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RESEARCH ARTICLE

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Distributed fault estimation for linear systems with actuator faults

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Summary

This article investigates the problem of designing a distributed fault estimation observer (DFEO) for a given linear time invariant observed system with disturbances. The DFEO consists of a network of local fault estimation observers. The local observers at the network nodes are physically distributed and hence each of them has access to only part of the output of the observed system. Each local fault estimation observer communicates with its neighbors as prescribed by the given network graph. Both full order and reduced order DFEO's are presented in this article. A systematic design procedure for DFEO gains is addressed, enabling the estimation error dynamics to be robust against the effects of the external process disturbance and the derivative of the fault. The numerical design of our DFEO is amounts to solving an optimization problem with constraints of a bank of linear matrix inequalities. Finally, we illustrate the effectiveness of the proposed distributed fault estimation approach by means of a number of simulation results.

KEYWORDS

distributed estimation, fault estimation, linear system observers, LMI's

1 | INTRODUCTION

Motivated by the requirement to improve the reliability of modern control systems, in the past two decades much research has been devoted to the development of fault diagnosis and fault-tolerant control (FTC) (see, eg, References 1-3 and the references therein). In general, fault diagnosis consists of fault detection and isolation (FDI) and fault estimation.[2,4] In practice, however, it turns out to be difficult to obtain accurate information on the size and shape of the faults using a FDI strategy only. Fortunately, there exist fault estimation techniques that are capable of providing exact information on

the size of the faults that occur, thereby helping to reconstruct the fault signals. As such, fault estimation is an important ingredient in improving active FTC. So far, considerable research attention has been devoted to research on the fault estimation problem, and in the literature a variety of fault estimation approaches have appeared (see, for instance, References 5-12). For example, both full order and reduced order fault estimation observers were proposed for discrete-time Takagi-Sugeno fuzzy systems in Reference 7. A centralized fault estimation and fault-tolerant control method using an augmented observer approach was studied in Reference 13. However, most of the fault estimation methods developed up to now assume that measurement outputs are obtained from sensors that are centrally located, that is, all output information is measured at a single node.

As the size and complexity of systems increase, several systems of practical interest are large-scale and/or physically output distributed. For these systems, some fault diagnosis approaches are proposed in the literature. For example, in References 14, a robust centralized fault estimation method based on the sliding mode observer technique was proposed for multiagent system exchanging relative information. Considering probabilistic performance, an FDI filter was designed for high dimensional nonlinear systems in Reference 15. We note that the fault diagnosis and fault estimation schemes proposed in the above literature are still in a centralized form. Some research on decentralized or distributed fault diagnosis and FTC has been carried out in the literature as well.[16-19] In Reference 20 fault tolerant decentralized H_∞ control for symmetric composite systems was presented. In Reference 21, a decentralized FDI/FTC scheme was proposed for a network system. A multilayer distributed FDI scheme was proposed for large-scale systems in Reference 22. In addition, a distributed fault detection approach for interconnected second-order systems was studied in Reference 23. The original plant discussed in the above literature can be separated into several interconnected subsystems. Each fault detection/estimation filter or observer is designed for the corresponding subsystem and only estimates the local fault of the subsystem. For a whole monitored system, a distributed FDI algorithm was proposed by using average-consensus techniques in Reference 24. It should be pointed out that, for a single monitored system, in contrast with many existing results on distributed FDI problems, distributed fault estimation problems have been little studied. In Reference 25, a distributed fault estimation problem was studied in which each node collects its neighbor's output measurements and constructs a local fault estimation observer. Some results on fault diagnosis have been proposed based on multiple sensors. The actuator fault was estimated by using interacting multiple models in Reference 26. A distributed integration method to achieve fault monitoring based on Kalman filter data fusion was proposed in Reference 27. An adaptive approximation-based distributed detection and isolation methodology was provided in Reference 28 for a class of nonlinear uncertain systems with multiple sensor faults. A sensor FTC was proposed for Takagi-Sugeno systems by using a dedicated observer scheme in Reference 29. Note that none of the methods in References 26-29 are able to detect the fault at the local nodes and they all have a decision center to achieve fault diagnosis.

Motivated by the above, this article studies the distributed fault estimation problem for continuous-time linear time invariant (LTI) systems. The measured output of the original plant is physically distributed and the proposed distributed fault estimation observer (DFEO) consists of a network of fault estimators with a priori given, fixed network graph (see Figure 1 for an illustration). Each fault estimator has access to only a portion of the output of the known LTI system, and communicates with its neighboring estimators. Each of the local fault estimators at the nodes of the network should estimate the size or amplitude of the faults occurring in the observed system. Such distributed fault estimation scheme could also serve as a basis for distributed active FTC. Note that the local measurement output and monitored system may be not observable, which pinpoints the challenge of distributed fault estimation. In this article, estimating all faults at each node is different from existing results, in which each local estimator only estimates part of the faults in References 18,21. The main contributions of this article are the following.

1. In our article, the local fault estimation observers at each node simultaneously estimate the fault and state of the entire system in the presence of disturbances, whereas in References 14,18,21 the local fault estimators only estimate the fault of their corresponding subsystems. Different from References 26-29, in our article there is no decision center to deal with fault diagnosis centrally.
2. Our article uses the observability decomposition in the context of distributed fault estimation. The estimation error of the local observer in the observable part is stabilized by the local output, while the estimate in the unobservable part reaches consensus with the other local observer's estimates. Compared with Reference 25, we use a different distributed fault estimation scheme and we do not impose the assumption of observability on the sensor network. Only joint observability is required, while in Reference 25 local observability is assumed.
3. Both full order as well as reduced order DFEO's with a given H_∞ performance level are proposed in order to restrict the effect of an extended disturbance, which consists of the unknown input disturbance and the derivative of the

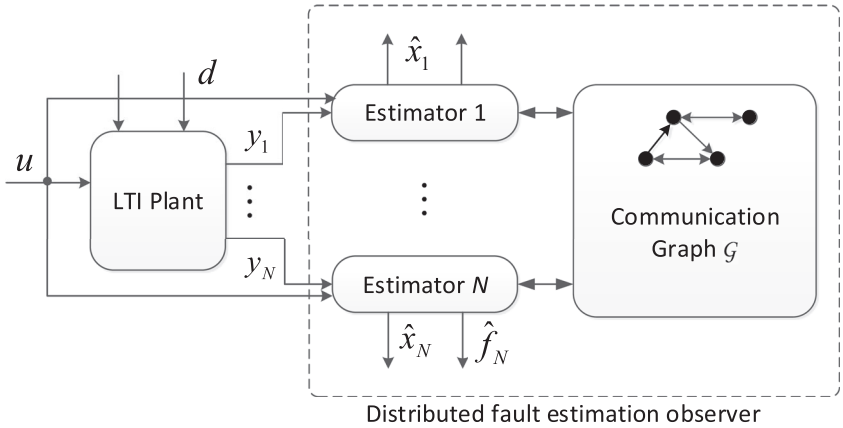

FIGURE 1 Framework of distributed fault estimation

fault. In addition, the existence of a suitable DFEO is expressed in terms of feasibility of linear matrix inequalities (LMI's).

The article is organized as follows. In Section 2 some preliminaries are given and the distributed fault estimation problem is formulated. A full order DFEO is proposed and a design procedure is established in Section 3. Then, we design a reduced order DFEO that estimates the state and the fault simultaneously in Section 4. In Section 5, simulation results illustrate the effectiveness of our fault estimation scheme. Finally, Section 6 contains our conclusions.

2 | PRELIMINARIES AND PROBLEM FORMULATION

2.1 | Preliminaries

Notation: For a given matrix M, its transpose is denoted by M^T and M^{-1} denotes its inverse. Sym(M) denotes the matrix Sym(M): $= M + M^T$. The rank of the matrix M is denoted by rank(M). If M has full column rank m then $M^\dagger = (M^TM)^{-1}M^T$ denotes its left Moore-Penrose inverse. The identity matrix of dimension N will be denoted by I_N. The vector $\mathbf{1}_N$ denotes the N-dimensional column vector comprising of all ones. For a symmetric matrix P, $P > 0$ ($P < 0$) means that P is positive (negative) definite. For a set $\{A_1, A_2, \dots, A_N\}$ of matrices, we use diag$\{A_1, A_2, \dots, A_N\}$ to denote the block diagonal matrix with the A_i's along the diagonal, and the matrix $\begin{bmatrix} A_1^T & A_2^T & \dots & A_N^T \end{bmatrix}^T$ is denoted by col$(A_1, A_2, \dots, A_N)$. The Kronecker product of the matrices M_1 and M_2 is denoted by $M_1 \otimes M_2$. For a linear map $A : \mathcal{X} \to \mathcal{Y}$, ker $(A) := \{x \in \mathcal{X} | Ax = 0\}$ and im $(A) := \{Ax | x \in \mathcal{X}\}$ will denote the kernel and image of A, respectively. For a real inner product space $\mathcal{X}$, if $\mathcal{V}$ is a subspace of $\mathcal{X}$, then $\mathcal{V}^\perp$ will denote the orthogonal complement of $\mathcal{V}$. For a signal x, $\|x\|_2$ denotes the L_2 norm of x, which is defined as $\|x\|_2 = \sqrt{\int_0^\infty x^T(t)x(t)\mathrm{d}t}$.

In this article, a weighted directed graph is denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$, where $\mathcal{N} = \{1, 2, \dots, N\}$ is a finite nonempty set of nodes, $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is an edge set of ordered pairs of nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ denotes the adjacency matrix. The (j,i)th entry a_{ji} is the weight associated with the edge (i,j). We have $a_{ji} \neq 0$ if and only if $(i,j) \in \mathcal{E}$. Otherwise $a_{ji} = 0$. An edge $(i,j) \in \mathcal{E}$ designates that the information flows from node i to node j. A graph is said to be undirected if it has the property that $(i,j) \in \mathcal{E}$ implies $(j,i) \in \mathcal{E}$ for all $i,j \in \mathcal{N}$. We will assume that the graph is simple, that is, $a_{ii} = 0$ for all $i \in \mathcal{N}$. For an edge (i,j), node i is called the parent node, node j the child node and j is a neighbor of i. A directed path from node i_1 to i_l is a sequence of edges (i_k, i_{k+1}), $k = 1, 2, \dots, l-1$ in the graph. A directed graph $\mathcal{G}$ is strongly connected if between any pair of distinct nodes i and j in $\mathcal{G}$, there exists a directed path from i to j, $i,j \in \mathcal{N}$.

The Laplacian $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$ is defined as $\mathcal{L} := \mathcal{D} - \mathcal{A}$, where the ith diagonal entry of the diagonal matrix $\mathcal{D}$ is given by $d_i = \sum_{j=1}^N a_{ij}$. By construction, $\mathcal{L}$ has a zero eigenvalue with a corresponding eigenvector $\mathbf{1}_N$ (ie, $\mathcal{L}\mathbf{1}_N = 0_N$), and if the graph is strongly connected, all the other eigenvalues lie in the open right-half complex plane.

For strongly connected graphs $\mathcal{G}$, we review the following lemma.

Lemma 1. 30-32 *Assume $\mathcal{G}$ is a strongly connected directed graph. Then there exists a unique positive row vector $r = \begin{bmatrix} r_1, \dots, r_N \end{bmatrix}$ such that $r\mathcal{L} = 0$ and $r\mathbf{1}_N = N$. Define R: = diag$\{r_1, \dots, r_N\}$. Then $\hat{\mathcal{L}} := R\mathcal{L} + \mathcal{L}^T R$ is positive semidefinite, $\mathbf{1}_N^T \hat{\mathcal{L}} = 0$ and $\hat{\mathcal{L}}\mathbf{1}_N = 0$.*

We note that $R\mathcal{L}$ is the Laplacian of the balanced digraph obtained by adjusting the weights in the original graph. The matrix $\hat{\mathcal{L}}$ is the Laplacian of the undirected graph obtained by taking the union of the edges and their reversed edges in this balanced digraph. This undirected graph is called the mirror of this balanced graph.[30]

2.2 | Problem formulation

In this article, we consider a continuous-time LTI system subject to actuator faults and disturbances represented by

$$\begin{aligned} \dot{x} &= Ax + Bu + Ff + Ed \\ y &= \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} x \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^r$ is the input, $f \in \mathbb{R}^q$ is the fault, $d \in \mathbb{R}^l$ is the disturbance, and $y \in \mathbb{R}^m$ is the aggregate measurement output. The output vector y has been partitioned into subvectors $y_i \in \mathbb{R}^{m_i}$, and the portion $y_i = C_i x$ is the only information that can be acquired by node i in the distributed fault estimator. Accordingly, we write $C = \mathrm{col}(C_1, \dots, C_N)$ with $C_i \in \mathbb{R}^{m_i \times n}$ and $\sum_{i=1}^{N} m_i = m$. Here, note that the second equation of (1) models centrally located sensors in the special case that the number of node is $N = 1$. We assume that $F \in \mathbb{R}^{n \times q}$ is a full column rank matrix, that is, $\mathrm{rank}(F) = q$.

Here, a distributed estimator is developed to simultaneously estimate the system state and actuator fault. For this purpose, we define

$$\zeta = \begin{bmatrix} x \\ f \end{bmatrix}, \quad \bar{d} = \begin{bmatrix} d \\ \dot{f} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & F \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{2}$$

Thus, $\zeta \in \mathbb{R}^{n_q}$ is an augmented state of dimension $n_q = n + q$ and $\bar{d} \in \mathbb{R}^{l_q}$ is the extended disturbance of dimension $l_q = l + q$. In this article, the disturbance d and the derivative $\dot{f}$ of the fault are assumed to be in $L_2(\mathbb{R}^+)$, the space of square integrable function on $[0, \infty)$. As a result, the signal $\bar{d}$ is in $L_2(\mathbb{R}^+)$ as well.

In terms of the notation introduced in (2), an augmented system is obtained as follows:

$$\begin{aligned} \dot{\zeta} &= \bar{A}\zeta + \bar{B}u + \bar{E}\bar{d} \\ y &= \bar{C}\zeta. \end{aligned} \tag{3}$$

Note that the augmented state vector ζ is composed of the original system state x and the actuator fault f. As a result, we get simultaneous estimation of the original system state and the actuator fault if we are able to construct a distributed estimator estimating the augmented state vector ζ and attenuating the effect of the extended disturbance $\bar{d}$.

In this article, a standing assumption will be that the communication graph is a strongly connected directed graph. Here, the graph is known and fixed. We will also assume that the pair $(\bar{C}, \bar{A})$ is observable. On the other hand, we will not impose any observability condition on the system (1) with the ith measured output, in other words, $(\bar{C}_i, \bar{A})$ is not assumed to be observable or detectable, where $\bar{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}$.

Remark 1. It is noted that the pair $(\bar{C}, \bar{A})$ is observable if and only if the following rank condition holds for all λ being the eigenvalues of A and $\lambda = 0$:[33]

$$\mathrm{rank} \begin{bmatrix} \lambda I - \bar{A} \\ \bar{C} \end{bmatrix} = \mathrm{rank} \begin{bmatrix} \lambda I - A & -F \\ 0 & \lambda I \\ C & 0 \end{bmatrix} = n + q.$$

In the next two sections, we will discuss robust full order and reduced order DFEO design, respectively. The state and the fault are estimated simultaneously. In addition, the distributed estimators will ensure a given H_∞ performance level $\beta > 0$.

3 | FULL ORDER DFEO DESIGN

In this section, we will design a full order DFEO with the given communication network for the system (3). The distributed fault estimator will consist of N local fault estimators, and the local fault estimator at node i will have dynamics of the following form:

$$\dot{\hat{\zeta}}_i = \bar{A}\hat{\zeta}_i + L_i(y_i - \overline{C}_i\hat{\zeta}_i) + \gamma r_i M_i \sum_{j=1}^{N} a_{ij}(\hat{\zeta}_j - \hat{\zeta}_i) + \overline{B}u, \quad i \in \mathcal{N}, \tag{4}$$

where $\hat{\zeta}_i \in \mathbb{R}^{n_q}$ is the augmented state estimation of the local estimator at node i, a_{ij} is the (i,j)th entry of the adjacency matrix $\mathcal{A}$ of the given network, r_i is defined as in Lemma 1, $\gamma \in \mathbb{R}$ is a coupling gain to be designed, and $L_i \in \mathbb{R}^{n_q \times m_i}$ and $M_i \in \mathbb{R}^{n_q \times n_q}$ are gain matrices to be designed.

The objective of distributed fault estimation is to design a network of observers that cooperatively estimate the state and the fault input signal of the system described by (1) in the presence of an unknown disturbance input.

To analyze and synthesize observer (4), we define the local estimation error of the ith observer as

$$e_i := \hat{\zeta}_i - \zeta. \tag{5}$$

By combining (3) and (4), we find that the error of the ith local observer is represented by

$$\begin{aligned} \dot{e}_i &= (\bar{A} - L_i\overline{C}_i)e_i + \gamma r_i M_i \sum_{j=1}^{N} a_{ij}(e_j - e_i) + \bar{E}\overline{d} \\ e_{fi} &= C_f e_i \end{aligned}, \quad i \in \mathcal{N}, \tag{6}$$

where e_{fi} is the local estimation error of the fault, $C_f = \begin{bmatrix} 0 & I_q \end{bmatrix}$.

Let $e := \mathrm{col}(e_1, e_2, \dots, e_N)$ be the joint vector of errors and $\tilde{d} := \mathbf{1}_N \otimes \overline{d}$ be the joint vector of disturbances. Then we obtain the global error system

$$\begin{aligned} \dot{e} &= \Lambda e - \gamma M(R\mathcal{L} \otimes I_n)e - \tilde{E}\tilde{d} \\ e_f &= \tilde{C}_f e, \end{aligned} \tag{7}$$

where

$$\begin{aligned} &\Lambda = \mathrm{diag}\{\bar{A} - L_1\overline{C}_1, \dots, \bar{A} - L_N\overline{C}_N\}, \\ &M = \mathrm{diag}\{M_1, \dots, M_N\}, \quad \tilde{C}_f = \mathrm{diag}\{C_f, \dots, C_f\}, \\ &\tilde{E} = I_N \otimes \bar{E}, \end{aligned}$$

and R is as defined in Lemma 1. It is noted that $\tilde{d}$ is in $L_2(\mathbb{R}^+)$ since $\overline{d}$ is in $L_2(\mathbb{R}^+)$.

Here, we will discuss how to design gain matrices for the DFEO (4) so that the joint error converges to zero while attenuating the effect of the extended disturbance signal on the estimation error. Note that we consider nonzero initial condition and use a new H_∞ performance motivated by Reference 34.

More specifically, we want to design a DFEO such that the following specifications hold:

(i) The error system (7) is internally stable, that is, it is asymptotically stable if the extended disturbance vector $\tilde{d} \equiv 0$.
(ii) The error system (7) satisfies the given H_∞ performance level $\beta > 0$, that is, for all $\tilde{d} \in L_2(\mathbb{R}^+)$, we have

$$\|e_f\|_2 \leqslant \sqrt{\beta^2\|\tilde{d}\|_2^2 + V(e(0))}, \tag{8}$$

where $V(e(0)) = e^T(0)\mathcal{P}e(0)$ is a quadratic function of $e(0)$ with $\mathcal{P} > 0$ to be specified later.

In order to design a suitable DFEO, we will use an orthogonal transformation that yields the observability decomposition for the pair $(\overline{C}_i, \bar{A})$. For $i \in \mathcal{N}$, let T_i be an orthogonal matrix, that is, a square matrix such that $T_iT_i^T = I_{n_q}$, such that the matrices $\bar{A}$ and $\overline{C}_i$ are transformed by the state space transformation T_i into the form

$$T_i^T\bar{A}T_i = \begin{bmatrix} A_{io} & 0 \\ A_{ir} & A_{iu} \end{bmatrix}, \quad \overline{C}_iT_i = \begin{bmatrix} C_{io} & 0 \end{bmatrix}, \tag{9}$$

where $A_{io} \in \mathbb{R}^{v_i\times v_i}$, $A_{ir} \in \mathbb{R}^{(n_q-v_i)\times v_i}$, $A_{iu} \in \mathbb{R}^{(n_q-v_i)\times(n_q-v_i)}$, $C_{io} \in \mathbb{R}^{m_i\times v_i}$, and $n_q - v_i$ is the dimension of the unobservable subspace of the pair $(\overline{C}_i, \bar{A})$. Clearly, by construction, the pair (C_{io}, A_{io}) is observable. In addition, if we partition

$$T_i = \begin{bmatrix} T_{i1} & T_{i2} \end{bmatrix}, \tag{10}$$

where T_{i1} consists of the first v_i columns of T_i, then the unobservable subspace is given by $\mathrm{im}T_{i2} = \ker(O_i)$, where $O_i = \mathrm{col}(\overline{C}_i, \overline{C}_i\bar{A}, \ldots, \overline{C}_i\bar{A}^{n_q-1})$. Note that $\mathrm{im}T_{i1} = (\ker(O_i))^\perp$.

Before presenting our main design procedure, we state the following lemmas, based on Lemma 1. Our first lemma is standard:

Lemma 2. 35 *For a strongly connected directed graph $\mathcal{G}$, zero is a simple eigenvalue of $\hat{\mathcal{L}} = R\mathcal{L} + \mathcal{L}^TR$ introduced in Lemma 1. Furthermore, its eigenvalues can be ordered as $\lambda_1 = 0 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_N$. Furthermore, there exists an orthogonal matrix $U = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & U_2 \end{bmatrix}$, where $U_2 \in \mathbb{R}^{N\times(N-1)}$, such that $U^T(R\mathcal{L} + \mathcal{L}^TR)U = \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_N\}$.*

Our second lemma was proven in Reference 36. The statement of the lemma is as follows:

Lemma 3. *Let $\mathcal{L}$ be the Laplacian matrix associated with the strongly connected directed graph $\mathcal{G}$. For all $g_i > 0$, $i \in \mathcal{N}$, there exists $\epsilon > 0$ such that*

$$T^T((R\mathcal{L} + \mathcal{L}^TR) \otimes I_n)T + G > \epsilon I_{nN}, \tag{11}$$

where $T = \mathrm{diag}\{T_1, \ldots, T_N\}$, R is defined as in Lemma 1, $G = \mathrm{diag}\{G_1, \ldots, G_N\}$, and $G_i = \begin{bmatrix} g_iI_{v_i} & 0 \\ 0 & 0_{n_q-v_i} \end{bmatrix}$, $i \in \mathcal{N}$.

To design a robust full order DFEO, we investigate the condition (i) first, that is, the requirement of internal stability. Let $r_i > 0$, $i \in \mathcal{N}$, be as in Lemma 1. Let $g_i > 0$, $i \in \mathcal{N}$, and $\epsilon > 0$ be such that (11) holds. We have the following lemma:

Lemma 4. *There exist a coupling gain γ, gain matrices L_i, and M_i, $i \in \mathcal{N}$, such that the error system (7) is internally stable if there exist positive definite matrices $\mathcal{P}_{io} \in \mathbb{R}^{v_i\times v_i}$, $\mathcal{P}_{iu} \in \mathbb{R}^{(n_q-v_i)\times(n_q-v_i)}$, and a matrix $\mathcal{W}_i \in \mathbb{R}^{v_i\times p_i}$ such that*

$$\begin{bmatrix} \Phi_i + \gamma g_iI_{v_i} & A_{ir}^T\mathcal{P}_{iu} \\ \mathcal{P}_{iu}A_{ir} & \mathrm{Sym}(\mathcal{P}_{iu}A_{iu}) \end{bmatrix} - \gamma\epsilon I_n < 0, \quad \forall i \in \mathcal{N}, \tag{12}$$

where $\Phi_i := \mathcal{P}_{io}A_{io} + A_{io}^T\mathcal{P}_{io} - \mathcal{W}_iC_{io} - C_{io}^T\mathcal{W}_i^T$. In that case, suitable gain matrices in the distributed observer (4) can be taken as

$$L_i := T_i\begin{bmatrix} L_{io} \\ 0 \end{bmatrix}, \quad M_i := T_i\begin{bmatrix} \mathcal{P}_{io}^{-1} & 0 \\ 0 & \mathcal{P}_{iu}^{-1} \end{bmatrix}T_i^T, \tag{13}$$

where $L_{io} = \mathcal{P}_{io}^{-1}\mathcal{W}_i$, $i \in \mathcal{N}$.

Proof. Choose a candidate Lyapunov function for the error system (7)

$$V(e_1, \dots, e_N) := \sum_{i=1}^{N} e_i^T \mathcal{P}_i e_i, \tag{14}$$

where $\mathcal{P}_i := T_i \begin{bmatrix} \mathcal{P}_{io} & 0 \\ 0 & \mathcal{P}_{iu} \end{bmatrix} T_i^T$. Clearly then $\mathcal{P}_i > 0$.

Taking $\tilde{d} \equiv 0$, the time-derivative of $V(e)$ is equal to

$$\dot{V}(e) = e^T(\mathcal{P}\Lambda + \Lambda^T\mathcal{P} - \gamma\mathcal{P}M(R\mathcal{L} \otimes I_n) - \gamma(\mathcal{L}^T R \otimes I_n)M^T\mathcal{P})e, \tag{15}$$

where $\mathcal{P} = \mathrm{diag}\{\mathcal{P}_1, \dots, \mathcal{P}_N\}$. Since the matrix M_i in (13) is chosen as $M_i = \mathcal{P}_i^{-1}$, we have $M = \mathcal{P}^{-1}$. Hence, the time-derivative of V becomes

$$\dot{V}(e) = e^T(\mathcal{P}\Lambda + \Lambda^T\mathcal{P} - \gamma(R\mathcal{L} + \mathcal{L}^T R) \otimes I_n)e. \tag{16}$$

On the other hand, we get the following inequality by (12) and (11) in Lemma 3.

$$\mathrm{diag}\{\mathcal{Q}_1, \dots, \mathcal{Q}_N\} - T^T\gamma((R\mathcal{L} + \mathcal{L}^T R) \otimes I_n)T < 0, \tag{17}$$

where $\mathcal{Q}_i = \begin{bmatrix} \Phi_i & A_{ir}^T\mathcal{P}_{iu} \\ \mathcal{P}_{iu}A_{ir} & \mathcal{P}_{iu}A_{iu} + A_{iu}^T\mathcal{P}_{iu} \end{bmatrix}$, $i \in \mathcal{N}$, with Φ_i as defined in the statement of the lemma.

By taking $L_{io} = \mathcal{P}_{io}^{-1}\mathcal{W}_i$ and pre- and post- multiplying the inequality (17) with T and its transpose, we get

$$\mathcal{P}\Lambda + \Lambda^T\mathcal{P} - \gamma(R\mathcal{L} + \mathcal{L}^T R) \otimes I_n < 0, \tag{18}$$

which implies $\dot{V}(e) < 0$. Hence the error system (7) is internally stable. ■

Based on Lemma 4, we now give our main theorem on designing a robust full order DFEO. To attenuate the effect of disturbances on the fault estimation error, the H_∞ performance index β can be minimized. A condition for its existence is expressed in terms of feasibility of an optimization problem. Solutions to the optimization problem yield required gain matrices. Let $r_i > 0$, $i \in \mathcal{N}$, be as in Lemma 1. Let $g_i > 0$, $i \in \mathcal{N}$, and $\epsilon > 0$ be such that (11) holds. We have the following:

Theorem 1. *There exist a coupling gain γ, gain matrices L_i, and M_i, $i \in \mathcal{N}$, such that the error system (7) satisfies (i) and (ii) if there exist positive definite matrices $\mathcal{P}_{io} \in \mathbb{R}^{\nu_i \times \nu_i}$, $\mathcal{P}_{iu} \in \mathbb{R}^{(n_q-\nu_i)\times(n_q-\nu_i)}$, and a matrix $\mathcal{W}_i \in \mathbb{R}^{\nu_i \times m_i}$ such that the following optimization problem is feasible*

$$\min \ \beta^2 \ \text{ s.t. } (20) \tag{19}$$

$$\begin{bmatrix} \Phi_i + \gamma g_i I_{\nu_i} & A_{ir}^T\mathcal{P}_{iu} & \mathcal{P}_{io}T_{i1}^T\bar{E} \\ \mathcal{P}_{iu}A_{ir} & \mathrm{Sym}(\mathcal{P}_{iu}A_{iu}) & \mathcal{P}_{iu}T_{i2}^T\bar{E} \\ \bar{E}^T T_{i1}\mathcal{P}_{io} & \bar{E}^T T_{i2}\mathcal{P}_{iu} & -\beta^2 I_{l_q} \end{bmatrix} + \begin{bmatrix} T_i^T C_f^T C_f T_i - \gamma\epsilon I_{n_q} & 0 \\ 0 & 0 \end{bmatrix} < 0, \quad \forall i \in \mathcal{N}, \tag{20}$$

where $\Phi_i := \mathcal{P}_{io}A_{io} + A_{io}^T\mathcal{P}_{io} - \mathcal{W}_i C_{io} - C_{io}^T\mathcal{W}_i^T$ and T_{i1},T_{i2} are defined in (10). In that case, the gain matrices in the distributed observer (4) can be taken as

$$L_i := T_i \begin{bmatrix} L_{io} \\ 0 \end{bmatrix}, \quad M_i := T_i \begin{bmatrix} \mathcal{P}_{io}^{-1} & 0 \\ 0 & \mathcal{P}_{iu}^{-1} \end{bmatrix} T_i^T, \tag{21}$$

where $L_{io} = \mathcal{P}_{io}^{-1}\mathcal{W}_i$, $i \in \mathcal{N}$.

Proof. For condition (i), it is clear that the inequality (12) follows from inequality (20). Hence condition (i) is satisfied.

In order to check condition (ii), from (12) and (11) in Lemma 3, we obtain

$$\begin{bmatrix} \mathrm{diag}\{\mathcal{Q}_1, \dots, \mathcal{Q}_N\} - T^T\gamma((R\mathcal{L} + \mathcal{L}^T R) \otimes I_n)T + T^T\tilde{C}_f^T\tilde{C}_f T & T^T\mathcal{P}\tilde{E} \\ \tilde{E}^T\mathcal{P}T & -\beta^2 I_{Nl_q} \end{bmatrix} < 0, \tag{22}$$

where $\mathcal{Q}_i = \begin{bmatrix} \Phi_i & A_{ir}^T\mathcal{P}_{iu} \\ \mathcal{P}_{iu}A_{ir} & \mathcal{P}_{iu}A_{iu} + A_{iu}^T\mathcal{P}_{iu} \end{bmatrix}$, $i \in \mathcal{N}$, with Φ_i as defined in the statement of the theorem.

By taking $L_{io} = \mathcal{P}_{io}^{-1}\mathcal{W}_i$ and pre- and post- multiplying the inequality (22) with $\mathrm{diag}\{T, I_{Nl_q}\}$ and its transpose, we get

$$\begin{bmatrix} \mathcal{P}\Lambda + \Lambda^T\mathcal{P} - \gamma(R\mathcal{L} + \mathcal{L}^T R) \otimes I_n + \tilde{C}_f^T\tilde{C}_f & \mathcal{P}\tilde{E} \\ \tilde{E}^T\mathcal{P} & -\beta^2 I_{Nl_q} \end{bmatrix} < 0. \tag{23}$$

Now choose the same candidate Lyapunov function as in (14). By pre- and post- multiplying the inequality (23) with $[e^T, \tilde{d}^T]$ and its transpose, we have

$$\dot{V}(e) < -e_f^T e_f + \beta^2\tilde{d}^T\tilde{d}. \tag{24}$$

for any $\tilde{d}$ in $L_2(\mathbb{R}^+)$.

Under the internal stability condition, that is, $\dot{V}(e) < 0$ and $V(e) \geqslant 0$ based on Lemma 4, it follows that $e \in L_2(\mathbb{R}^+)$ and hence

$$\int_0^\infty e_f^T(t)e_f(t)\mathrm{d}t + V(e(\infty)) - V(e(0)) \leqslant \beta^2 \int_0^\infty \tilde{d}^T(t)\tilde{d}(t)\mathrm{d}t \tag{25}$$

which implies

$$\|e_f\|_2^2 \leqslant \beta^2\|\tilde{d}\|_2^2 + V(e(0)). \tag{26}$$

We conclude that also condition (ii) is satisfied. ■

Next, we will give a design algorithm based on the above lemmas and theorem. Let $\beta > 0$. Assume that F has full column rank, (C,A) is an observable pair and 0 is not a zero of the system (A,F,C). Assume that the graph $\mathcal{G}$ is a strongly connected directed graph. Then a DFEO (4) that estimates the state and the fault simultaneously and attenuates the effect of the extended disturbance can be designed using the following algorithm:

Algorithm 1:

1 Let

$$\bar{A} = \begin{bmatrix} A & F \\ 0 & 0 \end{bmatrix}, \quad \overline{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}. \tag{27}$$

2 For each $i \in \mathcal{N}$, choose an orthogonal matrix T_i such that

$$T_i^T\bar{A}T_i = \begin{bmatrix} A_{io} & 0 \\ A_{ir} & A_{iu} \end{bmatrix}, \quad \overline{C}_i T_i = \begin{bmatrix} C_{io} & 0 \end{bmatrix} \tag{28}$$

with (C_{io},A_{io}) observable.

3 Compute the positive row vector $r = \begin{bmatrix} r_1, \dots, r_N \end{bmatrix}$ such that $r\mathcal{L} = 0$ and $r\mathbf{1}_N = N$.

4 Put $g_i = 1$, $i \in \mathcal{N}$ and take $\epsilon > 0$ such that (11) holds.

5 Solve the optimization problem (19) and get γ, $\mathcal{P}_{io}$, $\mathcal{P}_{iu}$, $\mathcal{W}_i$.

6 Define

$$L_i := T_i \begin{bmatrix} \mathcal{P}_{io}^{-1}\mathcal{W} \\ 0 \end{bmatrix}, M_i := T_i \begin{bmatrix} \mathcal{P}_{io}^{-1} & 0 \\ 0 & \mathcal{P}_{iu}^{-1} \end{bmatrix} T_i^T, i \in \mathcal{N}. \tag{29}$$

7 Build the distributed estimator (4) where the gain matrices are obtained in the above steps. As a result, the estimates of the state and fault are immediately given by

$$\hat{x}_i = \begin{bmatrix} I_n & 0 \end{bmatrix} \hat{\zeta}_i, \quad \hat{f}_i = \begin{bmatrix} 0 & I_q \end{bmatrix} \hat{\zeta}_i. \tag{30}$$

Remark 2. In the special case that the communication graph among the observers is a connected undirected graph, we have that $r = \mathbf{1}_N^T$ is the unique positive row vector such that $r\mathcal{L} = 0$ and $r\mathbf{1}_N = N$ in Lemma 1. In the design procedure, we can then take $r_i = 1$ for all $i \in \mathcal{N}$.

4 | REDUCED ORDER DFEO DESIGN

In the previous section, a full order DFEO design was presented to achieve robust distributed fault estimation. In order to reduce the order of the local estimators and computations at each node, in this section we investigate how to obtain a reduced order DFEO.

To design a reduced order DFEO, we make a full rank decomposition for each local augmented output matrix $\overline{C}_i$, that is, we decompose $\overline{C}_i = D_i W_i$ where $D_i \in \mathbb{R}^{m_i \times p_i}$ and $W_i \in \mathbb{R}^{p_i \times n_q}$ have full column rank and row rank, respectively, and $p_i = \text{rank}\ (\overline{C}_i)$.

Since $y_i = \overline{C}_i \zeta$, we have

$$\tilde{y}_i := D_i^{\dagger} y_i = W_i \zeta, \quad i \in \mathcal{N}, \tag{31}$$

where $\tilde{y}_i \in \mathbb{R}^{p_i}$ represents a virtual local output and $D_i^{\dagger} = (D_i^T D_i)^{-1} D_i^T$ denotes the Moore-Penrose left inverse, that is, $D_i^{\dagger} D_i = I_{p_i}$.

Denote $W = \text{col}(W_1, W_2, \dots, W_N)$. Clearly, since by assumption $(\overline{C}, \bar{A})$ is observable, $(W, \bar{A})$ is observable, but for $i \in \mathcal{N}$, $(W_i, \bar{A})$ is not necessarily observable. We use an orthogonal transformation that yields the observability decomposition for the pair $(W_i, \bar{A})$. For $i \in \mathcal{N}$, let T_i be an orthogonal matrix such that the matrices $\bar{A}$, $\bar{E}$, and W_i are transformed by the state space transformation T_i into the following form:

$$T_i^T \bar{A} T_i = \begin{bmatrix} A_{i11} & A_{i12} & 0 \\ A_{i21} & A_{i22} & 0 \\ A_{i31} & A_{i32} & A_{iu} \end{bmatrix}, \quad T_i^T \bar{E} = \begin{bmatrix} \bar{E}_{i1} \\ \bar{E}_{i2} \\ \bar{E}_{i3} \end{bmatrix}, \quad W_i T_i = \begin{bmatrix} J_i & 0 & 0 \end{bmatrix}, \tag{32}$$

where $A_{i11} \in \mathbb{R}^{p_i \times p_i}$, $A_{i12} \in \mathbb{R}^{p_i \times (v_i - p_i)}$, $A_{i21} \in \mathbb{R}^{(v_i - p_i) \times p_i}$, $A_{i22} \in \mathbb{R}^{(v_i - p_i) \times (v_i - p_i)}$, $A_{i31} \in \mathbb{R}^{(n_q - v_i) \times p_i}$, $A_{i32} \in \mathbb{R}^{(n_q - v_i) \times (v_i - p_i)}$, $A_{iu} \in \mathbb{R}^{(n_q - v_i) \times (n_q - v_i)}$, $\bar{E}_{i1} \in \mathbb{R}^{(n_q - v_i) \times l_q}$, $\bar{E}_{i2} \in \mathbb{R}^{(n_q - v_i) \times l_q}$, $\bar{E}_{i3} \in \mathbb{R}^{(n_q - v_i) \times l_q}$, $J_i \in \mathbb{R}^{p_i \times p_i}$ is a nonsingular matrix, and $n_q - v_i$ is the dimension of the unobservable subspace of the pair $(W_i, \bar{A})$.

For the sake of brevity, we denote

$$A_{io} = \begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix}, A_{ir} = \begin{bmatrix} A_{i31} & A_{i32} \end{bmatrix}, W_{io} = \begin{bmatrix} J_i & 0 \end{bmatrix}, \tag{33}$$

where $A_{io} \in \mathbb{R}^{v_i \times v_i}$, $A_{ir} \in \mathbb{R}^{(n_q - v_i) \times v_i}$, $W_{io} \in \mathbb{R}^{p_i \times v_i}$. Clearly, by construction, the pair (W_{io}, A_{io}) is observable. Furthermore, it can be checked, for example, by using Hautus test, that the pair (A_{i12}, A_{i22}) is also observable. Since J_i is nonsingular, then also the pair $(J_i A_{i12}, A_{i22})$ is observable.

Similarly, if we partition $T_i = \begin{bmatrix} T_{i1} & T_{i2} \end{bmatrix}$, where T_{i1} consists of the first v_i columns of T_i, then the unobservable subspace is given by $\text{im} T_{i2} = \ker(O_{Wi})$, where $O_{Wi} = \text{col}(W_i, W_i \bar{A}, \dots, W_i \bar{A}^{n_q - 1})$. Note that $\text{im} T_{i1} = (\ker(O_{Wi}))^{\perp}$.

We will now design a reduced order DFEO with the given communication network for the system (3). The reduced order DFEO will consist of N local fault estimators, and the local reduced order fault estimator at node i will have dynamics of the following form

$$\begin{aligned} \dot{z}_i &= N_i z_i + L_i y_i + U_i u + \gamma r_i M_i \sum_{j=1}^{N} a_{ij}(\hat{\zeta}_j - \hat{\zeta}_i) \\ \hat{\zeta}_i &= P_i z_i + Q_i y_i \end{aligned}, \quad i \in \mathcal{N}, \tag{34}$$

where $z_i \in \mathbb{R}^{n_q - p_i}$ is the state of local reduced order fault estimator, $\hat{\zeta}_i \in \mathbb{R}^{n_q}$ is the estimate of the plant state and fault at node i, a_{ij} is the (i,j)th entry of the adjacency matrix $\mathcal{A}$ of the given network, r_i is defined as in Lemma 1, $\gamma \in \mathbb{R}$ is a coupling gain to be designed, $N_i \in \mathbb{R}^{(n_q-p_i)\times(n_q-p_i)}$, $L_i \in \mathbb{R}^{(n_q-p_i)\times m_i}$, $M_i \in \mathbb{R}^{(n_q-p_i)\times n}$, $U_i \in \mathbb{R}^{(n_q-p_i)\times r}$, $P_i \in \mathbb{R}^{n_q\times(n_q-p_i)}$ and $Q_i \in \mathbb{R}^{n_q\times m_i}$ are gain matrices to be designed.

We now proceed with defining the gain matrices P_i and Q_i in the output equation of (34). For $i \in \mathcal{N}$, we define $S_i \in \mathbb{R}^{n_q\times(n_q-p_i)}$ and $K_i \in \mathbb{R}^{n_q\times m_i}$ by

$$S_i := \begin{bmatrix} 0 \\ I_{n_q-p_i} \end{bmatrix} \text{ and } K_i := \begin{bmatrix} J_i^{-1} \\ H_i \\ 0 \end{bmatrix} D_i^{\dagger}, \tag{35}$$

where $H_i \in \mathbb{R}^{(v_i-p_i)\times p_i}$ still needs to be designed. We denote

$$T_{is} := T_i S_i \tag{36}$$

as the $n_q \times (n_q - p_i)$ matrix consisting of the last $n_q - p_i$ columns of the orthogonal matrix T_i. Next define

$$P_i := T_{is} \text{ and } Q_i := T_i K_i. \tag{37}$$

To analyze and synthesize a local fault estimator (34), we define the local estimation error of the ith fault estimator as

$$e_i := \hat{\zeta}_i - \zeta. \tag{38}$$

Combining (3) and (34) yields the following error equation

$$\begin{aligned} \dot{e}_i &= P_i \dot{z}_i + Q_i \dot{y}_i - \dot{\zeta} \\ &= T_{is}\dot{z}_i + T_i K_i \dot{y}_i - \dot{\zeta} \\ &= T_i S_i (N_i z_i + L_i y_i + U_i u + \gamma r_i M_i \sum_{j=1}^{N} a_{ij}(\hat{\zeta}_j - \hat{\zeta}_i)) + (T_i K_i \overline{C}_i - I)(\bar{A}\zeta + \overline{B}u + \bar{E}\overline{d}) \\ &= T_i S_i (N_i S_i^T (T_i^T e_i - K_i y_i + T_i^T x) + L_i y_i + U_i u + \gamma r_i M_i \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i)) \\ &\quad + (T_i K_i \overline{C}_i - I)(\bar{A}\zeta + \overline{B}u + \bar{E}\overline{d}) \\ &= T_i S_i N_i S_i^T T_i^T e_i + \gamma r_i T_i S_i M_i \sum_{j=1}^{N} a_{ij}(e_j - e_i) \\ &\quad + T_i((S_i L_i - S_i N_i S_i^T K_i) D_i W_i T_i + S_i N_i S_i^T + (K_i D_i W_i T_i - I) T_i^T \bar{A} T_i) T_i^T \zeta \\ &\quad + T_i(S_i U_i + (K_i D_i W_i T_i - I) T_i^T \overline{B}) u + (T_i K_i \overline{C}_i - I)\bar{E}\overline{d}. \end{aligned} \tag{39}$$

It is required that the right-hand side of the differential equation (39) does not depend on the augmented state ζ and the input u. Hence the coefficient matrices to be defined should be such that the following equations are satisfied

$$(S_iL_i - S_iN_iS_i^{\dagger}K_i)D_iW_iT_i + S_iN_iS_i^{\dagger} + (K_iD_iW_iT_i - I)T_i^TAT_i = 0. \tag{40}$$

$$S_iU_i + (K_iD_iW_iT_i - I)T_i^T\overline{B} = 0. \tag{41}$$

It is checked by straightforward verification that (40) and (41) are achieved by choosing

$$N_i = \begin{bmatrix} A_{i22} - H_iJ_iA_{i12} & 0 \\ A_{i32} & A_{iu} \end{bmatrix}, \tag{42}$$

$$L_i = \begin{bmatrix} A_{i21} - H_iJ_iA_{i11} \\ A_{i31} \end{bmatrix} J_i^{-1}D_i^{\dagger} + N_iS_i^{\dagger}K_i, \tag{43}$$

$$U_i = \begin{bmatrix} -H_iJ_i & I_{v_i-q_i} & 0 \\ 0 & 0 & I_{n_q-v_i} \end{bmatrix} T_i^T\overline{B}. \tag{44}$$

Then the local error e_i satisfied the following differential equation

$$\begin{aligned} \dot{e}_i &= T_iS_iN_iS_i^{\dagger}T_i^Te_i + \gamma r_iT_iS_iM_i\sum_{j=1}^{N}a_{ij}(e_j - e_i) + (T_iK_i\overline{C}_i - I)\bar{E}\overline{d} \\ e_{fi} &= C_fe_i \end{aligned}, \quad i \in \mathcal{N}. \tag{45}$$

where e_{fi} is the local estimation error of the fault, $C_f = \begin{bmatrix} 0 & I_q \end{bmatrix}$.

Let $e := \mathrm{col}(e_1, e_2, \dots, e_N)$ be the joint vector of errors and $\tilde{d} := \mathbf{1}_N \otimes \overline{d}$ be the joint vector of disturbances. The joint error vector e satisfies

$$\begin{aligned} \dot{e} &= (T_s\tilde{N}T_s^T - \gamma T_sM(R\mathcal{L} \otimes I_n))e + \tilde{E}\tilde{d} \\ e_f &= \tilde{C}_fe, \end{aligned} \tag{46}$$

where

$$T_s := \mathrm{diag}\{T_{1s}, \dots, T_{Ns}\}, \quad \tilde{C}_f = \mathrm{diag}\{C_f, \dots, C_f\}, \tag{47}$$

$$M = \mathrm{diag}\{M_1, \dots, M_N\}, \quad \tilde{N} = \mathrm{diag}\{N_1, \dots, N_N\}, \tag{48}$$

$$\tilde{E} = \mathrm{diag}\{(T_1K_1\overline{C}_1 - I)\bar{E}, \dots, (T_NK_N\overline{C}_N - I)\bar{E}\}, \tag{49}$$

and R is as defined in Lemma 1.

It can be shown that $\mathrm{im}T_s$ is an invariant subspace for the differential Equation (46) in the sense that if $e(0) \in \mathrm{im}T_s$, then for each $\tilde{d}$ we have $e(t) \in \mathrm{im}T_s$ for all t. Even more, it can be shown that each feasible global error trajectory e lives in the subspace $\mathrm{im}T_s$. This fact was proven in Reference 37 and will be stated as a lemma here:

Lemma 5. *Assume that the gain matrices P_i, Q_i, N_i, L_i, and U_i are given by (37), (42), (43), and (44). Let $e := \mathrm{col}(e_1, e_2, \dots, e_N)$ be the joint vector of errors, with for $i \in \mathcal{N}$ the local error equal to $e_i = \hat{\zeta}_i - \zeta$, where ζ is a trajectory of the plant (3) and $\hat{\zeta}_i$ satisfies (34). Then $e(t) \in \mathrm{im}T_s$ for all $t \in \mathbb{R}$.*

Thus, we can conclude that each feasible global error trajectory e satisfies the differential equation (46) restricted to its invariant subspace $\mathrm{im}T_s$. Therefore, in the sequel we will study the restriction of (46) to the invariant subspace $\mathrm{im}T_s$.

The objective of the reduced order DFEO design is similar to the full order DFEO mentioned before. We want to design a reduced order DFEO such that the following specifications hold:

(i) The error system (46) restricted to $\mathrm{im}T_s$ is internally stable, that is, if the extended disturbance vector $\tilde{d} \equiv 0$ then for each initial condition $e(0) \in \mathrm{im}T_s$ we have $e(t) \to 0$ as t tends to ∞.

(ii) The error system (46) restricted to $\mathrm{im}T_s$ satisfies the given H_∞ performance level $\beta > 0$, that is, for all $\tilde{d} \in L_2(\mathbb{R}^+)$, we have

$$\|e_f\|_2 \leqslant \sqrt{\beta^2\|\tilde{d}\|_2^2 + V(e(0))}, \tag{50}$$

where $V(e(0)) = e^T(0)\mathcal{P}e(0)$ is a quadratic function of $e(0)$ with $\mathcal{P} > 0$ to be specified later.

Remark 3. Note that the fault estimation performance may not be satisfactory for high-frequency fault estimation. This is due to the fact that the joint disturbance $\tilde{d}$ in the H_∞ performances (8) and (50) includes the variation of the fault.

We will now first discuss the condition (i). Let $r_i > 0$, $i \in \mathcal{N}$, be as in Lemma 1. Let $g_i > 0$, $i \in \mathcal{N}$, and $\epsilon > 0$ be such that (11) holds. We have the following lemma:

Lemma 6. *There exist a coupling gain γ, gain matrices N_i, L_i, M_i, U_i, P_i, and Q_i, $i \in \mathcal{N}$, such that such that the error system (46) restricted to* $\mathrm{im}T_s$ *is internally stable if there exist positive definite matrices $\mathcal{P}_{ie} \in \mathbb{R}^{(\nu_i-p_i)\times(\nu_i-p_i)}$, $\mathcal{P}_{iu} \in \mathbb{R}^{(n_q-\nu_i)\times(n_q-\nu_i)}$, and a matrix $\mathcal{W} \in \mathbb{R}^{(\nu_i-p_i)\times p_i}$ such that*

$$\begin{bmatrix} \Phi_i + \gamma g_i I_{n_q-p_i} & A_{i32}^T\mathcal{P}_{iu} \\ \mathcal{P}_{iu}A_{i32} & \mathrm{Sym}(\mathcal{P}_{iu}A_{iu}) \end{bmatrix} - \gamma\epsilon I_{n_q-p_i} < 0, \quad \forall i \in \mathcal{N}, \tag{51}$$

where $\Phi_i := \mathcal{P}_{ie}A_{i22} + A_{i22}^T\mathcal{P}_{ie} - \mathcal{W}_iJ_iA_{i12} - A_{i12}^TJ_i^T\mathcal{W}_i^T$. In that case, suitable gain matrices in the reduced order DFEO (34) can be taken as

$$K_i := \begin{bmatrix} J_i^{-1} \\ H_i \\ 0 \end{bmatrix} D_i^\dagger, \quad M_i := \begin{bmatrix} \mathcal{P}_{ie}^{-1} & 0 \\ 0 & \mathcal{P}_{iu}^{-1} \end{bmatrix} T_{is}^T, \quad N_i := \begin{bmatrix} A_{i22} - H_iJ_iA_{i12} & 0 \\ A_{i32} & A_{iu} \end{bmatrix}, \tag{52}$$

$$L_i := \begin{bmatrix} A_{i21} - H_iJ_iA_{i11} \\ A_{i31} \end{bmatrix} J_i^{-1}D_i^\dagger + N_iS_i^\dagger K_i, \tag{53}$$

$$U_i = \begin{bmatrix} -H_iJ_i & I_{\nu_i-q_i} & 0 \\ 0 & 0 & I_{n_q-\nu_i} \end{bmatrix} T_i^T\overline{B}, \tag{54}$$

$$P_i := T_{is}, \; Q_i := T_iK_i, \tag{55}$$

where $H_i = \mathcal{P}_{ie}^{-1}\mathcal{W}_i$, $i \in \mathcal{N}$.

Proof. Choose a candidate Lyapunov function for the error system (46) restricted to $\mathrm{im}T_s$

$$V(e_1, \dots, e_N) := \sum_{i=1}^N e_i^T\mathcal{P}_ie_i, \tag{56}$$

where $\mathcal{P}_i := T_i \begin{bmatrix} I_{p_i} & 0 & 0 \\ 0 & \mathcal{P}_{ie} & 0 \\ 0 & 0 & \mathcal{P}_{iu} \end{bmatrix} T_i^T$. Clearly then $\mathcal{P}_i > 0$.

Taking $\tilde{d} \equiv 0$, the time-derivative of $V(e)$ is equal to

$$\dot{V}(e) = e^T(\mathcal{P}\Lambda + \Lambda^T\mathcal{P} - \gamma\mathcal{P}TSM(R\mathcal{L} \otimes I_n) - \gamma(\mathcal{L}^TR \otimes I_n)M^TS^TT^T\mathcal{P})e, \tag{57}$$

where $\mathcal{P} = \mathrm{diag}\{\mathcal{P}_1, \dots, \mathcal{P}_N\}$ and $\Lambda = T_s\tilde{N}T_s^T$.

Substituting $M_i := \begin{bmatrix} \mathcal{P}_{ie}^{-1} & 0 \\ 0 & \mathcal{P}_{iu}^{-1} \end{bmatrix} T_{is}^T$ into (57), the time-derivative of V becomes

$$\dot{V}(e) = e^T(\mathcal{P}\Lambda + \Lambda^T\mathcal{P} - \gamma T_s T_s^T(R\mathcal{L} \otimes I_n) - \gamma(\mathcal{L}^T R \otimes I_n)T_s^T T_s)e, \tag{58}$$

where $T_s = \mathrm{diag}\{T_{1s}, \dots, T_{Ns}\}$.

On the other hand, we get the following inequality by (51) and (11) in Lemma 3.

$$\mathrm{diag}\{\mathcal{Q}_1, \dots, \mathcal{Q}_N\} - T_s^T\gamma((R\mathcal{L} + \mathcal{L}^T R) \otimes I_n)T_s < 0, \tag{59}$$

where $\mathcal{Q}_i = \begin{bmatrix} \Phi_i & A_{i32}^T\mathcal{P}_{iu} \\ \mathcal{P}_{iu}A_{i32} & \mathcal{P}_{iu}A_{iu} + A_{iu}^T\mathcal{P}_{iu} \end{bmatrix}$, $i \in \mathcal{N}$, with Φ_i as defined in the statement of the lemma.

By substituting $H_i = \mathcal{P}_{ie}^{-1}\mathcal{W}_i$ into the inequality (59), we get

$$T_s^T(\mathcal{P}\Lambda + \Lambda^T\mathcal{P} - \gamma T_s T_s^T(R\mathcal{L} \otimes I_n) - \gamma(\mathcal{L}^T R \otimes I_n)T_s T_s^T)T_s < 0. \tag{60}$$

Since we have restricted the dynamics to the invariant subspace $\mathrm{im}T_s$, we have that e can be represented as $e = T_s z$ for some function z. Thus, we get

$$\begin{aligned}\dot{V}(e) &= e^T(\mathcal{P}\Lambda + \Lambda^T\mathcal{P} - \gamma T_s T_s^T(R\mathcal{L} \otimes I_n) - \gamma(\mathcal{L}^T R \otimes I_n)T_s T_s^T)e \\ &= z^T T_s^T(\mathcal{P}\Lambda + \Lambda^T\mathcal{P} - \gamma T_s T_s^T(R\mathcal{L} \otimes I_n) - \gamma(\mathcal{L}^T R \otimes I_n)T_s T_s^T)T_s z,\end{aligned}$$

and therefore $\dot{V}(e(t)) < 0$ whenever $e(t) \neq 0$.

Hence the solutions of the error system (46) restricted to $\mathrm{im}T_s$ is internally stable. ■

Based on Lemma 6, we now give our main theorem on designing a reduced order DFEO. Similarly, the H_∞ performance index β is minimized. A sufficient condition for its existence is expressed in terms of feasibility of an optimization problem. Solutions to the optimization problem yield required gain matrices. Let $r_i > 0$, $i \in \mathcal{N}$, be as in Lemma 1. Let $g_i > 0$, $i \in \mathcal{N}$, and $\epsilon > 0$ be such that (11) holds. We have the following theorem.

Theorem 2. *There exist a coupling gain γ, gain matrices N_i, L_i, M_i, U_i, P_i, and Q_i, $i \in \mathcal{N}$, such that the error system (46) restricted to $\mathrm{im}T_s$ satisfies (i) and (ii) if there exist positive definite matrices $\mathcal{P}_{ie} \in \mathbb{R}^{(v_i-p_i)\times(v_i-p_i)}$, $\mathcal{P}_{iu} \in \mathbb{R}^{(n_q-v_i)\times(n_q-v_i)}$, and a matrix $\mathcal{W}_i \in \mathbb{R}^{(v_i-p_i)\times p_i}$ for all $i \in \mathcal{N}$ such that the following optimization problem is feasible*

$$\min \ \beta^2 \ \text{ s.t. } (62) \tag{61}$$

$$\begin{bmatrix} \Phi_i + \gamma g_i I_{v_i-p_i} & A_{i32}^T\mathcal{P}_{iu} & \mathcal{W}_i J_i \bar{E}_{i1} - \mathcal{P}_{ie}\bar{E}_{i2} \\ \mathcal{P}_{iu}A_{i32} & \mathrm{Sym}(\mathcal{P}_{iu}A_{iu}) & \mathcal{P}_{iu}\bar{E}_{i3} \\ \bar{E}_{i1}^T J_i^T \mathcal{W}_i^T - \bar{E}_{i2}^T\mathcal{P}_{ie} & \bar{E}_{i3}^T\mathcal{P}_{iu} & -\beta^2 I_{l_q} \end{bmatrix} + \begin{bmatrix} T_{is}^T C_f^T C_f T_{is} - \gamma\epsilon I_{n_q-p_i} & 0 \\ 0 & 0 \end{bmatrix} < 0, \tag{62}$$

where $\Phi_i := \mathcal{P}_{ie}A_{i22} + A_{i22}^T\mathcal{P}_{ie} - \mathcal{W}_i J_i A_{i12} - A_{i12}^T J_i^T \mathcal{W}_i^T$. In that case, suitable gain matrices in the reduced order DFEO (34) can be taken as

$$K_i := \begin{bmatrix} J_i^{-1} \\ H_i \\ 0 \end{bmatrix} D_i^\dagger, \quad M_i := \begin{bmatrix} \mathcal{P}_{ie}^{-1} & 0 \\ 0 & \mathcal{P}_{iu}^{-1} \end{bmatrix} T_{is}^T, \quad N_i := \begin{bmatrix} A_{i22} - H_i J_i A_{i12} & 0 \\ A_{i32} & A_{iu} \end{bmatrix}, \tag{63}$$

$$L_i := \begin{bmatrix} A_{i21} - H_i J_i A_{i11} \\ A_{i31} \end{bmatrix} J_i^{-1} D_i^\dagger + N_i S_i^\dagger K_i, \tag{64}$$

$$U_i = \begin{bmatrix} -H_i J_i & I_{v_i-q_i} & 0 \\ 0 & 0 & I_{n_q-v_i} \end{bmatrix} T_i^T \overline{B}, \tag{65}$$

$$P_i := T_{is}, \ Q_i := T_i K_i, \tag{66}$$

where $H_i = \mathcal{P}_{ie}^{-1}\mathcal{W}_i$, $i \in \mathcal{N}$.

Proof. For condition (i), it is clear that the inequality (51) follows from inequality (62). Hence condition (i) is satisfied.

We take the same candidate Lyapunov function (56) in Lemma 6 for the error system (46).

The time-derivative of $V(e)$ is

$$\begin{aligned}\dot{V}(e) &= e^T(\mathcal{P}T_s\tilde{N}T_s^T + T_s\tilde{N}^T T_s^T\mathcal{P} - \gamma\mathcal{P}T_sM(R\mathcal{L}\otimes I_n) - \gamma(\mathcal{L}^TR\otimes I_n)M^TT_s^T\mathcal{P})e \\ &\quad + e^T\mathcal{P}\tilde{E}\tilde{d} + \tilde{d}^T\tilde{E}^T\mathcal{P}e\end{aligned} \tag{67}$$

with T_s, M, and N the block diagonal versions of the T_{is}, M_i, and N_i as defined by (47) and (48). By substituting $M_i := \begin{bmatrix}\mathcal{P}_{ie}^{-1} & 0\\ 0 & \mathcal{P}_{iu}^{-1}\end{bmatrix} T_{is}^T$ into (67), the time-derivative of $V(e)$ becomes

$$\dot{V}(e) = e^T\overline{\Lambda}e + e^T\mathcal{P}\tilde{E}\tilde{d} + \tilde{d}^T\tilde{E}^T\mathcal{P}e, \tag{68}$$

where we have defined

$$\overline{\Lambda} := \mathcal{P}T_s\tilde{N}T_s^T + T_s\tilde{N}^TT_s^T\mathcal{P} - \gamma T_sT_s^T(R\mathcal{L}\otimes I_n) - \gamma(\mathcal{L}^TR\otimes I_n)T_s^TT_s.$$

On the other hand, by combining (62) with (11) in Lemma 3 it can be verified that

$$\mathrm{diag}\{Q_1, \dots, Q_N\} - T_s^T\gamma((R\mathcal{L} + \mathcal{L}^TR)\otimes I_n)T_s + \frac{1}{\beta^2}\mathrm{diag}\{\Psi_1, \dots, \Psi_N\} < 0, \tag{69}$$

where

$$\Psi_i = \begin{bmatrix}\mathcal{W}_iJ_i\bar{E}_{i1} - \mathcal{P}_{ie}\bar{E}_{i2}\\ \mathcal{P}_{iu}\bar{E}_{i3}\end{bmatrix}\begin{bmatrix}\bar{E}_{i1}^TJ_i^T\mathcal{W}_i^T - \bar{E}_{i2}^T\mathcal{P}_{ie} & \bar{E}_{i3}^T\mathcal{P}_{iu}\end{bmatrix}, \quad i\in\mathcal{N},$$

$$Q_i := \begin{bmatrix}\Phi_i & A_{i32}^TP_{iu}\\ \mathcal{P}_{iu}A_{i32} & \mathcal{P}_{iu}A_{iu} + A_{iu}^T\mathcal{P}_{iu}\end{bmatrix} + T_{is}^TC_f^TC_fT_{is}, \quad i\in\mathcal{N},$$

and Φ_i as defined in the statement of the theorem.

Recall that we have defined $H_i := \mathcal{P}_{ie}^{-1}\mathcal{W}_i$. Hence $\mathcal{W}_i = \mathcal{P}_{ie}H_i$. By substituting this into the expression for Φ_i and Ψ_i, we can check that

$$\begin{aligned}Q_i &= T_{is}^T\mathcal{P}_iT_{is}N_i + N_i^TT_{is}^T\mathcal{P}_iT_{is} + T_{is}^TC_f^TC_fT_{is},\\ \mathrm{diag}\{\Psi_1, \dots, \Psi_N\} &= T_s^T\mathcal{P}\tilde{E}\tilde{E}^T\mathcal{P}T_s.\end{aligned}$$

Substituting these into the inequality (59), using that $T_s^TT_s$ is the identity matrix, we get

$$T_s^T(\mathcal{P}T_s\tilde{N}T_s^T + T_s\tilde{N}^TT_s^T\mathcal{P} - \gamma T_sT_s^T(R\mathcal{L}\otimes I_n) - \gamma(\mathcal{L}^TR\otimes I_n)T_sT_s^T + \tilde{C}_f^T\tilde{C}_f + \frac{1}{\beta^2}\mathcal{P}\tilde{E}\tilde{E}^T\mathcal{P})T_s < 0,$$

so, in other words,

$$\begin{bmatrix}T_s & 0\\ 0 & I_{Nl_q}\end{bmatrix}^T\begin{bmatrix}\Lambda + \tilde{C}_f^T\tilde{C}_f & \mathcal{P}\tilde{E}\\ \tilde{E}^T\mathcal{P} & \beta^2I_{Nl_q}\end{bmatrix}\begin{bmatrix}T_s & 0\\ 0 & I_{Nl_q}\end{bmatrix} < 0. \tag{70}$$

By taking the gain matrices (63), (64), (65), and (66), the global error $e(t)$ satisfies the differential Equation (46). Moreover, we have $e(t)\in\mathrm{im}T_s$ for all $t\in\mathbb{R}$. Hence e can be represented as $e = T_sz$ for some function z. Thus, we get

$$\dot{V}(e) + e^Te + \beta^2\tilde{d}^T\tilde{d} = e^T\Lambda e + e^T\mathcal{P}\tilde{E}\tilde{d} + \tilde{d}^T\tilde{E}^T\mathcal{P}e + e^Te + \beta^2\tilde{d}^T\tilde{d}$$

$$= \begin{bmatrix} T_s z \\ \tilde{d} \end{bmatrix}^T \begin{bmatrix} \Lambda + \tilde{C}_f^T \tilde{C}_f & \mathcal{P}\tilde{E} \\ \tilde{E}^T \mathcal{P} & \beta^2 I_{Nl_q} \end{bmatrix} \begin{bmatrix} T_s z \\ \tilde{d} \end{bmatrix},$$

and therefore

$$\dot{V}(e) + e_f^T e_f - \beta^2 \tilde{d}^T \tilde{d} < 0. \tag{71}$$

Under the internal stability condition, that is, $\dot{V}(e) < 0$ and $V(e) \geqslant 0$ based on Lemma 6, it follows that $e \in L_2(\mathbb{R}^+)$ and hence

$$\int_0^\infty e_f^T(t) e_f(t) \mathrm{d}t + V(e(\infty)) - V(e(0)) \leqslant \beta^2 \int_0^\infty \tilde{d}^T(t) \tilde{d}(t) \mathrm{d}t \tag{72}$$

which implies

$$\|e_f\|_2^2 \leqslant \beta^2 \|\tilde{d}\|_2^2 + V(e(0)). \tag{73}$$

We conclude that also condition (ii) is satisfied. ■

Next, we will give a conceptual algorithm to compute a reduced order DFEO based on the above lemmas and theorem. Let $\beta > 0$. Assume that F has full column rank, (C,A) is an observable pair and 0 is not a zero of the system (A,F,C). Assume the graph $\mathcal{G}$ is a strongly connected directed graph. Then a reduced order DFEO (34) that estimates the state and the fault simultaneously and attenuates the effect of the extended disturbance can be computed using the following algorithm:

Algorithm: 2

1 Let

$$\bar{A} = \begin{bmatrix} A & F \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}, \quad \bar{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}. \tag{74}$$

2 For each $i \in \mathcal{N}$, make a full rank decomposition $\bar{C}_i = D_i W_i$ where $D_i \in \mathbb{R}^{m_i \times p_i}$ and $W_i \in \mathbb{R}^{p_i \times n}$ have full column rank and row rank, respectively.

3 For each $i \in \mathcal{N}$, choose an orthogonal matrix T_i such that

$$T_i^T \bar{A} T_i = \begin{bmatrix} A_{i11} & A_{i12} & 0 \\ A_{i21} & A_{i22} & 0 \\ A_{i31} & A_{i32} & A_{iu} \end{bmatrix}, \quad T_i^T \bar{E} = \begin{bmatrix} \bar{E}_{i1} \\ \bar{E}_{i2} \\ \bar{E}_{i3} \end{bmatrix}, \quad W_i T_i = \begin{bmatrix} J_i & 0 & 0 \end{bmatrix} \tag{75}$$

with the pair $\left(\begin{bmatrix} J_i & 0 \end{bmatrix}, \begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix} \right)$ observable and J_i nonsingular. Then $(J_i A_{i12}, A_{i22})$ is also observable.

4 Compute the positive row vector $r = \begin{bmatrix} r_1, \ \dots \ , r_N \end{bmatrix}$ such that $r\mathcal{L} = 0$ and $r\mathbf{1}_N = N$.

5 Put $g_i = 1$, $i \in \mathcal{N}$ and take $\epsilon > 0$ such that (11) holds.

6 Solve the optimization problem (61) and get γ, $\mathcal{P}_{ie}$, $\mathcal{P}_{iu}$, $\mathcal{W}_i$.

7 Define

$$H_i := \mathcal{P}_{ie}^{-1} \mathcal{W}_i, \quad K_i := \begin{bmatrix} J_i^{-1} \\ H_i \\ 0 \end{bmatrix} D_i^\dagger, \quad S_i := \begin{bmatrix} 0 \\ I_{n_q - p_i} \end{bmatrix}, T_{is} := T_i S_i, \tag{76}$$

$$L_i := \begin{bmatrix} A_{i21} - H_i J_i A_{i11} \\ A_{i31} \end{bmatrix} J_i^{-1} D_i^{\dagger} + N_i S_i^T K_i, \quad U_i := \begin{bmatrix} -H_i J_i & I_{v_i - q_i} & 0 \\ 0 & 0 & I_{n_q - v_i} \end{bmatrix} T_i^T \overline{B}, \tag{77}$$

$$M_i := \begin{bmatrix} \mathcal{P}_{ie}^{-1} & 0 \\ 0 & \mathcal{P}_{iu}^{-1} \end{bmatrix} T_{is}^T, \; N_i := \begin{bmatrix} A_{i22} - H_i J_i A_{i12} & 0 \\ A_{i32} & A_{iu} \end{bmatrix}, \tag{78}$$

$$P_i := T_{is}, \; Q_i := T_i K_i. \tag{79}$$

8 Build the reduce order distributed estimator (34) where the gain matrices are obtained in the above steps. As a result, the estimates of the state and fault are immediately given by

$$\hat{x}_i = \begin{bmatrix} I_n & 0 \end{bmatrix} \hat{\zeta}_i, \quad \hat{f}_i = \begin{bmatrix} 0 & I_q \end{bmatrix} \hat{\zeta}_i. \tag{80}$$

Remark 4. In the special case that $\overline{C}$ has full row rank m, all local output matrices $\overline{C}_i$ have full row rank m_i as well, so $p_i = m_i$ for all $i \in \mathcal{N}$. In this case our reduced order DFEO has order $Nn_q - m$. In Algorithm 2, step 1 can be skipped since $W_i = \overline{C}_i$ and $D_i = I_{m_i}$.

Remark 5. Another special case occurs if $v_i = p_i$ for some ith node, which means that ker $(\overline{C}_i)$ coincides with the unobservable subspace of $(\overline{C}_i, \bar{A})$. In this case, the second block column and row in the transformation (32) are void, so in particular $A_{i12}, A_{i22}, A_{i32}, A_{i21}$, and $\bar{E}_{i2}$ do not appear. The inequality (62) in Theorem 2 reduces to:

$$\begin{bmatrix} \mathrm{Sym}(\mathcal{P}_{iu} A_{iu}) - \gamma \epsilon I_{n_q - v_i} & \mathcal{P}_{iu} \bar{E}_3 \\ \bar{E}_3^T \mathcal{P}_{iu} & -\beta^2 I_{l_q} \end{bmatrix} + \begin{bmatrix} T_{is}^T C_f^T C_f T_{is} & 0 \\ 0 & 0 \end{bmatrix} < 0, \quad \forall i \in \mathcal{N}. \tag{81}$$

The gain matrices in the local fault estimator (34) at node i are then given by

$$N_i := A_{iu}, \quad L_i := A_{i31} J_i^{-1} D_i^{\dagger}, \quad M_i := T_{is}^T, \quad P_i := T_{is}, \quad Q_i := T_i K_i, \quad K_i := \begin{bmatrix} J_i^{-1} \\ 0 \end{bmatrix} D_i^{\dagger}, \quad U_i := T_{is}^T \overline{B}. \tag{82}$$

Remark 6. For both full order and reduced order DFEO's, if there is no fault, that is, $f = 0$, the DFEO reduces to a robust distributed observer that estimates the original state in the presence of disturbances. Furthermore, if there are no fault and disturbance, the distributed estimator reduces to an ordinary distributed observer studied in References 36,37.

5 | SIMULATIONS

In this section, we study a four-tank system borrowed from References 38,39 to show the effectiveness of the proposed approaches to design robust full order and reduced order DFEO's, respectively. A schematic diagram of the plant is shown in Figure 2. As inputs, water is delivered to the upper tanks, 1 and 3, whose levels are coupled through a connection pipe.

System description:

The coupled-tanks system admits the linear model (1), around the equilibrium point water level in the tanks $h_0 = [1017814]^T$(cm) and voltages applied to the pumps $u_0 = [32]^T$ (V). The coefficient matrices are given by

$$A = \begin{bmatrix} -0.0540 & 0 & 0.0343 & 0 \\ 0.0286 & -0.0205 & 0 & 0 \\ 0.0243 & 0 & -0.0355 & 0 \\ 0 & 0 & 0.0211 & -0.0310 \end{bmatrix}, \quad B = F = \begin{bmatrix} 0.2918 & 0 \\ 0 & 0 \\ 0 & 0.2557 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0.1 \end{bmatrix}.$$

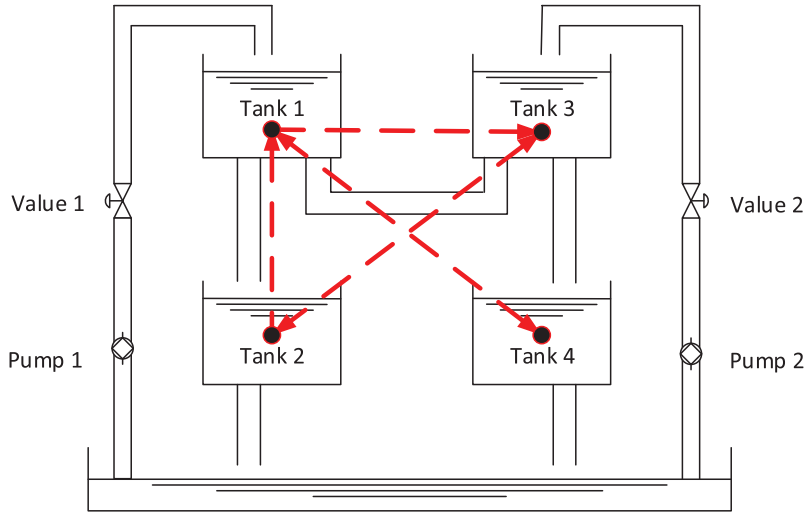

FIGURE 2 A diagram of four-tank system, assuming a node estimator attached to every tank. The dotted lines represent the communication links [Colour figure can be viewed at wileyonlinelibrary.com]

In our example, we will consider actuator additive faults. For a four-tank system, the control command acts on the actuator-pump. The additive actuator fault is an offset of the pump action during the process. The offset may be a constant or time-varying value in practice. The offset is one of the additive actuator faults. Such faults usually occur in the input channel. Therefore, the fault distribution matrix is assumed to be $F = B$.

A network with four agents is considered, each one labeled from 1 to 4 according to the number of the tank whose level is measured. As depicted in Figure 2, the communication network among measurement nodes is represented as red dotted graph. The digraph is strongly connected. The Laplacian matrix of this graph is given by

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

For our simulations, we take the following actuator faults:

$$f(t) = \begin{cases} \begin{bmatrix} 0 & 0 \end{bmatrix}^T & 0 \leqslant t < 300 \\ \begin{bmatrix} 5 & 4 + \sin(0.03(t-450)) \end{bmatrix}^T & 300 \leqslant t \leqslant 1000 \end{cases} \tag{83}$$

where the time units are seconds. The two actuator faults are injected at $t = 300$s.

Remark 7. Note that the faults considered in the example are a constant fault and a sinusoid fault, respectively. The injected sinusoid is a low frequency signal. It is not unreasonable to take a low frequency fault signal in our context of the tank system since the response of process control systems is usually slow, and therefore the faults in practice usually are in the low frequency domain.

It can be verified that $(\overline{C}_i, \bar{A})$ is not observable at any of the individual nodes. Obviously, $(\overline{C}, \bar{A})$ is observable by checking the rank condition in Remark 1. In the following we will compute robust full order and reduced order DFEO's using Algorithm 1 and Algorithm 2, respectively.

Full order DFEO design: We have computed a full order DFEO using Algorithm 1. The optimal H_∞ performance index is calculated as $\beta = 0.5638$. The fault estimator gain matrices are calculated by using the YALMIP toolbox in MATLAB. The observer parameters are given as

$$L_1 = \begin{bmatrix} 0.8403 \\ 0 \\ -0.1769 \\ 0 \\ 1.5846 \\ 2.6161 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 9.8324 \\ 1.0181 \\ -0.3034 \\ 0 \\ 5.6057 \\ 6.9316 \end{bmatrix}, \quad L_3 = \begin{bmatrix} -0.0129 \\ 0 \\ 0.8320 \\ 0 \\ 0.7722 \\ 1.3684 \end{bmatrix}, \quad L_4 = \begin{bmatrix} 0.1888 \\ 0 \\ 10.1303 \\ 1.0328 \\ 1.2941 \\ 4.5063 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0.0693 & 0 & -0.0078 & 0 & 0.0727 & 0.1174 \\ 0 & 10.6738 & 0.2993 & -0.3454 & -0.0352 & 0.0415 \\ -0.0078 & 0.2993 & 203.6346 & 16.6452 & -23.9621 & 28.0568 \\ 0 & -0.3454 & 16.6452 & 16.6401 & -1.9566 & 2.3109 \\ 0.0727 & -0.0352 & -23.9621 & -1.9566 & 3.0350 & -2.9289 \\ 0.1174 & 0.0415 & 28.0568 & 2.3109 & -2.9289 & 5.7788 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 6.2977 & 0.5716 & -0.2185 & 0 & 3.7851 & 4.7787 \\ 0.5716 & 0.0682 & -0.0188 & 0 & 0.3449 & 0.4273 \\ -0.2185 & -0.0188 & 303.2002 & 14.6812 & -35.7879 & 41.7448 \\ 0 & 0 & 14.6812 & 16.6799 & -1.7257 & 2.0383 \\ 3.7851 & 0.3449 & -35.7879 & -1.7257 & 6.6438 & -1.7790 \\ 4.7787 & 0.4273 & 41.7448 & 2.0383 & -1.7790 & 11.0160 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 104.5679 & 29.8460 & -0.0006 & -0.0038 & 19.2587 & -9.9403 \\ 29.8460 & 38.2941 & 0 & 0.0241 & 5.5232 & -2.8364 \\ -0.0006 & 0 & 0.0703 & 0 & 0.0380 & 0.0681 \\ -0.0038 & 0.0241 & 0 & 0.0532 & -0.0007 & 0.0004 \\ 19.2587 & 5.5232 & 0.0380 & -0.0007 & 4.1305 & -1.7092 \\ -9.9403 & -2.8364 & 0.0681 & 0.0004 & -1.7092 & 1.1495 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 113.2378 & 29.0144 & 0.1451 & 0.0143 & 20.8905 & -10.6930 \\ 29.0144 & 37.9140 & 0 & 0 & 5.3694 & -2.7573 \\ 0.1451 & 0 & 8.1872 & 0.7318 & 1.2002 & 3.8636 \\ 0.0143 & 0 & 0.7318 & 0.0832 & 0.1001 & 0.3454 \\ 20.8905 & 5.3694 & 1.2002 & 0.1001 & 4.5611 & -1.3098 \\ -10.6930 & -2.7573 & 3.8636 & 0.3454 & -1.3098 & 3.0509 \end{bmatrix}.$$

For this simulations, the disturbance is assumed to be $d(t) = 0.3\sin(t)e^{-0.01t}$, which belongs to $L_2(R^+)$.

Applying the corresponding distributed fault estimator (4), we get simulated curves of the faults and their estimates. The two actuator faults occur at $t = 300$. Figures 3 and 4 present the faults and their estimates at each node. The curves show that the fault is estimated successfully.

Figures 5 and 6 show the estimation errors of the states. In addition, Figure 7 presents the estimation errors of the two faults. It can be seen that all the errors tend to zero as time runs off to infinity. The estimation error results show that the effect of the joint disturbance is attenuated.

Reduced order DFEO design: We have computed a reduced order DFEO following Algorithm 2. Similarly, the optimal H_∞ performance index is calculated as $\beta = 0.5638$. The observer parameters are given as

$$L_1 = \begin{bmatrix} -2.2165 \\ -1.8806 \\ 0.5767 \\ 0.0006 \\ -0.1202 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -9.8693 \\ -12.4450 \\ -10.4420 \\ 0.0059 \\ 4.3914 \end{bmatrix}, \quad L_3 = \begin{bmatrix} -2.5336 \\ -1.0933 \\ -0.3845 \\ 0.0211 \\ 0.0687 \end{bmatrix}, \quad L_4 = \begin{bmatrix} -8.5797 \\ -9.5757 \\ -3.8644 \\ -2.6839 \\ 0.2735 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 & 0 & 0.0013 & 0 & 0.1309 & 0.1011 \\ 0 & 0 & -0.0903 & 0 & 0.1134 & 0.7465 \\ 0 & 0.0307 & 102.8003 & 6.5714 & -12.0838 & 14.2722 \\ 0 & 0.0047 & 6.6785 & 3.4860 & -0.7850 & 0.9272 \\ 0 & 0.0459 & -26.6726 & -1.7013 & 3.1353 & -3.7031 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.2652 & 0 & 0.0029 & 0 & 0.2143 & 0.1603 \\ 0.2132 & 0 & 0.0040 & 0 & 0.3204 & 0.2427 \\ 0.1618 & 0 & -0.0878 & 0 & 0.2571 & 0.8504 \\ 0 & 0 & 6.5025 & 3.2805 & -0.7643 & 0.9028 \\ 0 & 0 & 107.6796 & 6.6092 & -12.6573 & 14.9496 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0.0032 & 0 & 0 & 0 & 0.0491 & 0.1297 \\ -0.0685 & 0 & 0 & 0 & 0.3995 & 0.0567 \\ -32.7472 & -10.7611 & 0 & -0.0037 & -6.0601 & 3.1121 \\ 0.0039 & -0.0012 & 0 & 0.0517 & 0.0007 & -0.0004 \\ -3.4953 & -5.5842 & 0 & 0.0019 & -0.6468 & 0.3322 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0.0082 & 0 & 0.3492 & 0 & 0.0878 & 0.2573 \\ 0.0101 & 0 & 0.2569 & 0 & 0.1243 & 0.3478 \\ -0.0655 & 0 & 0.0893 & 0 & 0.4221 & 0.1332 \\ -30.5617 & -8.5620 & 0 & 0 & -5.6557 & 2.9044 \\ 12.0341 & 8.0309 & 0 & 0 & 2.2270 & -1.1436 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} -0.8475 & 0.0301 & 0 & 0 & 0 \\ -0.6661 & -0.0350 & 0 & 0 & 0 \\ -0.0039 & 0.2458 & -0.0013 & 0 & -0.0050 \\ 0.0025 & -0.0029 & 0.0201 & -0.0310 & -0.0052 \\ 0.0010 & -0.0638 & -0.0050 & 0 & -0.0192 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} -0.7109 & 0.2938 & 0 & 0 & 0 \\ -0.5799 & -0.0005 & 0.0301 & 0 & 0 \\ -0.4454 & 0.0006 & -0.0350 & 0 & 0 \\ 0 & 0.0025 & -0.0029 & -0.0310 & 0.0208 \\ 0.0239 & -0.0041 & 0.2540 & 0 & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} -0.8311 & 0.0281 & 0 & 0 & 0 \\ -0.3290 & -0.0535 & 0 & 0 & 0 \\ 0.0044 & -0.2847 & 0.0043 & 0 & -0.0046 \\ 0 & 0 & 0 & -0.0310 & 0 \\ -0.0036 & 0.0577 & 0.0234 & 0 & -0.0248 \end{bmatrix},$$

$$N_4 = \begin{bmatrix} -0.5691 & 0.2569 & 0 & 0 & 0 \\ -0.4097 & -0.0005 & 0.0281 & 0 & 0 \\ -0.1476 & 0.0009 & -0.0535 & 0 & 0 \\ -0.0334 & 0.0052 & -0.2897 & -0.0028 & -0.0017 \\ 0.0032 & 0.0022 & 0.0223 & -0.0297 & -0.0177 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2513 & 0 & 0.9679 \\ 0.1167 & -0.1357 & 0.9523 & 0 & -0.2472 \\ 0 & 0 & 0 & 1 & 0 \\ 0.9932 & 0.0159 & -0.1119 & 0 & 0.0291 \\ 0 & 0.9906 & 0.1322 & 0 & -0.0343 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1167 & -0.1357 & 0 & 0.9839 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.9932 & 0.0159 & 0 & -0.1156 \\ 0 & 0 & 0.9906 & 0 & 0.1366 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.0946 & -0.1804 & -0.9628 & 0 & 0.1774 \\ 0 & 0 & -0.1812 & 0 & -0.9835 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.9835 & -0.1782 & 0 & 0.0328 \\ 0.9955 & 0.0171 & 0.0915 & 0 & -0.0169 \end{bmatrix}, P_4 = \begin{bmatrix} 0 & 0.0946 & -0.1804 & -0.9746 & 0.0926 \\ 0 & 0 & 0 & 0.0946 & 0.9955 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9835 & -0.1804 & 0.0171 \\ 0 & 0.9955 & 0.0171 & 0.0926 & -0.0088 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1 \\ 0 \\ 0.0287 \\ 0 \\ 2.8994 \\ 2.2478 \end{bmatrix}, Q_2 = \begin{bmatrix} 22.9672 \\ 1 \\ 0.2817 \\ 0 \\ 20.4825 \\ 15.3125 \end{bmatrix}, Q_3 = \begin{bmatrix} 0.0743 \\ 0 \\ 1 \\ 0 \\ 1.2632 \\ 3.2414 \end{bmatrix}, Q_4 = \begin{bmatrix} 0.6431 \\ 0 \\ 25.2879 \\ 1 \\ 6.5895 \\ 19.5990 \end{bmatrix}.$$

For this simulation, we consider a large magnitude for the disturbance. It is assumed to be $d(t) = 0.5\sin(t)e^{-0.01t}$, which belongs to $L_2(R^+)$.

By using the reduced order DFEO (34), we obtain simulated curves of the faults and their estimates in Figures 8 and 9. From the simulation results, it can be seen that the proposed reduced order DFEO achieves robust distributed fault estimation.

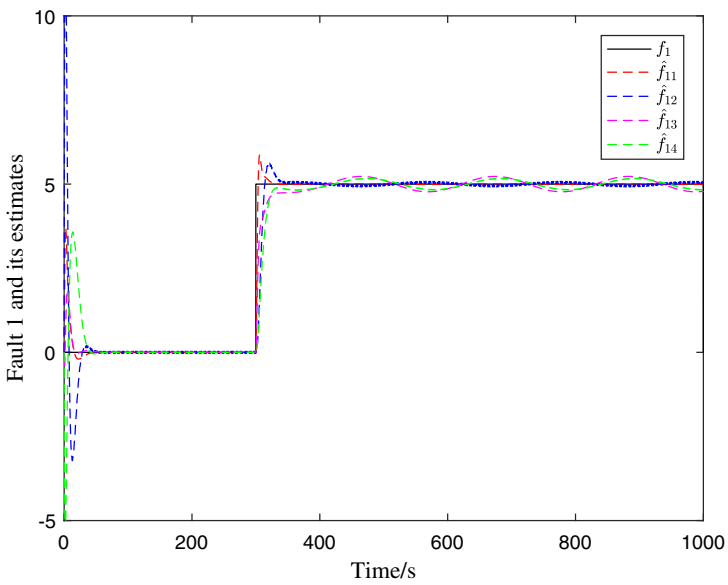

FIGURE 3 The fault 1 and its estimates at each node using full order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

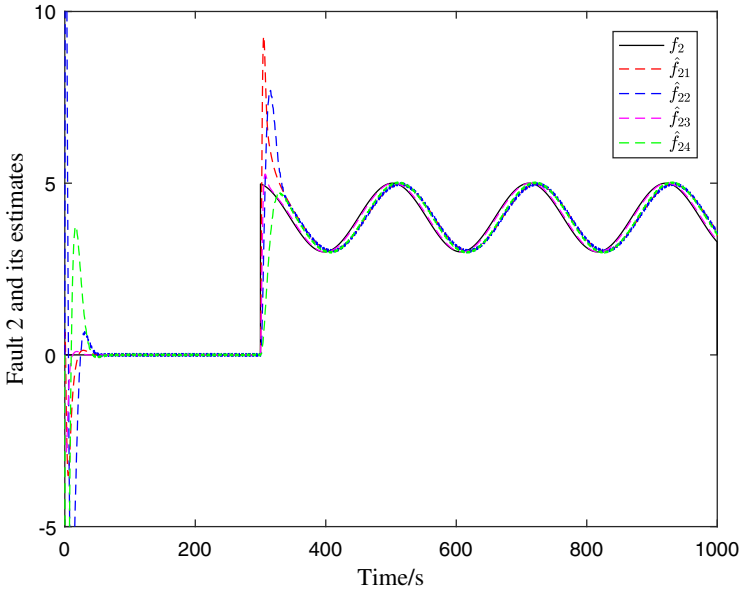

FIGURE 4 The fault 2 and its estimates at each node using full order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

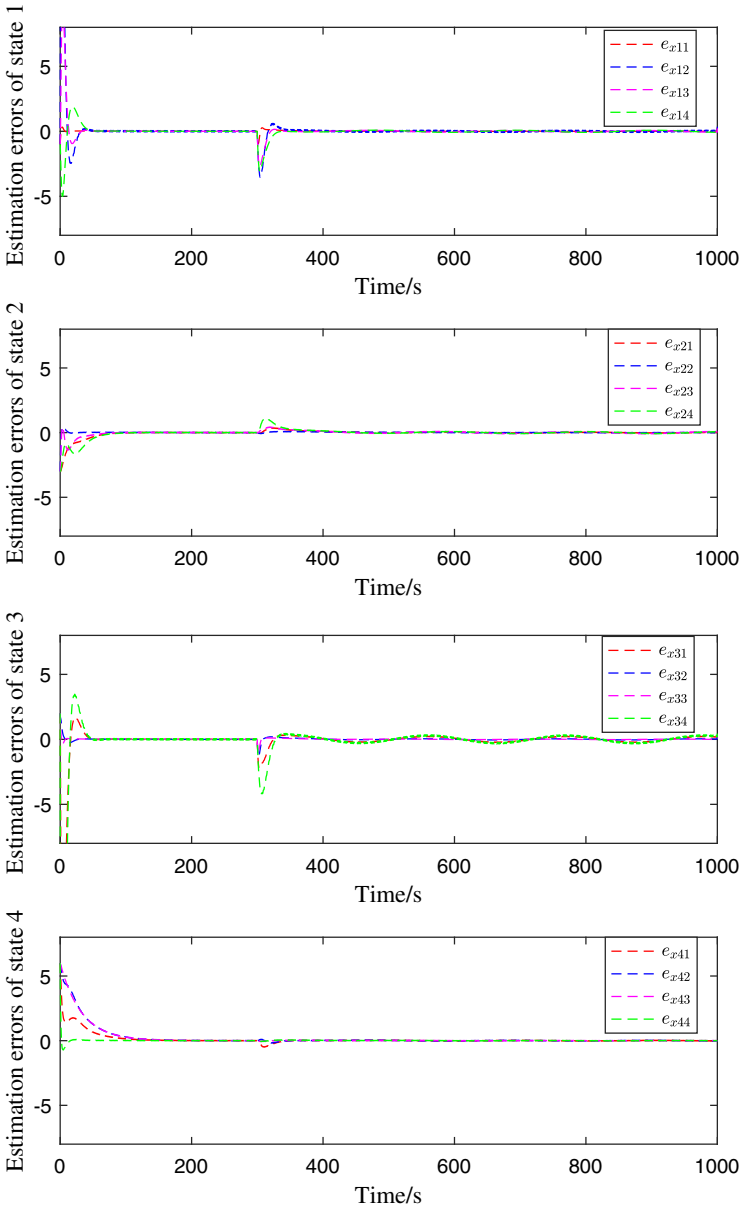

FIGURE 5 Estimation errors of the states 1 and 2 at each node using full order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 6 Estimation errors of the states 3 and 4 at each node using full order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

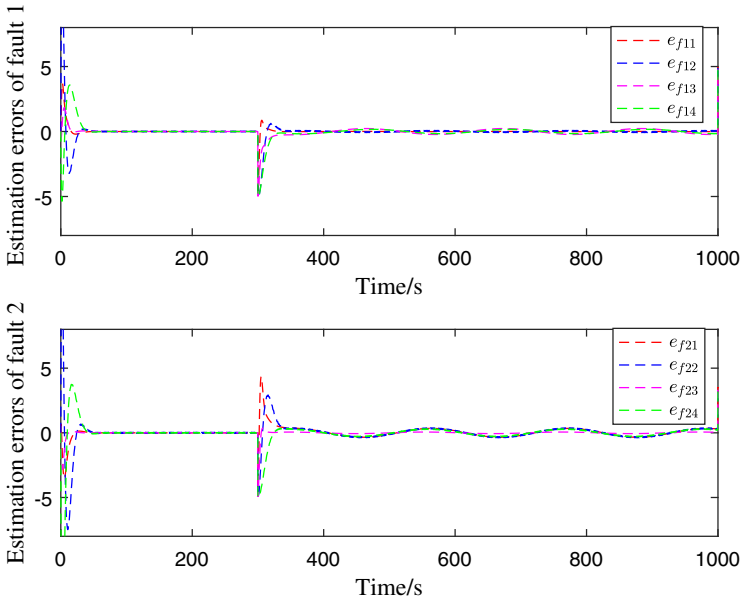

FIGURE 7 Estimation errors of the faults 1 and 2 at each node using full order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

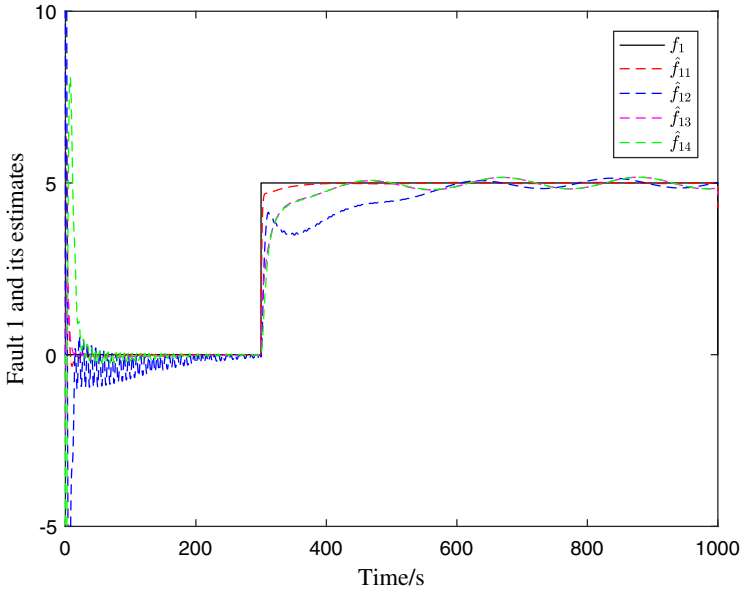

FIGURE 8 The fault 1 and its estimates at each node using reduced order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

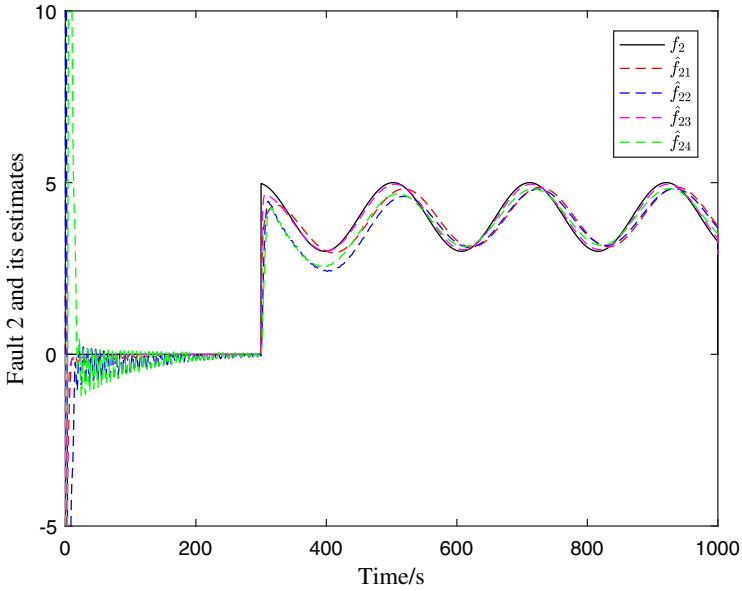

FIGURE 9 The fault 2 and its estimates at each node using reduced order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

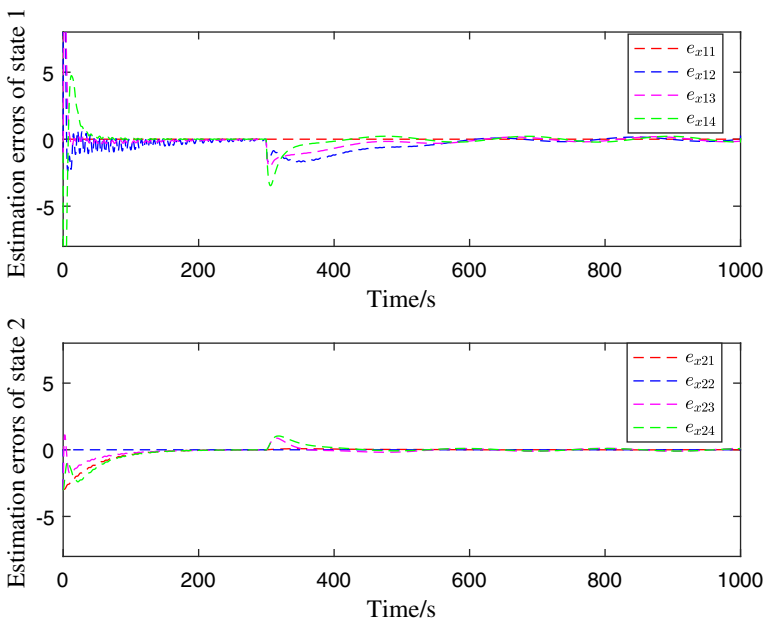

FIGURE 10 Estimation errors of the states 1 and 2 at each node using reduced order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

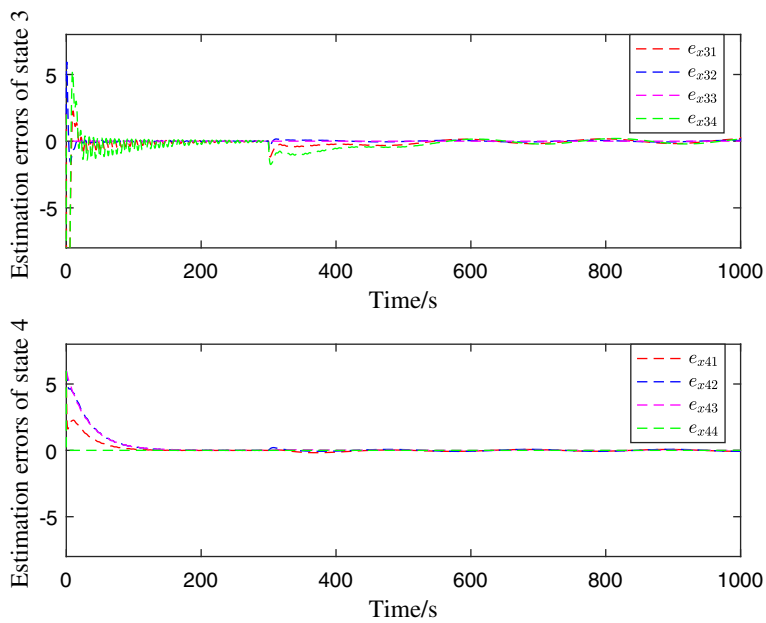

FIGURE 11 Estimation errors of the states 3 and 4 at each node using reduced order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

Figures 10 and 11 show the estimation errors of the states. Figure 12 presents the estimation errors of the two faults. The results show that the estimation errors of all states and faults reach the region around 0. It can be seen that the joint disturbance is attenuated successfully.

Remark 8. The fault and disturbance can be distinguished by differences of the matrices F and E. The fault matrix F and disturbance matrix E are usually different. These matrices depend on the application that is considered. We plan to attenuate the disturbances in H_∞ norm sense in this article. If the fault and disturbances affect the same channel, even $F = E$, it is hard to distinguish faults and disturbances. Finite-frequency fault estimation could be a possible way.[11]

Remark 9. The simulations show some advantages of the proposed distributed fault estimation method. In this article, the local fault estimation observers at each node simultaneously estimate the fault and state of the entire system in the presence of disturbances, whereas in References 14,18,21 the local fault estimators only estimate the fault of their corresponding subsystems. On the other hand, compared with Reference 25, we use a different

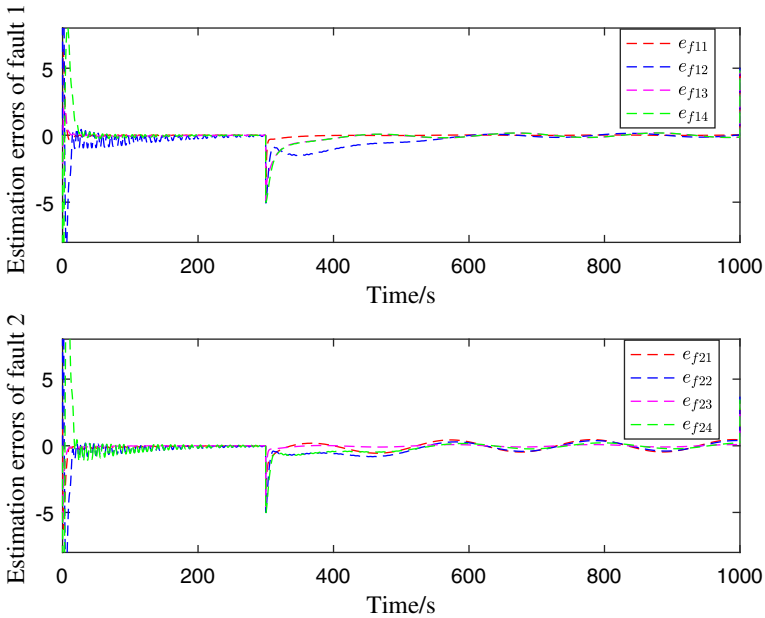

FIGURE 12 Estimation errors of the faults 1 and 2 at each node using reduced order DFEO. DFEO, distributed fault estimation observer [Colour figure can be viewed at wileyonlinelibrary.com]

distributed fault estimation scheme. Only joint observability is required, while in Reference 25 local observability is assumed.

6 | CONCLUSIONS

In this article, we have proposed a distributed fault estimation approach. The framework includes full order and reduced order DFEO's, respectively. Under some standard assumptions, conditions for the existence of suitable DFEO's have been expressed in terms of feasibility of LMI's. To calculate the gain matrices in our DFEO's, systematic design algorithms have been presented for full order and reduced order DFEO's, respectively. Finally, simulation results show the effectiveness of the proposed method.

Further results are anticipated by applying the proposed distributed fault estimation techniques to the problem of distributed FTC, which will be the focus of our future work.

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