that the negative dual of $H_{\mathcal{D}}$ satisfies:

$$\text{gr}(H_{\mathcal{D}}^-) = \begin{bmatrix} W^\top \\ Z^\top \end{bmatrix} \mathbb{R}_+^\ell = \left\{ (x, y) \mid [Y^\top \ -X^\top] \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}_-^T \right\}.$$

Then, we have that λ and ξ form an eigenpair of $H_{\mathcal{D}}^-$ if and only if $\xi \neq 0$ and $\xi^\top (Y - \lambda X) \leq 0$.

We can now combine the previous discussion with Theorem 1 and Theorem 3 to obtain the following characterization of informativity for reachability in terms of data:

Theorem 5: Let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a finite set. Suppose that

$$X\mathbb{R}_+^T + (W^{-1}Z)^n\{0\} = \mathbb{R}^n.$$

Then, \mathcal{D} is informative for reachability if and only if $(YX^{-1})^n\{0\} = \mathbb{R}^n$ and for all $\lambda \geq 0$:

$$\xi^\top (Y - \lambda X) \leq 0 \implies \xi = 0.$$

Remark 3: Note that $(YX^{-1})^n\{0\} = \mathbb{R}^n$ implies that $\mathcal{R}(L) = \mathbb{R}^n$ for all linear processes L such that $\mathcal{D} \subseteq \text{gr } L$. That is, all such linear processes are reachable.

Example 4: Let $n = 2$ and suppose that we measure the following 4-step trajectory:

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, x_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

If we define X and Y as before, we get

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

We can use these to find Z and W :

$$[Z \ -W] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

First, note that $X\mathbb{R}_+^4 = \mathbb{R}_+ \times \mathbb{R}$ and $(W^{-1}Z)^2\{0\} = \mathbb{R} \times \{0\}$. Therefore, we can now use Theorem 5 to check for informativity.

Now, it is straightforward to verify that $(YX^{-1})^2\{0\} = \mathbb{R}^2$. Lastly, let $\lambda \geq 0$ and

$$\begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda & 0 & -1 \\ 0 & 1 & -1-\lambda & \lambda \end{bmatrix} \leq 0.$$

By direct inspection, it is clear that this implies that

$$\xi_1 \leq 0, \quad \xi_2 \leq \lambda \xi_1, \quad 0 \leq (1 + \lambda)\xi_2, \quad \lambda \xi_2 \leq \xi_1.$$

These inequalities show that for any $\lambda \geq 0$ we have that $\xi_1 = \xi_2 = 0$. This proves that \mathcal{D} is informative for reachability.

In a similar fashion we can apply our discussion to Theorem 2 and Theorem 4 to obtain a characterization of informativity for null-controllability.

Theorem 6: Let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a finite set. Suppose that

$$X\mathbb{R}_+^T + (W^{-1}Z)^n\{0\} = \mathbb{R}^n$$

and

$$(YX^{-1})^n\{0\} = Y\mathbb{R}_+^T + (Z^{-1}W)^n\{0\} = \mathbb{R}^n.$$

Then \mathcal{D} is informative for null-controllability if and only if for all $\lambda > 0$:

$$\xi^\top (Y - \lambda X) \leq 0 \implies \xi = 0.$$

Remark 4: If H is a convex process whose graph is polyhedral, we can always find a finite set \mathcal{D} such that $H = H_{\mathcal{D}}$. This means that the results of Theorem 5 and Theorem 6 can be applied to any polyhedral convex process without loss of generality.

VI. CONCLUSION

In this letter, we have resolved a number of informativity problems for conically constrained linear systems. This means that we have formulated conditions on finite, exact, state measurements under which we can test whether the measured system is reachable or null-controllable. The resulting tests take the convenient form of subspace inclusions and spectral conditions.

Future work includes extending the ideas in this letter towards the more general class of linear systems with convex constraints. It is easy to see that these systems can be viewed as difference inclusions of convex set-valued maps. Similar to the approach in this letter, we can define the smallest set-valued map consistent with the data by taking the convex hull instead of the conic hull. As such, a characterization of reachability for such systems will lead to informativity results for this class of systems. Another direction of future work is investigating informativity for the analysis of other properties and for control. Interesting problems are for example dissipativity or feedback stabilization. Resolving such a problem would require formulating characterizations for a given convex process to have the aforementioned properties. Lastly, this letter considers only exact measurements of the state. However, many realistic scenarios will involve noisy measurements. Incorporating noisy data within this framework will lead to interesting informativity problems.

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