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Abstract

We present a trajectory tracking control design method for a class of mechanical systems in the port-Hamiltonian framework. The proposed solution is based on the virtual contractivity-based control (v-CBC) method, which employs the notions of *virtual systems* and of *contractivity.* This approach leads to a family of asymptotic tracking controllers that are *not limited* to those that preserve the pH structure of the closed-loop system nor require an *intermediate* change of coordinates. Nevertheless, structure preservation and other properties (e.g., passivity) are possible under sufficient conditions. The performance of the proposed v-CBC scheme is experimentally evaluated on a planar robot of two degrees of freedom (DoF).

Key words: Port-Hamiltonian systems; trajectory tracking; virtual systems; contractivity; mechanical systems.

1 Introduction

The control of electro-mechanical (EM) systems is a well-known problem in the systems and control literature. As an alternative to the Euler-Lagrange (EL) formalism for modeling EM systems, the port-Hamiltonian (pH) framework has been proposed (van der Schaft & Maschke 1995). This combines the physical systems *analysis* approach of analytical mechanics with the port-based network *modeling* point of view of complex physical systems. A number of set-point control design methods for pH systems have been proposed during the past two decades. For instance, the standard PI control (Jayawardhana et al. 2007), PID-PBC (Borja et al. 2021), the well-known Interconnection and Damping Assignment PBC (IDA-PBC) technique, the Control by Interconnection (CbI) method, and others expounded in van der Schaft & Jeltsema (2014). Nonetheless, for the tracking control problem it is not straightforward to design controllers for such (nonlinear) pH systems with an insightful energy interpretation of the closed-loop system. For instance, it is not trivial to obtain a passive incremental system via a change of coordinates (Fujimoto et al. 2003). This is the case for many mechanical pH systems that have a state-dependent inertia matrix or with some degrees of underactuation, because these systems *cannot* be put into *normal form*, in general; see Venkatraman et al. (2010).

In Fujimoto et al. (2003), the authors provide necessary and sufficient conditions based on generalized canonical transformations (GCTs) to construct an incremental system with pH structure. By using the new coordinates, the system can be stabilized via standard PBC. However, this method may easily lead to a non-tractable problem since nonlinear PDEs need to be solved.

The GCT approach is applied to mechanical pH systems in the works of Dirksz & Scherpen (2010) and Romero et al. (2015). In the former, the method is an adaptive control scheme; whereas in the latter, the authors use the GCT approach to obtain a pH error system with a constant inertia matrix. Then the controller is designed in a structure-preserving manner. While solving the PDEs that correspond to the existence of GCTs is, in general, *not trivial*, some characterizations are presented in Venkatraman et al. (2010) via partial linearization.

A different approach is taken in Yaghmaei & Yazdanpanah (2017), where the authors extend the structure-preserving IDA-PBC method to solve the tracking control problem of pH systems by means of *contractivity* (Lohmiller & Slotine 1998). They characterize a class of *contractive pH systems* that are later used in the IDA-PBC method as target dynamics. Still, similar to GCT approaches, nonlinear PDEs need to be solved.

A system is *contractive* if *any* pair of neighboring trajectories converge to each other. The contractivity property can be understood as a local (or *differential*) notion

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of incremental stability (Angeli 2002), and it does not require the construction of an incremental model. See Lohmiller & Slotine (1998), Forni & Sepulchre (2014) for further details. It allows one to analyze the behavior of systems subject to (time-varying) *inputs* (Sontag 2010), or to design *contractivity-based control* (CBC) schemes as in Manchester & Slotine (2014). Using the notion of *differential* dissipativity Forni & Sepulchre (2013), contractive systems enjoy dissipative-like intput/output and interconnection properties.

In this paper, we propose a constructive procedure to design a family of controllers based upon the notion of *virtual contractivity.* Such controllers are suitable to solve a tracking control problem of mechanical systems in the pH framework. This family of controllers is not limited to preserving the pH structure of the closed-loop system and it can also be applicable to other structural properties under some sufficient conditions. Furthermore, no (intermediate) change of (GCT) coordinates is needed.

Virtual contractivity is a generalization of the standard contractivity to include convergence properties of a specific behavior like a reference trajectory. This generalization exploits the notion of *virtual systems* to infer the convergence properties of a given *original system.* Roughly speaking, for a given (original) plant a virtual system can be understood as a system that can produce all plant's trajectories, i.e., the plant's is *embedded* in the virtual one (Reyes-Báez 2019). If the virtual system is contractive then all of its solutions will converge to any plant's trajectory (Wang & Slotine 2005), and the original plant is said to be *virtually contractive.* Analogous to CBC, virtual systems with inputs are suitable for *virtual contractivity-based control* (v-CBC) design, e.g., see Manchester et al. (2018), Reyes-Báez et al. (2020).

The v-CBC method consists of three main steps (Reyes-Báez 2019). Firstly, we define a *virtual system* which embeds all the solutions of a given *original system.* Secondly, a controller is designed such that the closed-loop virtual system is contractive and tracks a reference trajectory. Finally, the third step consists of closing the loop of the original system where the control law is given by the virtual system's controller with the virtual state being replaced by the state of original system.

The paper is organized as follows. In Section 2, we introduce the v-CBC method which is based on the *differential Lyapunov* framework for contraction analysis in Forni & Sepulchre (2014). Section 3 contains the solution to the tracking problem of fully-actuated mechanical systems in the pH framework via the v-CBC design method. We later apply the method to the control of a 2-DoF robot, where a detailed controller's construction and experiments are presented in Section 4.

2 Preliminaries

Throughout this paper all objects (manifolds, mappings, etc.) are assumed to be smooth. When it is clear from the context, arguments will be omitted from the function.

We consider a control system Σ_u given by

$$\Sigma_u : \begin{cases} \dot{x} = f(x,t) + \sum_{i=1}^{m} g_i(x,t)u_i, \\ y = h(x,t), \end{cases} \tag{1}$$

evolving on an N-dimensional state-space manifold $\mathcal{X}$ with tangent bundle $T\mathcal{X}$; where $x \in \mathcal{X}$, $u \in \mathcal{U} \subset \mathbb{R}^m$ and $y \in \mathcal{Y} \subset \mathbb{R}^m$. The sets $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be open subsets of $\mathbb{R}^m$. System Σ_u in closed-loop with the feedback (nonlinear) control law $u = \gamma(x,t)$ is

$$\Sigma : \begin{cases} \dot{x} = F(x,t), \\ y = h(x,t). \end{cases} \tag{2}$$

2.1 A differential Lyapunov method for contractivity

Contraction analysis aims at inferring incremental stability of a nonlinear system from a local analysis via the *linear* variational dynamics of a *pronlonged system.*

Definition 1 *The* prolonged control system Σ_u^δ *associated to the control system* Σ_u *in* (1) *is given by*

$$\begin{aligned} &\dot{x} = f(x,t) + \textstyle\sum_{i=1}^{n} g_i(x,t)u_i, \\ &y = h(x,t), \\ &\delta\dot{x} = \tfrac{\partial f}{\partial x}\delta x + \textstyle\sum_{i=1}^{n} u_i \tfrac{\partial g_i}{\partial x}\delta x + \sum_{i=1}^{n} g_i \delta u_i, \\ &\delta y = \tfrac{\partial h}{\partial x}(x,t)\delta x. \end{aligned} \tag{3}$$

with $(u, \delta u) \in T\mathcal{U}$, $(x, \delta x) \in T\mathcal{X}$, *and* $(y, \delta y) \in T\mathcal{Y}$. *The prolonged system* Σ^δ *of* Σ *in* (2) *is similarly defined.*

Similar to a standard Lyapunov function, a (Finsler) differential Lyapunov function (dL) can be introduced on $T\mathcal{X}$. Consider the following definition of dL function which is an adaption from Forni & Sepulchre (2014).

Definition 2 *A function* $V : T\mathcal{X} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *is a candidate* dL *function for* (2) *if it satisfies*

$$c_1\|\delta x\|_x^p \le V(x, \delta x, t) \le c_2\|\delta x\|_x^p, \tag{4}$$

uniformly in t, *for some* $c_1, c_2 > 0$, *and with* p *a positive integer where* $\|\delta x\|_x$ *is a Finsler metric*[2] *defined on* $T\mathcal{X}$.

[2] The reader is referred to (Reyes-Báez 2019, Chapter 2) and references therein for a definition of a Finsler metric.

For sake of clarity, along this work we will take $p = 2$ and $\|\delta x\|_x := \sqrt{\delta x^\top \delta x}$ that can be understood as an Euclidean norm in each tangent space $T_x\mathcal{X}$.

The following theorem is the key result of the dL-framework for contraction in Forni & Sepulchre (2014).

Theorem 1 (Differential Lyapunov method) *Consider the prolonged system Σ^δ in Definition 1, a connected and forward invariant set $\mathcal{C} \subseteq \mathcal{X}$, and a strictly increasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Let V be a candidate differential Lyapunov function satisfying*

$$\dot{V}(x, \delta x, t) \leq -\alpha(V(x, \delta x, t)) \tag{5}$$

for each $(x, \delta x) \in T\mathcal{X}$ and all t. Then, system Σ in (2) is

- *Incrementally Stable (IS) on $\mathcal{C}$ if $\alpha(s) = 0 \; \forall s \geq 0$;*
- *Asymptotically IS (AIS) if α is a $\mathcal{K}$ function[3] ;*
- *Exponentially IS (EIS) on $\mathcal{C}$ if $\alpha(s) = \beta s, \forall s > 0$.*

System Σ in (2) is said *contractive* if (5) holds with $\alpha(s)$ in AIS or EIS. Contractive systems exhibit the following inherent robustness property.

Lemma 1 (Zamani & Tabuada (2011)) *Consider the perturbed system*

$$\dot{x}_p = F(x_p, t) + p(x_p, t), \quad x_p \in \mathcal{X}, \tag{6}$$

where $p(x_p, t)$ is uniformly bounded for every t. Suppose the unperturbed system (2) is contractive with $\alpha(s) = \beta s$, then system (6) is input-to-state (ISS) EIS.

Lemma 1 is an ISS-EIS adaptation of (Zamani & Tabuada 2011, Definition 2.6), where the condition in terms of contraction metrics is replaced by a dL counterpart. This opens the door to use other metrics than Riemannian, e.g, logarithmic matrix measures Sontag (2010). Due to space limitations the proof is omitted.

2.2 Virtual contractivity

Virtual contractivity of an (original) system refers to deducing convergence properties of a system's particular solution by means of the contractivity of an auxiliary *virtual system*, which is defined below (Reyes-Báez 2019).

Definition 3 (Virtual system) *Consider the systems Σ_u and Σ in (1) and (2), respectively. A* virtual control system *associated to Σ_u is defined as the system*

$$\Sigma_u^v : \begin{cases} \dot{x}_v = \Gamma_v(x_v, x, u_v, t), \\ y_v = h_v(x_v, x, t), \qquad \forall t \geq t_0, \end{cases} \tag{7}$$

parametrized by x, with state $x_v \in \mathcal{X}$, and input $u_v \in \mathcal{U}$, where $h_v : \mathcal{X} \times \mathcal{X} \times \mathbb{R}_{\geq 0} \to \mathcal{Y}$ and $\Gamma_v : \mathcal{X} \times \mathcal{X} \times \mathcal{U} \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ are such that

$$\begin{aligned} \Gamma_v(x, x, u, t) &= f(x, t) + \sum_{i=1}^{n} g_i(x, t) u_i, \\ h_v(x, x, t) &= h(x, t), \qquad \forall u, \forall t \geq t_0, \end{aligned} \tag{8}$$

hold. Similarly, a virtual system *associated to Σ is*

$$\Sigma^v : \begin{cases} \dot{x}_v = \Phi_v(x_v, x, t), \\ y_v = h_v(x_v, x, t). \end{cases} \tag{9}$$

with state $x_v \in \mathcal{X}$ and parametrized by $x \in \mathcal{X}$, where $\Phi_v : \mathcal{X} \times \mathcal{X} \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ satisfies uniformly the conditions:

$$\Phi_v(x, x, t) = F(x, t) \quad \text{and} \quad h_v(x, x, t) = h(x, t). \tag{10}$$

Theorem 2 (Virtual contractivity) *Consider systems Σ and Σ^v in (2) and (9), respectively. Let $\mathcal{C}_v \subseteq \mathcal{X}$ (resp. $\mathcal{C}_x \subseteq \mathcal{C}_v$) be a connected and forward invariant set of Σ^v (resp. Σ). Suppose that Σ^v is contractive with respect to x_v for every x. Then, for all initial conditions $x_0 \in \mathcal{C}_x$ and $x_{v0} \in \mathcal{C}_v$, each solution of Σ^v converges asymptotically to the solution of Σ.*

2.3 Virtual contractivity-based control (v-CBC)

The design procedure of v-CBC is divided in three steps:

(1) Design of the virtual system (7) for system (1).
(2) Design a controller $u_v = \zeta(x_v, x, t)$ for the virtual system (7) such that the closed-loop system is contractive and tracks a reference behavior $x_d(t)$.
(3) Define the controller for system (1) as $u = \zeta(x, x, t)$.

By Theorem 2, it follows that the trajectories (starting at $x_0 \in \mathcal{C}_x$) of the *original* system (1) in closed-loop with $u = \zeta(x, x, t)$ will converge to $x_d(t)$ exponentially.

3 Control of mechanical pH systems via v-CBC

The dynamics of *fully-actuated* mechanical systems, with generalized position q on the configuration space $\mathcal{Q}$ of dimension n, is modeled by the pH system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0_n & I_n \\ -I_n & -D(x) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(x) \\ \frac{\partial H}{\partial p}(x) \end{bmatrix} + \begin{bmatrix} 0_n \\ I_n \end{bmatrix} u, \\ y &= \begin{bmatrix} 0_n & I_n \end{bmatrix} \frac{\partial H}{\partial x}(x), \end{aligned} \tag{11}$$

with Hamiltonian function given by the total energy

$$H(x) = \frac{1}{2} p^\top M^{-1}(q) p + P(q), \tag{12}$$

[3] α is of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

where $x = (q, p)$ evolves in $\mathcal{X} := T^*\mathcal{Q}$ (the cotangent bundle of $\mathcal{Q}$), $P(q)$ is the potential energy, $p := M(q)\dot{q}$ is the generalized momentum; the matrix $M(q) = M^\top(q) > 0_n$ represents the inertia of the system, while the matrix $D(q) = D^\top(q) \geq 0_n$ represents the damping. The matrices I_n and 0_n are the $n \times n$ identity and zero matrices, respectively. A fundamental structural property of (11) is that the map $u \mapsto y$ is *passive* with (12) as the storage function (van der Schaft 2017). As shown in (Reyes-Báez 2019, Appendix B.2.3), (11) can be equivalently rewritten as the system given by

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0_n & I_n \\ -I_n & -(E(x)+D(x)) \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(x) \end{bmatrix} + \begin{bmatrix} 0_n \\ I_n \end{bmatrix} u, \\ y &= \begin{bmatrix} 0_n & I_n \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix},\end{aligned} \tag{13}$$

where $E(x) := S_H(x) - \frac{1}{2}\dot{M}(q)$, and $S_H(x) := S_L(q, M^{-1}(q)p)$ is a skew-symmetric matrix with

$$S_{Lkj}(q,\dot{q}) := \frac{1}{2}\sum_{i=1}^{n}\left\{\frac{\partial M_{ki}}{\partial q_j}(q) - \frac{\partial M_{ij}}{\partial q_k}(q)\right\}\dot{q}_i. \tag{14}$$

The main motivation of (13) is that the *workless* forces of the differential of $H(x)$ in (11) can be decoupled and arranged into the matrix $E(x)$ in (13). Note that this decoupling is possible without a GCT as in Dirksz & Scherpen (2010), Romero et al. (2015). The map $u \mapsto y$ remains to be passive for the alternative system's form in (13) with the total energy in (12) as the storage function.

3.1 Tracking controller design using the v-CBC method

In this part, we follow the steps of the v-CBC method described in Section 2.3; one step per subsection. All the proofs are presented in Appendix A.

3.1.1 Step 1: Design of the virtual system

Following Definition 3, a virtual control system associated to the original system in (13) (equivalently (11)) is

$$\begin{aligned}\dot{x}_v &= \begin{bmatrix} 0_n & I_n \\ -I_n & -(E(x)+D(x)) \end{bmatrix} \begin{bmatrix} \frac{\partial H_v}{\partial q_v}(x_v,x) \\ \frac{\partial H_v}{\partial p_v}(x_v,x) \end{bmatrix} + \begin{bmatrix} 0_n \\ I_n \end{bmatrix} u_v \\ y_v &= \begin{bmatrix} 0_n & I_n \end{bmatrix} \begin{bmatrix} \frac{\partial H_v}{\partial q_v}(x_v,x) \\ \frac{\partial H_v}{\partial p_v}(x_v,x) \end{bmatrix},\end{aligned} \tag{15}$$

with state $x_v = (q_v, p_v) \in \mathcal{X}$ and parametrized by the solution $x = (q, p)$ of system (13) and

$$H_v(x_v, x) = \frac{1}{2}p_v^\top M^{-1}(q)p_v + P(q_v). \tag{16}$$

It is straightforward to verify that if $u_v = u$ and $x_v = x$, then we recover (13). The map $u_v \mapsto y_v$ is also passive with storage function (16) for any x.

3.1.2 Step 2: Design a controller for the virtual system

Proposition 1 *Consider a smooth reference position trajectory $q_d(t) \in \mathcal{Q}$ for system (15). Let us introduce the following error coordinates*

$$\tilde{x}_v := \begin{bmatrix} \tilde{q}_v \\ \sigma_v \end{bmatrix} = \begin{bmatrix} q_v - q_d(t) \\ p_v - p_r(\tilde{q}_v, t) \end{bmatrix}, \tag{17}$$

where the auxiliary momentum reference p_r is given by

$$p_r(\tilde{q}_v, t) := M(q)(\dot{q}_d - \phi(\tilde{q}_v)), \tag{18}$$

with function $\phi : \mathcal{Q} \to T_q\mathcal{Q}$ satisfying $\phi(0_n) = 0_n$ and

$$\dot{\Pi}(\tilde{q}_v) - \Pi(\tilde{q}_v)\frac{\partial \phi}{\partial \tilde{q}_v} - \frac{\partial \phi}{\partial \tilde{q}_v}^\top \Pi(\tilde{q}_v) \leq -2\beta_{q_v}\Pi(\tilde{q}_v), \tag{19}$$

for all $\tilde{q}_v$, with $\beta_{q_v} > 0$, and $\Pi(\tilde{q}_v)$ a positive definite metric tensor. Then, the system (15) in closed-loop with

$$\begin{aligned} u_v(x_v,x,t) &= u_v^{ff}(x_v,x,t) + u_v^{fb}(x_v,x,t), \\ u_v^{ff}(x_v,x,t) &= \dot{p}_r + \frac{\partial P}{\partial q_v}(q_v) + \big[E(x)+D(x)\big]M^{-1}(q)p_r, \\ u_v^{fb}(x_v,x,t) &= -\int_0^{\tilde{q}_v}\Pi(\varrho)d\varrho - K_d M^{-1}(q)\sigma_v, \end{aligned} \tag{20}$$

given by the equations

$$\dot{x}_v = \begin{bmatrix} -\phi(\tilde{q}_v) + M^{-1}(q)\sigma_v \\ -\int_0^{\tilde{q}_v}\Pi d\xi - [(E+D)(x) + K_d]M^{-1}\sigma_v \end{bmatrix}, \tag{21}$$

is contractive with differential Lyapunov function

$$V(\tilde{x}_v, \delta\tilde{x}_v, x) = \frac{1}{2}\delta\tilde{x}_v^\top \begin{bmatrix} \Pi(\tilde{q}_v) & 0_n \\ 0_n & M^{-1}(q) \end{bmatrix}\delta\tilde{x}_v, \tag{22}$$

for every $x(t)$, where $K_d > 0$ is a symmetric matrix gain.

One can immediately check that the closed-loop system satisfying the hypotheses in Proposition 1 is EIS. Analytically finding a non-constant contraction metric $\Pi(\tilde{q}_v)$ in (19) may be difficult (Kawano & Ohtsuka 2017). However, the problem can be simplified by taking a constant metric Π or using a numerical approach as in (Manchester & Slotine 2014, Sec.6). Alternatively, one can use the logarithmic measure (Sontag 2010) as in Reyes-Báez et al. (2020). The existence of the integral in (20) is guaranteed by the smoothness of all functions. A sufficient condition to compute the integral analytically is that the i-th row of $\Pi(\tilde{q}_v)$ is a conservative vector field for $i \in \{1, \dots, n\}$. With this the integrand becomes an exact differential.

3.1.3 Step 3: The controller for the original system

Corollary 1 *Consider the controller in* (20). *Then, all the solutions of the original mechanical pH system* (11) *in closed-loop with the control given by*

$$u(x, x, t) = u_v^{ff}(x, x, t) + u_v^{fb}(x, x, t) \tag{23}$$

converge exponentially to the reference trajectory $x_d(t)$ *with convergence rate given by*

$$\beta = \min\{\beta_{q_v}, \lambda_{\min}\{D(x) + K_d\}\lambda_{\min}\{M^{-1}(q)\}\}. \tag{24}$$

3.2 Properties of the closed-loop virtual system

Corollary 2 *Under the hypotheses of Proposition 1, define the contraction metric as* $\Pi(\tilde{q}_v) = \frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)$. *Then the closed-loop virtual system in* (21) *takes the form*

$$\dot{\tilde{x}}_v = [J_v(x) - R_v(x)] \frac{\partial \tilde{H}}{\partial \tilde{x}_v}(\tilde{x}_v, x) \tag{25}$$

where the matrices $J_v(x)$ *and* $R_v(x)$ *are given by*

$$J_v = \begin{bmatrix} 0_n & I_n \\ -I_n & -S_H \end{bmatrix}, R_v = \begin{bmatrix} I_n & 0_n \\ 0_n & (D + K_d - \frac{1}{2}\dot{M}) \end{bmatrix}, \tag{26}$$

and the x*-parametrized Hamiltonian-like function is*

$$\tilde{H}(\tilde{x}_v, x) = \frac{1}{2}\sigma_v^\top M^{-1}(q)\sigma_v + \int_{0_n}^{\tilde{q}_v} \phi(\varrho)\, d\varrho. \tag{27}$$

Moreover, if (20) *is modified as follows*

$$u_v = u_v^{ff}(x_v, x, t) + u_v^{fb}(x_v, x, t) + \omega, \tag{28}$$

with an external input ω, *then* $\omega \mapsto \tilde{y}_{\sigma_v} = \frac{\partial \tilde{H}}{\partial \sigma_v}(\tilde{x}_v, q)$ *is a passive map with storage function given by* (27).

The (line) integral term in (27), denoted by $\tilde{P}_v(\tilde{q}_v)$, acts as a potential energy-like function. Hence, the contraction rate in (24) is directly related to the Hessian of $\tilde{P}_v(q_v)$. When $(\tilde{q}_v, \sigma_v) = (\tilde{q}, \sigma)$, system (25) resembles the drift vector field of system (13), and therefore of the pH system (11). On the other hand, the second result of Corollary 2 shows that the closed-loop system with (28) is simultaneously passive and contractive (when $\omega = 0_n$). This, however, does not imply that the system is incrementally passive (van der Schaft 2017, Def. 4.7.1). Also, when $(\tilde{q}_v, \sigma_v) = (\tilde{q}, \sigma)$ the controller (28) is a PBC controller that solves the tracking problem for system (13), where $\tilde{P}_v(q_v)$ shapes the potential energy, and the sliding variable σ_v shapes the kinetic energy. This is possible because p_r in (18) adds an inner feedback loop, *mutatis mutandis*, v_r in Slotine & Li (1987).

Remark 1 *For set-point regulation, i.e., for a constant reference* q_d *in Proposition 1, our result is limited to fully-actuated mechanical systems. Whereas the works of Ortega et al. (2002), Romero et al. (2015), Yaghmaei & Yazdanpanah (2017) are applicable to a larger class of mechanical systems. However, we have presented an extension of Proposition 1 in Reyes-Báez et al. (2020) for a class of underactuated mechanial systems.*

Taking $\Pi(\tilde{q}_v) = \frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)$ restricts the functions $\phi(\cdot)$ that must satisfy (19). This condition is relaxed below.

Corollary 3 *Consider system* (15) *in closed-loop with* (28). *Let* $\Pi(\tilde{q}_v)$ *and* $\phi(\tilde{q}_v)$ *be such that*

$$\frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)\Pi_{\tilde{q}_v}^{-1}(\tilde{q}_v, t) = \left(\frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)\Pi_{\tilde{q}_v}^{-1}(\tilde{q}_v, t)\right)^\top. \tag{29}$$

Then the closed-loop variational dynamics is

$$\begin{aligned} \delta\dot{x}_v &= [J_v(x) - R_v(x)] \frac{\partial^2 H_v}{\partial x_v^2}(x_v, x)\delta x_v + g\delta\omega, \\ \delta y_v &= g^\top \frac{\partial^2 H_v}{\partial x_v^2}(x_v, x)\delta x_v, \end{aligned} \tag{30}$$

with

$$\begin{aligned} \frac{\partial^2 H_v}{\partial \tilde{x}_v^2}(\tilde{x}_v, x) &= \frac{\partial^2 V}{\partial \tilde{x}_v^2}(\tilde{x}_v, \delta\tilde{x}_v, x), \\ R_v(\tilde{x}_v, x) &= \text{diag}\left\{\frac{\partial \phi}{\partial \tilde{q}_v}\Pi_{\tilde{q}_v}^{-1}, D + K_d - \frac{1}{2}\dot{M}\right\}, \\ J_v(\tilde{x}_v, x) &= \begin{bmatrix} 0_n & I_n \\ -I_n & -S_H(x) \end{bmatrix}, \quad g = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}. \end{aligned} \tag{31}$$

It is clear that under Corollary 2, the condition in (29) holds, but no the converse implication. Interestingly, the form of (30) resembles the form of the variational dynamics of the pH system in (11). This does not imply that (30) is the variational dynamics of a pH system.

4 Case study: Tracking of a planar RR robot

Consider a 2-DoF planar robot from Quanser Consulting Inc. (2008), whose parameters are given in the Table 1.

Table 1
Robot parameters and controller gains.

Parameter	Value	Parameter	Value	Parameter	Value
m_1	$1.510kg$	I_1	$.039kgm^2$	m_2	$0.873kg$
I_2	$.0081kgm^2$	r_1	$.159m$	r_2	$.055m$
ℓ_1	$.343m$	ℓ_2	$.267m$	D_1	$0.8\ Ns/m$
D_2	$0.55\ Ns/m$	Π	$\text{diag}\{7,25\}$	K_d	$\text{diag}\{0.2,0.1\}$

The robot is modeled as the pH system in (11) with $q = [q_1, q_2]^\top$, $p = [p_1, p_2]^\top$. The inertia matrix is

$$M(q) = \begin{bmatrix} a_1 + a_2 + 2b\cos(q_2) & a_2 + b\cos(q_2) \\ a_2 + b\cos(q_2) & a_2 \end{bmatrix}, \quad (32)$$

where the constants $a_1 := m_1 r_1^2 + m_2 \ell_1^2 + I_1$ and $a_2 := m_2 r_2^2 + I_2; b := m_2 \ell_1 r_2$, with ℓ_i the length of the link i, and r_i the distance from the joint to the center of gravity of the link i; for $i = 1, 2$. The matrix $E(x)$ in (13) is

$$E(x) = b\sin(q_2) \begin{bmatrix} \dot{q}_2 & -\dot{q}_1 \\ \dot{q}_1 + \dot{q}_2 & 0 \end{bmatrix}_{\dot{q}=M^{-1}(q)p}. \quad (33)$$

4.1 Controller construction

Consider the following operators acting on $w \in \mathbb{R}^p$

$$\text{Tanh}(w) := \Big[\tanh(w_1), \ldots, \tanh(w_p)\Big]^\top \in \mathbb{R}^p,$$
$$\text{SECH}(w) = \begin{bmatrix} \text{sech}(w_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \text{sech}(w_p) \end{bmatrix} \in \mathbb{R}^{p\times p}. \quad (34)$$

The controller to be constructed is an example of Corollary 3. For illustration purposes, let Π to be constant. With $\phi(\tilde{q}_v) = \Pi \cdot \text{Tanh}(\tilde{q}_v)$, condition in (19) becomes

$$-2\Pi^2 \cdot \text{SECH}^2(\tilde{q}_v) \leq -\beta_{q_v}\Pi, \quad (35)$$

as the matrix product $\Pi \cdot \text{SECH}(\tilde{q}_v)$ satisfies (29). Note that $0_n < \text{SECH}(\tilde{q}_v) \leq I_n$ due to $\text{sech}(\cdot) \in (0, 1]$, and [4] $\lambda_{\min}(\Pi) I_n \leq \Pi \leq \lambda_{\max}(\Pi) I_n$. Then, it follows that

$$-2\Pi^2 \cdot \text{SECH}^2(\tilde{q}_v) \leq -2\lambda_{\min}(\Pi^2)\lambda_{\min}(\text{SECH}^2(\tilde{q}_v)) I_n. \quad (36)$$

Therefore, the contraction condition in (35) holds with

$$\beta_{\tilde{q}_v} := 2\lambda_{min}(\Pi^2) \cdot \lambda_{min}(\text{SECH}^2(\tilde{q}_v))/\lambda_{max}(\Pi). \quad (37)$$

The controller gains Π and K_d are given in Table 1.

[4] $\lambda_{\min}(\cdot)$ (resp. $(\lambda_{\max}(\cdot))$ denotes the minimum (rep. maximum) eigenvalue of its matrix argument.

4.2 Experimental evaluation

The reference $q_d(t)$ is given by $q_{1d}(t) = \sum_{k=0}^{18} a_{(18-k)} t^k$ and $q_{2d}(t) = \sum_{j=0}^{18} b_{(18-j)} t^j$, where $a_{(18-k)}$ and $b_{(18-j)}$, for $k, j \in \{0, 1, \ldots, 18\}$, are [5] such that the $q_{1d}(t)$ and $q_{2d}(t)$ are as in the first plot of Figure 1.
The experimental performance of the original closed-loop system is shown in Figure 1, where the first plot presents the reference trajectories for each joint versus the measured positions. The second and third plots show the error performance. The last plot shows the control.

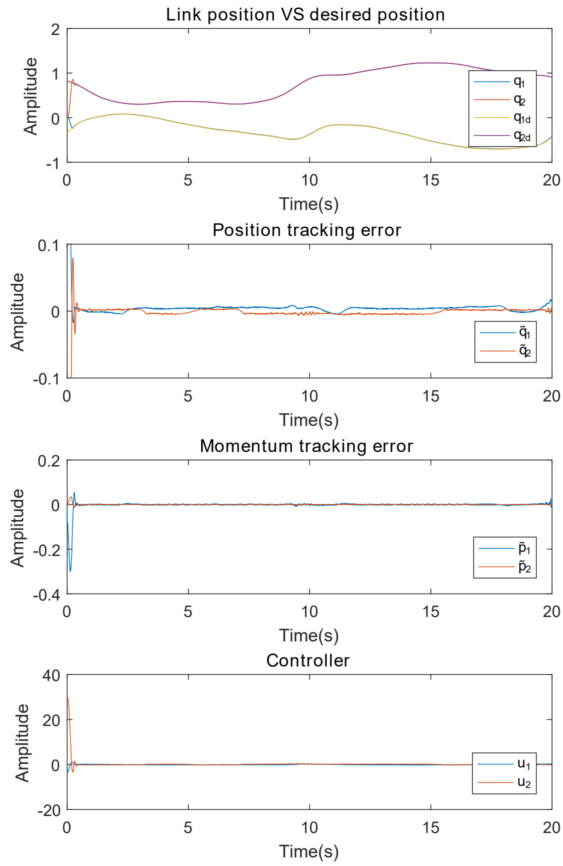

Fig. 1. Performance of the $(\Lambda, K_d, \Lambda\text{Tanh}(\cdot))$-controller.

The practical convergence of the signals in Figure 1 is mainly attributed to the accuracy of the encoder. In order to implement (28), we need the real-time measurement of q and $p = M^{-1}(q)\dot{q}$. As our experimental setup Quanser Consulting Inc. (2008) is not equipped with a momentum (or velocity) sensor, we rely on a (filtered) numerical approximation based on the available position measurement. This approximation also introduces noise in the feedback action due to the numerical differentiation, which is not contemplated in the robustness property of contractive systems of Lemma 1.

[5] The coefficients are in DOI: 10.13140/RG.2.2.29652.42880.

5 Conclusions

In this work, we have proposed a family of v-CBC schemes that solve the trajectory tracking control problem of fully-actuated mechanical systems in the pH framework. The closed-loop virtual system exhibits a number of structural properties by imposing sufficient conditions on the contraction metric $\Pi(\cdot)$ and $\phi(\cdot)$.
By exploiting the systems' structure, the proposed design procedure is simplified in comparison to the other methods in the literature that require an intermediate change of coordinates to do same.
We have applied the design procedure to construct a novel controller for a 2-DoF planar robot. It results in a PD + feedforward-like type of controller that includes feedback and feedforward actions, and an extra inner control loop that acts as a nonlinear contractive filter. The experimental results validate the design method.

Acknowledgements

The first author thanks Prof. Romeo Ortega, Dr. Alejandro Donaire, Dr. José Guadalupe Romero, and Dr. Pablo Borja for their insightful comments.

A Proofs

A.1 Proof of Proposition 1

For the first step, let us consider the position dynamics in (15) with $y_{\tilde{q}_v} = q_v - q_d(t)$ as output and p_v as an artificial "input". Define the control-like input as $p_v = \sigma_v + p_r$, where σ_v is a new state and p_r as in (18). Substituting this in the q_v-dynamics of (15) results in

$$\begin{cases} \dot{q}_v = M^{-1}(q_d)p_d - \phi(q_v - q_d) + M^{-1}(q)\sigma_v, \\ y_{\tilde{q}_v} = q_v - q_d(t), \end{cases} \tag{A.1}$$

whose prolonged system, in coordinates (17), is given by

$$\tilde{\Sigma}^\delta_{\sigma_v} : \begin{cases} \dot{\tilde{q}}_v = -\phi(\tilde{q}_v) + M^{-1}(q)\sigma_v \\ \delta\dot{\tilde{q}}_v = -\frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)\delta\tilde{q}_v + M^{-1}(q)\delta\sigma_v \\ y_{\tilde{q}_v} = \tilde{q}_v, \\ \delta y_{\tilde{q}_v} = \delta\tilde{q}_v. \end{cases} \tag{A.2}$$

Now, let the function

$$W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) = \frac{1}{2}\delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)\delta\tilde{q}_v. \tag{A.3}$$

be a candidate differential Lyapunov function. Then, the time derivative of (A.3) along solutions of $\tilde{\Sigma}^\delta_\sigma$ is

$$\dot{W}_{\tilde{q}_v} = \frac{1}{2}\delta\tilde{q}_v^\top \left[\dot{\Pi}_{\tilde{q}_v} - \Pi_{\tilde{q}_v}\frac{\partial \phi}{\partial \tilde{q}_v} - \frac{\partial \phi^\top}{\partial \tilde{q}_v}\Pi_{\tilde{q}_v}\right]\delta\tilde{q}_v + \delta\overline{y}_{\tilde{q}_v}^\top \delta\sigma_v. \tag{A.4}$$

where $\delta\overline{y}_{\tilde{q}_v}^\top = \delta\tilde{q}_v^\top \Pi_{\tilde{q}_v} M^{-1}(q)$. By (19), it follows that

$$\dot{W}_{\tilde{q}_v} \leq -2\beta_{q_v}(\tilde{q}_v, t)W_{\tilde{q}_v} + \delta\overline{y}_{\tilde{q}_v}^\top \delta\sigma_v. \tag{A.5}$$

Hence, system (A.1) is strictly differentially passive with differential input-output pair $(\delta\sigma_v, \delta\overline{y}_{\tilde{q}_v})$ and differential storage function (A.3). This implies contraction when $\delta\sigma_v = 0_n$ and it implies convergence to $\tilde{q}_v = 0_n$ if $\sigma = 0_n$.
For the second step, similar as before, let us consider now the whole system (15) and take $y_{\sigma_v} = p_v - p_r$ as its output. In the error coordinate (17), $y_{\sigma_v} = \sigma_v$ and system (15) are expressed as a system that is composed of the $\tilde{q}_v$ system in (A.2) and

$$\dot{\sigma}_v = -\frac{\partial P}{\partial q_v} - [E + D]\, M^{-1}(\sigma_v + p_r) + u - \dot{p}_r, \tag{A.6}$$

where u is given by (20). Direct substitution of the control action $u_v^{ff}(x_v, x, t)$ in (28) yields

$$\dot{\sigma}_v = -\left[E(x) + D(x)\right] M^{-1}(q)\sigma_v + u_v^{fb}. \tag{A.7}$$

Notice that with last substitution, $\sigma_v = 0_n$ is imposed as a particular solution of (A.7) when $u_v^{fb} = 0_n$, as desired. Thus, the prolonged system of (15), in error coordinate (17), is a system that is composed of (A.2) and

$$\tilde{\Sigma}^\delta_{u_v^{fb}} : \begin{cases} \dot{\sigma}_v = -\left[E + D\right] M^{-1}\sigma_v + u_{fbv}, \\ \delta\dot{\sigma}_v = -\left[E + D\right] M^{-1}\delta\sigma_v + \delta u_{fbv}, \\ y_{\sigma_v} = \sigma_v, \\ \delta y_{\sigma_v} = \delta\sigma_v. \end{cases} \tag{A.8}$$

Let us consider (22) as a candidate differential Lyapunov function for the complete prolonged system $\tilde{\Sigma}^\delta_{\sigma_v}$-$\tilde{\Sigma}^\delta_{u_{fbv}}$ and substitute the control action u_v^{fb} in (28). The derivative of (22) along prolonged system $\tilde{\Sigma}^\delta_{\sigma_v}$-$\tilde{\Sigma}^\delta_{u_v^{fb}}$ satisfies

$$\begin{aligned} \dot{W} \leq &-2\min\{\beta_{q_v}, \lambda_{\min}\{D + K_d\}\lambda_{\min}\{M^{-1}\}\}W \\ &+ \delta\overline{y}_{\sigma_v}^\top \delta\omega. \end{aligned} \tag{A.9}$$

The derivative (A.9) implies that system (15) in closed-loop with (20), given by the equation (21), is contractive with dL function (22). Therefore, the closed-loop system is contractive and $\tilde{x}_v$ exponentially converges to 0_n with

$$\beta = \min\{\beta_{q_v}, \lambda_{\min}\{D + K_d\}\lambda_{\min}\{M^{-1}(q)\}\}. \tag{A.10}$$

A.2 Proof of Corollary 1

The dynamics in (21) is a virtual system associated to the resulting closed-loop system in Corollary 1, with state $(\tilde{q}, \sigma) = (q - q_d, p - p_r(\tilde{q}, t))$ in error coordinates (17). That is, $(\tilde{q}_v, \sigma_v) = (\tilde{q}, \sigma)$ is a solution of (21). Clearly, $(\tilde{q}_v, \sigma_v) = (0_n, 0_n) = 0_{2n}$ is another solution of (21). Then, the conclusion follows by Proposition 1

A.3 Proof of Corollary 2

Consider system in (21) and take $\Pi_{\tilde{q}_v}(\tilde{q}_v, t) = \frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)$. Using matrices $J_v(x)$ and $R_v(x)$, and the Hamiltonian-like function (27), the system (21) can be written as in (25). To prove passivity, consider the modified input (28), and take (27) as the storage function.

A.4 Proof of Corollary 3

Compute the variational system of (21). Under the hypotheses of the corollary, the claim follows immediately.

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