# Angle Formation of Double Integrator with Bearing and Velocity Information

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# Angle formation of double integrator with bearing and velocity information

Ningbo Li, * Pablo Borja, * Arjan van der Schaft, *
Jacquelien M.A. Scherpen, * Liangming Chen **

* Jan C. Willems Center for Systems and Control, University of Groningen, The Netherlands (e-mail: ningbo.li@rug.nl, l.p.borja.rosales@rug.nl, a.j.van.der.schaft@rug.nl, and j.m.a.scherpen@rug.nl).
** School of Mechanical and Aerospace Engineering, Nanyang Technological University (e-mail: liangmingchen2018@gmail.com).

**Abstract:** This paper proposes a passivity-based approach using bearing and velocity information for a triangular formation control with the interaction topology constrained by angles. The controller framework is designed using virtual couplings on the relative measurements related to the edges. The different measurements associated with the edges are mapped by the measurement Jacobian, which is calculated by the time-evolution of the measurement. To avoid unavailable distance measurements in the control law, an estimator is designed based on port-Hamiltonian theory using bearing and velocity measurements. The stability analysis of the closed-loop system is provided and simulations are performed to illustrate the effectiveness of the approach.



*Keywords:* Angle formation, double integrator, port-Hamiltonian, distance estimator

## 1. INTRODUCTION

Recently, the passivity-based port-Hamiltonian (pH) approach has used for the design of formation controllers, such as, Vos et al. (2014), Stacey and Mahony (2015), Xu and Liang (2018). The advantages of this approach can be summarized as follows: on one hand, it allows for complex and heterogenous agent dynamics. For most of the existing literature where the agent is modeled as a single or double integrator and the measurement is limited to one kind of position, distance, and bearing, the pH approach can be applied to heterogenous systems where the agents are modeled as nonlinear dynamics with different kinds of measurements. In addition, it enables the flexibility and scalability of the network. Passivity-based decentralized controllers allow the agents to exert forces based on different types of information about their neighbors, such as relative position, distance and bearing.

In terms of the sensing capability, using partial information of the positions of agents requires less onboard sensors, which reduces the cost of hardware and introduces less measurement errors. Much research has been reported on this topic in recent years, such as Anderson et al. (2008), Cao et al. (2011) for distance measurement, Zhao et al. (2019), Trinh et al. (2018) for bearing measurement, and Chen et al. (2020), Jing et al. (2019) for angle measurement. In this paper, we study the case where the sensing capability of agents is based on bearing measurement and the interaction topology of agents is constrained by angles. Angle-based constraints are expressed by less information of the agents compared with position-, distance- and bearing-based approaches. Therefore, it is invariant to more group motions, such as, translation, rotation, scaling and reflection, which means the group of agents can achieve these corresponding maneuvers while satisfying angle-based constraints.

The control objectives are achieved by virtual couplings where the virtual springs determine the formation by shaping the energy function of the network, while the virtual dampers shape the transient response by injecting damping. However, the resulting control law in passivity-based approaches usually contains a negative gradient of the energy function, which implies that the agents need the full information of relative position even if the sensing capability and the interaction topology of agents are both only bearing or distance. To solve this problem, we extend the passive adaptive compensator proposed for bearing formation control in Stacey and Mahony (2015) to estimate the unavailable distance information by relative velocity.

The contributions of our approach can be summarized in two points.

- **(i)** We propose a control law for double integrator dynamics based on the virtual mechanical couplings and pH theory. Compared with the control law in Chen et al. (2020) which uses an intuitive design for single integrator dynamics, ours does better in control performance and is more suited to analyze the complex dynamics of the system, since gradient-like control law can avoid undesired equilibria.
- **(ii)** Compared with the gradient-like control law in Jing et al. (2019) which requires both bearing and distance measurement, we design an estimator to avoid the distance measurement by pH theory.

The rest of the paper is structured as follows. Some preliminaries and the problem formulation are introduced in Section 2. The control architecture is developed in Section 3 and the stability analysis is given in Section 4. Simulations are provided in Section 5, and concluding remarks appear in Section 6.


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## 2. PRELIMINARIES AND PROBLEM FORMULATION

### 2.1 Preliminaries

We consider the triangular formation as in Fig.1. For the link $k$ between the agent 1 and the agent 2, we have

$$z_k = q_1 - q_2, \tag{1}$$

where $q_1, q_2 \in \mathbb{R}^2$ are positions of the agents 1 and 2, $z_k \in \mathbb{R}^2$ is the relative position associated with the link $k$.

Note that according to different kinds of sensors, we have only access to partial information of position measurement, which can be distance or bearing. We define the distance and bearing between the agents 1 and 2 as

$$r_k = ||z_k||, \quad s_k = \frac{z_k}{||z_k||}. \tag{2}$$

We define the general form of partial measurement as $y_k \in \mathbb{Y}_k$. $\mathbb{Y}_k$ is the sensor space. The time-evolution of $y_k$ is given by

$$\dot{y}_k = L_{y_k}\dot{z}_k, \tag{3}$$

where $L_{y_k}(z_k) = \frac{\partial y_k}{\partial z_k}(z_k)$ is the measurement Jacobian. For a bearing measurement $s_k$ and a distance measurement $r_k$, we have the bearing Jacobian and distance Jacobian , respectively, as

$$L_{s_k} = \frac{1}{r_k}(I_2 - s_k {s_k}^T) \in \mathbb{R}^{2\times 2} \tag{4}$$

$$L_{r_k} = s_k^T \in \mathbb{R}^{1\times 2} \tag{5}$$

Define $y_k^*$ as the set of parameters that describe the desired formation in terms of the available sensor measurements $y_k$. The objective is to design a controller that ensures that the following error (6) is zero.

$$\tilde{y}_k = y_k - y_k^*. \tag{6}$$

The dynamics of edges are associated with virtual couplings. The controller assigns virtual couplings between the robots, where the virtual springs determine the formation shape, while the virtual dampers shape the transient response. The input to the control system $\omega_k \in T_{y_k}\mathbb{Y}_k$ is the velocity, where $T_{y_k}\mathbb{Y}_k$ is the tangent space of $\mathbb{Y}_k$.

Define the corresponding Hamiltonian as $H_k(\tilde{y}_k) = \frac{1}{2}\tilde{y}_k^T c_k \tilde{y}_k$, which denotes the energy stored in virtual spring $k$. $c_k$ is the virtual spring constant. The dynamics of the virtual coupling $k$ are given byvan der Schaft and Jeltsema (2014)

$$\begin{aligned} \dot{\tilde{y}}_k &= \omega_k \\ \gamma_k &= \frac{\partial H_k}{\partial \tilde{y}_k} + d_k\omega_k \end{aligned} \tag{7}$$

where the output $\gamma_k$ of the control system corresponds to the force exerted by the virtual spring. The first term of $\gamma_k$ is the spring term and the second one is the damping term. $d_k$ is the corresponding virtual dissipation matrix.

Note that the dynamics of the virtual coupling are expressed in the sensor space. However, they can be mapped to $\mathbb{R}^2$ by the measurement Jacobian

$$\epsilon_k = L_{y_k}^T \gamma_k, \tag{8}$$

where $\epsilon_k$ is the virtual force in $\mathbb{R}^2$.

### 2.2 Problem Formulation

We consider the triangular formation determined only by angle constraints. As shown in Fig.1, $1, 2, 3$ are agents. $i, j, k$ are the the edges. Angle $\theta$ and $\phi$ are the angles to be controlled. Since the summation of three angles is $\pi$, if angle $\theta$ and angle $\phi$ are both controlled to be desired one, the third angle also satisfy the angle constraint. For the more complicated cases, it is related the angle rigidity theory. Here we only consider this simple case angle rigid.

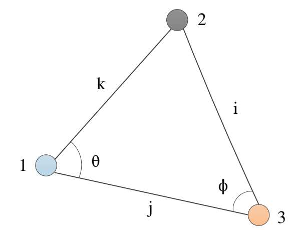


Fig. 1. Triangular formation

We assume the dynamics of the agents are given by a simple double integrator model in $\mathbb{R}^2$, which are given in Hamiltonian framework as

$$\begin{aligned} \begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} &= \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H_i}{\partial q_i} \\ \dfrac{\partial H_i}{\partial p_i} \end{pmatrix} + \begin{pmatrix} 0 \\ I_2 \end{pmatrix} U_i, \\ H_i(p_i) &= \frac{1}{2m_i} p_i^T p_i, \quad Y_i = \frac{\partial H_i}{\partial p_i}(p_i). \end{aligned} \tag{9}$$

Where $q_i = (q_{x_i}, q_{y_i}), i = 1, 2, 3$ is the position of agent $i$, $p_i = (p_{x_i}, p_{y_i}) = (m_i \dot{q}_{x_i}, m_i \dot{q}_{y_i})$ is the momentum, $m_i$ is the mass. $U_i = (U_{x_i}, U_{y_i})$ and $Y_i = (Y_{x_i}, Y_{y_i})$ are the input and output respectively. $I_2$ is two-dimensional identity matrix. $H_i$ is the Hamiltonian.

We assume that each agent has only the bearing sensors and linear velocity sensors. In addition, we assume the communication topology is connected, i.e. each agent has access to the bearing measurement of the non-adjacent edge. For example, the agent 1 has access to the bearing measurement $s_i$. Since the inner constraints of the agents in formation are give by angles, we use the cosine of the angle to represent the angle measurement, which can be easily calculated by bearing measurement. For angle $\theta$, it is given by

$$\cos\theta = s_k^T s_j \tag{10}$$

The objective of this paper is to design a controller using only bearing and linear velocity measurements that ensures the group of three agents modeled by (9) achieve a formation constrained by angles.

## 3. CONTROL DESIGN

### 3.1 Controller of the agent 1 for the angle $\theta$

Now we consider the controller of agent $a$. since the moving of the agent 1 affects both the angle $\theta$ and $\phi$, the controller of the agent 1 consist of two parts. One is to satisfy the constraint of the angle $\theta$, the other part is to satisfy the constraint of the angle $\phi$.

We assume the agent can measure the bearing and the inner constraints of the agents in formation are angles. We use cosine of the angle $\theta$ to represent the angle measurement. The time-evolution of the angle measurement can be derived as:

$$\begin{aligned}\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}t} &= (L_{s_k}\dot{z}_k)^T s_j + s_k^T(L_{s_j}\dot{z}_j) \\ &= -(s_j^T L_{s_k} + s_k^T L_{s_j})\dot{q}_1 + s_j^T L_{s_k}\dot{q}_2 + s_k^T L_{s_j}\dot{q}_3 \\ &= L_{\theta 1}\dot{q}_1 + L_{\theta 2}\dot{q}_2 + L_{\theta 3}\dot{q}_3\end{aligned} \tag{11}$$

Define $L_{\theta 1}, L_{\theta 2}, L_{\theta 3}$ as the angle Jacobian mapping from position of the agents 1,2, and 3, respectively, to the angle $\theta$. The expressions are given as follows

$$\begin{aligned}L_{\theta 1} &= -(s_j^T L_{s_k} + s_k^T L_{s_j}) \\ L_{\theta 2} &= s_j^T L_{s_k} \\ L_{\theta 3} &= s_k^T L_{s_j}\end{aligned} \tag{12}$$

Similarly, the angle Jacobian mapping from position of the agents 1,2, and 3, respectively, to the angle $\phi$ are given as

$$\begin{aligned}L_{\phi 3} &= -s_i^T L_{s_j} \\ L_{\phi 3} &= -s_j^T L_{s_i} \\ L_{\phi 3} &= s_j^T L_{s_i} + s_i^T L_{s_j}\end{aligned} \tag{13}$$

The control aim is to design a controller to ensure the cosine of the angle $\theta$, given by $s_k^T s_j$, to converge to the desired value $(s_k^T s_j)^*$. Hence, we define the error by

$$\widetilde{(s_k^T s_j)} = (s_k^T s_j) - (s_k^T s_j)^*. \tag{14}$$

In order to ensure that the system converges to the desired point, i.e., the error converges to zero, it is necessary to assign a potential energy of the angle to the closed-loop system. To this end, we propose the following Hamiltonian function as

$$H_{\theta 1} = \frac{1}{2}c_{\theta 1}\widetilde{(s_k^T s_j)}^2, \tag{15}$$

where $c_{\theta 1} > 0$ is a constant.

The corresponding controller with spring term and damping term can be derived as

$$\begin{aligned}\dot{\widetilde{(s_k^T s_j)}} &= \omega_{\theta 1}, \\ \gamma_{\theta 1} &= \frac{\partial H_{\theta 1}}{\partial \widetilde{(s_k^T s_j)}} + d_{\theta 1}\omega_{\theta 1}.\end{aligned} \tag{16}$$

where $\omega_{\theta 1}$ denotes the input of the controller. $d_{\theta 1} > 0$ is a positive constant.

The $\gamma_{\theta 1}$ actually is the resulting virtual force in the space of angle measurement. According to the port-Hamiltonian theory van der Schaft and Jeltsema (2014), we define the force and velocity as effort and flow. Hence, the power of the port can be derived as

$$< \gamma_{\theta 1}|\frac{\mathrm{d}(s_k^T s_j)}{\mathrm{d}t} >= \gamma_{\theta 1}^T\frac{\mathrm{d}(s_k^T s_j)}{\mathrm{d}t}. \tag{17}$$

Here, we only consider the relation between the agent 1 and the angle $\theta$. To transform the power from angle measurement space to $\mathbb{R}^2$ space, we have

$$\begin{aligned}< \gamma_{\theta 1}|\frac{\mathrm{d}(s_k^T s_j)}{\mathrm{d}t} > &=< \gamma_{\theta 1}|L_{\theta 1}\dot{q}_1 > \\ &=< L_{\theta 1}^T\gamma_{\theta 1}|\dot{q}_1 > \\ &=< -L_{s_k}^T s_j\gamma_{\theta 1}|\dot{q}_1 > + < -L_{s_j}^T s_k\gamma_{\theta 1}|\dot{q}_1 > .\end{aligned} \tag{18}$$

The effort of the port in (18) relies on the distance information which is not measurable. In order to avoid distance measurement, we use the relative velocity measurement to estimate the unknown distance Duindam et al. (2009).

Note that the estimated distance is used, the angle Jacobian also needs to be modified. Therefore, the estimated angle Jacobian is given by

$$\begin{aligned}\hat{L}_{\theta 1} &= s_j^T\hat{L}_{\theta s_k} + s_k^T\hat{L}_{\theta s_j} \\ &= s_j^T\frac{1}{\hat{r}_{\theta k}}(I_2 - s_k{s_k}^T) + s_k^T\frac{1}{\hat{r}_{\theta j}}(I_2 - s_j{s_j}^T),\end{aligned} \tag{19}$$

where $\hat{r}_{\theta k}$ is the estimate of the edge $k$ using the measurement of the angle $\theta$, $\hat{r}_{\theta j}$ is the estimate of the edge $j$ using the measurement of the angle $\theta$. Correspondingly, $\hat{L}_{\theta s_k}, \hat{L}_{\theta s_j}$ are the estimated bearing Jacobian using the measurement of the angle $\theta$.

However if $\hat{L}_{\theta 1}$ is used to replace $L_{\theta 1}$ in the right side of (18), the equation is not satisfied because the effort $\gamma_{\theta 1}$ corresponds to the real flow $\dot{(s_k^T s_j)}$ in the angle space. It causes the discrepancy of the power through the virtual coupling due to the error between the estimated distance and the real unknown distance. Define the distance error as: $\bar{r}_{\theta k} = \hat{r}_{\theta k} - r_k, \bar{r}_{\theta j} = \hat{r}_{\theta j} - r_j$. To calculate the estimated effort in $\mathbb{R}^2$, we have

$$-(\hat{L}_{\theta s_k}^T s_j + \hat{L}_{\theta s_j}^T s_k)\gamma_{\theta 1} = -L_{s_k}^T s_j\alpha_{\theta k} - L_{s_j}^T s_k\alpha_{\theta j}, \tag{20}$$

where $\alpha_{\theta k} = \frac{r_k}{\hat{r}_{\theta k}}\gamma_{\theta 1}$ is the estimated effort related to $\hat{r}_{\theta k}$ and $\alpha_{\theta j} = \frac{r_j}{\hat{r}_{\theta j}}\gamma_{\theta 1}$ is the estimated effort related to $\hat{r}_{\theta j}$.

Furthermore, considering the ports in different spaces, we have that

$$\begin{aligned}< \hat{L}_{\theta 1}^T\gamma_{\theta 1}|\dot{q}_1 > &=< -L_{s_k}^T s_j\alpha_{\theta k} - L_{s_j}^T s_k\alpha_{\theta j}|\dot{q}_1 > \\ &=< \alpha_{\theta k}|\frac{\mathrm{d}(s_k^T s_j)}{\mathrm{d}t} > + < \alpha_{\theta j}|\frac{\mathrm{d}(s_k^T s_j)}{\mathrm{d}t} > .\end{aligned} \tag{21}$$

Comparing (18) with (21), the discrepancy between the real effort and the estimated effort can be derived as

$$\begin{aligned}\beta_{\theta k} &= \alpha_{\theta k} - \gamma_{\theta 1} = -\frac{\bar{r}_k}{\hat{r}_{\theta k}}\gamma_{\theta 1}, \\ \beta_{\theta j} &= \alpha_{\theta j} - \gamma_{\theta 1} = -\frac{\bar{r}_j}{\hat{r}_{\theta j}}\gamma_{\theta 1}.\end{aligned} \tag{22}$$

The power of ports with $\beta_{\theta k}, \beta_{\theta j}$ as the efforts are given by

$$< \beta_{\theta k}| - s_j^T L_{s_k}\dot{q}_1 >, \quad < \beta_{\theta j}| - s_k^T L_{s_j}\dot{q}_1 > . \tag{23}$$

To account for the power associated with the ports in distance space, we define the corresponding Hamiltonian as

$$H_{\theta k} = \frac{1}{2}c_{\theta k}\bar{r}_{\theta k}^2, \quad H_{\theta j} = \frac{1}{2}c_{\theta j}\bar{r}_{\theta j}^2, \tag{24}$$

where $c_{\theta k}, c_{\theta j} > 0$ are constants.

The power of the ports in distance space are given by

$$\begin{aligned}&< \frac{\partial H_{\theta k}}{\partial \bar{r}_{\theta k}}|\dot{\bar{r}}_{\theta k} >=< c_{\theta k}\bar{r}_{\theta k}|\dot{\bar{r}}_{\theta k} >, \\ &< \frac{\partial H_{\theta j}}{\partial \bar{r}_{\theta j}}|\dot{\bar{r}}_{\theta j} >=< c_{\theta j}\bar{r}_{\theta j}|\dot{\bar{r}}_{\theta j} > .\end{aligned} \tag{25}$$

Since the energy is coordinate free, the power in angle space and distance space are the same. Therefore, comparing (23) and (25), we have

$$
\begin{aligned}
&< \beta_{\theta k}| - s_j^T L_{s_k}\dot{q}_1 > = < c_{\theta k}\bar{r}_{\theta k}|\dot{\bar{r}}_{\theta k} > \\
\Rightarrow \quad & c_{\theta k}\bar{r}_{\theta k}^T\dot{\bar{r}}_{\theta k} = -\frac{\bar{r}_k}{\hat{r}_{\theta k}}\gamma_{\theta 1}^T(-s_j^T L_{s_k}\dot{q}_1) \\
\Rightarrow \quad & \dot{\bar{r}}_{\theta k} = -\frac{\gamma_{\theta 1}^T}{c_{\theta k}\hat{r}_{\theta k}}(-s_j^T L_{s_k}\dot{q}_1).
\end{aligned} \tag{26}
$$

Similarly,

$$
\dot{\bar{r}}_{\theta j} = -\frac{\gamma_{\theta 1}^T}{c_{\theta j}\hat{r}_{\theta j}}(-s_k^T L_{s_j}\dot{q}_1). \tag{27}
$$

Furthermore, the dynamics of the estimators are given by

$$
\begin{aligned}
\dot{\hat{r}}_{\theta k} &= \dot{r}_k + \dot{\bar{r}}_{\theta k} \\
&= s_k^T\dot{z}_k - \frac{\gamma_{\theta 1}^T}{c_{\theta k}\hat{r}_{\theta k}}(-s_j^T L_{s_k}\dot{q}_1).
\end{aligned} \tag{28}
$$

Similarly,

$$
\begin{aligned}
\dot{\hat{r}}_{\theta j} &= \dot{r}_j + \dot{\bar{r}}_{\theta j} \\
&= s_j^T\dot{z}_j - \frac{\gamma_{\theta 1}^T}{c_{\theta j}\hat{r}_{\theta j}}(-s_k^T L_{s_j}\dot{q}_1).
\end{aligned} \tag{29}
$$

Note that we only use the information of relative velocity and bearing measurement in above estimators, while the information of distance measurement is not used.

The control law of the agent 1 for the angle $\theta$ is given by

$$
\begin{aligned}
U_{\theta 1} =& [\frac{1}{\hat{r}_{\theta k}}(I_2 - s_k s_k{}^T)^T s_j + \frac{1}{\hat{r}_{\theta j}}(I_2 - s_j s_j{}^T)^T s_k] \\
& \times [c_{\theta 1}\widetilde{(s_k^T s_j)} + d_{\theta 1}\dot{\widetilde{(s_k^T s_j)}}].
\end{aligned} \tag{30}
$$

*3.2 Controller of the agent 1 for the angle $\phi$*

Now we design the controller of the agent 1 to control the angle $\phi$. Define the corresponding Hamiltonian as

$$
H_{\phi 1} = \frac{1}{2}c_{\phi 1}\widetilde{(s_i^T s_j)}^2, \tag{31}
$$

where $c_{\phi 1} > 0$ is a constant. The controller with spring and damping term is given by

$$
\begin{aligned}
\dot{\widetilde{(s_i^T s_j)}} &= \omega_{\phi 1}, \\
\gamma_{\phi 1} &= \frac{\partial H_{\phi 1}}{\partial \widetilde{(s_i^T s_j)}} + d_{\phi 1}\omega_{\phi 1}.
\end{aligned} \tag{32}
$$

Where $\omega_{\phi 1}$ denotes the input of the controller. $d_{\phi 1} > 0$ is a constant.

Considering the ports in different spaces, we have that

$$
\begin{aligned}
& < \alpha_{\phi 1}|(-s_i^T L_{s_j}\dot{q}_1) > = < \hat{L}_{\phi 1}^T\gamma_{\phi 1}|\dot{q}_1 >, \\
& \hat{L}_{\phi 1} = s_i^T\hat{L}_{\phi s_k} = s_i^T\frac{1}{\hat{r}_{\phi k1}}(I_2 - s_k s_k{}^T).
\end{aligned} \tag{33}
$$

Where $\hat{L}_{\phi 1}$ is the estimated angle Jacobian mapping from position of the agent 1 to the angle $\phi$ and $\hat{L}_{\phi s_k}$ is the estimated bearing Jacobian using the measurement of the angle $\phi$. $\hat{r}_{\phi k1}$ is the estimated distance of the edge $k$ by the agent 1 using the measurement of the angle $\phi$.

Furthermore, taking the same steps as in Section 3.1, we have the following estimator as

$$
\dot{\hat{r}}_{\phi k1} = \dot{r}_k + \dot{\bar{r}}_{\phi k1} = s_k^T\dot{z}_k - \frac{\gamma_{\phi 1}^T}{c_{\phi k}\hat{r}_{\phi k1}}(-s_i^T L_{s_j}\dot{q}_1). \tag{34}
$$

where $c_{\phi k} > 0$ is a constant.

Correspondingly, the control law of the agent $a$ for the angle 2 is given by

$$
U_{\phi 1} = \frac{1}{\hat{r}_{\phi k1}}(I_2 - s_k s_k{}^T)^T s_i[c_{\phi 1}\widetilde{(s_i^T s_j)} + d_{\phi 1}\dot{\widetilde{(s_i^T s_j)}}] \tag{35}
$$

The power from the port 1 is given by

$$
< \hat{L}_{\theta 1}^T\gamma_{\theta 1} + \hat{L}_{\phi 1}^T\gamma_{\phi 1}|\dot{q}_1 > \tag{36}
$$

The control law of the agent 1 is given by

$$
U_1 = U_{\theta 1} + U_{\phi 1}. \tag{37}
$$

*3.3 Controllers for the agents 2 and 3*

Since the design process of the agents 2 and 3 is similar to the agent 1, we only give the conclusions here.

For the agent 3, there are two parts in the control law. The first part is to control the angle $\phi$, whose corresponding Hamiltonian and controller in angle space are given by

$$
H_{\phi 3} = \frac{1}{2}c_{\phi 3}\widetilde{(s_i^T s_j)}^2, \tag{38}
$$

$$
\begin{aligned}
\dot{\widetilde{(s_i^T s_j)}} &= \omega_{\phi 3}, \\
\gamma_{\phi 3} &= \frac{\partial H_{\phi 3}}{\partial \widetilde{(s_i^T s_j)}} + d_{\phi 3}\omega_{\phi 3}.
\end{aligned} \tag{39}
$$

Where $\omega_{\phi 3}$ denotes the input of the controller. $c_{\phi 3} > 0, d_{\phi 3} > 0$ are constants. The corresponding estimated angle Jacobian and distance estimators are given

$$
\hat{L}_{\phi 3} = s_j^T\frac{1}{\hat{r}_{\phi i}}(I_2 - s_i s_i{}^T) + s_i^T\frac{1}{\hat{r}_{\phi j}}(I_2 - s_j s_j{}^T), \tag{40}
$$

$$
\dot{\hat{r}}_{\phi i} = \dot{r}_i + \dot{\bar{r}}_{\phi i} = s_i^T\dot{z}_i - \frac{\gamma_{\phi 3}^T}{c_{\phi i}\hat{r}_{\phi i}}(s_j^T L_{s_i}\dot{q}_3), \tag{41}
$$

$$
\dot{\hat{r}}_{\phi j} = \dot{r}_j + \dot{\bar{r}}_{\phi j} = s_j^T\dot{z}_j - \frac{\gamma_{\phi 3}^T}{c_{\phi j}\hat{r}_{\phi j}}(s_i^T L_{s_j}\dot{q}_3). \tag{42}
$$

Where $c_{\phi i} > 0, c_{\phi j} > 0$ are constants.

The second part is to control the angle $\theta$, whose corresponding Hamiltonian and controller in angle space are given by

$$
H_{\theta 3} = \frac{1}{2}c_{\theta 3}\widetilde{(s_k^T s_j)}^2, \tag{43}
$$

$$
\begin{aligned}
\dot{\widetilde{(s_k^T s_j)}} &= \omega_{\theta 3}, \\
\gamma_{\theta 3} &= \frac{\partial H_{\theta 3}}{\partial \widetilde{(s_k^T s_j)}} + d_{\theta 3}\omega_{\theta 3}.
\end{aligned} \tag{44}
$$

Where $\omega_{\theta 3}$ denotes the input of the controller. $c_{\theta 3} > 0, d_{\theta 3} > 0$ are constants. The corresponding estimated angle Jacobian and distance estimators are given by

$$
\hat{L}_{\theta 3} = s_k^T\frac{1}{\hat{r}_{\theta i}}(I_2 - s_i s_i{}^T), \tag{45}
$$

$$
\dot{\hat{r}}_{\theta i3} = \dot{r}_i + \dot{\bar{r}}_{\theta i3} = s_i^T\dot{z}_i - \frac{\gamma_{\theta 3}^T}{c_{\theta i}\hat{r}_{\theta i3}}(s_k^T L_{s_j}\dot{q}_3). \tag{46}
$$

Where $c_{\theta i} > 0$ is a constant.

In general, the power from the port 3 is given by

$$
< \hat{L}_{\phi 3}^T\gamma_{\phi 3} + \hat{L}_{\theta 3}^T\gamma_{\theta 3}|\dot{q}_3 > \tag{47}
$$

Correspondingly, the controller of the agent 3 is given by

$$
\begin{aligned}
U_3 &= U_{\theta 3} + U_{\phi 3} \\
&= [\frac{1}{\hat{r}_{\phi i}}(I_2 - s_i {s_i}^T)^T s_j + \frac{1}{\hat{r}_{\phi j}}(I_2 - s_j {s_j}^T)^T s_i] \\
&[c_{\phi 3}\widetilde{(s_i^T s_j)} + d_{\phi 3}\dot{\widetilde{(s_i^T s_j)}}] + \\
&\frac{1}{\hat{r}_{\theta i3}}(I_2 - s_i {s_i}^T)^T s_k [c_{\theta 3}\widetilde{(s_k^T s_j)} + d_{\theta 3}\dot{\widetilde{(s_k^T s_j)}}]
\end{aligned} \tag{48}
$$

For the agent 2, there are also two parts in the control law. The first part is to control the angle $\theta$, whose corresponding Hamiltonian and controller in the angle space are given by

$$
H_{\theta 2} = \frac{1}{2} c_{\theta 2} \widetilde{(s_k^T s_j)}^2, \tag{49}
$$

$$
\begin{aligned}
\dot{\widetilde{(s_k^T s_j)}} &= \omega_{\theta 2}, \\
\gamma_{\theta 2} &= \frac{\partial H_{\theta 2}}{\partial \widetilde{(s_k^T s_j)}} + d_{\theta 2}\omega_{\theta 2}.
\end{aligned} \tag{50}
$$

Where $\omega_{\theta 2}$ denotes the input of the controller. $c_{\theta 2} > 0, d_{\theta 2} > 0$ are constants. The corresponding estimated angle Jacobian and distance estimators are given by

$$
\hat{L}_{\theta 2} = s_j^T \frac{1}{\hat{r}_{\theta i}}(I_2 - s_i {s_i}^T), \tag{51}
$$

$$
\dot{\hat{r}}_{\theta i2} = \dot{r}_i + \dot{\bar{r}}_{\theta i2} = s_i^T \dot{z}_i - \frac{\gamma_{\theta 2}^T}{c_{\theta i}\hat{r}_{\theta i2}}(s_j^T L_{s_k} \dot{q}_2). \tag{52}
$$

Note that the expressions (52) and (46) are both distance estimators of the edge $i$, but (52) is estimated by the agent 2, while (46) is estimated by the agent 3. The second part is to control the angle $\phi$, whose corresponding Hamiltonian and controller are given by

$$
H_{\phi 2} = \frac{1}{2} c_{\phi 2} \widetilde{(s_i^T s_j)}^2, \tag{53}
$$

$$
\begin{aligned}
\dot{\widetilde{(s_i^T s_j)}} &= \omega_{\phi 2}, \\
\gamma_{\phi 2} &= \frac{\partial H_{\phi 2}}{\partial \widetilde{(s_i^T s_j)}} + d_{\phi 2}\omega_{\phi 2}.
\end{aligned} \tag{54}
$$

Where $\omega_{\phi 2}$ denotes the input of the controller. $c_{\phi 2} > 0, d_{\phi 2} > 0$ are constants. The corresponding estimated angle Jacobian and distance estimators are given

$$
\hat{L}_{\phi 2} = s_j^T \frac{1}{\hat{r}_{\phi k}}(I_2 - s_k {s_k}^T), \tag{55}
$$

$$
\dot{\hat{r}}_{\phi k2} = \dot{r}_k + \dot{\bar{r}}_{\phi k2} = s_k^T \dot{z}_k - \frac{\gamma_{\phi 2}^T}{c_{\phi k}\hat{r}_{\phi k2}}(-s_j^T L_{s_i} \dot{q}_2). \tag{56}
$$

Note that the expressions (34) and (56) are both distance estimators of the edge $k$, but (34) is estimated by the agent 1, while (56) is estimated by the agent 2.

The power from the port 2 is given by

$$
< \hat{L}_{\theta 2}^T \gamma_{\theta 2} + \hat{L}_{\phi 2}^T \gamma_{\phi 2} | \dot{q}_2 > \tag{57}
$$

Correspondingly, the control law of the agent 2 is given by

$$
\begin{aligned}
U_2 &= U_{\theta 2} + U_{\phi 2} \\
&= \frac{1}{\hat{r}_{\theta i2}}(I_2 - s_i {s_i}^T)^T s_j [c_{\theta 2}\widetilde{(s_k^T s_j)} + d_{\theta 2}\dot{\widetilde{(s_k^T s_j)}}] \\
&+ \frac{1}{\hat{r}_{\phi k2}}(I_2 - s_k {s_k}^T)^T s_j [c_{\phi 2}\widetilde{(s_i^T s_j)} + d_{\phi 2}\dot{\widetilde{(s_i^T s_j)}}]
\end{aligned} \tag{58}
$$

## 4. STABILITY ANALYSIS

The main result of this paper is given by the following theorem.

*Theorem 1.* Consider the three agents modeled as in Section 2.2. Moreover, assume that the matrix

$$
L := \begin{bmatrix} \hat{L}_{\theta 1} & \hat{L}_{\phi 1} \\ \hat{L}_{\theta 2} & \hat{L}_{\phi 2} \\ \hat{L}_{\theta 3} & \hat{L}_{\phi 3} \end{bmatrix}. \tag{59}
$$

is full column rank. Hence, using the control law (37) for the agent 1, the control law (58) for the agent 2, and the control law (48) for the agent 3, the three agents converge to the formation constrained by the desired angles.

*Proof:* Take the following Hamiltonian as a candidate Lyapunov function

$$
\begin{aligned}
H =& \frac{1}{2}(m_1 \dot{q}_1^T \dot{q}_1 + m_2 \dot{q}_2^T \dot{q}_2 + m_3 \dot{q}_3^T \dot{q}_3) \\
& \frac{1}{2}(c_{\theta 1} + c_{\theta 2} + c_{\theta 3})\widetilde{(s_k^T s_j)}^2 + \\
& \frac{1}{2}(c_{\phi 1} + c_{\phi 2} + c_{\phi 3})\widetilde{(s_i^T s_j)}^2 + \\
& \frac{1}{2}c_{\theta i}\bar{r}_{\theta i2}^2 + \frac{1}{2}c_{\theta i}\bar{r}_{\theta i3}^2 + \frac{1}{2}c_{\theta j}\bar{r}_{\theta j}^2 + \frac{1}{2}c_{\theta k}\bar{r}_{\theta k}^2 + \\
& \frac{1}{2}c_{\phi k}\bar{r}_{\phi k1}^2 + \frac{1}{2}c_{2k}\bar{r}_{\phi k2}^2 + \frac{1}{2}c_{\phi i}\bar{r}_{\phi i}^2 + \frac{1}{2}c_{\phi j}\bar{r}_{\phi j}^2.
\end{aligned} \tag{60}
$$

It follows that $H$ is positive definite. Now we consider the time derivative of (60)

$$
\begin{aligned}
\dot{H} =& m_1 \dot{q}_1^T \ddot{q}_1 + m_2 \dot{q}_2^T \ddot{q}_2 + m_3 \dot{q}_3^T \ddot{q}_3 \\
& (c_{\theta 1} + c_{\theta 2} + c_{\theta 3})[\widetilde{(s_k^T s_j)}(\hat{L}_{\theta 1}\dot{x}_1 + \hat{L}_{\theta 2}\dot{x}_2 + \hat{L}_{\theta 3}\dot{x}_3)] + \\
& (c_{\phi 1} + c_{\phi 2} + c_{\phi 3})[\widetilde{(s_i^T s_j)}(\hat{L}_{\phi 1}\dot{x}_1 + \hat{L}_{\phi 2}\dot{x}_2 + \hat{L}_{\phi 3}\dot{x}_3)] + \\
& c_{\theta i}\bar{r}_{\theta i2}\dot{\bar{r}}_{\theta i} + c_{\theta i}\bar{r}_{\theta i3}\dot{\bar{r}}_{\theta i} + c_{\theta j}\bar{r}_{\theta j}\dot{\bar{r}}_{\theta j} + c_{\theta k}\bar{r}_{\theta k}\dot{\bar{r}}_{\theta k} + \\
& c_{\phi k}\bar{r}_{\phi k1}\dot{\bar{r}}_{\phi k1} + c_{\phi k}\bar{r}_{\phi k2}\dot{\bar{r}}_{\phi k2} + c_{\phi i}\bar{r}_{\phi i}\dot{\bar{r}}_{\phi i} + c_{\phi j}\bar{r}_{\phi j}\dot{\bar{r}}_{\phi j}.
\end{aligned} \tag{61}
$$

Note that

$$
\ddot{q}_n = \frac{U_n}{m_n}, \quad n = 1, 2, 3. \tag{62}
$$

Substituting (62) into the first line of (61), substituting (19), (51), (45) into the second line of (61), substituting (33), (55), (40) into the third line of (61), substituting (52), (46), (29) and (28) into the fourth line of (61), and substituting (56), (34), (41) and (42) into the fifth line of (61), we can simplify (61). For simplicity, we omit the process and only give the results as follows

$$
\begin{aligned}
\dot{H} =& -(d_{\theta 1} + d_{\theta 2} + d_{\theta 3})\dot{\widetilde{(s_k^T s_j)}}^2 \\
& -(d_{\phi 1} + d_{\phi 2} + d_{\phi 3})\dot{\widetilde{(s_i^T s_j)}}^2 \leq 0.
\end{aligned} \tag{63}
$$

By invoking LaSalle's invariance principle, we get that the trajectories of the closed-loop system converge to the largest invariant set where $\dot{H} = 0$. On this set $\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0$, $\dot{\widetilde{(s_k^T s_j)}} = 0$ and $\dot{\widetilde{(s_i^T s_j)}} = 0$.

Furthermore, we can conclude that $\ddot{q}_1 = \ddot{q}_2 = \ddot{q}_3 = 0$, according to (62) it follows that $U_1 = U_2 = U_3 = 0$ on this invariant set.

Table 1. Model parameters

| Parameter | Value |
|---|---|
| $m_n, n = 1,2,3$ | 1 |
| $c_{\theta n}, n = 1,2,3$ | 10 |
| $c_{\phi n}, n = 1,2,3$ | 10 |
| $d_{\theta n}, n = 1,2,3$ | 1 |
| $d_{\phi n}, n = 1,2,3$ | 1 |
| $c_{\theta l}, l = i,j,k$ | 1 |
| $c_{\phi l}, l = i,j,k$ | 1 |

Next we prove $\widetilde{(s_k^T s_j)} = 0$ and $\widetilde{(s_i^T s_j)} = 0$.

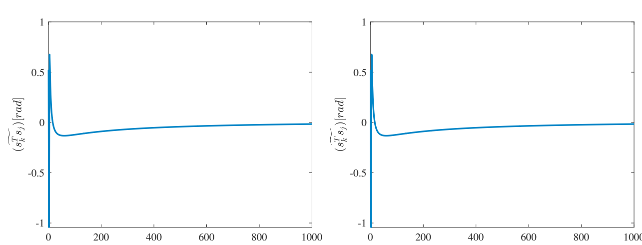
$$\begin{aligned} U_n &= \hat{L}_{\theta n}^T \gamma_{\theta n} + \hat{L}_{\phi n}^T \gamma_{\phi n} \\ &= \hat{L}_{\theta n}^T (c_{\theta n} \widetilde{(s_k^T s_j)} + d_{\theta n} \dot{\widetilde{(s_k^T s_j)}}) + \\ &\quad \hat{L}_{\phi n}^T (c_{\phi n} \widetilde{(s_i^T s_j)} + d_{\phi n} \dot{\widetilde{(s_i^T s_j)}}) = 0, \quad n = 1,2,3. \end{aligned} \tag{64}$$

Substituting $\dot{\widetilde{(s_k^T s_j)}} = 0$ and $\dot{\widetilde{(s_i^T s_j)}} = 0$ into (64) of $1,2,3$ respectively, we have

$$\begin{aligned} 0 &= c_{\theta 1} \hat{L}_{\theta 1}^T \widetilde{(s_k^T s_j)} + c_{\phi 1} \hat{L}_{\phi 1}^T \widetilde{(s_i^T s_j)} \\ 0 &= c_{\theta 2} \hat{L}_{\theta 2}^T \widetilde{(s_k^T s_j)} + c_{\phi 2} \hat{L}_{c\phi 2}^T \widetilde{(s_i^T s_j)} \\ 0 &= c_{\theta 3} \hat{L}_{\theta 3}^T \widetilde{(s_k^T s_j)} + c_{\phi 3} \hat{L}_{\phi 3}^T \widetilde{(s_i^T s_j)} \end{aligned} \tag{65}$$

Note that $c_{\theta 1}, c_{\theta 2}, c_{\theta 3}, c_{\phi 1}, c_{\phi 2}, c_{\phi 3} > 0$. Since $L$ is full column rank, we conclude that $\widetilde{(s_k^T s_j)} = \widetilde{(s_i^T s_j)} = 0$, which means that the three agents achieve the desired formation, thus completing the proof. □

*Remark 1.* The assumption that $L$ has full rank is not restrictive. If the three agents are neither coincident nor collinear, according to the expression of $L$, all the elements in the same column cannot be zero simultaneously, so it is column full rank.




(a) Error of angle 1 (b) Error of angle 2

Fig. 2. Angle errors

## 5. SIMULATIONS

Consider three agents modeled by double integrator as shown in Fig. 1. The related parameters are given in Table 1, and the initial positions and desired angles are given in Table 2. The simulation are performed using MATLAB.

The error curves of the angle 1 and the angle 2 are given in Fig. 2. It can be seen that the errors converge to zero and the desired formation constrained by angles is achieved.

## 6. CONCLUSIONS AND FUTURE RESEARCH

In this paper, we have developed a passivity-based approach for three agents modeled as double integrators to achieve a formation constrained by the desired angles. The resulting control law based on dynamic virtual couplings contains the unavailable distance, which is obtained by an estimator designed based on the pH theory. The effectiveness of the approach is validated by stability analysis and simulations.

Table 2. Initial positions and desired angles

| Parameter | Value |
|---|---|
| $q_1$ | (1,1) |
| $q_2$ | (1,3) |
| $q_3$ | (3,1) |
| $(s_k^T s_j)^*$ | $\pi/3$ |
| $(s_i^T s_j)^*$ | $\pi/3$ |

However, since we only consider the angle, as the error of angle goes to zero, the distance between the agents and the velocities become very large. Hence although the formation constrained by angles is achieved, the velocity and the scale are still unspecified. Therefore, for the practical application, we need to design a new controller to track the velocity and to control the scale of the shape of the formation. This is left for future research.

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