On modeling social diffusion under the impact of dynamic norms

Zino, Lorenzo; Ye, Mengbin; Cao, Ming

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On modeling social diffusion under the impact of dynamic norms

Lorenzo Zino, Mengbin Ye, and Ming Cao

Abstract— **We develop and analyze a collective decision-making model concerning the adoption and diffusion of a novel product, convention, or behavior within a population. Motivated by the growing social psychology literature on dynamic norms, under which an individual is influenced by changing trends in the population, we propose a stochastic model for the decision-making process encompassing two behavioral mechanisms. The first is social influence, which drives coordination among individuals. Consistent with the literature on social diffusion modeling, we capture such a mechanism through an evolutionary game-theoretic framework for a network of interacting individuals. The second, which is the main novelty of our model, represents the impact of dynamic norms, capturing the tendency of individuals to be attracted to products or behaviors with growing popularity. We analytically determine sufficient conditions under which a novel alternative spreads to the majority of the population. Our findings provide insights into the unique and nontrivial role of human sensitivity to dynamic norms in facilitating social diffusion.**

I. Introduction

The past decade has witnessed an increasing interest from the systems and control community in the mathematical modeling and analysis of social dynamics [1]–[4]. *Social diffusion* is a fundamental phenomenon of interest [5], referring to the process whereby a novel *alternative* product, idea, or behavior spreads across a population to replace the *status quo* currently adopted by the majority [6]–[8]. Evolutionary game-theoretic models of networks of interacting agents have been used to capture the impact of social influence on human decision-making [2], [9]–[12]. These works show that social influence occurring through individual-level interactions can lead to the emergence of collective phenomena. Importantly, it has been shown that social diffusion can occur when the alternative has a payoff advantage over the status quo [11].

However, in addition to social influence, human decisions made during social interactions are often affected by other salient behavioral mechanisms. A mechanism of particular note is the tendency for an individual to be attracted to products, ideas, or behaviors with growing popularity, even if the product/behavior is currently in the minority [13]. Recent literature refers to this as *dynamic norms* [14], [15]. Apart from our recent developments [16], [17], current evolutionary game-theoretic models for social diffusion (including those mentioned above) have largely omitted consideration of the impact of dynamic norms.

L. Zino and M. Cao are with the Faculty of Science and Engineering, ENTEG, University of Groningen, Groningen 9747 AG, Netherlands. Emails: {lorenzo.zino, m.cao}@rug.nl. M. Ye is with the Faculty of Science and Engineering, Curtin University, Perth, Australia. Email: mengbin.ye@curtin.edu.au. The work of L. Zino and M. Cao is partially supported by the European Research Council (ERC-CoG-771687), and the Netherlands Organization for Scientific Research (NWO-vidi-14134). M. Ye is supported by the Western Australian Government, under the Premier's Science Fellowship Program.

This paper aims to fill this gap, and demonstrate the nontrivial impact of dynamic norms on social diffusion. We propose a mathematical model for social diffusion that incorporates both social influence and dynamic norms. In our model, a population of interacting individuals decide between a status quo and a novel alternative, updating their decisions in a stochastic fashion. Specifically, the stochastic process includes a game-theoretic mechanism that captures social influence as in existing models and a novel trend-seeking mechanism that captures dynamic norms. The impact of these two mechanisms is regulated by a parameter, that represents the population's *sensitivity* to dynamic norms.

Through theoretical analysis of the proposed model, we shed light on the key role of dynamic norms in triggering social diffusion. In fact, our findings reveal that when people are only affected by social influence, diffusion occurs if and only if the alternative has a sufficiently large payoff advantage over the status quo. On the other hand, when the effect of dynamic norms is present, we show that the alternative may spread even in the absence of any advantage. By leveraging tools from probability theory and dynamical systems theory, we establish a sufficient condition for social diffusion to occur with high probability as a threshold of the population's sensitivity to dynamic norms. Simulations are provided to support and extend our theoretical findings.

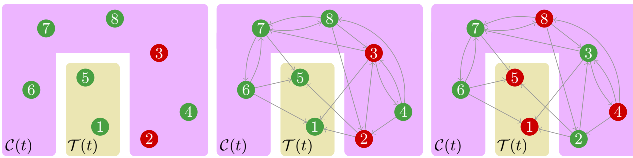
The rest of the paper is organized as follows. In Section II, we provide some mathematical preliminaries. In Section III, we introduce the model and formalize the problem. Section IV discusses general properties of the model. Section V presents our main results. Conclusions are drawn in Section VI.

II. Mathematical Preliminaries

We denote the set of nonnegative and strictly positive integer numbers by $\mathbb{N}$ and $\mathbb{N}_+$, respectively. A vector $\boldsymbol{x}$ is denoted with bold font, with ith entry x_i. The all-0 column vector is denoted by $\mathbf{0}$ (with dimension determined in the context). For a stochastic event E, $\mathbb{P}[E]$ is its probability.

Definition 1 (High probability)**.** *Given a family of events E_n, $n \in \mathbb{N}_+$, we say that E_n occurs with high probability (w.h.p.) with respect to n if and only if there exists a constant $K > 0$ such that $\mathbb{P}[E_n] \geq 1 - K/n$, for all $n \in \mathbb{N}_+$.*

Definition 2 (Markov chain)**.** *A process $\boldsymbol{x}(t)$ that takes values in $\mathcal{A}$, with $t \in \mathbb{N}$, is a Markov chain if and only if, for any $t \in \mathbb{N}$ and $\mathcal{B} \subseteq \mathcal{A}$, there holds $\mathbb{P}[\boldsymbol{x}(t+1) \in \mathcal{B} \,|\, \mathcal{F}_t] = \mathbb{P}[\boldsymbol{x}(t+1) \in \mathcal{B} \,|\, \boldsymbol{x}(t)]$, where $\mathcal{F}_t$ is the natural filtration associated to $\boldsymbol{x}(t)$ [18].*



(a) Partitioning (b) Network formation (c) Action update

Fig. 1: Example of the action revision process; nodes in red (green) adopt the alternative (status quo). In (a), we illustrate a realization of the stochastic partitioning process at time t with $\gamma = 1/4$. In (b), we illustrate a realization of the time-varying network formation process in the social influence mechanism with $k = 3$. In (c), we illustrate the action update process with $\alpha = 0.5$, assuming $z(t) > z(t-1)$.

Proposition 1 (Hoeffding's inequalities [19])**.** *Let $b_1, \dots, b_n$ be a sequence of $n \in \mathbb{N}_+$ independent and identically distributed (i.i.d.) Bernoulli random variables (r.v.s), each with the mean equal to $p > 0$. Then,*

$$\mathbb{P}[\tfrac{1}{n}\textstyle\sum_{i=1}^{n} b_i \leq p - \delta] \leq \exp\left\{-2\delta^2 n\right\}, \tag{1a}$$
$$\mathbb{P}[\tfrac{1}{n}\textstyle\sum_{i=1}^{n} b_i \geq p + \delta] \leq \exp\left\{-2\delta^2 n\right\}. \tag{1b}$$

III. Model and Problem Formulation

We consider a population of $n \in \mathbb{N}_+$ individuals, denoted by the set $\mathcal{V} = \{1, \dots, n\}$. At each discrete time-step $t \in \mathbb{N}$, each individual $i \in \mathcal{V}$ makes a binary decision on the *action* to adopt in the following time-step $t+1$. Specifically, they can choose whether to adopt the *status quo* action (denoted by 0) or the *alternative* action (denoted by 1). We denote by $x_i(t) \in \{0,1\}$ the action adopted by individual $i \in \mathcal{V}$ at time $t \in \mathbb{N}$. The actions of all the individuals in the population at time t are gathered in an n-dimensional vector $\boldsymbol{x}(t) \in \{0,1\}^n$, which we term the *action state*. We further define the variable $z(t) := \frac{1}{n}\sum_{i\in\mathcal{V}} x_i(t)$, which counts the fraction of adopters of the alternative action 1 in the population at time $t \in \mathbb{N}$.

At each time-step, individuals decide on how to revise their actions according to a stochastic procedure, which follows three steps. First, the population is stochastically partitioned into two sets. Second, depending on the set they belong to, individuals may apply a decision-making mechanism based on either social influence or dynamic norms. Third, actions are revised according to the selected mechanism. We now describe each step in detail, with an example in Fig. 1.

Partitioning. The population is partitioned into two time-varying sets of $\mathcal{T}(t)$ and $\mathcal{C}(t)$; the former denotes those individuals that at time $t \in \mathbb{N}$ base their decision on the observation of *dynamic norms*, and the latter captures those individuals that at time t base their decision on *social influence*. In more detail, at each discrete time-step $t \in \mathbb{N}$, a constant fraction $\gamma \geq 0$ of the population is selected uniformly at random and assigned to the set $\mathcal{T}(t)$. For simplicity of notation, we henceforth assume that $\gamma n \in \mathbb{N}$. The remaining population is assigned to the set $\mathcal{C}(t) = \mathcal{V} \setminus \mathcal{T}(t)$. In general, $\mathcal{T}(t)$ and $\mathcal{C}(t)$ contain different individuals at different time-steps, and determine only whether an individual at a given time-step uses dynamic norms or social influence in the decision-making process. Figure 1a illustrates an example realization of this partitioning process.

Social influence. Each individual $i \in \mathcal{C}(t)$ contacts k individuals from the entire population $\mathcal{V}$, selected uniformly at random and each one independent of the others; we call this set of contacted individuals the neighbors of individual i at time t, denoted by $\mathcal{N}_i(t)$. Such a mechanism generates a directed time-varying (multi-)graph of interactions $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where $(i,j) \in \mathcal{E}(t) \iff j \in \mathcal{N}_i(t)$ (possibly, with multiple occurrences), as shown in Fig. 1b. Note that the network formation process is similar to the one of directed discrete-time activity-driven networks [20], [21]. Then, each individual $i \in \mathcal{C}(t)$ makes their decision according to a *network coordination game* on $\mathcal{G}(t)$.

Specifically, $i \in \mathcal{C}(t)$ engages in k symmetric 2-player coordination games with each of their k neighbors $j \in \mathcal{N}_i(t)$ (multiple occurrences of the same individuals are treated as distinct games) [2]. The payoff that individual i would receive for selecting action 0 and 1 at time t, denoted $\pi_i^{(0)}(\boldsymbol{x}(t))$ and $\pi_i^{(1)}(\boldsymbol{x}(t))$ respectively, is the sum of the payoffs from all k coordination games, yielding:

$$\pi_i^{(0)}(\boldsymbol{x}(t)) := \textstyle\sum_{j\in\mathcal{N}_i(t)}(1 - x_j(t)), \tag{2a}$$
$$\pi_i^{(1)}(\boldsymbol{x}(t)) := \textstyle\sum_{j\in\mathcal{N}_i(t)}(1 + \alpha)x_j(t). \tag{2b}$$

where $\alpha \geq 0$ represents the evolutionary advantage of the alternative with respect to the status quo, which is discussed later in this section. Individuals revise their actions to maximize their payoffs, according to *best-response dynamics*, which is a standard protocol adopted in evolutionary game theory [22]. Specifically, $i \in \mathcal{C}(t)$ revises its action to

$$x_i(t+1) = \begin{cases} 1 & \text{if } \pi_i^{(1)}(\boldsymbol{x}(t)) > \pi_i^{(0)}(\boldsymbol{x}(t)), \\ 0 & \text{if } \pi_i^{(1)}(\boldsymbol{x}(t)) < \pi_i^{(0)}(\boldsymbol{x}(t)), \\ x_i(t) & \text{if } \pi_i^{(1)}(\boldsymbol{x}(t)) = \pi_i^{(0)}(\boldsymbol{x}(t)). \end{cases} \tag{3}$$

Note that we have adopted a "conservative" best-response, where individual i will not change action if the two payoffs are equal, consistent with the social psychology literature on the presence of inertia in decision-making [23]. Figure 1c illustrates an example of this revision protocol.

Dynamic norms. Each individual $i \in \mathcal{T}(t)$ chooses to adopt the action whose number of adopters has increased in the previous time-step. If the fraction of adopters of both actions are unchanged in the previous time-step, then the player does not revise their action. Hence, for all $i \in \mathcal{T}(t)$, the decision is made according to the following rule:

$$x_i(t+1) = \begin{cases} 1 & \text{if } z(t) > z(t-1), \\ 0 & \text{if } z(t) < z(t-1), \\ x_i(t) & \text{if } z(t) = z(t-1). \end{cases} \tag{4}$$

Figure 1c shows an example of this revision protocol. Thus, Eq. (4) captures individual i adopting the action that has increased in popularity over the previous time-step (is trending upward), even if the action is currently only adopted by the minority in the population. This closely reflects recent empirical studies and social psychology literature on dynamic norms [14], [15], which showed individuals could be influenced to take up a behavior/action currently adopted by a minority, simply by being made aware that the behavior/action was becoming increasingly popular.

The proposed model is characterized by three parameters. The parameter $\gamma \in [0,1]$, termed *trend-seeking*, represents the fraction of the population that makes its decision on the basis of dynamic norms. The parameter $k \in \mathbb{N}_+$ is the number of *social contacts* for each individual, and we have assumed homogeneity in the population. The value k captures the limited amount of social information that individuals generally use in their decision-making processes. Finally, the parameter $\alpha \geq 0$ models the (possible) *evolutionary advantage* of the alternative action with respect to the status quo action.

Remark 1. *In the absence of trend-seeking $\gamma = 0$, the model reduces to a coordination game on a (time-varying) network [9]–[11] or, equivalently, to a (biased) k-majority dynamic [24]. In the special case $k = 1$, the dynamics reduces to an unbiased voter model [25].*

We are interested in studying the role of dynamic norms in social diffusion. To this aim, we will specifically consider the scenario in which the population begins with all individuals adopting the status quo action 0. At time 0, the alternative action is introduced into the system by selecting individuals in the set $\mathcal{T}(0)$ to act as *early adopters* of the alternative action 1 [5]. To do this, we initialize the system by setting $\boldsymbol{x}(t) = \mathbf{0}$, for all $t < 0$, and $\boldsymbol{x}(0)$ such that $x_i(0) = 1$ if $i \in \mathcal{T}(0)$ and $x_j(0) = 0$ if $j \in \mathcal{C}(0)$. Note that this implies $z(0) = \gamma$. In the special case where trend-seeking is absent ($\gamma = 0$), we will instead select a small amount of early adopters uniformly at random in the population, so that $z(0) > 0$.

We wish to determine which number of social contacts $k \in \mathbb{Z}_+$, evolutionary advantage $\alpha \geq 0$, and trend-seeking $\gamma \geq 0$ ensure that the alternative action diffuses across the population. We restrict our analysis to scenarios in which social influence is the dominant mechanisms, that is, $\gamma \in [0, 1/2]$. This is because the opposite scenario is less interesting and unrealistic; if $\gamma > 1/2$, then the alternative is immediately adopted by the majority of the population. In particular, we establish sufficient conditions that guarantee social diffusion to occur. To this aim, for any $\varepsilon \in (0,1)$, we define the random times:

$$T_n := \inf\{t \in \mathbb{N} : z(t) \geq (1-\varepsilon)n\}\,, \tag{5a}$$

$$S_n := \inf\{t \in \mathbb{N} : z(t) \leq z(t-1)\}\,, \tag{5b}$$

writing them as explicit functions of the population size n. Thus, T_n and S_n define the times at which the alternative action reaches a fraction $1-\varepsilon$ of adopters, and the first time-instant at which the number of adopters of the alternative decreases, respectively (where $\inf(\emptyset) = +\infty$). Clearly, if $T_n < S_n$, then not only does diffusion occurs, but it occurs in a monotonic fashion. Hence, determining conditions for which $T_n < S_n$ yields sufficient conditions for social diffusion to occur. We formalize our research question as follows.

Problem 1 (Achieving social diffusion)**.** *Consider the triple (k, α, γ). Determine whether $T_n < S_n$ w.h.p., for any $\varepsilon > 0$.*

In the rest of this work, we make the following assumption.

Assumption 1. *We assume that $k/(2+\alpha) \notin \mathbb{N}$.*

When the assumption is not satisfied, the direct analysis of the system is made more complicated by the possibility of having an individual $i \in \mathcal{C}(t)$ that receives the same payoff for the two actions. In that scenario, the current action of the individual i would play a role in the decision-making process. Thus, the derivation of probabilistic bounds would be significantly more complicated, but nonetheless, the arguments used in the following analytical derivations can be used with appropriate extensions. Moreover, sufficient conditions to guarantee social diffusion can be directly established from our findings through standard coupling arguments between Markov chains [18].

IV. General Properties of the Model

We start by observing that the action update depends on the time-varying set to which the individual belongs at time t. If $i \in \mathcal{T}(t)$, then their next action is fully determined by Eq. (4). In the following, we explicitly compute the probability that $i \in \mathcal{C}(t)$ updates their state to $x_i(t+1) = 1$.

Proposition 2. *Let us define $\Pi_{k,\alpha}(z) := \sum_{\ell=k^*}^{k} \binom{k}{\ell} z^\ell (1-z)^{k-\ell}$, with $k^* := \lceil k/2 + \alpha \rceil$. If Assumption 1 holds, then*

$$\mathbb{P}[x_i(t+1) = 1 | i \in \mathcal{C}(t)] = \Pi_{k,\alpha}(z(t))\,. \tag{6}$$

Proof. For the sake of readability, we drop the time-index t, so $\boldsymbol{x}$ and z stand for $\boldsymbol{x}(t)$ and $z(t)$, respectively. First, we rewrite the payoff in Eq. (2a) as $\pi_i^{(0)}(\boldsymbol{x}) = k - \sum_{j \in \mathcal{N}_i(t)} x_j$. By substituting this expression and Eqs. (2b) into the condition in Eq. (3), we observe that $\pi_i^{(1)}(\boldsymbol{x}) > \pi_i^{(0)}(\boldsymbol{x})$ if and only if $\sum_{j \in \mathcal{N}_i(t)} x_j > k/(2+\alpha)$. That is, the payoff for playing $x_i = 1$ is greater than the payoff for playing $x_i = 0$ if there are at least k^* neighbors of i playing 1. Note that, because $k/(2+\alpha) \notin \mathbb{N}$, the payoffs $\pi_i^{(0)}$ and $\pi_i^{(1)}$ can never be equal.

Then, we compute the probability that an individual $i \in \mathcal{C}(t)$ selects k individuals $j \in \mathcal{V}$ (with repetition) during the network formation process at time t, and at least k^* of them have state $x_j = 1$. Since each sample consists of a Bernoulli r.v. with the success probability equal to z, independent of the others, the number of neighbors $j \in \mathcal{N}_i(t)$ playing $x_j = 1$ is a binomial r.v. with k trials and success probability equal to z. Hence, the desired quantity is the probability that such a binomial r.v. is greater than or equal to k^* (that is, its complementary cumulative distribution evaluated for $k^* - 1$), yielding Eq. (6). Note that this probability is independent of $i \in \mathcal{V}$ and depends on $\boldsymbol{x}$ only through z. □

Next, we establish some of the key properties of the function $\Pi_{k,\alpha}(z)$ that plays a key role in the revision process of individual $i \in \mathcal{C}(t)$. The proofs, omitted due to space constraints, follow from the properties of the binomial cumulative probability distribution [26].

Proposition 3. *The function $\Pi_{k,\alpha}(z) : [0,1] \to [0,1]$ is a monotonically nondecreasing function of α, and a monotonically increasing function of z. Moreover, $\Pi_{k,\alpha}(1/2) \geq 1/2$, and the following properties hold true:*

1) *for any $\alpha \geq 0$, then $\Pi_{1,\alpha}(z) = z$;*
2) *if $k \geq 2$ and $\alpha > k-2$, then $\Pi_{k,\alpha} > z$, $\forall\, z \in (0,1)$;*
3) *if $k \geq 2$ and $\alpha < k-2$, then there exists a unique $z^* \in (0, 1/2]$ such that $\Pi_{k,\alpha}(z) < z$ for all $z \in (0, z^*)$, and $\Pi_{k,\alpha}(z) > z$ for all $z \in (z^*, 1)$.*

It is worth noticing that the dynamics $\boldsymbol{x}(t)$, defined by Eq. (3) and Eq. (4), is in general not a Markov process, except for $\gamma = 0$. In fact, from Eq. (4), we observe that the state transitions governed by the trend-seeking mechanism do not only depend on $\boldsymbol{x}(t)$, but also depend on the state of the system at the previous time-step, namely $\boldsymbol{x}(t-1)$. To address this issue, and as we will later require Markov chain theory, we define new dynamics with an augmented state space. To do this, we introduce two new vector variables $\hat{\boldsymbol{x}}(t) \in \{0,1\}^n$ and $\hat{\boldsymbol{y}}(t) \in \{0,1\}^n$, which roughly speaking have dynamics equal to $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t-1)$, respectively. More precisely, we define a $2n$-dimensional Markov process $(\hat{\boldsymbol{x}}(t), \hat{\boldsymbol{y}}(t))$, which evolves according to the following rules. For all $i \in \mathcal{V}$, we have $y_i(t+1) = \hat{x}_i(t)$. Meanwhile, $\hat{x}_i(t)$ is updated according to the mechanisms described above. Specifically, if $i \in \mathcal{C}(t)$, then the state is updated according to Eqs. (2) and (3) but with $\hat{x}_i$ replacing x_i for all $i \in \mathcal{V}$. If $i \in \mathcal{T}(t)$, then Eq. (4) is substituted by the following update rule:

$$\hat{x}_i(t+1) = \begin{cases} 1, & \text{if } \sum \hat{x}_i(t) > \sum \hat{y}_i(t)\,, \\ 0, & \text{if } \sum \hat{x}_i(t) < \sum \hat{y}_i(t)\,, \\ \hat{x}_i(t), & \text{if } \sum \hat{x}_i(t) = \sum \hat{y}_i(t)\,. \end{cases} \tag{7}$$

Finally, observe that the processes $\boldsymbol{x}(t)$ and $\hat{\boldsymbol{x}}(t)$ are equivalent when we initialize $\hat{\boldsymbol{x}}(t) = \boldsymbol{x}_0$ and $\hat{\boldsymbol{y}}(t) = \boldsymbol{0}$. Hence, and as we shall do in the sequel, one can use the Markov process $(\hat{\boldsymbol{x}}(t), \hat{\boldsymbol{y}}(t))$ to study Problem 1, by observing that the random times in Eq. (5) can be equivalently defined as $T_n := \inf\{t \in \mathbb{N} : \sum_{i\in\mathcal{V}} \hat{x}_i(t) \geq (1-\varepsilon)n\}$ and $S_n := \inf\{t \in \mathbb{N} : \sum_{i\in\mathcal{V}} \hat{x}_i(t) \leq \sum_{i\in\mathcal{V}} \hat{y}_i(t)\}$.

We further define the stopping time $Q_n := \min\{T_n, S_n\}$. The following result guarantees that Q_n is always well defined (that is, $Q_n < \infty$) and provides a bound on such a quantity.

Proposition 4. *For any $\gamma \in [0, 1/2]$, $\varepsilon \in (0,1)$, and $z(0) > 0$, the stopping time $Q_n = \min\{T_n, S_n\}$ satisfies $Q_n \leq \max\{(1-\varepsilon-\beta)n, 1\}$, where $\beta = \max\{z(0), \gamma\}$.*

Proof. Our problem formulation has assumed that γn individuals initially play action 1 (or if $\gamma = 0$, then some predetermined amount $z(0) > 0$). This implies that $z(0) \geq \beta$. Thus, if $\varepsilon \geq 1-\beta$, then $T \leq 1$. If $\varepsilon < 1-\beta$, we prove the claim by contradiction. Let us hypothesize that $Q_n > (1-\varepsilon-\beta)n$. Then the definition of Q_n yields $S_n > (1-\varepsilon-\beta)n$, which in turn implies from the definition of S_n that $z(t) > z(t-1)$ for all $t \leq (1-\varepsilon-\beta)n$. Since $Q_n > 1$, it follows that $z(1) \geq \beta$ and $z(t) \geq z(t-1) + 1/n$ because for $t < Q_n$, $z(t)$ is increasing at each time-step and such an increase is lower bounded by $1/n$ (being $z(t)$ a process taking values on $\{0, \frac{1}{n}, \dots, 1\}$). As a consequence, $z((1-\varepsilon-\beta)n) \geq z(0) + (1-\varepsilon-\beta)n \cdot 1/n \geq 1-\varepsilon$. However, this would imply that $T_n \leq (1-\varepsilon-\beta)n$, and thus $Q_n \leq (1-\varepsilon-\beta)n$, which contradicts our hypothesis. $\square$

V. Main Results

We will start our analysis from the case without trend-seeking ($\gamma = 0$), with all individuals revising their action based on social influence and $\mathcal{T}(t) = \emptyset$. Since social influence as in Eq. (3) is often used to study social diffusion [9]–[11], this case will serve as a baseline for comparison, to better understand the effect of dynamic norms in facilitating social diffusion, which is studied in the second part of this section.

A. Absence of trend-seeking with $\gamma = 0$

We assume that the system starts with a positive (small) fraction of early adopters of the innovation $z(0) = \beta \in (0, 1/2]$. In the following, we establish a condition for the alternative to spread for any $\beta > 0$ (solving Problem 1). Consistent with the literature of social diffusion and coordination games on networks [9]–[11], we find that the alternative spreads if and only if it has a sufficiently large evolutionary advantage α (dependent on the number of social contacts, k).

Theorem 1. *Under Assumption 1, a triple $(k, \alpha, 0)$ solves Problem 1 for any $\beta > 0$ if and only if $k \geq 2$ and $\alpha > k-2$.*

Proof. First, we observe that $z(t) = \frac{1}{n}\sum_{i\in\mathcal{V}} \hat{x}_i(t)$ is a Markov chain [18]. In fact, since all the individuals revise their action according to social influence, $i \in \mathcal{C}(t)$ for all $i \in \mathcal{V}$ and $t \in \mathbb{N}$. Based on Proposition 2, we can write

$$z(t+1) = \tfrac{1}{n}\textstyle\sum_{\ell=1}^{n} b_\ell(t)\,, \tag{8}$$

where $b_1(t), \dots, b_n(t)$ is a sequence of i.i.d. Bernoulli r.v.s, each one with the mean equal to $\Pi_{k,\alpha}(z(t))$. Hence, the probability distribution of $z(t+1)$ depends only on t through $\Pi_{k,\alpha}(z(t))$, and, ultimately, on $z(t)$.

For $k = 1$, the system reduces to a voter model, for which the probability of $z(t)$ reaching $1-\varepsilon$ before reaching 0 is $\beta/(1-\varepsilon)$ [25]. However, the definitions of T_n and S_n yield $\mathbb{P}[T_n < S_n] \leq \beta/(1-\varepsilon)$. Unless $\beta \geq 1-\varepsilon$ (which is not verified by a generic $\beta \in (0, 1/2]$ and $\varepsilon > 0$), $\mathbb{P}[T_n < S_n]$ never converges to 1, and therefore does not occur w.h.p.

For $k \geq 2$, and $\alpha > k-2$, item 2) of Proposition 3 guarantees that $\Pi_{k,\alpha}(z) > z$, for all $z \in (0,1)$. Hence, by continuity, there exists a constant $\delta > 0$ such that $\Pi_{k,\alpha}(z) > z + \delta$, for all $z \in [\beta, 1-\varepsilon]$, for any $\varepsilon > 0$. Then, Hoeffding's inequality in Eq. (1a) applied to Eq. (8) yields

$$\mathbb{P}\left[z(t+1) \leq z(t) | \mathcal{F}_t\right] \leq \exp\{-2\delta^2 n\}\,, \tag{9}$$

for all $t < Q_n$, where $\mathcal{F}_t$ is the natural filtration associated to $(\hat{\boldsymbol{x}}(t), \hat{\boldsymbol{y}}(t))$ at time t. We use the law of total probability with respect to the conditioning on Q_n, and we write

$$\begin{aligned} \mathbb{P}[T_n < S_n] &= \mathbb{P}[\nexists\, t < Q_n : z(t+1) \leq z(t)] \\ &= \sum_{s=0}^{\infty} \mathbb{P}[Q_n = s]\mathbb{P}[\nexists\, t < s : z(t+1) \leq z(t)] \\ &= \sum_{s=0}^{\infty} \mathbb{P}[Q_n = s] \prod_{k=0}^{s} \mathbb{P}[z(k+1) > z(k) | \mathcal{F}_k] \\ &= \sum_{s=0}^{(1-\beta-\varepsilon)n} \mathbb{P}[Q_n = s] \prod_{k=0}^{s} \left(1 - \mathbb{P}[z(k+1) \leq z(k) | \mathcal{F}_k]\right), \end{aligned} \tag{10}$$

where the last equality holds since $Q_n \leq (1-\beta-\varepsilon)n$ (Proposition 4). Finally, substituting Eq. (9) into Eq. (10), and then bounding the convex combination on the right-hand-side with its minimum, we obtain

$$\begin{aligned}\mathbb{P}[T_n < S_n] &\geq \textstyle\sum_{s=0}^{(1-\beta-\varepsilon)n} \mathbb{P}[Q_n = s]\big(1-e^{-2\delta^2 n}\big)^s \\ &\geq \big(1-e^{-2\delta^2 n}\big)^{(1-\beta-\varepsilon)n} \geq 1 - K/n\,, \end{aligned} \tag{11}$$

for some constant $K > 0$, which yields the claim.

For $k \geq 2$ and $\alpha < k-2$, item 3) of Proposition 3 guarantees that $\Pi_{k,\alpha}(z) < z$, for all $z \in (0, z^*)$. Hence, for any $\beta < z^*$, by continuity, there exists a constant $\delta > 0$ such that $\Pi_{k,\alpha}(\beta) < \beta - \delta$. Then, we bound $\mathbb{P}[T_n < S_n]$ with the probability of not changing the trend at the first step (which is a necessary condition for $T_n < S_n$), obtaining

$$\mathbb{P}[T_n < S_n] \leq \mathbb{P}\left[z(1) > z(0)\right] \leq \exp\{-2\delta^2 n\}\,, \tag{12}$$

where the last inequality is obtained by applying Hoeffding's inequality in Eq. (1b) to Eq. (8). Note that Eq. (12) never converges to 1, and therefore does not occur w.h.p. $\square$

B. Presence of trend-seeking with $\gamma > 0$

In the previous section, we concluded that in the absence of trend-seeking, the coordination mechanism induced by social influence hinders social diffusion, unless the novel alternative provides a sufficiently large evolutionary advantage α. Here, we will show how sensitivity to dynamic norms —captured by the parameter γ— can facilitate to unlock social diffusion, even in the absence of any evolutionary advantage.

When $\gamma > 0$, we can study the dynamics of the stochastic process $z(t)$ up to the stopping time Q_n by using a similar approach to the one used for $\gamma = 0$. In fact, we will observe that $z(t)$ is a Markov process, when conditioning to $t < Q_n$. Such an observation allows us to use Hoeffding's inequalities to bound $z(t+1)$ through a function of $\Pi_{k,\alpha}$, and of the parameter γ. The following result establishes our findings.

Lemma 1. *If Assumption 1 is verified, then, for any $t < Q_n$ and for any constant $\delta > 0$,*

$$\mathbb{P}[z(t+1) \leq (1-\gamma)\Pi_{k,\alpha}(z(t)) + \gamma - \delta | \mathcal{F}_t] \leq e^{-Kn},$$

with constant $K = 2\delta^2/(1-\gamma) > 0$.

Proof. Proposition 4 guarantees that the stopping time Q_n is always well defined. Up to the stopping time Q_n, $z(t)$ is a Markov chain on $\{\gamma, \gamma+\frac{1}{n}, \dots, 1-\varepsilon\}$. In fact, the definition of Q_n yields that $\sum_{i\in\mathcal{V}} \hat{x}_i(t) > \sum_{i\in\mathcal{V}} \hat{y}_i(t)$ for any $t < Q_n$. Along with Eq. (7), this implies that $\hat{x}_i(t+1) = 1$, for all $i \in \mathcal{T}(t)$, independent of the previous history of the process. We now focus our analysis on the Markov chain $z(t)$, up to the stopping time Q_n. Similar to the case without trend-seeking, and because $z(0) = \gamma$, we can write

$$z(t+1) = \gamma + \tfrac{1}{n}\textstyle\sum_{\ell=1}^{(1-\gamma)n} b_\ell(t), \tag{13}$$

where $b_1(t), \dots, b_{(1-\gamma)n}(t)$ is a sequence of i.i.d. Bernoulli r.v.s, with the mean equal to $\Pi_{k,\alpha}(z(t))$. Hoeffding's inequality in Eq. (1a) applied to the second term of the right-hand-side of Eq. (13) yields

$$\begin{aligned}&\mathbb{P}\Big[\tfrac{1}{n}\textstyle\sum_{\ell=1}^{(1-\gamma)n} b_\ell(t) \leq (1-\gamma)\Pi_{k,\alpha}(z(t)) - \delta \big| \mathcal{F}_k\Big] \\ &\quad= \mathbb{P}\Big[\tfrac{1}{(1-\gamma)n}\textstyle\sum_{\ell=1}^{(1-\gamma)n} b_\ell(t) \leq \Pi_{k,\alpha}(z(t)) - \tfrac{\delta}{1-\gamma}\big|\mathcal{F}_k\Big] \\ &\quad\leq \exp\{-2\delta^2 n/(1-\gamma)\}\,,\end{aligned} \tag{14}$$

for any constant $\delta > 0$. Using Eq. (13) and Eq. (14), we prove the claim. $\square$

Thus, for a given number of social contacts $k \in \mathbb{N}_+$ and evolutionary advantage $\alpha \geq 0$, we can define the function:

$$f_\gamma(z) := (1-\gamma)\Pi_{k,\alpha}(z) - z + \gamma\,, \tag{15}$$

which allows us to establish the following result.

Theorem 2. *Let the number of social contacts $k \in \mathbb{N}_+$ and evolutionary advantage $\alpha \geq 0$ be given, satisfying Assumption 1. Then (k, α, γ) solves Problem 1 if*

$$\gamma > \gamma^*_{k,\alpha} := \inf\{\gamma \in [0, 1/2] : f_\gamma(z) > 0,\ \forall z \in (0,1)\}, \tag{16}$$

where $f_\gamma(z)$ is defined in Eq. (15). Moreover, $\gamma^*_{k,\alpha} = 0$ if $k = 1$ or $\alpha > k-2$, otherwise $\gamma^*_{k,\alpha} > 0$.

Proof. Given γ such that $f_\gamma > 0$, $\forall z \in (0,1)$, there then exists by continuity a $\delta > 0$ such that $f_\gamma(z) > \delta$, for all $z \in [\gamma, 1-\varepsilon]$, for any $\varepsilon > 0$. This implies that $(1-\gamma)\Pi_{k,\alpha}(z) + \gamma - \delta > z$. Hence, Lemma 1 guarantees that

$$\mathbb{P}[z(t+1) \leq z(t) | \mathcal{F}_t] \leq \exp\{-Kn\} \tag{17}$$

for each $t < Q_n$. Hence, we can prove that the triple (k, α, γ) solves Problem 1, by using arguments and calculations similar to those used in Eqs. (10) and (11) in the proof of Theorem 1. Briefly speaking, the arguments rely on conditioning on the time-instant Q_n, using the law of total probability to express $\mathbb{P}[T_n < S_n]$, to bound the sum obtained using Proposition 4, and substituting in Eq. (17).

Then, we observe that f_γ in Eq. (15) is monotonically nondecreasing with respect to γ. Therefore, if a triple (k, α, γ) solves Problem 1, then (k, α, γ') also solves Problem 1, for any $\gamma' \geq \gamma$. Thus, either the set in Eq. (16) is empty, or it is an interval from $\gamma^*_{k,\alpha}$ to $1/2$.

Finally, we prove that the set in Eq. (16) is always nonempty. For $k = 1$, we recall that $\Pi_{k,\alpha}(z) = z$. Notice that $f_\gamma(z) > 0$ reduces to $(1-\gamma)z + \gamma - z > 0$, which holds for any $\gamma > 0$. Hence $\gamma^*_{1,\alpha} = 0$, for any $\alpha \geq 0$. For $k > 1$ and $\alpha > k-2$, Theorem 1 guarantees that $(k, \alpha, 0)$ solves Problem 1, and thus $\gamma^*_{k,\alpha} = 0$. Finally, we analyze the case $k > 1$ and $\alpha < k-2$ (which implies that $k^* \geq 2$) by observing that $\gamma = 1/2$ always verifies the condition $f_\gamma > 0$. In fact, for $z < \gamma = 1/2$, we observe that both the first term and the sum of the second and third terms in f_γ are positive, that is, $-z+\gamma > 0$. This implies that $f_\gamma > 0$ for $z < \gamma = 1/2$. For $z = 1/2$, we have $f_{1/2}(1/2) = \Pi_{k,\alpha}(1/2) > 0$. Lastly, for $z \in (1/2, 1)$, the function $f_{1/2}$ is strictly concave, and one further has $f_{1/2}(1/2) > 0$ and $f_{1/2}(1) = 0$, which implies

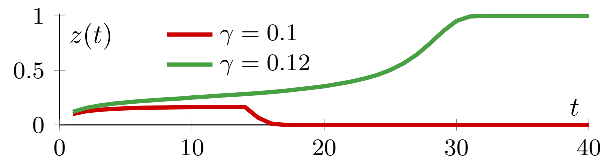


Fig. 2: Simulated trajectories of the diffusion process with $n = 100,000$, $k = 3$, $\alpha = 0$, and two different values of γ.

that $f_{1/2}(z) > 0$ for $z \in (1/2, 1)$. Hence $\gamma = 1/2$ always belongs to the set in Eq. (16), which is not empty. □

Remark 2. *When Assumption 1 is not satisfied, standard coupling arguments between Markov chains [18] can be used to show that if a triple (k, α, γ) solves Problem 1, then (k, α', γ) also solves Problem 1, for any $\alpha' \geq \alpha$ (even if α' does not satisfy Assumption 1).*

Consider the case in which $k = 1$. In the absence of trend-seeking $\gamma = 0$, Problem 1 has no solution, while a nonzero presence of trend-seeking $\gamma > 0$ is sufficient to guarantee that the alternative spreads across the population, since $\gamma^*_{1,\alpha} = 0$. For $k = 2$ and $\alpha > 0$, the alternative always spread, even in the absence of trend-seeking (and thus $\gamma^*_{2,\alpha} = 0$). As k increases, the behavior of the system becomes more complex. Indeed, complex emergent phenomena can be already found for $k = 3$, as shown in the following corollary and supporting simulations. In particular, the alternative requires a sufficiently large evolutionary advantage ($\alpha > 1$) to guarantee diffusion in the absence of trend-seeking. However, when trend-seeking is present, diffusion of the alternative can be achieved even for $\alpha \leq 1$, provided γ is sufficiently large.

Corollary 1 (The case of $k = 3$)**.** *A triple $(3, \alpha, \gamma)$ solves Problem 1 if i) $\alpha > 1$ or ii) $\gamma > 1/9$.*

Proof. For $\alpha > 1 = k-2$, Theorem 1 ($\gamma = 0$) and Theorem 2 ($\gamma > 0$) yield the claim. For $\alpha \in [0, 1)$, we compute $k^* = \lfloor 3/(2+\alpha) \rfloor + 1 = 2$ and $\Pi_{k,\alpha}(z) = 3z^2(1-z) + z^3$. Hence, $f_\gamma(z) = (1-\gamma)(3z^2(1-z)+z^3) - z + \gamma$ is a cubic curve that is positive for all $z \in (0, 1)$ if and only if $\gamma > \gamma^*_{3,\alpha} = 1/9$. Theorem 2 and Remark 2 (for $\alpha = 1$) yield the claim. □

The simulations in Fig. 2 illustrate the behaviors described in Corollary 1 in the absence of evolutionary advantage. Interestingly, these simulations suggest that the sufficient condition $\gamma > \gamma^*_{3,0}$ might define a sharp transition, as below such a threshold (red curve) diffusion seems not to occur.

VI. Conclusions

In this paper, we proposed and analyzed a model for social diffusion under the impact of dynamic norms. Our findings revealed that individual sensitivity to emerging trends plays a key role into enabling social diffusion. Building on these preliminary results, several avenues of future research may be outlined. First, a complete characterization of the asymptotic behavior of the diffusion process is still missing. Second, the impact of a heterogeneous population should be investigated, whereby individuals may have different tendency to follow dynamic norms, social contacts, and, possibly, a pattern of preferred interactions. Third, the proposed model should be validated against real-world data on social diffusion processes.

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