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Strong Structural Controllability of Systems on Colored Graphs

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Abstract—**This article deals with strong structural controllability of leader–follower networks. The system matrix defining the network dynamics is a pattern matrix, in which *a priori* given entries are equal to zero, while the remaining entries take nonzero values. These nonzero entries correspond to edges in the network graph. The network is called strongly structurally controllable if for all choices of real values for the nonzero entries in the pattern matrix, the system is controllable in the classical sense. The novelty of this article is that we consider the situation that pre-specified nonzero entries in the system's pattern matrix are constrained to take identical (nonzero) values. These constraints can be caused by symmetry properties or physical constraints on the network. Restricting the system matrices to those satisfying these constraints yields a new notion of strong structural controllability. The aim of this article is to establish graph-theoretic conditions for this more general property of strong structural controllability.**

Index Terms—**Colored graphs, controllability, network analysis, strong structural controllability, zero forcing set.**

I. INTRODUCTION

THE past two decades have shown an increasing research effort in networked dynamical systems. To a large extent, this increase has been caused by technological developments, such as the emergence of the Internet and the growing relevance of smart power grids. The spreading interest in social networks and biological systems has also contributed to this surge [1]–[4].

A fundamental issue in networked systems is that of controllability. This issue deals with the question whether all parts of the global network can be adequately influenced or manipulated by applying control inputs only locally to the network. A vast amount of literature has been devoted to several variations on this issue (see [5]–[10] and the references therein). In most of the literature, a networked system is a collection of input–state–output systems, called agents, together with an interconnection structure between them. Some of these systems

can also receive input from outside the network and are called leaders. The remaining systems are called followers. At a higher level of abstraction, a networked system can be described by a directed graph, called the network graph, where the vertices represent the input–state–output systems and the edges represent the interactions between them. Controllability of the networked system then deals with the question whether the states of all agents can be steered from any initial state to any final state in finite time by applying suitable input signals to the network through the leaders.

Based on the observation that the underlying graph plays an essential role in the controllability properties of the networked system [9], an increasing amount of literature has been devoted to uncovering this connection (see [11]–[13] and the references therein). In order to allow zooming in on the role of the network graph, it is common to proceed with the simplest possible dynamics at the vertices of the graph, and to take the agents to be single integrators, with a one-dimensional state space. These single integrators are interconnected through the network graph, and the interconnection strengths are given by the weights on the edges. Based on this, the overall networked system can be represented by a linear input–state–output system of the form

$$\dot{x} = Ax + Bu$$

where the system matrix $A \in \mathbb{R}^{n \times n}$ represents the network structure with the given edge weights, and the matrix $B \in \mathbb{R}^{n \times m}$ encodes which m vertices are the leaders. The n-dimensional state vector x consists of the states of the n agents, and the m-dimensional vector u collects the input signals to the m leader vertices.

Roughly speaking, the research on network controllability based on the above model can be subdivided into three directions. The first direction deals with the situation that the values of the edge weights in the network are known exactly. In this case, the matrix A is a given constant matrix, and specific dynamics are considered for the network. For example, the system matrix can be defined as the adjacency matrix of the graph [14] or the graph Laplacian matrix [5], [7], [9], [15]–[17]. Furthermore, a framework for controllability was also introduced in [18], offering tools to treat controllability of complex networks with arbitrary structure and edge weights. Related results can be found in [19] and [20]. We also refer to [21] and [22].

A second research direction deals with the situation, where the exact values of the edge weights are not known, but only information on whether these weights are zero or nonzero is available. In this case, the system matrix is not a known, given, matrix, but

rather a matrix with a certain zero/nonzero pattern: some of the entries are known to be equal to zero; the other entries are unknown. This framework deals with the concept of *structural controllability*. Up to now, two types of structural controllability have been studied, namely *weak* structural controllability and *strong* structural controllability. A networked system of the form above is called weakly structurally controllable if there exists at least one choice of values for the nonzero unknown entries in the system matrices such that the corresponding matrix pair (A, B) is controllable. The networked system is called strongly structurally controllable if, roughly speaking, for *all* choices of nonzero values for the unknown entries, the matrix pair (A, B) is controllable. Conditions for weak and strong structural controllability have been expressed entirely in terms of the underlying network graph, using concepts like cactus graphs, maximal matchings, and zero forcing sets (see [8] and [23]–[28]).

A third, more recent, research direction again deals with weak and strong structural controllability. However, the nonzero entries in the pattern matrices defining the networked system can no longer take arbitrary nonzero real values, independently of each other. Instead, this framework considers the situation that there are certain constraints on some of the nonzero entries. These constraints can require that some of the nonzero entries have given values (see, e.g., [29]), or that there are given linear dependencies between some of the nonzero entries (see [30]). In both cases, these constraints lead to a subclass of the family of systems dealt with in the second research direction mentioned above. A networked system with such constraints is called weakly (strongly) structurally controllable if almost all (all) members in the corresponding subclass are controllable. In [30], necessary and sufficient conditions for weak structural controllability were established in terms of *multicolored subgraphs*. Later on, Mousavi *et al.* [29] studied weak and strong structural controllability of undirected networks. In addition, Menara *et al.* [31] studied weak structural controllability of networks with symmetric weights. In the present article, we will focus on a special constraint, in which the values of certain *a priori* specified nonzero entries in the system matrix are constrained to be identical. In order to formalize this, the corresponding network structure is represented by so-called *colored graphs*, where edges with identical weights have identical colors.

Indeed, it is a typical situation that certain edge weights are equal, either by symmetry considerations or by the physics of the underlying problem. One application domain is provided by real-world networks modeled as homogeneous multiagent systems, such as those used in formation control. In such networks, agents can be considered as identical subnetworks of smaller order, which lead to identical edge weights in the overall network. Such a situation can be considered as a so-called network of networks [32], which is obtained by taking the Cartesian product of smaller factor networks. For each factor network, the internal edge weights are independent. However, by applying the Cartesian product, some edge weights in the overall network will become identical.

Another application domain consists of physical networks, such as power grids, traffic networks, and water distribution networks. For example, in power networks, certain physical components typically appear multiple times, leading to identical edge weights in the network models. The same holds for water distribution networks. As for traffic networks, two-directional traffic flow sharing the same channel leads to symmetry properties of the network models. An example is also provided by real-world networks modeled as undirected networks [29]–[31], in which the network graph has to be symmetric.

In this article, strong structural controllability of networked systems defined on such colored graphs will be called *colored strong structural controllability*. This version of strong structural controllability has not been studied in the literature before. The aim of this article is to establish graph-theoretic tests for this property of networked systems.

The main contributions of this article are the following.

1) We introduce a new color change rule and define the corresponding notion of a zero forcing set. To do this, we consider colored bipartite graphs and establish a necessary and sufficient graph-theoretic condition for nonsingularity of the pattern class associated with this bipartite graph.

2) We provide a sufficient graph-theoretic condition for our new notion of strong structural controllability in terms of zero forcing sets.

3) We introduce so-called elementary edge operations that can be applied to the original network graph and that preserve the property of strong structural controllability.

4) A sufficient graph-theoretic condition for strong structural controllability is developed based on the notion of edge-operations-color-change derived set, which is obtained by applying elementary edge operations and the color change rule iteratively.

The organization of this article is as follows. In Section II, some preliminaries are presented. In Section III, we give a formal definition of the main problem treated in this article in terms of systems defined on colored graphs. In Section IV, we establish our main result, which gives a sufficient graph-theoretic condition for strong structural controllability of systems defined on colored graphs. Section V provides two additional sufficient graph-theoretic conditions. For this, we introduce the concept of elementary edge operations and the associated notion of edge-operations-color-change derived set. This set is obtained from the initial coloring set by successively applying elementary edge operations and the color change rule. Finally, Section VI formulates the conclusions of this article. We note that a preliminary version [33] of this article has appeared in the proceedings of NecSys 2018. In that note, the condition for strong structural controllability in terms of our new concept of zero forcing set was stated without giving proofs. The present article provides these proofs and, in addition, provides new conditions for strong structural controllability in terms of elementary edge operations and the concept of edge-operations-color-change derived set that were not yet given in [33].

II. PRELIMINARIES

In this article, we will use standard notation. We denote by $\mathbb{C}$ and $\mathbb{R}$ the fields of complex and real numbers, respectively.

The vector spaces of n-dimensional real and complex vectors are denoted by $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. Likewise, the spaces of $n \times m$ real and complex matrices are denoted by $\mathbb{R}^{n \times m}$ and $\mathbb{C}^{n \times m}$, respectively. For a given $n \times m$ matrix A, the entry in the ith row and jth column is denoted by A_{ij}. For a given $m \times n$ matrix A and for given subsets $S = \{s_1, s_2, \ldots, s_k\} \subseteq \{1, 2, \ldots, m\}$ and $T = \{t_1, t_2, \ldots, t_l\} \subseteq \{1, 2, \ldots, n\}$, we define the $k \times l$ submatrix of A associated with S and T by $A_{S,T}$, with $(A_{S,T})_{ij} := A_{s_i t_j}$. Similarly, for a given n-dimensional vector x, we denote by x_T the subvector of x consisting of the entries of x corresponding to T. For a given square matrix A, we denote its determinant by $\det(A)$. Finally, I and $\mathbf{0}$ will denote the identity and zero matrix of appropriate dimensions, respectively.

A. Elements of Graph Theory

Let $\mathcal{G} = (V, E)$ be a directed graph, with vertex set $V = \{1, 2, \ldots, n\}$, and the edge set E a subset of $V \times V$. In this article, we will only consider simple graphs, that is, the edge set E does not contain edges of the form (i, i). In our article, the phrase "directed graph" will always refer to a simple directed graph. We call vertex j an out-neighbor of vertex i if $(i, j) \in E$. We denote the set of all out-neighbors of i by $N(i) := \{j \in V \mid (i, j) \in E\}$. Given a subset S of the vertex set V and a subset $X \subseteq S$, we denote by

$$N_{V \setminus S}(X) = \{j \in V \setminus S \mid \exists\, i \in X \text{ such that } (i, j) \in E\}$$

the set of all vertices outside S, but an out-neighbor of some vertex in X. A directed graph $\mathcal{G}_1 = (V_1, E_1)$ is called a subgraph of $\mathcal{G}$ if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Associated with a given directed graph $\mathcal{G} = (V, E)$, we consider the set of matrices

$$\mathcal{W}(\mathcal{G}) := \{W \in \mathbb{R}^{n \times n} \mid W_{ij} \neq 0 \text{ iff } (j, i) \in E\}.$$

For any such W and $(j, i) \in E$, the entry W_{ij} is called the weight of the edge (j, i) and W is called a *weighted adjacency matrix* of the graph. For a given directed graph $\mathcal{G} = (V, E)$, we denote the associated graph with weighted adjacency matrix W by $\mathcal{G}(W) = (V, E, W)$. This is then called the *weighted graph* associated with the graph $\mathcal{G} = (V, E)$ and weighted adjacency matrix W. Finally, we define the graph $\mathcal{G} = (V, E)$ to be an *undirected graph* if $(i, j) \in E$ whenever $(j, i) \in E$. In that case, the order of i and j in (i, j) does not matter, and we interpret the edge set E as the set of unordered pairs $\{i, j\}$, where $(i, j) \in E$.

An undirected graph $\mathcal{G} = (V, E)$ is called *bipartite* if there exist nonempty disjoint subsets X and Y of V such that $X \cup Y = V$ and $\{i, j\} \in E$ only if $i \in X$ and $j \in Y$. Such a *bipartite graph* is denoted by $G = (X, Y, E_{XY})$, where we denote the edge set by E_{XY} to stress that it contains edges $\{i, j\}$ with $i \in X$ and $j \in Y$. In this article, we will use the symbol $\mathcal{G}$ for arbitrary directed graphs and G for bipartite graphs.

A set of t edges $m \subseteq E_{XY}$ is called a *t-matching* in G, if no two distinct edges in m share a vertex. In the special case that $|X| = |Y| = t$, such a t-matching is called a *perfect matching*.

For a bipartite graph $G = (X, Y, E_{XY})$, with vertex sets X and Y given by $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$,

we define *the pattern class* of G by

$$\mathcal{P}(G) = \{M \in \mathbb{C}^{t \times s} \mid M_{ji} \neq 0 \text{ iff } \{x_i, y_j\} \in E_{XY}\}.$$

Note that, in the context of pattern classes for undirected bipartite graphs, we allow complex matrices.

B. Controllability of Systems Defined on Graphs

For a directed graph $\mathcal{G} = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$, the *qualitative class* of $\mathcal{G}$ is defined as the family of matrices

$$\mathcal{Q}(\mathcal{G}) = \{A \in \mathbb{R}^{n \times n} \mid \text{for } i \neq j : A_{ij} \neq 0 \text{ iff } (j, i) \in E\}.$$

Note that the diagonal entries of $A \in \mathcal{Q}(\mathcal{G})$ do not depend on the structure of $\mathcal{G}$ and can take arbitrary real values.

Next, we specify a subset $V_L = \{v_1, v_2, \ldots, v_m\}$ of V, called the *leader set*, and consider the following family of leader–follower systems defined on the graph $\mathcal{G}$ with dynamics:

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. The systems (1) have the distinguishing feature that the matrix A belongs to $\mathcal{Q}(\mathcal{G})$ and $B = B(V; V_L)$ is defined as the $n \times m$ matrix given by

$$B_{ij} = \begin{cases} 1, & \text{if } i = v_j \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

An important notion associated with systems defined on a graph $\mathcal{G}$ as in (1) is the notion of strong structural controllability.

Definition 1: Let $\mathcal{Q}' \subseteq \mathcal{Q}(\mathcal{G})$. The system defined on the directed graph $\mathcal{G} = (V, E)$ with dynamics (1) and leader set $V_L \subseteq V$ is called *strongly structurally controllable with respect to $\mathcal{Q}'$* if the pair (A, B) is controllable for all $A \in \mathcal{Q}'$. In that case, we will simply say that $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}'$.

One special case of the above notion is that $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}(\mathcal{G})$. In that case, we will simply say that $(\mathcal{G}; V_L)$ is controllable. Another special case is that $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}'$, where, for a given weighted adjacency matrix $W \in \mathcal{W}(\mathcal{G})$, $\mathcal{Q}'$ is the subclass of $\mathcal{Q}(\mathcal{G})$ defined by

$$\mathcal{Q}_W(\mathcal{G}) = \{A \in \mathcal{Q}(\mathcal{G}) \mid \text{for } i \neq j : A_{ij} = W_{ij}\}.$$

This subclass is called the *weighted qualitative class* associated with W. Note that the off-diagonal elements of $A \in \mathcal{Q}_W(\mathcal{G})$ are fixed by those of the given adjacency matrix, while, again, the diagonal entries of $A \in \mathcal{Q}_W(\mathcal{G})$ can take arbitrary real values. Obviously

$$\mathcal{Q}(\mathcal{G}) = \bigcup_{W \in \mathcal{W}(\mathcal{G})} \mathcal{Q}_W(\mathcal{G}).$$

Since there is a unique weighted graph $\mathcal{G}(W) = (V, E, W)$ associated with the graph $\mathcal{G} = (V, E)$ and weighted adjacency matrix W, we will simply say that $(\mathcal{G}(W); V_L)$ *is controllable* if $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}_W(\mathcal{G})$.

C. Zero Forcing Set and Controllability of $(\mathcal{G}; V_L)$

Let $\mathcal{G} = (V, E)$ be a directed graph with vertices colored either black or white. We now review the concept of color change rule [34]: if v is a black vertex in $\mathcal{G}$ with exactly one white out-neighbor u, then we change the color of u to black and write $v \xrightarrow{c} u$. Such a color change is called a *force*. A subset C of V is called a *coloring set* if the vertices in C are initially colored black and those in $V \setminus C$ initially colored white. Given a coloring set $C \subseteq V$, the derived set $\mathcal{D}(C)$ is the set of black vertices obtained after repeated application of the color change rule, until no more changes are possible. It was shown in [34] that the derived set is indeed uniquely defined, in the sense that it does not depend on the order in which the color changes are applied to the original coloring set C. A coloring set $C \subseteq V$ is called a *zero forcing set for $\mathcal{G}$* if $\mathcal{D}(C) = V$.

It was shown in [26] that controllability of $(\mathcal{G}; V_L)$ can be characterized in terms of zero forcing sets.

Proposition 2: Let $\mathcal{G} = (V, E)$ be a directed graph and $V_L \subseteq V$ be the leader set. Then, $(\mathcal{G}; V_L)$ is controllable if and only if V_L is a zero forcing set.

D. Balancing Set and Controllability of $(\mathcal{G}(W); V_L)$

Consider the weighted graph $\mathcal{G}(W) = (V, E, W)$ associated with the directed graph $\mathcal{G} = (V, E)$ and the weighted adjacency matrix $W \in \mathcal{W}(\mathcal{G})$. For $i = 1, \ldots, n$, let x_i be a variable assigned to vertex i. For a given subset of vertices $C \subseteq V$, we put $x_j = 0$ for all $j \in C$. We call C *the set of zero vertices*. The values of the other vertices of $\mathcal{G}(W)$ are initially undetermined. To every vertex $j \in C$, we assign a so-called *balance equation*

$$\sum_{k \in N_{V \setminus C}(\{j\})} x_k W_{kj} = 0. \tag{3}$$

Note that for weighted undirected graphs, in which case $W = W^T$, the balance equation (3) coincides with the one introduced in [29]. If for a given subset X of the set of zero vertices C, the system of $|X|$ balance equations corresponding to the vertices in X implies that $x_k = 0$ for all $k \in Y$ with $C \cap Y = \emptyset$, we say that *zeros extend from X to Y*. We denote this by $X \xrightarrow{z} Y$. The updated set of zero vertices is now defined as $C' = C \cup Y$.

This one-step procedure of making the values of possibly additional vertices equal to zero is called *the zero extension rule*. We define the *derived set* $\mathcal{D}_z(C)$ to be the set of zero vertices obtained after repeated application of the zero extension rule until no more zero vertices appear. Although not explicitly stated in [29], it can be shown that the derived set is uniquely defined, in the sense that it does not depend on the particular zero extensions that are applied to the original set of zero vertices C. An initial zero vertex set $C \subseteq V$ is called a *balancing set* if the derived set $\mathcal{D}_z(C)$ is V.

A necessary and sufficient condition for strong structural controllability with respect to $\mathcal{Q}_W(\mathcal{G})$ for the special case that $W = W^T$ was given in [29].

Proposition 3: Let $\mathcal{G}$ be a simple undirected graph, $V_L \subseteq V$ be the leader set, and $W \in \mathcal{W}(\mathcal{G})$ be a weighted adjacency matrix with $W = W^T$. Then, $(\mathcal{G}(W); V_L)$ is controllable if and only if V_L is a balancing set.

III. PROBLEM FORMULATION

In this section, we will introduce the main problem to be considered in this article. At the end of the section, we will also formulate two preliminary results that will be needed in the following. In order to proceed, we will now first formalize the constraint that the weights of *a priori* given edges in the network graph are equal. This is equivalent to saying that given off-diagonal entries in the matrices belonging to the qualitative class $\mathcal{Q}(\mathcal{G})$ are equal. To do this, we introduce a partition

$$\pi = \{E_1, E_2, \ldots, E_k\}$$

of the edge set E into disjoint subsets E_r whose union is the entire edge set E. The edges in a given cell E_r are constrained to have identical weights. We then define the *colored qualitative class* associated with π by

$$\mathcal{Q}_\pi(\mathcal{G}) = \{A \in \mathcal{Q}(\mathcal{G}) \mid A_{ij} = A_{kl}$$
$$\text{if } (j, i), (l, k) \in E_r \text{ for some } r\}.$$

In order to visualize the partition π of the edge set in the graph, two edges in the same cell E_r are said to have the same color. The colors will be denoted by the symbols $c_1, c_2, \ldots, c_k$, and the edges in cell E_r are said to have color c_r. This leads to the notion of *colored graph*. A *colored graph* is a directed graph together with a partition π of the edge set, which is denoted by $\mathcal{G}(\pi) = (V, E, \pi)$.

In the following, sometimes, the symbols c_i will also be used to denote independent nonzero variables. A set of real values obtained by assigning to each of these variables c_i a particular real value is called a *realization* of the color set.

Example 4: Consider the colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ associated with the directed graph $\mathcal{G} = (V, E)$ and edge partition $\pi = \{E_1, E_2, E_3\}$, where $E_1 = \{(1, 4), (1, 6)\}$, $E_2 = \{(2, 4), (2, 5)\}$ and $E_3 = \{(3, 5), (3, 6)\}$, as depicted in Fig. 1. Edges having the same color mean that the weight of these edges are constrained to be equal. In this example, the edges in E_1 have color c_1 (blue), those in E_2 have color c_2 (green), and those in E_3 have color c_3 (red). The corresponding colored qualitative class consists of all matrices of the form

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & \lambda_4 & 0 & 0 \\ 0 & c_2 & c_3 & c_1 & \lambda_5 & c_3 \\ c_1 & 0 & c_3 & 0 & 0 & \lambda_6 \end{bmatrix}$$

where λ_i is an arbitrary real number for $i = 1, 2, \ldots, 6$ and c_i is an arbitrary *nonzero* real number for $i = 1, 2, 3$.

Given a colored directed graph $\mathcal{G}(\pi) = (V, E, \pi)$ with edge partition $\pi = \{E_1, E_2, \ldots, E_k\}$, we define the corresponding family of weighted adjacency matrices

$$\mathcal{W}_\pi(\mathcal{G}) := \{W \in \mathcal{W}(\mathcal{G}) \mid W_{ij} = W_{kl}$$
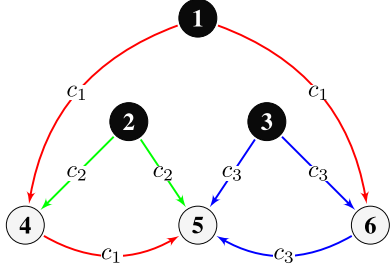$$\text{if } (j, i), (l, k) \in E_r \text{ for some } r\}.$$

Fig. 1.   Colored directed graph with leader set $\{1, 2, 3\}$.

Note that any weighted adjacency matrix $W \in \mathcal{W}_\pi(\mathcal{G})$ is associated with a unique realization of the color set. Obviously, the colored qualitative class $\mathcal{Q}_\pi(\mathcal{G})$ is equal to the union of all the subclasses $\mathcal{Q}_W(\mathcal{G})$ with $W \in \mathcal{W}_\pi(\mathcal{G})$, i.e.,

$$\mathcal{Q}_\pi(\mathcal{G}) = \bigcup_{W \in \mathcal{W}_\pi(\mathcal{G})} \mathcal{Q}_W(\mathcal{G}). \tag{4}$$

If $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}' = \mathcal{Q}_\pi(\mathcal{G})$ (see Definition 1), we will simply say that $(\mathcal{G}(\pi); V_L)$ *is controllable*. In that case, we call the system *colored strongly structurally controllable*. For example, the system with graph depicted in Fig. 1 is colored strongly structurally controllable, as will be shown later in this article.

The aim of this article is to establish graph-theoretic tests for colored strong structural controllability of a given graph. In order to obtain these, we now first make the observation that conditions for strong structural controllability can be expressed in terms of balancing sets. Generalizing Proposition 3 to the case of weighted directed graphs, we have the following lemma.

Lemma 5: Let $\mathcal{G} = (V, E)$ be a directed graph with leader set V_L and let $W \in \mathcal{W}(\mathcal{G})$. Then, $(\mathcal{G}(W); V_L)$ is controllable if and only if V_L is a balancing set.

The proof can be found in Appendix I.

The following lemma follows immediately from Lemma 5 by noting that (4) holds.

Lemma 6: Let $\mathcal{G} = (V, E)$ be a directed graph with leader set V_L and let π be a partition of the edge set. Then, $(\mathcal{G}(\pi); V_L)$ is controllable if and only if V_L is a balancing set for all weighted graphs $\mathcal{G}(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$.

Obviously, the necessary and sufficient conditions presented in Lemma 6 cannot be verified easily, as the set $\mathcal{W}_\pi(\mathcal{G})$ contains infinitely many elements. Therefore, we aim at establishing graph-theoretic conditions, under which $(\mathcal{G}(\pi); V_L)$ is controllable.

IV. ZERO FORCING SETS FOR COLORED GRAPHS

In order to provide a graph-theoretic condition for colored strong structural controllability, in this section, we introduce a new color change rule and define the corresponding notion of zero forcing set. To do this, we first consider colored bipartite graphs and establish a necessary and sufficient graph-theoretic condition for nonsingularity of the associated pattern class.

A. Colored Bipartite Graphs

Consider the bipartite graph $G = (X, Y, E_{XY})$, where the vertex sets X and Y are given by $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$. We will now introduce the notion of *colored* bipartite graph. Let $\pi_{XY} = \{E_{XY}^1, E_{XY}^2, \ldots, E_{XY}^\ell\}$ be a partition of the edge set E_{XY} with associated colors $c_1, c_2, \ldots, c_\ell$. This partition is used to formalize that certain entries in the pattern class $\mathcal{P}(G)$ are constrained to take identical values. Again, the edges in a given cell E_{XY}^r are said to have the same color. The pattern class of the colored bipartite graph $G(\pi) = (X, Y, E_{XY}, \pi_{XY})$ is then defined as the following set of complex $t \times s$ matrices:

$$\mathcal{P}_\pi(G) = \{M \in \mathcal{P}(G) \mid M_{ji} = M_{hg}$$
$$\text{if } \{x_i, y_j\}, \{x_g, y_h\} \in E_{XY}^r \text{ for some } r\}.$$

Assume now that $|X| = |Y|$ and let $t = |X|$. Suppose that p is a perfect matching of $G(\pi)$. The *spectrum* of p is defined to be the set of colors (counting multiplicity) of the edges in p. More specifically, if the perfect matching p is given by $p = \{\{x_1, y_{\gamma(1)}\}, \ldots, \{x_t, y_{\gamma(t)}\}\}$, where γ denotes a permutation of $(1, 2, \ldots, t)$, and $c_{i_1}, c_{i_2}, \ldots, c_{i_t}$ are the respective colors of the edges in p, then the spectrum of p is $\{c_{i_1}, c_{i_2}, \ldots, c_{i_t}\}$, where the same color can appear multiple times.

In addition, we define the *sign* of the perfect matching p as $\text{sign}(p) = (-1)^m$, where m is the number of swaps needed to obtain $(\gamma(1), \gamma(2), \ldots, \gamma(t))$ from $(1, 2, \ldots, t)$. Since every perfect matching is associated with a unique permutation, with a slight abuse of notation, we sometimes use the perfect matching p to represent its corresponding permutation.

Two perfect matchings are called *equivalent* if they have the same spectrum. Obviously, this yields a partition of the set of all perfect matchings of $G(\pi)$ into *equivalence classes* of perfect matchings. We denote these equivalence classes of perfect matchings by $\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_l$, where perfect matchings in the same class $\mathbb{P}_i$ are equivalent. Clearly, $\mathbb{P}_i \cap \mathbb{P}_j = \emptyset$ for $i \neq j$. Correspondingly, we then define the *spectrum of the equivalence class* $\mathbb{P}_i$ to be the (common) spectrum of the perfect matchings in this class and denote it by $\text{spec}(\mathbb{P}_i)$. Finally, we define the *signature of the equivalence class* $\mathbb{P}_i$ to be the sum of the signs of all perfect matchings in this class, which is given by

$$\text{sgn}(\mathbb{P}_i) = \sum_{p \in \mathbb{P}_i} \text{sign}(p).$$

Example 7: Consider the colored bipartite graph $G(\pi)$ depicted in Fig. 2(a). It contains three perfect matchings, p_1, p_2, and p_3, respectively, depicted in Fig. 2(b)–(d). Clearly, p_1 and p_3 are equivalent. The equivalence classes of perfect matchings are then $\mathbb{P}_1 = \{p_1, p_3\}$ and $\mathbb{P}_2 = \{p_2\}$. Clearly, $\text{sgn}(\mathbb{P}_1) = 0$ and $\text{sgn}(\mathbb{P}_2) = -1$.

We are now ready to state a necessary and sufficient condition for nonsingularity of all matrices in the colored pattern class $\mathcal{P}_\pi(G)$.

Theorem 8: Let $G(\pi) = (X, Y, E_{XY}, \pi_{XY})$ be a colored bipartite graph and $|X| = |Y|$. Then, all matrices in $\mathcal{P}_\pi(G)$
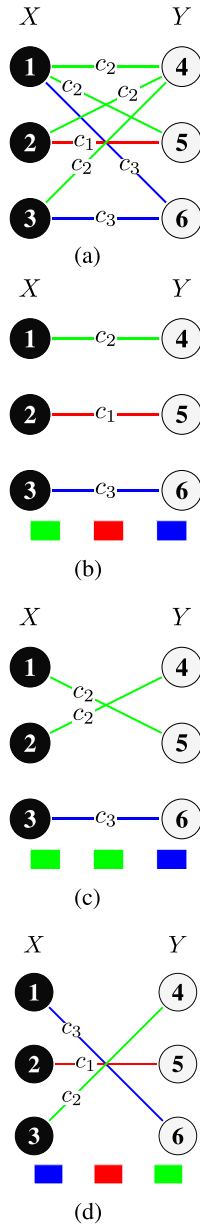
Fig. 2. Example of a colored bipartite graph and its perfect matchings. (a) Colored bipartite graph $G(\pi)$. (b) Perfect matching p_1 with $\mathrm{sign}(p_1) = 1$. (c) Perfect matching p_2 with $\mathrm{sign}(p_2) = -1$. (d) Perfect matching p_3 with $\mathrm{sign}(p_3) = -1$.

are nonsingular if and only if there exists at least one perfect matching and exactly one equivalence class of perfect matchings has nonzero signature.

Proof: Denote the cardinality of X and Y by t. Let $A \in \mathcal{P}_\pi(G)$. By the Leibniz formula for the determinant, we have

$$\det(A) = \sum_{\gamma} \mathrm{sign}(\gamma) \prod_{i=1}^{t} A_{i\gamma(i)}$$

where the sum ranges over all permutations γ of $(1, 2, \ldots, t)$ and where $\mathrm{sign}(\gamma) = (-1)^m$, with m being the number of swaps needed to obtain $(\gamma(1), \gamma(2), \ldots, \gamma(t))$ from $(1, 2, \ldots, t)$. Note that $\prod_{i=1}^{t} A_{i\gamma(i)} \neq 0$ if and only if there exists at least one

perfect matching $p = \{\{x_1, y_{\gamma(1)}\}, \ldots, \{x_{|X|}, y_{\gamma(t)}\}\}$ in $G(\pi)$. In that case, we have

$$\det(A) = \sum_{p} \mathrm{sign}(p) \prod_{i=1}^{t} A_{ip(i)}$$

where p ranges over all perfect matchings and $\mathrm{sign}(p)$ denotes the sign of the perfect matching (we now identify perfect matchings with their permutations). Suppose now there are l equivalence classes of perfect matchings $\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_l$. Then, we obtain

$$\det(A) = \sum_{j=1}^{l} \left(\mathrm{sgn}(\mathbb{P}_j) \prod_{i=1}^{t} A_{ip(i)} \right) \qquad (5)$$

where, for $j = 1, 2, \ldots l$, in the product appearing in the jth term, p is an arbitrary matching in $\mathbb{P}_j$. We will now prove the "if" part. Assume that there exists at least one perfect matching, and exactly one equivalence class of perfect matchings has nonzero signature. Without loss of generality, assume that the equivalence class $\mathbb{P}_1$ has nonzero signature. Obviously, for every $A \in \mathcal{P}_\pi(G)$, we then have

$$\det(A) = \mathrm{sgn}(\mathbb{P}_1) \prod_{i=1}^{t} A_{ip(i)} \neq 0$$

where $p \in \mathbb{P}_1$ is arbitrary; in other words, every $A \in \mathcal{P}_\pi(G)$ is nonsingular.

Next, we prove the "only if" part. For this, assume that all $A \in \mathcal{P}_\pi(G)$ are nonsingular, but one of the following holds.

  i) There does not exist any perfect matching.
  ii) No equivalence class of perfect matchings with nonzero signature exists.
  iii) There exist at least two equivalence classes of perfect matchings with nonzero signature.

We will show that all these cases lead to a contradiction.

In case (i), we must obviously have $\det(A) = 0$ for any $A \in \mathcal{P}_\pi(G)$, which gives a contradiction. For case (ii), it follows from (5) that $\det(A) = 0$, since all equivalence classes have zero signature. Therefore, we reach a contradiction again. Finally, consider case (iii). Without loss of generality, assume that $\mathbb{P}_1$ and $\mathbb{P}_2$ have nonzero signature. The signatures of the remaining equivalence classes can be either zero or nonzero. In the following, we associate the colors $c_1, c_2, \ldots, c_\ell$ of the cells $E_{XY}^1, E_{XY}^2, \ldots E_{XY}^\ell$ with independent, nonzero, variables $c_1, c_2, \ldots, c_\ell$ that can take values in $\mathbb{C}$. The spectrum of an equivalence class $\mathbb{P}_j$ then uniquely determines a monomial $c_1^{i_1} c_2^{i_2} \ldots c_\ell^{i_\ell}$, where the powers $i_1, i_2, \ldots i_k$ correspond to the multiplicities of the colors $c_1, c_2, \ldots, c_\ell$ in the perfect matchings in $\mathbb{P}_j$. We also identify each entry of a matrix A in $\mathcal{P}_\pi(G)$ with the color of its corresponding edge. In particular, for such A, we have

$$A_{ij} = \begin{cases} c_r, & \text{if } (j, i) \in E_r \text{ for some } r \\ 0, & \text{otherwise.} \end{cases}$$

From the expression (5) for the determinant of A, it can be seen that the perfect matchings in the equivalence class $\mathbb{P}_j$ yield a contribution $\mathrm{sgn}(\mathbb{P}_j) c_1^{i_1} c_2^{i_2} \ldots c_\ell^{i_\ell}$, where the degrees correspond

to the multiplicities of the colors of the perfect matchings in $\mathbb{P}_j$. By assumption, we have that $\text{spec}(\mathbb{P}_1)$ and $\text{spec}(\mathbb{P}_2)$ are not equal. Without loss of generality, we assume that the multiplicity of c_1 as an element of $\text{spec}(\mathbb{P}_1)$ is unequal to the multiplicity of c_1 as an element of $\text{spec}(\mathbb{P}_2)$. Denote these multiplicities by j_1 and j_2, respectively, with $j_1 \neq j_2$. Then, for all values of $c_2, \ldots, c_\ell$, the determinant of A has the form

$$\det(A) = \text{sgn}(\mathbb{P}_1)a_1 c_1^{j_1} + \text{sgn}(\mathbb{P}_2)a_2 c_1^{j_2} + f(c_1) \quad (6)$$

where a_1 and a_2 depend on $c_2, \ldots, c_k$ and $f(c_1)$ is a polynomial in c_1. The polynomial $f(c_1)$ corresponds to the remaining equivalence classes. It can happen that some of these equivalence classes also contain the color c_1 in their spectrum with multiplicity j_1 or j_2. By moving the corresponding monomials to the first two terms in (6), we obtain

$$\det(A) = b_1 c_1^{j_1} + b_2 c_1^{j_2} + f'(c_1) \quad (7)$$

with b_1 and b_2 depending on $c_2, \ldots, c_k$. Note that the first term in (7) corresponds to the equivalence classes containing c_1 in their spectrum with multiplicity j_1, and likewise the second term with multiplicity j_2. The remaining polynomial $f'(c_1)$ does not contain monomials with $c_1^{j_1}$ and $c_1^{j_2}$. It is now easily verified that nonzero $c_2, \ldots, c_\ell$ can be chosen such that $b_1 \neq 0$ and $b_2 \neq 0$. By the fundamental theorem of algebra, we then have that the polynomial equation $b_1 c_1^{j_1} + b_2 c_1^{j_2} + f'(c_1) = 0$ has at least one nonzero root, since both b_1 and b_2 are nonzero. This implies that for some choice of nonzero complex values $c_1, c_2, \ldots, c_\ell$, we have $\det(A) = 0$. In other words, not all $A \in \mathcal{P}_\pi(G)$ are nonsingular. This is a contradiction. ∎

Example 9: For the colored bipartite graph in Fig. 2(a), the pattern class consists of all matrices of the form

$$\begin{bmatrix} c_2 & c_2 & c_2 \\ c_2 & c_1 & 0 \\ c_3 & 0 & c_3 \end{bmatrix}$$

where c_1, c_2, and c_3 are arbitrary nonzero complex numbers. In Example 7, we saw that there is exactly one equivalence class of perfect matchings with nonzero signature. By Theorem 8, we thus conclude that all these matrices are nonsingular.

B. Color Change Rule and Zero Forcing Sets

In this subsection, we will introduce a tailor-made zero forcing notion for colored graphs. Let $\mathcal{G}(\pi) = (V, E, \pi)$ be a colored directed graph with $\pi = \{E_1, E_2, \ldots, E_k\}$ the partition of E. For given disjoint subsets $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$ of V, we define an associated colored bipartite graph $G(\pi) = (X, Y, E_{XY}, \pi_{XY})$ as follows:

$$E_{XY} := \{\{x_i, y_j\} \mid (x_i, y_j) \in E, \ x_i \in X, \ y_j \in Y\}.$$

Obviously, the partition π induces a partition π_{XY} of E_{XY} by defining

$$E_{XY}^r := \{\{x_i, y_j\} \in E_{XY} \mid (x_i, y_j) \in E_r\}, \quad r = 1, 2 \ldots, k.$$

Note that for some r, this set might be empty. Removing these, we get a partition

$$\pi_{XY} = \{E_{XY}^{i_1}, E_{XY}^{i_1}, \ldots, E_{XY}^{i_\ell}\}$$

of E_{XY}, with associated colors $c_{i_1}, c_{i_2}, \ldots, c_{i_\ell}$, with $\ell \leq k$. Without loss of generality, we renumber $c_{i_1}, c_{i_2}, \ldots, c_{i_\ell}$ as $c_1, c_2, \ldots, c_\ell$, and the edges in cell E_{XY}^r are said to have color c_r.

As before, a subset C of V is called a *coloring set* if the vertices in C are initially colored black and those in $V \setminus C$ initially colored white. We will now define the notion of *color-perfect white neighbor*.

Definition 10: Let $X \subseteq C$ and $Y \subseteq V$ with $|Y| = |X|$. We call Y a *color-perfect white neighbor* of X if:
1) $Y = N_{V \setminus C}(X)$, i.e., Y is equal to the set of white outneighbors of X;
2) in the associated colored bipartite graph $G = (X, Y, E_{XY}, \pi_{XY})$, there exists a perfect matching, and exactly one equivalence class of perfect matchings has nonzero signature.

Based on the notion of color-perfect white neighbor, we now introduce the following color change rule: if $X \subseteq C$ and Y is a color-perfect white neighbor of X, then we change the color of all vertices in Y to black and write $X \xrightarrow{c} Y$. Such a color change is called a *force*. We define *a derived set* $\mathcal{D}_c(C)$ as a set of black vertices obtained after repeated application of the color change rule, until no more changes are possible. In contrast with the original color change rule (see Section II-C), under our new color change rule, derived sets will no longer be uniquely defined and may depend on the particular list of forces that is applied to the original coloring set C. This is illustrated by Example 26 in Appendix II.

A coloring set $C \subseteq V$ is called a *zero forcing set for* $\mathcal{G}(\pi)$ if there exists a derived set $\mathcal{D}_c(C)$ such that $\mathcal{D}_c(C) = V$.

Before illustrating the new color change rule, we remark on its relation to the one defined earlier.

Remark 11: Given a directed graph $\mathcal{G} = (V, E)$, one can obtain a colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ by assigning to every edge a different color, i.e., $|\pi| = |E|$. Clearly, the colored qualitative class $\mathcal{Q}_\pi(\mathcal{G})$ coincides with the qualitative class $\mathcal{Q}(\mathcal{G})$. In addition, the original color change rule for $\mathcal{G}$ introduced in Section II-C can be seen to be a special case of the new one for $\mathcal{G}(\pi)$. This observation in mind, we will use the same terminology for these two color change rules, and it will be clear from the context which one is employed.

We now illustrate the new color change rule by means of an example.

Example 12: Fig. 3 illustrates the repeated application of zero forcing in the context of colored graphs. In Fig. 3(a), initially, vertices $\{1, 2, 3\}$ are black and the remaining vertices are white. As shown in Example 7, $\{4, 5, 6\}$ is a color-perfect white neighbor of $\{1, 2, 3\}$. Therefore, we have $\{1, 2, 3\} \xrightarrow{c} \{4, 5, 6\}$, as depicted in Fig. 3(b). Next, observe that the colored bipartite graph associated with $X = \{4, 5, 6\}$ and $Y = \{7, 8, 9\}$ has two perfect matchings, with identical spectrum and the same sign 1. Hence, the single equivalence class has signature 2. As
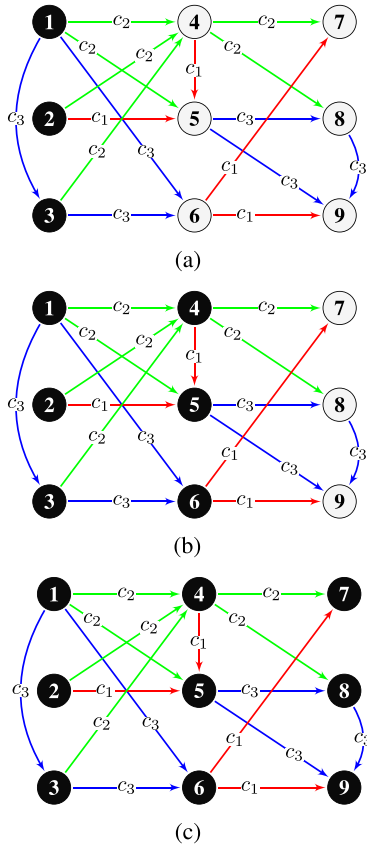
Fig. 3. Example of a zero forcing set. (a) Initial. (b) Step 1. (c) Step 2.

such, $\{7, 8, 9\}$ is a color-perfect white neighbor of $\{4, 5, 6\}$. Therefore, we have $\{4, 5, 6\} \overset{c}{\to} \{7, 8, 9\}$, as shown in Fig. 3(c). Consequently, we conclude that the vertex set $\{1, 2, 3\}$ is a zero forcing set for $\mathcal{G}(\pi)$.

Next, we explore the relationship between zero forcing sets and controllability of $(\mathcal{G}(\pi); V_L)$. First, we show that color changes do not affect the property of controllability. This is stated in the following theorem.

Theorem 13: Let $\mathcal{G}(\pi)$ be a colored directed graph and let $C \subseteq V$ be a coloring set. Suppose that $X \overset{c}{\to} Y$ with $X \subseteq C$ and $Y \subseteq V \setminus C$. Then, $(\mathcal{G}(\pi); C)$ is controllable if and only if $(\mathcal{G}(\pi); C \cup Y)$ is controllable.

Proof: Due to Lemma 6, it suffices to show that $\mathcal{D}_z(C) = V$ if and only if $\mathcal{D}_z(C \cup Y) = V$ for all weighted graphs $\mathcal{G}(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. Here, C and $C \cup Y$ are taken as zero vertex sets.

Let $W \in \mathcal{W}_\pi(\mathcal{G})$ and $\mathcal{G}(W) = (V, E, W)$. By definition of the color change rule, $X \overset{c}{\to} Y$ means that $Y = N_{V \setminus C}(X)$, and there exists exactly one equivalence class of perfect matchings with nonzero signature in the colored bipartite graph $G = (X, Y, E_{XY}, \pi_{XY})$. By applying Theorem 8, we then find that all matrices in the pattern class of G are nonsingular. Now, let $x_1, x_2, \ldots, x_n$ be variables assigned to the vertices in V, with $x_j = 0$ for $j \in C$ and x_j undetermined for the remaining vertices. For the vertices $j \in C$, consider the balance equations (3). By the fact that $W_{kj} = 0$ for all $k \in V \setminus C$ with
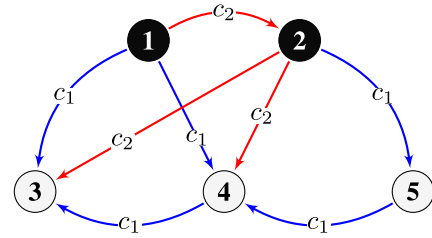
$k \notin N_{V \setminus C}(\{j\})$, the system of balance equations (3) for the vertices $j \in X$ can be written as

$$x_Y^T W_{Y,X} = 0. \tag{8}$$

We now observe that the submatrix $W_{Y,X}$ of W belongs to the pattern class of G. Using the fact that all matrices in this pattern class are nonsingular, we obtain that $x_Y^T = 0$. By the definition of the zero extension rule, we have that $X \overset{z}{\to} Y$ for $\mathcal{G}(W)$ with the set of zero vertices C. It then follows immediately that $C \cup Y \subseteq \mathcal{D}_z(C)$ and thus $\mathcal{D}_z(C \cup Y) = \mathcal{D}_z(C)$. As a consequence, C is a balancing set for $G(W)$ if and only if $C \cup Y$ is a balancing set for $G(W)$. Since this holds for arbitrary choice of W in $\mathcal{W}_\pi(G)$, the result follows immediately from Lemma 6. ∎

By Theorem 13, colored strong structural controllability is invariant under application of the color change rule. We then obtain the following corollary.

Corollary 14: Let $\mathcal{G}(\pi)$ be a colored directed graph, let $V_L \subseteq V$ be a leader set, and let $\mathcal{D}_c(V_L)$ be a derived set. Then, $(\mathcal{G}(\pi); V_L)$ is controllable if and only if $(\mathcal{G}(\pi); \mathcal{D}_c(V_L))$ is controllable.

As an immediate consequence of Corollary 14, we arrive at the main result of this section, which provides sufficient graph-theoretic condition for controllability of $(\mathcal{G}(\pi); V_L)$.

Theorem 15: Let $\mathcal{G}(\pi) = (V, E, \pi)$ be a colored directed graph with leader set $V_L \subseteq V$. If V_L is a zero forcing set, then $(\mathcal{G}(\pi); V_L)$ is controllable.

Proof: The proof follows immediately from Corollary 14 and the fact that, trivially, $(\mathcal{G}(\pi); V)$ is controllable. ∎

To conclude this section, we will provide a counterexample to show that the condition in Theorem 15 is not a necessary condition.

Example 16: Consider the colored graph $\mathcal{G}(\pi)$ depicted in Fig. 4 with a leader set $V_L = \{1, 2\}$. It will turn out that $(\mathcal{G}(\pi); V_L)$ is controllable because V_L is a balancing set for all weighted graphs $\mathcal{G}(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. Yet, V_L is not a zero forcing set.

Clearly, since none of the subsets $\{1, 2\}$, $\{1\}$ and $\{2\}$ have color-perfect white neighbors, there does not exist a derived set $\mathcal{D}_c(V_L)$ that equals V. Hence, V_L is not a zero forcing set. We will show that, however, $(\mathcal{G}(\pi); V_L)$ is controllable. Due to Lemma 6, it is sufficient to show that V_L is a balancing set for all weighted graphs $\mathcal{G}(W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. To do this, let $W \in \mathcal{W}_\pi(\mathcal{G})$ correspond to a realization $\{c_1, c_2\}$ of the color set, with c_1 and c_2 nonzero real numbers. Assign variables $x_1, \ldots, x_5$

Fig. 4. Example to show that V_L being a zero forcing set is not a necessary condition for controllability of $(\mathcal{G}(\pi); V_L)$.

to the vertices in V. Let $x_1 = x_2 = 0$ and let x_3, x_4, and x_5 be undetermined. The system of balance equations (3) for the vertices 1 and 2 in V_L is then given by

$$c_1 x_3 + c_1 x_4 = 0$$
$$c_2 x_3 + c_2 x_4 + c_1 x_5 = 0. \qquad (9)$$

Since $c_1 \neq 0$ and $c_2 \neq 0$, the homogeneous system (9) is equivalent to the system

$$c_1 x_3 + c_1 x_4 = 0$$
$$c_1 x_5 = 0 \qquad (10)$$

which yields $x_5 = 0$. By the definition of the zero extension rule, we therefore have $\{1, 2\} \xrightarrow{z} \{5\}$. Repeated application of the zero extension rule yields that V_L is a balancing set. Since the matrix $W \in \mathcal{W}_\pi(\mathcal{G})$ was taken arbitrary, we conclude that V_L is a balancing set for all weighted graphs $\mathcal{G}(W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. Thus, we have found a counterexample for the necessity of the condition in Theorem 15.

Remark 17: In this remark, we provide some intuition on why the colored graph of Example 16 leads to a controllable system, while $V_L = \{1, 2\}$ is not a zero forcing set for $\mathcal{G}(\pi)$. The main observation is that the balance equations (9) are equivalent to the equations in (10), which correspond to a new colored graph $\mathcal{G}'(\pi)$, in which the edges (2, 3) and (2, 4) have been removed. In other words, we see that V_L is a balancing set for all weighted graphs associated with $\mathcal{G}(\pi)$ if and only if the same holds for $\mathcal{G}'(\pi)$. Since V_L is a zero forcing set for the new graph $\mathcal{G}'(\pi)$, we have that $(\mathcal{G}'(\pi), V_L)$ is controllable, so also $(\mathcal{G}(\pi), V_L)$ is controllable. In fact, we will generalize this idea in the next section.

V. ELEMENTARY EDGE OPERATIONS AND DERIVED COLORED GRAPHS

In the previous section, in Theorem 15, we have established a sufficient condition for colored strong structural controllability. In this section, we will establish another sufficient graph-theoretic condition. This new condition is based on the so-called *elementary edge operations*. These are operations that can be performed on the original colored graph, and that preserve colored strong structural controllability. These edge operations on the graph are motivated by the observation that elementary operations on the systems of balance equations appearing in the zero extension rule do not modify the set of solutions to these linear equations. Indeed, in Example 16, we verified that $\{1, 2\} \xrightarrow{z} \{5\}$ for all weighted graphs $\mathcal{G}(W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. As explained in Remark 17, this is due to the fact that the system of balance equations (9) is equivalent to (10), implying that $x_5 = 0$ for all nonzero values c_1 and c_2. To generalize and visualize this idea on the level of the colored graph, we now introduce the following two types of elementary edge operations.

Let $C \subseteq V$ be a coloring set, i.e., the set of vertices initially colored black. The complement $V \setminus C$ is the set of white vertices. For two vertices $u, v \in C$ (where u and v can be the same vertex), we define

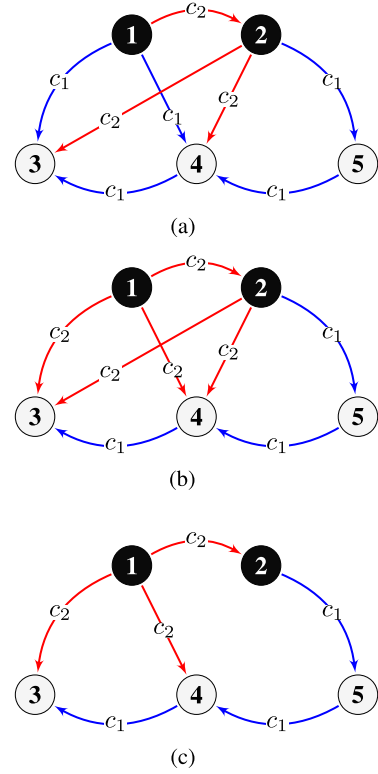$$\mathcal{E}_u(v) := \{(v, j) \in E \mid j \in N_{V \setminus C}(u)\}$$

Fig. 5. Example of performing elementary edge operations. (a) Initial colored graph $\mathcal{G}(\pi) = (V, E, \pi)$. (b) Derived colored graph $\mathcal{G}_1(\pi_1) = \mathcal{G}(\pi, C, o_1)$ where o_1 represents "turning the colors of (1, 3) and (1, 4) to c_2." (c) Derived colored graph $\mathcal{G}_2(\pi_2) = \mathcal{G}_1(\pi_1, C, o_2)$ where o_2 represents "removing all the edges in $\mathcal{E}_1(2) = \{(2,3),(2,4)\}$."

as the subset of all edges between v and white out-neighbors of u. We now introduce the following two elementary edge operations.

1) (*Turn color*) If all edges in $\mathcal{E}_u(u)$ have the same color, say c_i, then change the color of these edges to any other color in the color set.
2) (*Remove edges*) Assume $N_{V \setminus C}(u) \subseteq N_{V \setminus C}(v)$. If for any $k \in N_{V \setminus C}(u)$, the two edges (u, k) and (v, k) have the same color, then remove all edges in $\mathcal{E}_u(v)$.

The above elementary edge operations can be applied sequentially and, obviously, will not introduce new colors or add new edges. In the following, we will denote an edge operation by the symbol o. Applying the edge operation o to $\mathcal{G}(\pi)$, we obtain a new colored graph $\mathcal{G}'(\pi') = (V, E,' \pi')$. We then call $\mathcal{G}'(\pi')$ a derived graph of $\mathcal{G}(\pi)$ associated with C and o. We denote such a derived graph by $\mathcal{G}(\pi, C, o)$. An application of a sequence of elementary edge operations is illustrated in the following example.

Example 18: For the colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ depicted in Fig. 5(a), let $C = \{1, 2\}$ be the coloring set. For the vertex $1 \in C$, we have $\mathcal{E}_1(1) = \{(1, 3), (1, 4)\}$, in which both edges have the same color c_1. We apply the turn color operation to change the colors of (1, 3) and (1, 4) to c_2. Denote this operation by o_1. We then obtain the derived colored graph $\mathcal{G}(\pi, C, o_1)$ of $\mathcal{G}(\pi)$ with respect to C and o_1, which is denoted by $\mathcal{G}_1(\pi_1)$ and shown in Fig. 5(b). In addition, for nodes 1 and 2 in $\mathcal{G}_1(\pi_1)$, we have $N_{V \setminus C}(1) \subseteq N_{V \setminus C}(2)$, where $N_{V \setminus C}(1) = \{3, 4\}$ and $N_{V \setminus C}(2) = \{3, 4, 5\}$. Besides, for any $k \in N_{V \setminus C}(1)$, the two edges $(1, k)$ and $(2, k)$ have the same color. Performing the edge

removal operation denoted by o_2, we then remove all the edges in $\mathcal{E}_1(2) = \{(2,3),(2,4)\}$. Thus, we obtain the derived colored graph $\mathcal{G}_1(\pi_1, C, o_2)$ of $\mathcal{G}_1(\pi_1)$ with respect to C and o_2, which is denoted by $\mathcal{G}_2(\pi_2)$ and depicted in Fig. 5(c).

Each elementary edge operation o corresponds to a single vertex $u \in C$ or a pair of vertices $u, v \in C$. In the following, we will denote this subset of C corresponding to o by $C(o)$. Thus, $C(o)$ is either a singleton containing one vertex or a subset of V consisting of two vertices.

Next, we study the relationship between elementary edge operations and controllability of $(\mathcal{G}(\pi); V_L)$. We first show that elementary edge operations preserve zero extension. This issue is addressed in the following lemma.

Lemma 19: Let $\mathcal{G}(\pi)$ be a colored directed graph and C be a coloring set. Let o represent an edge operation, and let $\mathcal{G}'(\pi') = \mathcal{G}(\pi, C, o)$ be a derived graph with respect to C and o. Let $W \in \mathcal{W}_\pi(\mathcal{G})$ be a weighted adjacency matrix, and let $W' \in \mathcal{W}_{\pi'}(\mathcal{G}')$ be the corresponding matrix associated with the same realization of the colors. Let $X \subseteq C \setminus C(o)$ and define $X' := C(o) \cup X$. Then, interpreting C as the set of zero vertices, for any $Y \subseteq V$, we have $X' \xrightarrow{z} Y$ in the weighted graph $\mathcal{G}(W)$ if and only if $X' \xrightarrow{z} Y$ in the weighted graph $\mathcal{G}'(W')$.

Proof: By suitably relabeling the vertices, we may assume that W has the form

$$W = \begin{bmatrix} W_{1,1} & W_{1,2} & \ldots & W_{1,6} \\ W_{2,1} & W_{2,2} & \ldots & W_{2,6} \\ W_{3,1} & W_{3,2} & \ldots & W_{3,6} \\ W_{4,1} & W_{4,2} & \ldots & W_{4,6} \\ W_{5,1} & W_{5,2} & \ldots & W_{5,6} \\ W_{6,1} & W_{6,2} & \ldots & W_{6,6} \end{bmatrix}$$

where the first row block corresponds to the vertices indexed by $C(o)$, the second row block corresponds to the vertices indexed by X, the third row block corresponds to the vertices indexed by $C \setminus X'$, the fourth row block corresponds to the vertices indexed by $N_{V \setminus C}(C(o))$, the fifth row block corresponds to the vertices indexed by $N_{V \setminus C}(X') \setminus N_{V \setminus C}(C(o))$, and the last row block corresponds to the remaining white vertices. The column blocks of W result from the same labeling. Correspondingly, the matrix W' must then be equal to

$$W' = \begin{bmatrix} W_{1,1} & W_{1,2} & \ldots & W_{1,6} \\ W_{2,1} & W_{2,2} & \ldots & W_{2,6} \\ W_{3,1} & W_{3,2} & \ldots & W_{3,6} \\ W'_{4,1} & W_{4,2} & \ldots & W_{4,6} \\ W_{5,1} & W_{5,2} & \ldots & W_{5,6} \\ W_{6,1} & W_{6,2} & \ldots & W_{6,6} \end{bmatrix}$$

for some matrix $W'_{4,1}$. Since the fourth and fifth row blocks correspond to the vertices indexed by $N_{V \setminus C}(C(o))$ and $N_{V \setminus C}(X') \setminus N_{V \setminus C}(C(o))$, respectively, it follows easily

that $W_{5,1} = \mathbf{0}$, $W_{6,1} = \mathbf{0}$, and $W_{6,2} = \mathbf{0}$. Consider the submatrices $W_{N_{V \setminus C}(X'), X'} = \begin{bmatrix} W_{4,1} & W_{4,2} \\ \mathbf{0} & W_{5,2} \end{bmatrix}$ and $W'_{N_{V \setminus C}(X'), X'} = \begin{bmatrix} W'_{4,1} & W_{4,2} \\ \mathbf{0} & W_{5,2} \end{bmatrix}$ of W and W', respectively. We then distinguish two cases.

1) Suppose the edge operation o represents a color turn operation. In that case, $C(o)$ only contains one vertex; in other words, both $W_{4,1}$ and $W'_{4,1}$ consist of only one column. Hence, it follows that $W'_{4,1} = \alpha W_{4,1}$ for a suitable nonzero real number α.

2) Suppose the edge operation o represents an edge removal operation. In that case, $C(o)$ contains two vertices, say u and v, and both $W_{4,1}$ and $W'_{4,1}$ consist of two columns. We may assume that u and v correspond to the first and second columns of these matrices, respectively, and the edges in $\mathcal{E}_u(v)$ are removed. This implies that

$$W'_{4,1} = W_{4,1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Clearly, $W_{N_{V \setminus C}(X'), X'}$ and $W'_{N_{V \setminus C}(X'), X'}$ are column equivalent. Next, again assign variables $x_1, \ldots, x_n$ to every vertex in V, where x_i is equal to 0 if $i \in C$ and otherwise undetermined. For the vertex $j \in C$, we consider the balance equation (3). By the fact that $W_{kj} = 0$ for all $k \in V \setminus C$ with $k \notin N_{V \setminus C}(\{j\})$ and $N_{V \setminus C}(\{j\}) \subseteq N_{V \setminus C}(X')$, the balance equation (3) is equivalent to

$$\sum_{k \in N_{V \setminus C}(X')} x_k W_{kj} = 0. \tag{11}$$

Again using the notation for the submatrix $W_{N_{V \setminus C}(X'), X'}$ and subvector $x_{N_{V \setminus C}(X')}$, we can rewrite the system of balance equations (11) for $j \in X'$ as

$$x^T_{N_{V \setminus C}(X')} W_{N_{V \setminus C}(X'), X'} = 0. \tag{12}$$

Similarly, for the graph $\mathcal{G}'(W')$, we obtain the following system of balance equations for $j \in X'$:

$$x^T_{N_{V \setminus C}(X')} W'_{N_{V \setminus C}(X'), X'} = 0. \tag{13}$$

Since $W'_{N_{V \setminus C}(X'), X'}$ and $W_{N_{V \setminus C}(X'), X'}$ are column equivalent, the solution sets of (12) and (13) coincide. By definition of the zero extension rule, we therefore have that, for any vertex set Y, $X' \xrightarrow{z} Y$ in $\mathcal{G}(W)$ if and only if $X' \xrightarrow{z} Y$ in $\mathcal{G}'(W')$. This completes the proof. ∎

It follows from the previous that colored strong structural controllability is preserved under elementary edge operations. Indeed, we have the following.

Theorem 20: Let $\mathcal{G}(\pi)$ be a colored directed graph, $V_L \subseteq V$ be a leader set, and o an elementary edge operation. Let $\mathcal{G}'(\pi') = \mathcal{G}(\pi, V_L, o)$ be a derived colored graph of $\mathcal{G}(\pi)$ with respect to V_L and o. Then, we have that $(\mathcal{G}(\pi); V_L)$ is controllable if and only if $(\mathcal{G}'(\pi'); V_L)$ is controllable.

Proof: The proof follows from Lemmas 6 and 19. ∎

As an immediate consequence of Theorems 15 and 20, we see that if the leader set V_L of the original colored graph $\mathcal{G}(\pi)$ is a zero forcing set for the derived graph $\mathcal{G}'(\pi') = \mathcal{G}(\pi, V_L, o)$, then $(\mathcal{G}(\pi); V_L)$ is controllable. Obviously, this result can be readily

extended to derived graphs obtained by applying not only one, but a finite *sequence* of edge operations.

This immediately leads to the following sufficient graph-theoretic condition for controllability of $(\mathcal{G}(\pi); V_L)$.

Corollary 21: Let $\mathcal{G}(\pi)$ be a colored directed graph, and let V_L be a leader set. Let $\mathcal{G}'(\pi')$ be a colored graph obtained by applying finitely many elementary edge operations. Then, $(\mathcal{G}(\pi), V_L)$ is controllable if V_L is a zero forcing set for $\mathcal{G}'(\pi')$.

Example 22: We now apply Corollary 21 to the colored graph of Example 16. We already saw that $V_L = \{1, 2\}$ is not a zero forcing set, but we showed that we do have strong structural controllability for this colored graph. This can now also be shown graph theoretically by means of Corollary 21: the leader set V_L is a zero forcing set for the derived graph in Fig. 5(c), so the original colored graph in Fig. 5(a) yields a controllable system.

By combining Theorem 20 and Corollary 14, we are now in the position to establish yet another procedure for checking controllability of a given colored graph $(\mathcal{G}(\pi); V_L)$. First, distinguish the following two steps.

1) As the first step, apply the color change operation to compute a derived set $\mathcal{D}_c(V_L)$. If this derived set is equal to V, we have controllability. If not, we cannot yet decide whether we have controllability or not.

2) As a next step, then, apply an edge operation o to $\mathcal{G}(\pi)$ to obtain $\mathcal{G}_1(\pi_1)$, where $\mathcal{G}_1(\pi_1) = \mathcal{G}(\pi, \mathcal{D}_c(V_L), o)$ is a derived graph of $\mathcal{G}(\pi)$ with coloring set $\mathcal{D}_c(V_L)$ and edge operation o.

By Theorem 20 and Corollary 14, it is straightforward to verify that $(\mathcal{G}(\pi); V_L)$ is controllable if and only if $(\mathcal{G}_1(\pi_1); \mathcal{D}_c(V_L))$ is controllable.

We can now repeat steps 1 and 2, applying them to $\mathcal{G}_1(\pi_1)$. Successive and alternating application of these two steps transforms the original leader set V_L using several color change operations associated with the several derived graphs appearing in the process. After finitely many iterations, we thus arrive at a so-called *edge-operations-color-change derived set* of V_L, which will be denoted by $\mathcal{D}_{ec}(C)$. This set will remain unchanged in case we again apply step 1 or step 2. Since controllability is preserved, we arrive at the following theorem, which gives yet another sufficient condition for colored strong structural controllability.

Theorem 23: Let $\mathcal{G}(\pi)$ be a colored directed graph, and let $V_L \subseteq V$ be a leader set. Let $\mathcal{D}_{ec}(V_L)$ be an edge-operations-color-change derived set of V_L. We then have that $(\mathcal{G}(\pi); V_L)$ is controllable if $\mathcal{D}_{ec}(V_L) = V$.

Remark 24: Obviously, a derived set $\mathcal{D}_c(V_L)$ of V_L in $\mathcal{G}(\pi)$ is always contained in an edge-operations-color-change derived set $\mathcal{D}_{ec}(V_L)$ of V_L. Hence, the condition in Theorem 23 is weaker than the conditions in Theorem 15 and Corollary 21.

In the following example, we illustrate the application of Theorem 23 to check controllability of a given colored graph and leader set.

Example 25: Consider the colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ depicted in Fig. 6(a) with $V_L = \{1, 2, 3\}$ the leader set. To start with, we compute a derived set $\mathcal{D}_c(V_L) = \{1, 2, 3, 4, 5, 6\}$ of V_L in $\mathcal{G}(\pi)$, as depicted in Fig. 6(b), and denote it by $\mathcal{D}_0$. For the vertices $5, 4 \in \mathcal{D}_0$, in $\mathcal{G}(\pi)$, we have $N_{V \setminus \mathcal{D}_0}(6) = \{8, 9\} \subseteq N_{V \setminus \mathcal{D}_0}(4)$. Since the edges $(6, 8)$ and $(6, 9)$ have the same color

(a)

(b)

(c)

(d)

(e)

(f)
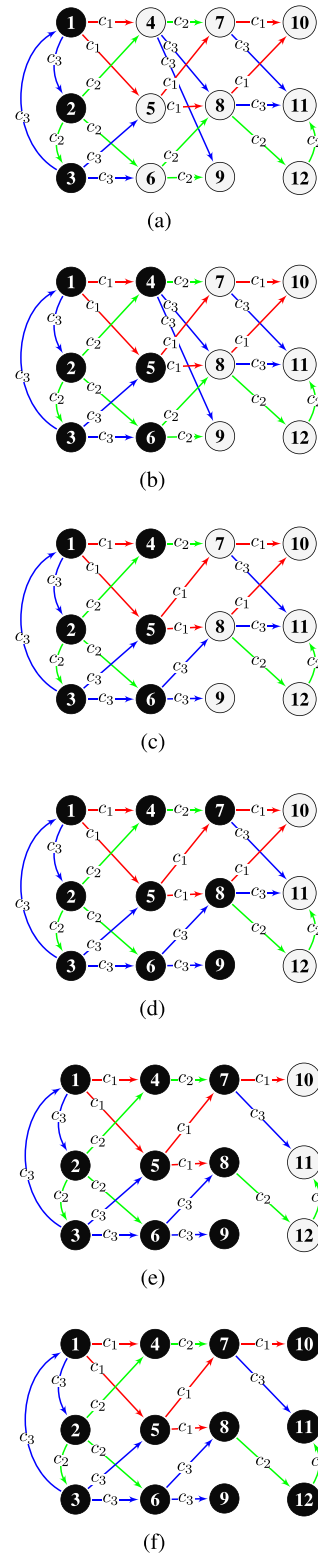
Fig. 6. Example of application of Theorem 23. (a) Initial colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ with coloring set $V_L = \{1, 2, 3\}$. Let $\mathcal{G}_0(\pi_0) = \mathcal{G}(\pi)$. (b) Compute a derived set $\mathcal{D}_c(V_L) = \{1, 2, 3, 4, 5, 6\}$ of V_L in $\mathcal{G}_0(\pi_0)$ and set $\mathcal{D}_0 = \mathcal{D}_c(V_L)$. (c) Derived colored graph $\mathcal{G}_1(\pi_1) = \mathcal{G}_0(\pi_0, \mathcal{D}_0, o_0)$ with $\mathcal{D}_0 = \{1, 2, 3, 4, 5, 6\}$ and o_0 representing "turning colors of edges $(6, 8)$ and $(6, 9)$ to c_3 and removing edges $(4, 8)$ and $(4, 9)$." (d) Compute a derived set $\mathcal{D}_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of $\mathcal{D}_0$ in the colored graph $\mathcal{G}_1(\pi_1)$. (e) Derived colored graph $\mathcal{G}_2(\pi_2) = \mathcal{G}_1(\pi_1, \mathcal{D}_1, o_1)$ with $\mathcal{D}_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and o_1 representing "removing edges $(8, 10)$ and $(8, 11)$." (f) Compute a derived set $\mathcal{D}_2 = V$ of $\mathcal{D}_1$ in the colored graph $\mathcal{G}_2(\pi_2)$. Return that $(\mathcal{G}(\pi); V_L)$ is controllable.

c_1, their color can be changed to any arbitrary color. Here, we change the colors of $(6, 8)$ and $(6, 9)$ to c_3. Then, for any $k \in N_{V \setminus \mathcal{D}_0}(4)$, the two edges $(4, k)$ and $(6, k)$ have the same color c_3. Thus, we remove the edges in $\mathcal{E}_4(6) = \{(4, 8), (4, 9)\}$, and we denote the above two edge operations by o_0. In this way, we obtain a derived colored graph $\mathcal{G}_1(\pi_1) = \mathcal{G}(\pi, \mathcal{D}_0, o_0)$ of $\mathcal{G}(\pi)$ with respect to $\mathcal{D}_0$ and o_0, that is depicted in Fig. 6(c). We proceed to compute a derived set $\mathcal{D}_c(\mathcal{D}_0) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of $\mathcal{D}_0$ in $\mathcal{G}_1(\pi_1)$, as shown in Fig. 6(d), and denote this derived set by $\mathcal{D}_1$. Since $\mathcal{D}_1 \neq V$ and $\mathcal{D}_1 \neq \mathcal{D}_0$, the procedure will continue. For the nodes $7, 8 \in \mathcal{D}_1$ in the graph $\mathcal{G}_1(\pi_1)$, we have $N_{V \setminus \mathcal{D}_1}(7) \subseteq N_{V \setminus \mathcal{D}_1}(8)$, and for any $k \in N_{V \setminus \mathcal{D}_1}(7)$, the two edges $(7, k)$ and $(8, k)$ have the same color. Thus, we remove the edges in $\mathcal{E}_7(8) = \{(8, 10), (8, 11)\}$ and denote this operation by o_1. We then obtain a derived colored graph $\mathcal{G}_2(\pi_2) = \mathcal{G}_1(\pi_1, \mathcal{D}_1, o_1)$ of $\mathcal{G}_1(\pi_1)$ with respect to $\mathcal{D}_1$ and o_1, which is depicted in Fig. 6(e). Finally, we compute a derived set $\mathcal{D}_c(\mathcal{D}_1)$ of $\mathcal{D}_1$ in $\mathcal{G}_2(\pi_2)$, as shown in Fig. 6(f). This derived set is denoted by $\mathcal{D}_2$ and turns out to be equal to the original vertex set V. Thus, we obtain that an edge-operations-color-change derived set $\mathcal{D}_{ec}(V_L)$ is equal to V and conclude that $(\mathcal{G}(\pi); V_L)$ is controllable.

VI. CONCLUSION

In this article, we have studied strong structural controllability of leader–follower networks. In contrast to existing work, in which the nonzero off-diagonal entries of matrices in the qualitative class are completely independent, in this article, we have studied the general case that there are equality constraints among these entries, in the sense that *a priori* given entries in the system matrix are restricted to take arbitrary but identical nonzero values. This has been formalized using the concept of colored graph and by introducing the new concept of colored strong structural controllability. In order to obtain conditions for colored strong structural controllability of leader–follower networks, we have introduced a new color change rule and a new concept of zero forcing set. These have been used to formulate a sufficient condition for controllability of the colored graph with a given leader set. We have shown that this condition is not necessary, by giving an example of a colored strong structurally controllable colored graph and leader set, for which our sufficient condition is not satisfied.

Motivated by this example, we have established the concept of elementary edge operations on colored graphs. It has been shown that these edge operations preserve colored strong structural controllability. Based on these elementary edge operations and the color change rule, a second sufficient graph-theoretic condition for colored strong structural controllability has been provided.

Finally, we have established a condition for colored strong structural controllability in terms of the new notion of edge-operations-color-change derived set. This derived set is obtained from the original leader set by applying edge operations and the color change rule sequentially in an alternating manner. This iterative procedure has been illustrated by means of an example.

The main new ideas of this article are a new color change rule and the concept of elementary edge operations for colored directed graphs. We have established several conditions for colored strong structural controllability using these new concepts. The conditions that we provided are not necessary, and finding necessary and sufficient conditions is still an open problem. Another open problem is to establish methods to characterize strong structural controllability for the case that given entries in the system matrices satisfy linear relations (instead of requiring them to take identical values). For *weak* structural controllability, this was studied in [30].

In this article, we have focused on finding graph-theoretic conditions rather than providing suitable algorithms (see, e.g., [27]). Establishing an efficient algorithm to check colored strong structural controllability could also be a future research problem. Finally, other system-theoretic concepts like strong targeted controllability [10], [35] and identifiability [36] for systems defined on colored graphs are possible future research directions.

APPENDIX I
PROOF OF LEMMA 5

Proof: By the Hautus test [37], $(\mathcal{G}(W); V_L)$ is controllable if and only if $[A - \lambda I \; B]$ has full row rank for all $A \in \mathcal{Q}_W(\mathcal{G})$ and all $\lambda \in \mathbb{C}$ with $B = B(V; V_L)$ given by (2). Let $V = \{1, 2, \ldots, n\}$.

We first prove the "if" part. Suppose that V_L is a balancing set for $\mathcal{G}(W)$. Without loss of generality, we may assume that there is a chronological list of zero extensions

$$(C_1 \xrightarrow{z} Y_1, C_2 \xrightarrow{z} Y_2, \ldots, C_s \xrightarrow{z} Y_s)$$

where, for $r = 1, 2, \ldots, s$, C_r represents the current set of zero vertices before the rth zero extension and $Y_r \subseteq V \setminus C_r$, and $C_s \cup Y_s = V$. Assign variables $x_1, x_2, \ldots, x_n$ to every vertex in V, with $x_i = 0$ if $i \in C_r$ and x_i undetermined otherwise. To every vertex $j \in C_r$, we then assign a balance equation given by (3). By definition of the zero extension rule, we have the following implications:

$$x_{V \setminus C_i}^T W_{V \setminus C_i, C_i} = 0 \Rightarrow x_{Y_i}^T = 0 \quad \text{for } i = 1, 2, \ldots, s. \quad (14)$$

For any $A \in \mathcal{Q}_W(\mathcal{G})$ and $\lambda \in \mathbb{C}$, there exists a diagonal matrix $D \in \mathbb{C}^{n \times n}$ such that $A - \lambda I = W + D$. It then follows immediately that

$$(A - \lambda I)_{V \setminus C_i, C_i} = W_{V \setminus C_i, C_i} \quad \text{for } i = 1, 2, \ldots, s.$$

Recalling (14), we have that

$$x_{V \setminus C_i}^T (A - \lambda I)_{V \setminus C_i, C_i} = 0 \Rightarrow x_{Y_i}^T = 0 \quad \text{for } i = 1, 2, \ldots, s.$$

Since $x^T B = 0 \Rightarrow x_{V_L}^T = 0$ and $V_L \cup (\bigcup_{j=1}^{s} Y_j) = V$, we then have that

$$x^T [A - \lambda I \; B] = 0 \Rightarrow x^T = 0$$

which implies that $[A - \lambda I \; B]$ has full row rank. Since the A and λ are arbitrary, $(\mathcal{G}(W); V_L)$ is controllable. Thus, we have proved the "if" part.

To prove the converse, suppose that $(\mathcal{G}(W); V_L)$ is controllable, while V_L is not a balancing set. It follows immediately
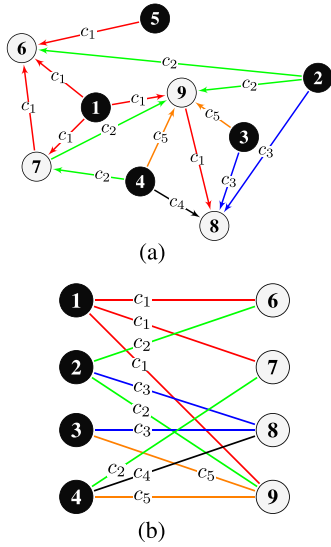
Fig. 7. Example of nonuniqueness of derived sets. (a) Colored directed graph $\mathcal{G}(\pi)$. (b) Colored bipartite graph G.

that $[A - \lambda I \ B]$ has full row rank for all $A \in \mathcal{Q}_W(\mathcal{G})$ and all $\lambda \in \mathbb{C}$, with $B = B(V; V_L)$ given by (2), and the derived set $D = \mathcal{D}_z(V_L)$ is not equal to V. Again assign variables x_i to the vertices $i \in V$ such that $x_i = 0$ if $i \in D$ and x_i is undetermined otherwise. Let $D' = V \setminus D$. By definition of the zero extension rule, we conclude that there exists a vector x such that $x_D = 0$, $x_{D'} \neq 0$ and $x^T W = 0$, where x_D and $x_{D'}$ are the subvectors corresponding to the components in D and D', respectively. Recalling that $V_L \subseteq D$, it follows that $x^T [W \ B] = 0$. This implies that the matrix $[W \ B]$ does not have full row rank. Thus, we have reached a contradiction, and the proof is complete. ∎

APPENDIX II
EXAMPLE OF NONUNIQUENESS OF DERIVED SETS

Example 26: Consider the colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ depicted in Fig. 7(a). Take as coloring set $C = \{1, 2, 3, 4, 5\}$. Consider the colored bipartite graph $G = (X, Y, E_{XY}, \pi_{XY})$ associated with $X = \{1, 2, 3, 4\}$ and $Y = \{6, 7, 8, 9\}$, as depicted in Fig. 7(b). It can be shown that there exists exactly one equivalence class of perfect matchings in G with nonzero signature. Since $X \subset C$ and $Y = N_{V \setminus C}(X)$, we have that $X \xrightarrow{c} Y$. After applying this force, we arrive at the derived set $\mathcal{D}_1(C) = V$.
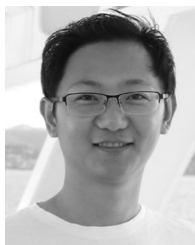
On the other hand, obviously, $X_1 \xrightarrow{c} Y_1$, with $X_1 = \{5\}$ and $Y_1 = \{6\}$. After applying this force, no other forces are possible. Indeed, it can be verified that there does not exist a subset of $\{1, 2, 3, 4, 5, 6\}$ that forces any subset of $\{7, 8, 9\}$. In this way, we arrive at the derived set $\mathcal{D}_2(C) = \{1, 2, 3, 4, 5, 6\}$.

We conclude that there exist two different derived sets in $\mathcal{G}(\pi)$ with coloring set C. Thus, we have found an example for the nonuniqueness of derived sets for a given colored graph and coloring set.
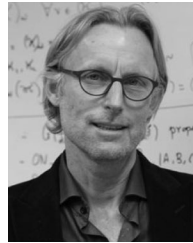
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