A geometric framework for constraints and data

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DOI:
10.33612/diss.177749387

Publication date:
2021

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A GEOMETRIC FRAMEWORK FOR CONSTRAINTS AND DATA

FROM LINEAR SYSTEMS TO CONVEX PROCESSES

The research described in this thesis has been carried out at the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, Faculty of Science and Engineering, University of Groningen, the Netherlands.

The research described in this thesis is part of the research program of the Dutch Institute of Systems and Control (DISC). The author has successfully completed the educational program of the Graduate School of DISC.

Printed by Ipskamp printing

rijksuniversiteit
groningen

A geometric framework for constraints and data

From linear systems to convex processes

Proefschrift

ter verkrijging van de graad van doctor aan de
Rijksuniversiteit Groningen
op gezag van de
rector magnificus prof. dr. C. Wijmenga
en volgens besluit van het College voor Promoties.

De openbare verdediging zal plaatsvinden op

vrijdag 3 september 2021 om 14:30 uur

door

Jaap Eising

geboren op 28 juni 1992
te Eelde

Promotores

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ACKNOWLEDGEMENTS

Before getting to the interesting part of the thesis, I have some things to address. I think I can safely say I'm not the same person that started studying mathematics in 2010, or even the person that started a PhD 4 years ago. Therefore, at least some of the goals behind writing a thesis like this one seem to have been a succes. However, while obtaining a PhD is officially a solitary matter, I could not have done it alone. On the scientific side of things, I have noticed that a single five minute discussion can lead to better ideas than a whole day of thinking on your own. Additionally, while it may seem counter-intuitive, it can not be overstated how useful it is to get away from research for a while. This thesis would not have been possible without a large group of people around me helping in either of these regards.

First of all, I'd like to thank my supervisors. Kanat, thanks for everything. I remember the first time you explained the concept of a convex process to me, at a time where any systems theory courses I'd followed were 5 years in the past. Even though I still thought I wanted to do 'pure' mathematics or formal logic at the time, your explanation convinced me to give control theory a try, leading to this thesis. I'm in awe of the way you attack problems in creative and yet mathematically precise ways, and you have taught me a lot. However, I do not think our collaboration has been purely scientific: We've had discussions on almost any topic under the sun, and it's great to see how you have reacted to becoming a father.

Harry, your no-nonsense approach to science and life has been an inspiration. I can't remember the number of times you've told me things were too complicated and how no-one would understand any of it. I especially enjoyed our collaboration on what is now Chapter 6. Apart from any mathematical nonsense, your presence at the department was always enjoyable, and you've made clear that there's more important things in life than research (even though I still don't think cycling is the answer).

I also want to thank my reading committee, professors Arjan van der Schaft, Maurice Heemels and Helene Frankowska for their useful comments on this thesis. In addition, I also want to thank Arjan, for the discussions and for the fact that your office was always open for questions.

Of my other colleagues, special mention should be made my long term office mates, Mark and Henk. Between fish, (grape)fruit, music, boardgames and mathematics, I've had a great time sharing office 389 with you. Mark, you are one of the most all-round creative people I know. While I always pretend not to enjoy your creativity regarding terrible jokes, I've always been amazed by your musical and mathematical ideas. Also, without the inside information you provided I would not have known of this position in the first place. Thanks for sharing more than 10 years of mathematics with me. As for Henk, I'm always amazed by your persistence and insight. Our collaboration leading to the paper on informativity was one of the highlights of this PhD project, and the trip to Nice was great. Let's keep working together in the future.

In addition, I would like to specifically thank Junjie, Jiajia and Bart for the warm welcome I received in the research group SCAA. Thanks for the great time and all the conversations. Brayan, Anne-men and Paul, even before you all started we were organising activities together. I immediately thought you were all great fits for the group, and I was right in thinking that. I hope to see your theses soon as well, good luck! For the great times at lunch, DISC courses, Benelux meetings and other conferences I want to thank the other members of the SCAA group and our colleagues at ENTEG.

Some further people need to be addressed in Dutch. Zoals gezegd waren ontspanning en afleiding van essentieel belang in het afronden van dit project. Naast collega's is er een grote groep mensen die nog genoemd moet worden, maar aangezien ik hoogstwaarschijnlijk mensen zal vergeten doe ik het in de vorm van een aantal favoriete herinneringen aan de afgelopen vier jaar: De maandagavondgroep, LANdal, pubquizzen, etentjes met Pieter, Dizy, Relaksen in Saksen, Plezant in Salland, Piekfijn online, Highchurch, de bierpas, speciaalbier- en whiskeyfestivals, Shadowrun, Dnd, bordspellen, en een eindeloze stroom concerten, housewarmings en bruiloften. Iedereen enorm bedankt!

Redelijk toevallig ben ik in hetzelfde vakgebied terechtgekomen waarin mijn oom ook gepromoveerd is. Rikus, bedankt voor alle interesse en alle interessante anekdotes over de vroegere dagen van de systeem- en regeltechniek.

Derk, Kristien en Teun, Frauke, Corné, Lia en Abe, wat waren de afgelopen jaren leuk! Er is zo ontzettend veel gebeurt qua verbouwingen, bruiloften, vakanties, en natuurlijk als klap op de vuurpijl het feit dat ik net zeven namen mocht noemen. Het was super om dit met jullie mee te mogen maken. Ten slotte Albert en Sita. Ik heb geen idee wat ik hier moet zeggen. Zonder jullie eindeloze aanmoediging, vertrouwen, geduld, advies en goede voorbeeld heb ik geen idee hoe ik, laat staan dit PhD project, terecht gekomen was. Dank jullie wel.

Jaap Eising *Groningen, 9 August 2021*

CONTENTS

1 INTRODUCTION

Mathematical models play a central role in many scientific disciplines. From the elaborate models of electrical grids used in power engineering, to models of opinion dynamics used in sociological research, the applications of mathematical modeling are endless. Models can aid understanding of complicated phenomena and can be used to obtain predictions of the future. In fact, on a daily basis we are heavily influenced by such models, as evidenced by the weather forecast and the, at time of writing extremely relevant, models of disease spreading.

The use of mathematics for modeling is very natural. Without elaborating into a discussion of mathematical philosophy, the use of mathematical models has a few obvious benefits: First of all, it is very precise. This is illustrated by the fact that the Dutch word for mathematics is 'wiskunde', which translates to 'the art of what is certain'. Aside from leaving no room for misinterpretation, this also enables the use of computers in modeling. Secondly, relatively simple mathematical models can describe surprisingly complex phenomena. In particular, linear dynamical systems have found a plethora of applications. Lastly, mathematical modeling allows for recycling of results: If two phenomena can be modeled in the same way, results for one can be applied to the other. For example, electrical circuits with resistors, inductors and capacitors admit the same mathematical description as mass-spring-damper networks. This observation has led to strong links between electrical and mechanical engineering.

This motivates the following question: Given a dynamical system, how do we decide whether it exhibits certain qualitative or quantitative properties? An interesting example of this is characterizing stability, that is, determining whether the system comes to a rest if it is left alone. For dynamical systems with inputs we will also consider control problems, an example of which is to find a choice of input such that the resulting system comes to a rest. These problems are at the core of the field of systems and control and this thesis. In particular, we will extend the theories of analysis and control of dynamical systems towards broader classes of systems.

1.1 LINEAR SYSTEMS

Of course, the use of a mathematical model is to describe reality in some meaningful way. This means that the properties of the phenomenon under consideration need to be reflected in the dynamical systems we employ. Often, this motivates the assumptions of *linearity* and *time-invariance*.

Indeed, for linear time-invariant systems, the field of systems and control is very mature. For a very thorough introduction to most of the topics below, we refer to [165]. Here we will discuss some early results from the theory of linear systems that will play a recurring role in the remainder of this thesis.

1.1.1 Properties of systems

With the introduction of state space by Kalman (see e.g. [88, 89, 91, 94]), the properties of *controllability* and *observability* were defined. The duality between these notions was shown, and characterizations were given: Both in terms of positive definiteness of the controllability Gramian and in terms of the rank of the controllabilty matrix. It was also shown that controllability of linear systems is equivalent to the related concept of *reachability*.

In addition to studying these properties for their own sake, these papers revealed the links between these system-theoretic properties and solvability of the linear quadratic regulator (LQR) problem. In fact, analysis of system theoretic properties is often an important stepping stone in the development of new control results.

Further investigation of these properties led, among others, to a *spectral* characterization of controllability. In other words, a test in terms of the *eigenvalues* of the state map. This result is known as the Hautus or Popov-Belevitch-Hautus lemma, as the result by Hautus in [73] was also independently found by Popov in [135] and by Belevitch in [20]. One of the most appealing properties of the Hautus lemma is that in [74] it was shown to lead in a natural way to characterizations for *stabilizability* and *detectability* as well.

A popular and versatile tool in control theory is the *Lyapunov function*. These are based on the observation that, if there exists a function of the state which is bounded below, and which decreases along each trajectory, then the system must be stable. The methods of Lyapunov, as described in [109], were first applied to stabilization in a system theory setting in the works of Kalman [92, 93]. Lyapunov functions arise in many settings as an analogue to 'energy' in physical systems. In addition to its use in stability analysis, the study of Lyapunov functions has many applications in control. A particularly relevant result is the Kalman-Yakubovich-Popov (or bounded real) lemma, discovered independently in [90, 134, 185]. In a modern reformulation, this result links the study of Lyapunov functions and linear matrix inequalities.

1.1.2 Geometric control theory

It should be observed that the previously mentioned methods are seemingly dependent on the representation of the system, as they are stated in either system matrices or by explicit functions. However, properties like controllability and stabilizability are clearly independent of any choice of basis. This suggests that it may be more natural to work with linear maps and subspaces than to work with explicit matrices. More eloquently, Wonham, one of the proponents of geometric control, stated in the introduction of [183] that:

> *The geometry was first brought in out of revulsion against the orgy of matrix manipulation which linear control theory mainly consisted of. Whereas the main advantage of the theory is that it works coordinate free, and hence, successfully captures the essence of many analysis and synthesis problems.*

The geometric theory of control views properties of the system in terms of subspaces of the state space. Here, the central concepts are *controlled invariant* subspaces and their duals, *conditioned invariant* spaces. These were independently introduced by Basile and Marro in [18] and Wonham and Morse in [184]. In layman's terms: A subspace is controlled invariant if, given a state inside it, there exists an input such that the resulting trajectory stays inside the subspace. For example, by definition the subspace containing all stabilizable points has this property. As such, characterizations of for example reachability and stabilizability were given in terms of geometric properties.

These and related notions can be applied in a myriad of analysis and control problems. Some noteworthy examples include the disturbance decoupling problem and the (algebraic) regulator problem. These, and many other results from geometric control may be found in the books [19, 165, 183].

1.2 CONSTRAINED SYSTEMS AND CONVEX PROCESSES

Recall that the most important property of a mathematical model is that it reflects the properties of the (natural) phenomena under investigation. However, in many real applications the assumption of linearity is too restrictive. As such, many of the above-mentioned concepts have been generalized towards a nonlinear setting. For an overview of different approaches, we point towards the works of Nijmeijer and Van der Schaft [124], Isidori [84] and Khalil [98]. These works largely deal with differentiable nonlinear systems, and as such, these systems can be linearized locally. This means that these theories need to recover the linear theory both as a special case, and as a limit case when viewed locally.

However, not all nonlinear systems exhibit such properties. A simple example is the following: Consider a mass on which a force is applied by means of a rope. With an idealized rope, forces can only be applied in one direction, which means that this input is constrained to be nonnegative. As such, the system is not linear. Similarly, many models use state variables that are physically incapable of being less than zero. In the presence of such constraints, it is no longer possible to locally linearize the system. This means that these situations are not (easily) captured by the general classes of nonlinear systems considered in the greater literature.

In addition, there is always a trade-off between generality of the system class and strength of the corresponding analysis results. Since results hold more readily for all linear systems than for all nonlinear systems, it seems prudent to consider a class of systems that is (in some sense) as small as possible. In the first part of this thesis, we will focus on developing analysis results for constrained linear systems. To be precise, we will focus on *convex conic constraints*.

1.2.1 Constrained linear systems

Of course, the problem of characterizing system-theoretic properties for constrained linear systems is not a new one. In broad strokes, the problem of analysis of constrained linear systems has been attacked from two sides: From either a control theory or a convex analysis standpoint.

Early papers from the control theory side have focused on *input constraints* only. To the best of our knowledge, the first work in this regard is Brammer's work characterizing controllability and null-controllability in [36]. To be precise, this paper considers nonnegative input constraints for continuous time systems. With regard to controllability, this work was extended by Saperstone in [149] and the problem was viewed geometrically in [79]. Another important consideration was the discrete time version of the same problem. This was resolved by Evans and Murthy in [52]. Later works, [51, 122], considered more general input constraints. Since, for discrete time linear systems controllability and null-controllability are not equivalent, the latter was more extensively investigated in discrete time by Sontag [160] and Nguyen [123].

Stepping away from input constraints, there are a number of types of systems with state constraints that are well studied. First among these are systems where the state is constrained to a subspace of the state space. Such systems are referred to as implicit, singular, descriptor, differential-algebraic (or difference-algebraic) systems. Due to the many applications of such systems, they have attracted significatn attention. Relating to the work in this thesis, we note the thorough overview on controllability of such systems in [24].

Another class of systems with state constraints is the class of *positive systems*. Simply put, these are systems whose states evolve in the nonnegative orthant of the state space. For a general overview, we refer to the work of Farina and Rinaldi in [54], and the references therein. With particular relevance to this thesis are the investigations of the geometry of the reachable set for such systems given in [21, 53, 128]. A recent paper in this vein, [187], points towards an interesting complication: Even in the presence of polyhedral state constraints, the structure of the reachable set might not be polyhedral.

One particularly interesting property of linear positive systems is with regard to Lyapunov functions. In papers by Willems [179] and Barker, Berman and Plemmons [17, 25] (see for a more modern approach [155]) it was shown that these systems are stabilizable if and only if they admit a *diagonal* quadratic Lyaponov function. As such, the amount of variables in the design of a Lyapunov function is linear, instead of quadratic, in the dimension of the state. Indeed, in recent papers by Rantzer [137–139] it is shown that this allows for well-scaling control schemes. This result was extended to the more general setting of the Kalman-Yakubovich-Popov lemma in [163, 164].

For other types of state constraints, invariance is also used as a central concept in the so-called set-theoretic methods of e.g. Blanchini and Miani [31, 32]. Based on these ideas, a currently popular method for control of constrained systems is that of control barrier functions. For an overview of this topic, see for instance [6]. Lastly, stabilizability was also investigated with more general input and state constraints in [145, 146, 153, 176, 177].

The last type of constraints discussed in the literature are *mixed constraints*, that is, constraints on the input and state simultaneously. In this setting the first characterization of reachability is the work by Heemels and Camlibel in [76]. To be precise, this paper considers right-invertible systems, and constraints that are polyhedral and solid, that is, they span the entire space. In [77], the same authors characterized null-controllability under similar conditions.

1.2.2 Convex processes

As said, the second field attacking the analysis of constrained systems is convex analysis. These methods use the language of convex processes. A *convex process* is a *set-valued map* whose graph is a convex cone. This thesis will explicitly work within a framework of *difference inclusions* with convex processes. This class of system has a few properties worth discussing.

First of all, this class can be used to describe any linear system with convex conic constraints on the input, state or both. This link can easily be made explicit, and therefore it is straightforward to translate any analysis result for convex processes into one for conically constrained systems and vice-versa. In addition, in a behavioral sense (see e.g. [180, 181], convex processes are a first step away from linearity. To be precise, note that linear systems are such that any linear combination of two trajectories is another trajectory. This property is reflected in the fact that for convex processes, any conic (or nonnegative) combination of trajectories results in a trajectory. As such, any investigation of properties of convex processes serves as a stepping stone towards a theory for more general nonlinear dynamics.

The analysis of system-theoretic properties of convex processes can be said to have started with the seminal paper by Aubin, Frankowska and Olech [14]. In this paper, the authors characterize reachability of *strict* (that is, nonempty everywhere) *convex processes* in continuous time. Following this this, Phat and Dieu investigated reachability and null-controllability in discrete time in the paper [132]. More recently, in the works of Kaba and Camlibel [85, 86] it was shown that these conditions could be generalized towards certain classes of nonstrict convex processes. This revealed that a characterization of e.g. reachability for linear systems with state constraints was possible in a context of convex processes. Similarly, works by Phat [131] and Smirnov [159] have characterized stabilizability for strict convex processes.

These works have a few properties in common. First of all, each of them relies heavily on *duality* of convex processes. Secondly, these characterizations are all *spectral*, that is, in line with the Hautus Lemma. However, the theories behind these concepts are not as mature for nonstrict convex processes as they are for the strict case. As such, the first part of this thesis will discuss duality, eigenvalues, invariance, and the links between these concepts.

Of course, this all stands in a wider setting of analysis of convex processes. The history of convex processes starts with their definition by Rockafellar in [141, 142]. With regard to this thesis, we note a number of relevant works. Duality of convex processes in general was investigated by Borwein [33, 34] (see for a modern approach [8]). Leizarowitz [104] and Seeger [151] deal with eigenvalues and eigenvectors. Extending on the aforementioned characterizations of controllability, Lewis [105] investigates the robustness of this property in terms of distance to uncontrollability. Lastly, we note the work of Henrion [78], which deals with sequences and boundedness of convex processes. In a context of more general set-valued maps, a topic of particular interest is Viability theory [10, 13], which focuses on what we call the feasible set in this work.

Of course, any convex process is a linear process. This means that any result for the analysis of convex processes has a special case pertaining to linear systems with linear constraints, that is, difference algebraic systems. Controllability and observability of linear processes have also been studied in their own right, in e.g. [62].

1.2.3 Contributions of the first part

In Chapter 2 we will look at the following problem: Given a convex process and a convex cone, develop conditions under which we can guarantee the existence of an eigenvector of the process within the cone. Our solution to this problem will be shown to generalize all earlier similar results, and will be used as a stepping stone towards analysis of convex processes in Chapters 3 and 4.

This chapter also highlights an important difference between linear systems and convex processes: The latter cannot be represented as easily. This is due to the fact that, while a subspace can be represented by a (finite) number of linear equations, representing a convex cone requires a possibly infinite number of linear inequalities. As such, working with explicit representations would be very burdensome. Therefore, our main approach when developing analysis results is geometric in nature.

In particular, the conditions of the main theorems are stated in terms of cones and subspaces relating to the given convex process and cone. In fact, one of the main assumptions that is required is that the given convex cone is *weakly invariant* under the convex process. This notion precisely generalizes that of controlled invariance. This relation between weak invariance and controlled invariance is illustrative for the second part of Chapter 2, where we will formulate the conditions of the main theorem in terms of classical geometric control. This will rely on a realization linear processes as linearly constrained linear systems.

Chapter 3 contains the bulk of the new results on analysis of convex processes. In this chapter, we will use the results of Chapter 2, in order to obtain Hautus-like characterizations of reachability, null-controllability and stabilizability. This chapter will use analogues of many of the techniques mentioned above:

First, we will generalize the geometric characterizations of these properties. This will allow us to properly define e.g. reachability in terms of convex cones. We will consider weakly and strongly invariant cones, which we correspond to controlled and conditioned invariant subspaces respectively, and show that under certain conditions theses notions are dual. As for linear systems we will see that the reachable, stabilizable and null-controllable sets naturally have such invariance properties.

Note that any cone contains a largest subspace and is contained in a smallest subspace. Using this with respect to the graph of a convex process gives rise to the definitions of the minimal and maximal linear processes. Clearly, in order for a convex process to be reachable, it is required that the maximal linear process is reachable. In addition, as we will show in Chapter 2, the study of linear processes is equivalent to that of linearly constrained systems. As such, we can characterize the reachable set of the maximal process. Similar to the classical Kalman decomposition of linear systems, we will consider the convex process within and outside of the reachable set of the maximal linear process.

Under a condition on the *domain* of the convex process, this will allow us to characterize reachability and stabilizability in the vein of the Hautus lemma. We further prove that under these conditions a reachable convex process is guaranteed to be null-controllable and therefore controllability. However, as for unconstrained linear systems, null-controllability does not imply reachability. Under a further condition on the *image*, we can also characterize null-controllability. These results will generalize all previously known results.

Lastly, in Chapter 4, we develop Lyapunov theory for nonstrict convex processes. The results of this chapter also come in two parts. At first, we develop a framework of extended real-valued Lyapunov functions. Using functions that are allowed to be infinite outside of the constraints of the system, will allow us to better reflect the constraints in the design of our Lyapunov functions. In particular, we will define notions of weak and strong Lyapunov functions, corresponding to the specific notions of *uniform exponential stabilizability* and *strong stability*. One of the main results of this part, is that a convex process is uniformly exponentially stabilizable if and only if there exists a weak Lyapunov function in a certain class.

The second part of Chapter 4 is motivated by two related observations. First, there is the duality that is central in Chapter 3: In the characterization of stabilizability it can be seen that stabilizability of a convex process and strong stability of its dual are related. Furthermore, earlier work like [159] employed

Lyapunov functions in proving the relation between stabilizability of a primal system and strong stability of the dual. This relation was made much more explicit in [69]. As such, we will generalize the aforementioned results to work for nonstrict convex process. After this, we will bring together a number of results on duality into a result that links different stability notions, Lyapunov functions and duality.

1.3 DATA INFORMATIVITY

So far, we have dealt with problems in *model-based* systems and control theory: Given a dynamical system as a model of some (natural) phenomenon, we developed tests for certain properties the system might have. A clear prerequisite of any such problem is the construction of a suitable model, for example from first-principles modeling. However, this may not always be tractable. As such, the second part of this thesis deals with situations where such a model is unknown. To offset this lack of knowledge, we assume that we have access to *measurements* of the system. The goal of this part is performing analysis and designing controllers for the unknown system based on these measured data.

1.3.1 Data-driven analysis and control

Recently, the field of data-driven control has been extremely active. While it is impossible to list all contributions, we note the works of [1,2,16,35,43,58,59,63, 71,95,121,129,154,158] with regard to control. In addition to data-driven control we also mention the data-driven analysis methods of e.g. [106, 125, 178, 188].

A particularly relevant result is Willems' fundamental lemma, which was proven in the paper by Willems et al. [182]. For a recent proof using state space methods see [171]. This result shows that all trajectories of a linear time invariant system can be written in terms of a *single* measured trajectory, provided this measurement is *sufficiently exciting*. This work immediately had repercussions in the field of data-driven control.

The first results building on Willems' lemma were developed in a behavioral context by Markovsky and Rapisarda [111, 112]. More recently, the work was brought into the context of state space systems in for instance [15, 23, 30, 42, 72, 83, 118]. Also data-driven analysis has benefited from the development of the fundamental lemma. A particularly active area is the study of dissipativity on the basis of measured system trajectories, as seen in e.g. [22, 113, 143, 144]. Extensions with regard to controllability are provided in [186].

Since a large part of this thesis is on constrained systems, we specifically note the developments in model predictive control (MPC). Many methods arising from MPC are particularly well suited to constrained systems. For an overview of such methods, we refer to the book by Mayne et al. [116] or the survey paper by Mayne [115]. Using Willems' lemma, MPC has recently been brought into a data-based context in [3, 4, 41, 133].

In a sense, these works all use a single trajectory of a system as a model for the system. In this thesis we will be interested in problems where potentially a model can not be uniquely identified from the data. Inspired by the concept of data informativity in system identification [66, 67, 108], we will look at informativity for other system-theoretic properties. This means that we will present conditions *on the data* under which it is guaranteed that the measured system has a given property. Clearly, informativity is a prerequisite for *any* type of control problem: Before trying to obtain a controller from data, it should be checked whether such controller can exist. Of course, previous methods of data-driven analysis and control have (necessarily) assumed that the data are informative, but the property itself has not been studied in the literature explicitly. As such, many earlier works have employed assumptions on the data that are more restrictive than necessary.

The work on informativity presented in this thesis has led to applications of the framework to a myriad of analysis and control problems, model classes, and data structures. In particular, suboptimal LQR and $\mathcal{H}_2$ control were discussed in [173], control based on noisy data in [169, 170], the paper [166] considers tracking and regulation, and model reduction in [38]. Additionally, the paper [168] considers the essential problem of obtaining informative data from experiments.

1.3.2 Contributions of the second part

In Chapter 5 we introduce the *data informativity framework*. This framework is based on the following simple observation: Suppose that we wish to assess whether an unknown system has a certain property based only on measurements performed on the system. We can only guarantee that the property holds for the unknown system if it holds for *all* systems that could have generated the data. As such, we will focus on developing methods that guarantee analysis and control properties for sets of systems.

After setting up the data informativity framework, we will resolve a number of informativity problems. To be precise we will consider measured input and state data, and characterize controllability, stabilizability and stability. In addition to analysis results, we will also resolve a number of control problems. Here we will characterize stabilization by static state feedback and the LQR problem. The latter of these is interesting in the fact that informativity for LQR design implies that we can uniquely recover the true system from data. This means that, from an information standpoint, data-driven LQR cannot outperform system identification combined with model-based methods. The last results of this chapter will involve also measured outputs, with which we will characterize informativity for stabilization using dynamic measurement feedback.

Chapter 6 will deal with informativity problems involving noisy data. This means that we will assume the true system is unknown and we will develop tests characterizing structural properties based on noisy measurements of the state, input, and outputs. Specifically, we will consider unbounded, structured noise. As such, we will assume that our true system is linear, where the state map is unknown but the other system maps are known. This situation arises in, for example, situations of networks: It is known how the disturbances and inputs enter the system, but unknown what happens internally. Mathematically, we will show that the set of such systems consistent with given measurements is an affine set of a certain form.

After this, we will give conditions under which the Rosenbrock system matrix satisfies a rank condition for all systems in such a set. This will naturally allow us to characterize (strong) controllability, stabilizability, observability and detectability using the same result. In a similar fashion, we will consider subspaces that are invariant for all systems in a specific affine set of systems. This will provide geometric characterizations of the aforementioned properties.

Lastly, in Chapter 7 we will bring together results from both parts of this thesis, by developing informativity conditions for reachability and null-controllability the class of convex processes. In order to do this we develop a few extensions of our previous work.

As before, developing informativity conditions will require us to develop analysis results for a set of systems. Here, the framework of set-valued maps will prove very useful: Given a set of measurements on the state, we will show that the set of all convex processes consistent with the data can be conveniently written in terms of the graph. To be precise, this set is equal to all convex processes which are greater than (with respect to the graph) a given convex process. We will then prove that, under certain conditions, if this 'lower bound' is reachable or null-controllable, so are all "larger" convex processes.

A further development is that we will specify the analysis results of Chapter 3 and formulate them specifically for polyhedral convex processes. Bringing these together will allow us to develop informativity conditions for reachability of convex processes.

1.4 PUBLICATIONS

This thesis is based on the following publications.
Journal articles.

1. J. Eising and M.K. Camlibel "On reachability and null-controllability of nonstrict convex processes", *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 751-756, 2019 (**Ch. 3**, [45]).

2. H.J. van Waarde, J. Eising, H.L. Trentelman, and M.K. Camlibel, "Data informativity: a new perspective on data-driven analysis and control", *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4753-4768, 2020 (**Ch. 5**, [172]).

3. J. Eising and H.L. Trentelman, "Informativity of noisy data for structural properties of linear systems", submitted for publication, 2020, arXiv: 2010.15564 (**Ch. 6**, [49]).

4. J. Eising and M.K. Camlibel, "Data informativity for analysis of linear systems with convex conic constraints", accepted for publication in *IEEE Control Systems Letters*, 2020, arXiv: 2103.02430 (**Ch. 7**, [47])

5. J. Eising and M.K. Camlibel, "On eigenvalues/eigenvectors of convex processes", submitted for publication, 2021, arXiv: 2106.05004 (**Ch. 2**, [48]).

6. J. Eising and M.K. Camlibel, "A geometric approach to reachability and stabilizability of convex processes", submitted for publication, 2021, arXiv: 2106.05128 (**Ch. 3**, [44]).

Conference articles.

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Part I

CONVEX PROCESSES

2 ON EIGENVALUES AND EIGENVECTORS OF CONVEX PROCESSES

In this chapter, we will begin our investigation of convex processes. We will focus on proving that under certain conditions there exists an eigenvector in a given cone corresponding to a nonnegative eigenvalue. This result will prove instrumental in the results of coming chapters. The second part of this chapter will illustrate how linear processes and linear systems are linked. In particular, this will reveal geometric characterizations for the conditions of the main result.

2.1 INTRODUCTION

Eigenvalues and eigenvectors of convex processes have been studied in the literature from different angles and for different purposes. In particular, existence of eigenvectors of convex processes within invariant cones has been investigated (see e.g. [14, Thm. 4.1], [132, Thm. 2.1], [159, Thm. 2.13]). More precisely, it is shown (see [86, Thm. 3.2]) that a closed convex process H admits an eigenvector corresponding to a nonnegative eigenvalue within a nonzero closed convex pointed cone $\mathcal{K}$ if $H(0) \cap \mathcal{K} = \{0\}$ and $H(x) \cap \mathcal{K} \neq \varnothing$ for all $x \in \mathcal{K}$. This result and its variants have been employed in the study of controllability and stabilizability of differential/difference inclusions with *strict* closed convex processes in [14, 131, 132, 159] and with particular *nonstrict* convex processes in [45, 86].

The main result of this chapter, Theorem 2.3, deals with the case for which $\mathcal{K}$ may contain a line. Under certain assumptions, Theorem 2.3 establishes not only existence of eigenvectors but also provides information about the locations of them. In addition, we prescribe a way to verify the assumptions of Theorem 2.3. The main contribution of the chapter is twofold. On the one hand, the results we present shed a new light on the spectral properties of convex processes by extending the existing results. On the other hand, they enable spectral characterizations of reachability, (null-)controllability, and stabilizability of difference inclusions with *nonstrict* convex processes, as will be studied in Chapter 3.

Of course, this chapter stands in a broader context of spectral analysis of set-valued maps. An introduction to this topic can be found in [151]. The links between stability and eigenvalues were explored in [5, 40, 65, 103]. However, as was also noted in [151], if the set-valued map under consideration is not a convex process, then the chances of obtaining an extension of known results are remote. With regard to the related work on eigenvalues of convex processes, the paper [104] studies extremal characterizations of eigenvalues and [64] studies "higher-order" eigenvalues in the context of weak asymptotic stability.

Similarly relevant works are those developing different generalizations of the Perron-Frobenius theorem regarding linear maps with eigenvectors in given cones. Here, we specifically note [29, 174]. Furthermore, there is the Kreĭn-Rutman theorem [102], which generalizes this to the context of Banach spaces. An extensive overview of this topic can be found in [162].

The organization of this chapter is as follows. Section 2.2 is devoted to the preliminaries whereas Section 2.3 presents the main results. In Section 2.4, we discuss how the assumptions of main results can be verified. Finally, Section 2.5 closes the chapter with conclusions.

2.2 PRELIMINARIES

For two sets $\mathcal{S}, \mathcal{T} \subseteq \mathbb{R}^n$ and a scalar $\rho \in \mathbb{R}$, we define $\mathcal{S} + \mathcal{T} := \{s + t \mid s \in \mathcal{S}, t \in \mathcal{T}\}$ and $\rho\mathcal{S} := \{\rho s \mid s \in \mathcal{S}\}$. By convention $\mathcal{S} + \varnothing = \varnothing$ and $\rho\varnothing = \varnothing$. We denote the closure of a set $\mathcal{S}$ by $\mathrm{cl}(\mathcal{S})$. For a convex set $\mathcal{S}$, we denote its relative interior by $\mathrm{ri}(\mathcal{S})$. We let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product and $|\cdot|$ the Euclidean norm.

A nonempty set $\mathcal{C}$ is said to be a *cone* if $\rho x \in \mathcal{C}$ for all $x \in \mathcal{C}$ and $\rho \geqslant 0$. Given a convex cone $\mathcal{C} \subseteq \mathbb{R}^n$, we define $\mathrm{lin}(\mathcal{C}) = \mathcal{C} \cap -\mathcal{C}$ and $\mathrm{Lin}(\mathcal{C}) = \mathcal{C} - \mathcal{C}$. These are respectively the largest subspace contained in $\mathcal{C}$ and the smallest subspace containing $\mathcal{C}$. A cone $\mathcal{C}$ is said to be *pointed* if it does not contain a line, i.e. $\mathrm{lin}(\mathcal{C}) = \{0\}$.

We can identify any set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with a subset of $\mathbb{R}^n \times \mathbb{R}^n$ by considering the *graph*:

$$\mathrm{gr}(H) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y \in H(x)\}.$$

Using this identification, we say that the map H is *closed*, *convex*, a *process* or a linear process if its graph is closed, convex, a cone or a subspace respectively. Direct application of this definition shows that if H is a convex process, then $H(x)+H(y) \subseteq H(x+y)$ for all $x, y \in \operatorname{dom}(H)$. In addition, $H(\rho x) = \rho H(x)$ for all $x \in \mathbb{R}^n$ and all $\rho > 0$. We define the *domain*, *image* and *kernel* of a set-valued map by

$$\begin{aligned} \operatorname{dom}(H) &:= \{x \in \mathbb{R}^n \mid H(x) \neq \varnothing\}, \\ \operatorname{im}(H) &:= \{y \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^n \text{ s.t. } y \in H(x)\}, \\ \ker(H) &:= \{x \in \mathbb{R}^n \mid 0 \in H(x)\}, \end{aligned}$$

respectively. If H is a convex process, the domain, image and kernel are all convex cones. However, these sets are not necessarily closed even if H is closed.

A set-valued map is said to be *strict* if $\operatorname{dom}(H) = \mathbb{R}^n$. For any set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we define the inverse H^{-1} by letting $x \in H^{-1}(y)$ if and only if $y \in H(x)$. This makes it clear that the domain of H is equal to the image of H^{-1} and vice versa. In terms of the graph, the inverse can be expressed as

$$\operatorname{gr}(H^{-1}) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \operatorname{gr}(H). \tag{2.1}$$

We will denote the image of the set $\mathcal{S}$ under H by $H(\mathcal{S}) := \{y \in \mathbb{R}^n \mid \exists x \in \mathcal{S} \text{ s.t. } y \in H(x)\}$. A direct application of the definitions shows that

$$H(\mathcal{S}) = \begin{bmatrix} 0 & I \end{bmatrix} \big(\operatorname{gr}(H) \cap (\mathcal{S} \times \mathbb{R}^n) \big). \tag{2.2}$$

For $q > 1$, we define the q-th power of a set-valued map, $H^q : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$H^q(x) := H(H^{q-1}(x)) \quad \forall x \in \mathbb{R}^n, \tag{2.3}$$

with the convention that H^0 is equal to the identity map. For given $\lambda \in \mathbb{R}$, we define $H - \lambda I$ as the set-valued map such that $(H - \lambda I)(x) = \{y - \lambda x \in \mathbb{R}^n \mid y \in H(x)\}$.

We define the negative dual of a convex process H by

$$p \in H^-(q) \iff \langle p, x \rangle \geqslant \langle q, y \rangle \quad \forall\, (x, y) \in \operatorname{gr}(H). \tag{2.4}$$

The negative dual is a *closed* convex process, regardless of whether H is closed. For a nonempty set $\mathcal{C} \subseteq \mathbb{R}^n$, we define the *negative polar cone* by

$$\mathcal{C}^- := \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leqslant 0 \quad \forall x \in \mathcal{C}\}.$$

This allows us to characterize the negative dual in terms of the graph as

$$\mathrm{gr}(H^-) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} (\mathrm{gr}(H))^- . \tag{2.5}$$

It is straightforward to check that if H is a convex or linear process, then so are powers H^q for all q, the inverse H^{-1}, $H - \lambda I$ for all $\lambda \in \mathbb{R}$ and H^-.

Using the graph, we define the *minimal* and *maximal linear processes* of a convex process H, denoted by L_- and L_+ respectively, as

$$\mathrm{gr}(L_-) := \mathrm{lin}(\mathrm{gr}(H)) \quad \text{and} \quad \mathrm{gr}(L_+) := \mathrm{Lin}(\mathrm{gr}(H)).$$

Clearly, L_- and L_+ are, respectively, the largest and smallest (with respect to graph inclusion) linear processes that satisfy

$$\mathrm{gr}(L_-) \subseteq \mathrm{gr}(H) \subseteq \mathrm{gr}(L_+).$$

In the case that the process H is not clear from context, we will denote these processes by $L_-(H)$ and $L_+(H)$ to avoid confusion.

If H is a convex process whose graph contains a nontrivial subspace, we can apply the following lemma to simplify its structure.

Lemma 2.1. Let H be a convex process and let L be a linear process such that $\mathrm{gr}(L) \subseteq \mathrm{gr}(H)$. For all $x \in \mathrm{dom}(H)$, $y \in \mathrm{dom}(L)$, we have

$$H(x+y) = H(x) + L(y).$$

Proof. Let $x \in \mathrm{dom}(H)$ and $y \in \mathrm{dom}(L)$. We will prove the equality by mutual inclusion. Note that, as $\mathrm{gr}(L) \subseteq \mathrm{gr}(H)$, we know that $L(y) \subseteq H(y)$, and therefore

$$H(x) + L(y) \subseteq H(x) + H(y) \subseteq H(x+y).$$

For the reverse inclusion, first observe that $y \in \mathrm{dom}(L)$ implies that $-y \in \mathrm{dom}(L)$ as L is a linear process. Then, we have

$$H(x+y) + L(-y) \subseteq H(x+y) + H(-y) \subseteq H(x).$$

This shows that $H(x+y) \subseteq H(x) - L(-y) = H(x) + L(y)$, where the last equality follows from L being a linear process. ■

A central role in this chapter will be played by weakly H invariant cones:

Definition 2.1. Let H be a convex process, we say that a convex cone $\mathcal{C}$ is weakly H invariant if $H(x) \cap \mathcal{C} \neq \varnothing$ for all $x \in \mathcal{C}$. Equivalently, $\mathcal{C}$ is weakly H invariant if $\mathcal{C} \subseteq H^{-1}(\mathcal{C})$.

A real number λ and vector $\xi \in \mathbb{R}^n \setminus \{0\}$ form an *eigenpair* of H if $\lambda\xi \in H(\xi)$. In this case λ is called an *eigenvalue* and ξ is called an *eigenvector* of H. For each real number λ and convex process H, it is easily verified that the convex cone $\ker(H - \lambda I)$ contains all eigenvectors corresponding to λ and the vector 0. This set is called the *eigencone* corresponding to λ. This means that λ is an eigenvalue of H if and only if $\ker(H - \lambda I) \neq \{0\}$.

If H is a convex process and $\lambda \geqslant 0$, then the eigencone corresponding to λ is a weakly H invariant cone, as $\lambda x \in H(x) \cap \ker(H - \lambda I)$ for all $x \in \ker(H - \lambda I)$.

As noted in the introduction, we will investigate the eigenvalues of H with corresponding eigenvectors in a weakly H invariant cone. For this, we define the *spectrum of* H *with respect to* $\mathcal{K}$ as

$$\sigma(H, \mathcal{K}) := \{\lambda \in \mathbb{R} \mid \exists \xi \in \mathcal{K} \setminus \{0\} \text{ such that } \lambda\xi \in H(\xi)\}.$$

If $\mathcal{C}, \mathcal{K}$ are cones such that $\mathcal{C} \subseteq \mathcal{K}$, then it is clear that $\sigma(H, \mathcal{C}) \subseteq \sigma(H, \mathcal{K})$. Note that, unlike the common definition of the spectrum of a linear map, we consider only real eigenvalues.

2.3 MAIN RESULTS

Our goal is to study eigenvalues and eigenvectors of convex processes. Before stating our main theorem, we make a few observations on properties of spectra. We begin with elementary results on the closedness and boundedness of the spectrum of a closed convex process.

Lemma 2.2. Let H be a closed convex process and $\mathcal{K}$ be a convex cone. Then, $\sigma(H, \mathcal{K})$ is

i. closed if $\mathcal{K}$ is closed.

ii. bounded above if $H(0) \cap \mathrm{cl}(\mathcal{K}) = \{0\}$.

Proof. Closedness of $\sigma(H, \mathcal{K})$ readily follows from those of H and $\mathcal{K}$. For the boundedness, suppose that $\sigma(H, \mathcal{K})$ is not bounded above. Then, we can take a sequence of eigenvalues of H, $(\lambda_k)_{k \in \mathbb{N}}$, such that $\lambda_k > k$ for each k. Let ξ_k be an eigenvector corresponding to the eigenvalue λ_k with $|\xi_k| = 1$. Note that

$$\left(\frac{1}{\lambda_k}\xi_k, \xi_k\right) \in \mathrm{gr}(H), \tag{2.6}$$

since H is a convex process. It follows from the Bolzano–Weierstrass theorem that $(\xi_k)_{k \in \mathbb{N}}$ converges, say to ξ, on a subsequence. Clearly, we have $|\xi| = 1$.

By taking the limit in (2.6) on that subsequence, we see that $(0, \xi) \in \mathrm{gr}(H)$ as H is closed. Therefore, we have $\xi \in H(0) \cap \mathrm{cl}(\mathcal{K})$. From the hypothesis, we obtain $\xi = 0$ which is a contradiction. Consequently, $\sigma(H, \mathcal{K})$ must be bounded above. ∎

Next, we deal with finiteness of spectra. Linear transformations mapping $\mathbb{R}^n$ to $\mathbb{R}^n$ are particular instances of linear (and hence convex) processes. Both a linear transformation and its dual have finitely many eigenvalues. A curious question to ask whether there are other convex processes that enjoy a similar finiteness property. It turns out that linearity is a crucial property for the spectra of a convex process and its dual to be finite at the same time.

We say a set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an *n-dimensional linear process* if its graph is an n-dimensional subspace. Typical examples of n-dimensional linear processes are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Note that inverse of an n-dimensional linear process is also an n-dimensional linear process.

Theorem 2.1. Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex process. Suppose that H is not an n-dimensional linear process. Then, any real number is an eigenvalue of either H or H^-.

Proof. Suppose λ is an eigenvalue of neither H nor H^-. This means that $\ker(H^- - \lambda I) = \{0\}$. By [12, Proposition 2.5.6], we know that $\ker(H^- - \lambda I) = \mathrm{im}(H - \lambda I)^-$. Therefore, we have that $\mathrm{im}(H - \lambda I) = \mathbb{R}^n$. This implies that $\mathrm{dom}(H - \lambda I)^{-1} = \mathbb{R}^n$. In other words, $(H - \lambda I)^{-1}$ is strict. On the other hand, as λ is not an eigenvalue of H, we see that $\ker(H - \lambda I) = \{0\}$. Therefore, we have that $(H - \lambda I)^{-1}(0) = \{0\}$. Then, it follows from [142, Theorem 39.1] that $(H - \lambda I)^{-1}$ is a linear transformation and hence an n-dimensional linear process. Consequently, $(H - \lambda I)$ is also an n-dimensional linear process. Note that

$$\mathrm{gr}(H - \lambda I) = \begin{bmatrix} I & 0 \\ -\lambda I & I \end{bmatrix} \mathrm{gr}(H).$$

Since the matrix on the right hand side is nonsingular, we see that H is an n-dimensional linear process as well. ∎

Example 2.1. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by

$$H(x) := \begin{cases} [\frac{1}{2}x, 2x] & x \geqslant 0, \\ \varnothing & x < 0. \end{cases}$$

Clearly, any $\lambda \in [\frac{1}{2}, 2]$ is an eigenvalue of H. We can find the dual to be:

$$H^-(x) := \begin{cases} [2x, \infty) & x \geqslant 0, \\ \left[\frac{1}{2}x, \infty\right) & x < 0. \end{cases}$$

Indeed, any $\lambda \notin (\frac{1}{2}, 2)$ is an eigenvalue of H^-.

The converse of this theorem is not true in general: Not all n-dimensional linear processes have only finitely many eigenvalues. For instance, let H be given by $\operatorname{gr}(H) := (\{0\} \times \mathbb{R}) \times (\{0\} \times \mathbb{R})$. Then H is a 2-dimensional linear process and all real numbers are eigenvalues of H.

We now approach our main result. As stated in the introduction, we will first discuss the result we aim to generalize. The following proposition provides conditions for the existence of eigenvectors contained in weakly invariant cones under convex processes.

Proposition 2.1 ([86, Thm. 3.2])**.** Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a closed convex process and $\{0\} \neq \mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex pointed cone. Suppose that $\mathcal{K}$ is weakly H invariant and $H(0) \cap \mathcal{K} = \{0\}$. Then, $\mathcal{K}$ contains an eigenvector of H corresponding to a nonnegative eigenvalue.

This proposition is a slight generalization of similar statements that appeared in the literature before (e.g. [14, Thm. 4.1], [132, Thm. 2.1], and [159, Thm. 2.13]). These results were employed in the study of differential/difference inclusions involving *strict* convex processes. Based on them, [14, Thm. 0.4], [132, Thm. 3.1], and [159, Ch. 6] characterize reachability and [131, Thm. 3.1] and [159, Thm. 8.10] weak asymptotic stability of *strict* convex processes in terms of the spectral properties of their dual processes. In Chapter 3, we will develop a framework to study similar system theoretic properties of *nonstrict* convex processes. It turns out that the pointedness hypothesis of Proposition 2.1 is typically not satisfied in the context of *nonstrict* convex processes. This calls for a study of existence of eigenvectors contained in weakly invariant cones that *may* contain lines.

However, the proof of Proposition 2.1 heavily relies on the assumption that $\mathcal{K}$ is pointed. Our approach to resolve this issue is based on the following decomposition: Let $\mathcal{K}$ be a convex cone and $\mathcal{W}$ be a subspace such that $\mathcal{W} \subseteq \mathcal{K}$. Then, we can express $\mathcal{K}$ (see e.g. [142, page 65]) as the direct sum

$$\mathcal{K} = \mathcal{W} \oplus \left(\mathcal{K} \cap \mathcal{W}^\perp\right). \tag{2.7}$$

We will investigate the behavior of H within $\mathcal{K}$ by looking at the behavior in $\mathcal{W}$ and $\mathcal{K} \cap \mathcal{W}^{\perp}$ separately. For this, we will require two convex processes associated to H. We define the restriction of H to $\mathcal{K}$ by

$$\mathrm{gr}(H_{\mathcal{K}}) := \mathrm{gr}(H) \cap (\mathcal{K} \times \mathcal{K}). \tag{2.8}$$

Based on (2.7), we define the convex process $H_{\mathcal{K},\mathcal{W}}$ by

$$\mathrm{gr}(H_{\mathcal{K},\mathcal{W}}) := \big(\mathrm{gr}(H_{\mathcal{K}}) + (\{0\} \times \mathcal{W})\big) \cap \Big((\mathcal{K} \cap \mathcal{W}^{\perp}) \times (\mathcal{K} \cap \mathcal{W}^{\perp}) \Big). \tag{2.9}$$

In the following, we will describe how eigenvectors of H in $\mathcal{K} \setminus \mathcal{W}$ are related to eigenvectors of $H_{\mathcal{K},\mathcal{W}}$ in the set $\mathcal{K} \cap \mathcal{W}^{\perp}$. The main benefit of using this relations is found for the particular choice of $\mathcal{W} = \mathrm{lin}(\mathcal{K})$. As $\mathcal{K} \cap \mathrm{lin}(\mathcal{K})^{\perp}$ is a pointed cone. for any $\mathcal{K}$, the existence of eigenvectors of $H_{\mathcal{K},\mathcal{W}}$ in $\mathcal{K} \cap \mathrm{lin}(\mathcal{K})^{\perp}$ can be analyzed by employing Proposition 2.1. This line of reasoning will allow us to weaken the assumptions made in Proposition 2.1, allowing for cones $\mathcal{K}$ that may contain a line.

To relate the eigenvalues of H with those of $H_{\mathcal{K},\mathcal{W}}$, we need the subsequent technical result.

Lemma 2.3. Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a closed convex process and $\{0\} \neq \mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex cone. Suppose that $\mathcal{K}$ is weakly H invariant and $H(0) \cap \mathcal{K}$ is a subspace. Let $\mathcal{W}$ be a subspace such that $H(0) \cap \mathcal{K} \subseteq \mathcal{W} \subseteq \mathcal{K}$. Then, we have:

i. $H_{\mathcal{K},\mathcal{W}}$ is closed,

ii. $H_{\mathcal{K},\mathcal{W}}(0) = \{0\}$,

iii. $\mathcal{K} \cap \mathcal{W}^{\perp}$ is weakly $H_{\mathcal{K},\mathcal{W}}$ invariant.

Proof. (i): It suffices to verify the closedness of the set $\mathrm{gr}(H_{\mathcal{K}}) + (\{0\} \times \mathcal{W})$ since $\mathcal{K} \cap \mathcal{W}^{\perp}$ is closed and the intersection of closed sets is closed. In view of [142, Corollary 9.1.1], it is enough to show that $\mathrm{gr}(H_{\mathcal{K}}) \cap (\{0\} \times \mathcal{W})$ is a subspace. Note that

$$\begin{aligned} \mathrm{gr}(H_{\mathcal{K}}) \cap (\{0\} \times \mathcal{W}) &= \mathrm{gr}(H) \cap (\mathcal{K} \times \mathcal{K}) \cap (\{0\} \times \mathcal{W}) \\ &= \mathrm{gr}(H) \cap (\{0\} \times \mathcal{W}) \\ &= \{0\} \times (H(0) \cap \mathcal{W}). \end{aligned}$$

Now, as $(H(0) \cap \mathcal{K}) \subseteq \mathcal{W}$, and both are subspaces by assumption, we see that $\mathrm{gr}(H_{\mathcal{K}}) \cap (\{0\} \times \mathcal{W})$ is a subspace. Therefore, $\mathrm{gr}(H_{\mathcal{K}}) + (\{0\} \times \mathcal{W})$ is closed and hence $H_{\mathcal{K},\mathcal{W}}$ is closed.

(ii): Note that $H_{\mathcal{K},\mathcal{W}}(0) = (H_{\mathcal{K}}(0) + \mathcal{W}) \cap (\mathcal{K} \cap \mathcal{W}^{\perp})$. As $H_{\mathcal{K}}(0) \subseteq \mathcal{W}$ by assumption, we see that $H_{\mathcal{K},\mathcal{W}}(0) = \mathcal{W} \cap \mathcal{K} \cap \mathcal{W}^{\perp} = \{0\}$.
(iii): Let $\xi \in \mathcal{K} \cap \mathcal{W}^{\perp}$ and $\eta \in H_{\mathcal{K}}(\xi)$. By the definition of $H_{\mathcal{K}}$, we see that $\eta \in \mathcal{K}$. Due to (2.7) we can write $\eta = \zeta + \theta$, where $\zeta \in \mathcal{W}$ and $\theta \in \mathcal{K} \cap \mathcal{W}^{\perp}$. Note that $(\xi, \theta) = (\xi, \eta) + (0, -\zeta)$. Since $(\xi, \eta) \in \mathrm{gr}(H_{\mathcal{K}})$, $-\zeta \in \mathcal{W}$, and $\theta \in \mathcal{K} \cap \mathcal{W}^{\perp}$, we see that $(\xi, \theta) \in \mathrm{gr}(H_{\mathcal{K},\mathcal{W}})$ and hence $\mathcal{K} \cap \mathcal{W}^{\perp}$ is weakly $H_{\mathcal{K},\mathcal{W}}$ invariant. ■

We are in a position to relate the eigenvectors of H and $H_{\mathcal{K},\mathcal{W}}$.

Theorem 2.2. Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a closed convex process and $\mathcal{K} \subseteq \mathbb{R}^n$ be a weakly H invariant closed convex cone such that $H(0) \cap \mathcal{K}$ is a subspace. Let $\mathcal{W}$ be a subspace such that

(a) $H(0) \cap \mathcal{K} \subseteq \mathcal{W} \subseteq \mathcal{K}$,

(b) $\mathcal{W}$ is weakly $L_-(H)$ invariant and

(c) $\mathcal{W} \subseteq \big(L_-(H) - \lambda I\big)\mathcal{W}$ for all $\lambda \geqslant 0$.

Then the following hold:

1. $\sigma(H, \mathcal{K} \setminus \mathcal{W}) \cap \mathbb{R}_+ = \sigma(H_{\mathcal{K},\mathcal{W}}, \mathcal{K} \cap \mathcal{W}^{\perp}) \cap \mathbb{R}_+$ and the set $\sigma(H_{\mathcal{K},\mathcal{W}}, \mathcal{K} \cap \mathcal{W}^{\perp})$ is closed and bounded above.

2. If $\ker(H - \lambda I) \subseteq \mathcal{W}$ for $\lambda \geqslant 0$ then $\ker(H - \lambda I)$ is a subspace.

Proof. Before starting the proof, we define a linear process $L_{\mathcal{W}}$ by taking:

$$\mathrm{gr}(L_{\mathcal{W}}) = \mathrm{gr}(L_-(H)) \cap (\mathcal{W} \times \mathcal{W}).$$

Clearly $\mathrm{gr}(L_{\mathcal{W}}) \subseteq \mathrm{gr}(H_{\mathcal{K}})$ and it is straightforward to show that $\mathrm{dom}\, L_{\mathcal{W}} = \mathcal{W}$ if and only if $\mathcal{W}$ is weakly $L_-(H)$ invariant. Furthermore, by definition we know that

$$(L_{\mathcal{W}} - \lambda I)\mathcal{W} \subseteq \mathcal{W} \quad \forall \lambda \in \mathbb{R}. \tag{2.10}$$

As we can write $(L_{\mathcal{W}} - \lambda I)\mathcal{W} = \big((L_-(H) - \lambda I)\mathcal{W}\big) \cap \mathcal{W}$, we know that (c) and (2.10) imply that

$$(L_{\mathcal{W}} - \lambda I)\mathcal{W} = \mathcal{W} \quad \forall \lambda \geqslant 0. \tag{2.11}$$

We can now prove the claims of the theorem in order.

To prove 1, we note that by Lemma 2.3 $H_{\mathcal{K},\mathcal{W}}$ is closed. As $\mathcal{K} \cap \mathcal{W}^{\perp}$ is closed, we know the set $\sigma(H_{\mathcal{K},\mathcal{W}}, \mathcal{K} \cap \mathcal{W}^{\perp})$ is closed by Lemma 2.2. From Lemma 2.3 we also know that $H_{\mathcal{K},\mathcal{W}}(0) = 0$ and therefore by Lemma 2.2 we know that this spectrum is bounded above. We will prove the equality of the two spectra by mutual inclusion.

Let $\lambda \in \sigma(H, \mathcal{K} \setminus \mathcal{W}) \cap \mathbb{R}_+$. Then $\lambda \geqslant 0$ and there exists $\xi \in \mathcal{K} \setminus \mathcal{W}$ such that $\lambda\xi \in H(\xi)$. Clearly (λ, ξ) is then also an eigenpair of $H_{\mathcal{K}}$. By the direct sum (2.7) we can write $\xi = \zeta + \eta$ where $\zeta \in \mathcal{W}$ and $\eta \in \mathcal{K} \cap \mathcal{W}^{\perp}$. Using Lemma 2.1 and the fact that $\mathrm{dom}(L_{\mathcal{W}}) = \mathcal{W}$ we can use this decomposition to show that $\lambda(\zeta + \eta) \in H_{\mathcal{K}}(\eta) + L_{\mathcal{W}}(\zeta)$. By (2.11), we know $L_{\mathcal{W}}(\zeta) - \lambda\zeta \subseteq \mathcal{W}$ and we can conclude that

$$\lambda\eta \in H_{\mathcal{K}}(\eta) + \mathcal{W} \implies \lambda\eta \in H_{\mathcal{K},\mathcal{W}}(\eta).$$

Now, as $\xi \in \mathcal{K} \setminus \mathcal{W}$, we know that $\eta \neq 0$, and therefore

$$\sigma(H, \mathcal{K} \setminus \mathcal{W}) \subseteq \sigma(H_{\mathcal{K},\mathcal{W}}, \mathcal{K} \cap \mathcal{W}^{\perp}).$$

It now suffices to prove the reverse. For this, let $\lambda \in \sigma(H_{\mathcal{K},\mathcal{W}}, \mathcal{K} \cap \mathcal{W}^{\perp}) \cap \mathbb{R}_+$. In other words, $\lambda \geqslant 0$ and there exists $0 \neq \xi \in \mathcal{K} \cap \mathcal{W}^{\perp}$ such that $\lambda\xi \in H_{\mathcal{K},\mathcal{W}}(\xi)$. Using the definition of $H_{\mathcal{K},\mathcal{W}}$, we know there exists $\eta \in \mathcal{W}$ such that $\lambda\xi \in H_{\mathcal{K}}(\xi) + \eta$. Using (2.11), we can find $\zeta \in \mathcal{W}$ such that $\eta \in (L_{\mathcal{W}} - \lambda I)\zeta$. Now we can apply Lemma 2.1 to show that

$$\lambda\xi + \lambda\zeta \in H_{\mathcal{K}}(\xi) + L_{\mathcal{W}}(\zeta) = H_{\mathcal{K}}(\xi + \zeta) \subseteq H(\xi + \zeta).$$

As $\zeta \in \mathcal{W}$ and $\xi \neq 0$ we can conclude that $\xi + \zeta \in \mathcal{K} \setminus \mathcal{W}$. Combined with the first part, this proves the claim.

Next we prove 2. Let $\ker(H - \lambda I) \subset \mathcal{W}$ and let (λ, ξ) be an eigenpair of H with $\lambda \geqslant 0$. Then $\xi \in \mathcal{W} \subseteq \mathcal{K}$ and therefore (λ, ξ) is an eigenpair of $H_{\mathcal{K}}$. In fact by Lemma 2.1 we know that $H(\xi) = H_{\mathcal{K}}(\xi) = H_{\mathcal{K}}(0) + L_{\mathcal{W}}(\xi)$. As we assumed that $H_{\mathcal{K}}(0) = H(0) \cap \mathcal{K}$ is a subspace, we know that

$$\lambda(-\xi) \in H_{\mathcal{K}}(0) + L_{\mathcal{W}}(-\xi) = H_{\mathcal{K}}(-\xi) \subseteq H(-\xi).$$

Therefore $\xi \in \ker(H - \lambda I)$. As the set $\ker(H - \lambda I)$ is a closed convex cone, this implies that the set is also subspace. ■

The pointedness assumption of Proposition 2.1 can be weakened with the help of Lemma 2.3.

Theorem 2.3. Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a closed convex process and $\mathcal{K} \subseteq \mathbb{R}^n$ be a weakly H invariant closed convex cone such that $H(0) \cap \mathcal{K}$ is a subspace, $\mathrm{lin}(\mathcal{K})$ is weakly $L_{-}(H)$ invariant and $\mathrm{lin}(\mathcal{K}) \subseteq (L_{-}(H) - \lambda I)\,\mathrm{lin}(\mathcal{K})$ for all $\lambda \geqslant 0$. Then $\mathcal{K} = \mathrm{lin}(\mathcal{K})$ if and only if any eigenvector of H in K corresponding to an eigenvalue $\lambda \geqslant 0$ belongs to $\mathrm{lin}\, K$.

Proof. Taking $\mathcal{W} = \mathrm{lin}(\mathcal{K})$ in Theorem 2.2 , we see that $(a) - (c)$ hold. Note that $\mathrm{lin}(\mathcal{K}) = \mathcal{K}$ if and only if $\mathcal{K} \cap (\mathrm{lin}(\mathcal{K}))^{\perp} = \{0\}$. As $\mathcal{K} \cap (\mathrm{lin}(\mathcal{K}))^{\perp}$ is pointed, it follows from the results of Lemma 2.3 and Proposition 2.1 that $\mathcal{K} \cap (\text{lin}(\mathcal{K}))^{\perp} \neq \{0\}$ if and only if $\mathcal{K} \cap (\text{lin}(\mathcal{K}))^{\perp}$ does not contain an eigenvector of $H_{\mathcal{K},\mathcal{W}}$ that corresponds to $\lambda \geqslant 0$. By Theorem 2.2.1 these eigenvectors correspond to those of H in $\mathcal{K} \setminus \text{lin}(\mathcal{K})$, therefore this proves the claim. ∎

If the cone $\mathcal{K}$ is pointed, we know that $\text{lin}(\mathcal{K}) = \{0\}$. As in addition $\{0\}$ is weakly L_- invariant and $\{0\} \subseteq (L_- - \lambda I)\{0\} = L_-(0)$ for any H and λ, we see that this theorem generalizes Proposition 2.1.

Theorem 2.3 has two useful applications. The first of these is a spectral test for a given cone to be equal to a subspace. In general, testing whether this is true is nontrivial. This application is used in Chapter 3 (Theorems 3.1, 3.2, 3.4) to obtain necessary and sufficient conditions for reachability, stabilizability and null-controllability of nonstrict convex processes.

On the other hand, the negation of this theorem gives an existence result: Under the assumptions of Theorem 2.3, if $\mathcal{K} \neq \text{lin}(\mathcal{K})$ then there exists an eigenvector of H in $K \setminus \text{lin}(\mathcal{K})$.

Example 2.2. A noteworthy observation is that Theorem 2.3 leads to a generalization of a part of the well-known Perron-Frobenius theorem when applied to (single-valued) linear maps. To show this, let $H(x) = \{Ax\}$, where A is a matrix with nonnegative elements. Let $\mathcal{K}$ be the cone of nonnegative vectors. Clearly, $\mathcal{K}$ is a closed, pointed cone, and is weakly H invariant. Furthermore, since $H(0) = \{0\}$, all assumptions are satisfied. Therefore, since $\mathcal{K}$ is not a subspace, there exists a nonnegative real eigenvector corresponding to a nonnegative real eigenvector of A.

2.4 SATISFYING THE ASSUMPTIONS

At this point, one might wonder how to satisfy the assumptions of Theorem 2.2. It might seem that the assumptions $(a) - (c)$ are difficult to check. We will show here that neither is the case.

In fact, we will find the largest subspace $\mathcal{W}$ satisfying these assumptions, if one exists. As shown in Theorem 2.3, taking a larger $\mathcal{W}$ that satisfies the assumptions results in more information on the location of eigenvectors of H. In addition, if the assumption $H(0) \cap \mathcal{K} \subseteq \mathcal{W}$ does not hold for the largest subspace satisfying $(a) - (c)$, it does not hold for any such subspace. Therefore, we are interested in finding the largest subspace $\mathcal{W}$ that satisfies $(a) - (c)$.

It is straightforward to check that in the assumptions $(b) - (c)$, the process $L_-(H)$ can be replaced by any linear process L such that $\text{gr}(L_{\mathcal{W}}) \subseteq \text{gr}(L) \subseteq$

$\mathrm{gr}(H)$ without changing the proof. In particular the linear process $\widehat{L}$, defined by

$$\mathrm{gr}(\widehat{L}) = \mathrm{gr}(L_-(H)) \cap \mathrm{lin}(\mathcal{K}) \times \mathrm{lin}(\mathcal{K}), \tag{2.12}$$

satisfies this property and for this choice, (a) holds immediately. This means that we are interested in finding the largest subspace $\mathcal{W}$ such that $\mathcal{W}$ is weakly $\widehat{L}$ invariant and such that $\mathcal{W} \subseteq (\widehat{L} - \lambda I)\mathcal{W}$ for all $\lambda \geqslant 0$.

The main result of this section is a characterization of this subspace in terms of stabilizability subspaces of linear systems. As a consequence of this, we present an algorithm which finds this subspace in a finite amount of steps.

Next, we study the relation between linear processes and linear systems. Consider the discrete-time linear input/state/output system $\Sigma = \Sigma(A, B, C, D)$ given by

$$x_{k+1} = Ax_k + Bu_k, \tag{2.13a}$$
$$y_k = Cx_k + Du_k, \tag{2.13b}$$

where $k \in \mathbb{N}$, $u_k \in \mathbb{R}^m$ is the input, $x_k \in \mathbb{R}^m$ is the state, $y_k \in \mathbb{R}^m$ is the output, and A, B, C, D are matrices of appropriate dimensions.

We define L_Σ, the *linear process associated with* Σ by:

$$\mathrm{gr}(L_\Sigma) := \begin{bmatrix} I_n & 0 \\ A & B \end{bmatrix} \ker \begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} A & -I_n \\ C & 0 \end{bmatrix}^{-1} \mathrm{im} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{2.14}$$

where $M^{-1}(\mathcal{Y})$ denotes the preimage of the set $\mathcal{Y}$ under M, that is $M^{-1}(\mathcal{Y}) = \{x \mid Mx \in \mathcal{Y}\}$. Direct inspection shows that the second equality holds for any quadruple (A, B, C, D) with appropriate dimensions.

We say that a linear system Σ is a *realization* of a linear process L if $L = L_\Sigma$. Given a linear process $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, a realization Σ of L can be constructed as follows.

Example 2.3. Let L be a linear process, then we can construct a realization as follows: Let $T = 2n - \dim(\mathrm{gr}(L))$, then we can find $C, D \in \mathbb{R}^{T \times n}$ such that $\mathrm{gr}(L) = \ker \begin{bmatrix} C & D \end{bmatrix}$. Now, we take $A = 0_{n \times n}$ and $B = I_{n \times n}$. It is clear to see that $\Sigma(A, B, C, D)$ is a realization of L.

Example 2.4. Let L be a linear process, then we can construct a realization as follows: Let $T = \dim(\mathrm{gr}(L))$, then we can find $B, D \in \mathbb{R}^{n \times T}$ such that $\mathrm{gr}(L) = \mathrm{im} \begin{bmatrix} B \\ D \end{bmatrix}$. Now, we take $A = 0_{n \times n}$ and $C = 0_{n \times n}$. It is clear to see that $\Sigma(A, B, C, D)$ is a realization of L.

As the previous examples show, a linear process L admits many realizations. In the following section, we will discuss a few subspace algorithms that benefit from having lower input and output dimensions. In this regard, the realization in Example 2.3 has an obvious downside: The dimensions of its input is always equal to n, and the dimension of the output is equal to T. As such, we are interested in obtaining a representation whose input and output dimensions are minimal.

Lemma 2.4. Let $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a linear process. Denote $m = \dim(L(0))$, $p = n - \dim(\operatorname{dom}(L))$. Then, there exist matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ such that $\Sigma(A, B, C, 0_{p\times m})$ is a realization of L.

Proof. Let B, C be full rank matrices such that $\operatorname{im} B = L(0)$ and $\ker C = \operatorname{dom}(L)$. Let $\mathcal{V}$ be any subspace such that $\mathcal{V} \oplus L(0) = \mathbb{R}^n$. Let π denote the projection of $\mathbb{R}^n$ onto $\mathcal{V}$ along $L(0)$. We will first prove that for each $x \in \operatorname{dom} H$, we have that $\pi(L(x))$ is a singleton.

Let $y, \hat{y} \in \pi(L(x))$. By the definition of π we have that $y, \hat{y} \in \mathcal{V}$ and there exist $z, \hat{z} \in L(0)$ such that $y + z, \hat{y} + \hat{z} \in L(x)$. By linearity we have that $(y - \hat{y}) + (z - \hat{z}) \in L(0)$. Therefore $y - \hat{y} \in \mathcal{V} \cap L(0)$, which proves that $y = \hat{y}$.

As any projection is a linear map, we can take A any matrix such that $Ax = \pi(L(x))$ for all $x \in \operatorname{dom}(L)$. Then, it is straightforward to see that $\Sigma(A, B, C, 0_{p\times m})$ is a realization of L. ∎

Let $\widehat{\Sigma} = \Sigma(A, B, C, D)$ be a realization of the linear process $\widehat{L}$ as in (2.12). Note that

(i) $\mathcal{W}$ is weakly $\widehat{L}$ invariant if and only if

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{W} \subseteq \left((\mathcal{W} \times \{0\}) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \right). \tag{2.15}$$

(ii) $\mathcal{W} \subseteq (\widehat{L} - \lambda I)\mathcal{W}$ for all $\lambda \geqslant 0$ if and only if

$$\mathcal{W} \times \{0\} \subseteq \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \mathcal{W} + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \qquad \text{for all } \lambda \geqslant 0. \tag{2.16}$$

Let $\mathcal{V}_g$ denote the *stabilizable weakly unobservable subspace* with respect to the *stability domain* $\mathbb{C}_g = \mathbb{C} \setminus \mathbb{R}_+$ of the system $\widehat{\Sigma}$ (see e.g. [165, Sec. 7 and Ex. 7.16-7.17]). By definition, $\mathcal{V}_g$ is the largest of the subspaces $\mathcal{W}$ satisfying both (2.15) and (2.16). Therefore, we have $\mathcal{W}^* = \mathcal{V}_g$.

The subspace $\mathcal{V}_g$ (and hence $\mathcal{W}^*$) can be computed in terms of certain other subspaces associated with $\widehat{\Sigma}$. Indeed, it is well-known (see e.g. [165, Ex. 7.17c, Cor. 4.27, and Thm. 8.22]) that

$$\mathcal{V}_g = \big(\chi_g(A+BF) \cap \mathcal{V}\big) + (\mathcal{T} \cap \mathcal{V}). \tag{2.17}$$

Here $\mathcal{V}$ is the *weakly unobservable subspace* of $\widehat{\Sigma}$, F is a *friend* of $\mathcal{V}$, $\mathcal{T}$ is the *strongly reachable subspace* of $\widehat{\Sigma}$, and $\chi_g(A+BF)$ is the $\mathbb{C}_g$*−stable subspace* of $A+BF$. In what follows, we will discuss these ingredients further.

The weakly unobservable subspace $\mathcal{V}$ of $\widehat{\Sigma}$ can be computed via the following subspace algorithm:

$$\mathcal{V}_0 := \mathbb{R}^n \tag{2.18a}$$

$$\mathcal{V}_{\ell+1} := \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\mathcal{V}_\ell \times \{0\} + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \right) \quad \text{for } \ell \geqslant 0. \tag{2.18b}$$

It is well-known (see e.g. [165, Thm. 7.12]) that

$$\mathcal{V}_0 \supset \mathcal{V}_1 \supset \cdots \supset \mathcal{V}_r = \mathcal{V}_{r+1} = \mathcal{V} \tag{2.19}$$

for some $r \leqslant n$ where '$\supset$' denotes strict inclusion.

Note that

$$\mathcal{V} = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\mathcal{V} \times \{0\} + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \right) \tag{2.20}$$

since $\mathcal{V} = \mathcal{V}_r = \mathcal{V}_{r+1}$. From this property of $\mathcal{V}$, one can show that there exists $F \in \mathbb{R}^{m \times n}$ such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker(C+DF)$. Such an F matrix is called a *friend* of $\mathcal{V}$. One can find a friend as follows: If $\mathcal{V}$ is the zero subspace, then every $m \times n$ matrix is clearly a friend. If $\mathcal{V} = \mathbb{R}^n$, then (2.20) implies that $\operatorname{im} C \subseteq \operatorname{im} D$. Hence, there exists F such that $C + DF = 0$ and every such F is a friend. If $\mathcal{V}$ is a proper subspace, let $n > q = \dim(V) \geqslant 1$. Also, let $x_1, x_2, \ldots, x_n$ be a basis for $\mathbb{R}^n$ such that $x_1, x_2, \ldots, x_q$ is a basis for $\mathcal{V}$. From (2.20), we see that for $i \in \{1, 2, \ldots, q\}$ $Ax_i = v_i + Bu_i$ and $Cx_i + Du_i = 0$ where $v_i \in \mathcal{V}$ and $u_i \in \mathbb{R}^m$. Now, one can construct a friend F by taking $Fx_i = u_i$ for $i \in \{1, 2, \ldots, q\}$ and $Fx_i = 0$ for $i \in \{q+1, \ldots, n\}$.

The strongly reachable subspace $\mathcal{T}$ of $\widehat{\Sigma} = \Sigma(A, B, C, D)$ is the *dual* of $\mathcal{V}$ in the sense that $(\mathcal{T})^\perp$ is the weakly unobservable subspace of the dual system

$$\widehat{\Sigma}^\top = \Sigma(A^\top, C^\top, B^\top, D^\top).$$

As such, the subspace algorithm (2.18) can be used to compute $\mathcal{T}$ of $\widehat{\Sigma}$.

Finally, the $\mathbb{C}_g-$stable subspace of $A+BF$, $\chi_g(A+BF)$, is in order. Let χ be the characteristic polynomial of $A+BF$. Factorize χ as $\chi = \chi_g\chi_b$ where all roots of χ_g are in $\mathbb{C}_g = \mathbb{C}\setminus\mathbb{R}_+$ and those of χ_b are in $\mathbb{R}_+$. Then, we have

$$\chi_g(A+BF) = \ker \chi_g(A+BF).$$

Summarizing, existence of a subspace $\mathcal{W}$ satisfying the hypotheses (a)-(c) of Theorem 2.2 can be verified by first finding a realization for the linear process $\widehat{L}$ given in (2.12), then finding $\mathcal{V}_g$ from (2.17), and finally checking if $H(0)\cap\mathcal{K}\subseteq\mathcal{V}_g$. If this is the case, then applying Theorem 2.2 by taking $\mathcal{W}=\mathcal{V}_g$ results in the sharpest statements that can be achieved by this theorem.

2.5 CONCLUSION

In this chapter, we investigated the existence of eigenvectors of convex processes within a weakly invariant cone. For this we made some assumptions on the considered convex process, under which we revealed the link between the eigenvalues of the convex process and those of a related process. Using this allowed us to prove a generalization of all known related results.

In the second part of the chapter, the aforementioned assumptions were explained in terms of classical geometric control theory. For this we made explicit the link between, among others, weakly invariant and weakly unobservable subspaces. In particular this revealed that the assumptions of the main results are satisfied by the stabilizable weakly unobservable subspace of a linear system associated to the convex process. In particular, this results allows easy verification of the assumptions.

3 REACHABILITY AND STABILIZABILITY OF CONVEX PROCESSES

In the previous chapter, we investigated eigenvalues of convex processes. This chapter will apply these results to a number of analysis problems. To be precise, we will develop Hautus-like conditions for reachability, stabilizability and null-controllability. In addition, we will prove results for controllability. The results of this chapter will play a key role later in Chapters 4 and 7.

3.1 INTRODUCTION

This chapter deals with a class of nonlinear systems, namely the class of difference inclusions of the form

$$x_{k+1} \in H(x_k), \tag{3.1}$$

where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex process, that is, a set-valued maps whose graph is a convex cone. These maps and their associated difference inclusions are encountered in various contexts, and have many applications. As such, analysis of these systems has attracted attention.

Convex processes were first introduced by Rockafellar in [141, 142] in the context of convex analysis. As such, many of the results on the analysis of convex processes have been within this context. While giving a full overview of works is impossible, we mention a number of works of interest. The paper [140] specifically considers normed convex processes. Duality of convex processes was investigated in [33, 34] and the focus of [104, 151] was on eigenvalues and eigenvectors. Lastly, we note [78], which deals with sequences and boundedness of convex processes.

Of particular relevance is the investigation of system-theoretic properties of convex processes. The study of reachability for convex processes was started in the seminal paper [14]. In the case that H is a *strict* convex process, that is, nonempty everywhere, this paper provides necessary and sufficient conditions for reachability in continuous-time. For discrete-time systems, in a Banach space setting, the paper [132] provides similar conditions. To the best of our knowledge, the stabilizability problem was first resolved under this assumption of strictness in [131] and [159] in discrete- and continuous-time respectively. The study of stabilizability has been extended to encompass Lyapunov functions in [69].

One of the main sources of applications of systems of the form (3.1) arises from the fact that these maps capture the dynamics of any linear system with convex conic input/state constraints. In this context, the assumption of strictness corresponds to the absence of state constraints. Similar to the situation for convex processes, the study of constrained linear systems has also mainly focused on input constraints (see e.g. [27, 28, 37, 55, 56, 100, 161]). This has developed to the point where reachability and null-controllability are rather well understood in the presence of input constraints. State constraints have only been considered under rather restrictive assumptions, in e.g. [76, 77, 101, 107]. With regard to stabilizability, we note the works [145, 177]. The first work to combine the approaches of convex processes and constrained linear systems, were the reachability results in the recent paper [86]. This paper provides conditions for reachability of certain convex processes induced by linear systems with constraints.

In this chapter, we will study system theoretic properties of general nonstrict convex processes. In particular, we are interested in characterizing reachability, null-controllability, controllability, and stabilizability. One of the main observations from the so-called geometric approach (see e.g. [19, 165, 183]) is that for unconstrained linear systems, these properties can be naturally described in terms of invariant subspaces of the state space. Therefore, with an aim of generalizing the geometric approach towards convex processes, we will develop a framework of invariant cones of convex processes.

A second key observation is that the previously mentioned characterizations of reachability of convex processes rely on a relation between the reachable set of the convex process H and the feasible set of the dual process of H. When H is nonstrict, however, this duality relation breaks down in general as shown by [152]. Therefore, this paper will focus particularly on duality properties of these invariant cones.

The framework we will develop leads to necessary and sufficient conditions for reachability, null-controllability, and stabilizability of nonstrict convex processes under a certain domain condition. These characterizations have a few appealing properties. First of all, they resemble the well-known Hautus test, that is, they are formulated in terms of eigenvalues and eigenvectors of the dual convex process. Secondly, the domain condition we will work with is shown to be easily verified. Finally, we show that they generalize all previously known results. Aside from these characterizations, the framework that is developed is interesting in its own right, and opens up new research lines.

The chapter is organized as follows. We begin with a formal problem statement in Section 3.2. To be able to state the main results, we have certain required preliminaries, which are introduced in Section 3.3. This is followed by the main results in Section 3.4. Before being able to prove these results, we introduce the novel framework in Section 3.5, after which Section 3.6 contains the proofs. We provide conclusions in Section 3.7.

3.2 PROBLEM FORMULATION

Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map. Its *graph* is defined by

$$\operatorname{gr} H := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y \in H(x)\}.$$

We say that H is *closed*, *convex*, a *process* or a *linear process* if its graph is closed, convex, a cone or a subspace, respectively.

Consider the difference inclusion

$$x_{k+1} \in H(x_k). \tag{3.2}$$

A *trajectory* of (3.2) is a sequence $(x_k)_{k\in\mathbb{N}}$ satisfying (3.2) for every $k \geqslant 0$. In what follows, we will introduce several sets associated with (3.2).

The *behavior* (see e.g. [180]) is the set of all trajectories:

$$\mathfrak{B}(H) := \{(x_k)_{k\in\mathbb{N}} \mid (3.2) \text{ is satisfied for all } k \in \mathbb{N}\}.$$

For an integer $q \geqslant 1$, we define the q-*step behavior* as

$$\mathfrak{B}_q(H) := \{(x_k)_{k=0}^q \mid (3.2) \text{ is satisfied for all } k \in \{0, 1, \dots, q-1\}\}.$$

The *feasible set* $\mathcal{F}(H)$ is the set of points from which a trajectory emanates:

$$\mathcal{F}(H) := \{\xi \mid \exists (x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H) \text{ with } x_0 = \xi\}. \tag{3.3}$$

The *reachable set* $\mathcal{R}(H)$ is the set of points that can be reached from the origin in finite steps:

$$\mathcal{R}(H) := \left\{\xi \mid \exists (x_k)_{k=0}^q \in \mathfrak{B}_q(H) \text{ s.t. } x_0 = 0, x_q = \xi\right\}. \tag{3.4}$$

The *stabilizable set* $\mathcal{S}(H)$ is the set of points from which a stable trajectory exists:

$$\mathcal{S}(H) := \{\xi \mid \exists (x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H) \text{ with } x_0 = \xi, \lim_{k\to\infty} x_k = 0\}. \tag{3.5}$$

The *exponentially stabilizable set* $\mathcal{S}_e(H)$ is the set of points from which an exponentially stable trajectory exists:

$$\mathcal{S}_e(H) := \{\xi \mid \exists (x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H), \alpha > 0, \mu \in [0,1) \text{ s.t. } x_0 = \xi \text{ and } |x_k| \leqslant \alpha\mu^k|\xi|\} \tag{3.6}$$

where $|\cdot|$ denotes the Euclidean norm.
The *null-controllable set* $\mathcal{N}(H)$ is the set of points that can be steered to the origin in finite steps:

$$\mathcal{N}(H) := \left\{\xi \mid \exists (x_k)_{k=0}^{q} \in \mathfrak{B}_q(H) \text{ s.t. } x_0 = \xi, x_q = 0\right\}. \tag{3.7}$$

All the sets defined above inherit algebraic properties of the set-valued map H. In particular, they are all convex cones if H is a convex process and subspaces if H is a linear process. However, they do not retain topological properties from the underlying set-valued map in general. Indeed, none of these sets would be necessarily closed even if H is closed.

We say H is *reachable*, *stabilizable*, *exponentially stabilizable*, *null-controllable* if $\mathcal{F}(H) \subseteq \mathcal{R}(H)$, $\mathcal{F}(H) \subseteq \mathcal{S}(H)$, $\mathcal{F}(H) \subseteq \mathcal{S}_e(H)$, $\mathcal{F}(H) \subseteq \mathcal{N}(H)$, respectively.

Also, we say H is *controllable* if for all $\xi, \eta \in \mathcal{F}(H)$ there exist $\ell \geqslant 0$ and $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}$ such that $x_0 = \xi$ and $x_\ell = \eta$. As the origin belongs to $\mathcal{F}(H)$, we see that H is controllable if and only if it is both reachable and null-controllable.

The problems we study are to find necessary and sufficient conditions for reachability, (exponential) stabilizability, null-controllability, and controllability of convex processes.

One of the motivations to study convex processes stems from their link to *constrained linear systems*. To elaborate further on this connection, consider the discrete-time linear input/state/output system given by

$$x_{k+1} = Ax_k + Bu_k \tag{3.8a}$$

$$y_k = Cx_k + Du_k \tag{3.8b}$$

where $k \in \mathbb{N}$, $u_k \in \mathbb{R}^m$ is the input, $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^p$ is the output and the matrices A, B, C, D are of appropriate dimensions. Suppose that the output of this system is constrained by

$$y_k \in \mathcal{Y} \tag{3.8c}$$

for all $k \in \mathbb{N}$ where $\mathcal{Y} \subseteq \mathbb{R}^p$ is a convex cone. Now, define the set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$H(x) := \{Ax + Bu \mid Cx + Du \in \mathcal{Y}\}. \tag{3.9}$$

Since $\mathcal{Y}$ is a convex cone, the set-valued map H is a convex process. This shows that we can view the linear constrained system (3.8) as a *difference inclusion* of the form (3.2) where H is given by (3.9).

3.3 PRELIMINARIES

In this section, we will introduce the notational conventions that will be in force throughout the chapter as well as the notions that will be employed in the study of reachability and stabilizability.

3.3.1 Convex cones

Let $\mathcal{S}, \mathcal{T} \subseteq \mathbb{R}^q$ be nonempty sets and $\rho \in \mathbb{R}$. We define $\mathcal{S} + \mathcal{T} := \{s + t \mid s \in \mathcal{S}, t \in \mathcal{T}\}$ and $\rho\mathcal{S} := \{\rho s \mid s \in \mathcal{S}\}$. By convention $\mathcal{S} + \varnothing = \varnothing$ for every $\mathcal{S}$ and $\rho\varnothing = \varnothing$ for every ρ. The relative interior of $\mathcal{S}$ is denoted by $\mathrm{ri}(\mathcal{S})$. We say that $\mathcal{S}$ is a *cone* if $\rho x \in \mathcal{S}$ whenever $x \in \mathcal{S}$ and $\rho \geqslant 0$. The conic hull of $\mathcal{S}$ will be denoted by $\mathrm{cone}(\mathcal{S})$. We say that a cone is *finitely generated* if it is a conic hull of finitely many vectors.

Next, we state four auxiliary results that will be used later. First, we prove a relation between sums and intersections of cones and subspaces.

Lemma 3.1. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex cone and let $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathbb{R}^n$ be subspaces. Then, $(\mathcal{C} \cap \mathcal{W}) + \mathcal{V} = (\mathcal{C} + \mathcal{V}) \cap \mathcal{W}$ and the following statements are equivalent:

1. $(\mathcal{C} \cap \mathcal{W}) + \mathcal{V} = \mathcal{W}$.

2. $(\mathcal{C} + \mathcal{V}) \cap \mathcal{W} = \mathcal{W}$.

3. $\mathcal{W} \subseteq \mathcal{C} + \mathcal{V}$.

4. $\mathcal{C} + \mathcal{V} = \mathcal{C} + \mathcal{W}$.

Proof. We will prove this case by case.
(1) $\Leftrightarrow$(2): As $V \subseteq W$, $(C + \mathcal{V}) \cap \mathcal{W} = (C \cap \mathcal{W}) + (\mathcal{V} \cap \mathcal{W}) = (C \cap \mathcal{W}) + \mathcal{V}$.
(2)$\Rightarrow$(3): This implication is immediate.

(3)⇒(4) : If $\mathcal{W} \subseteq C + \mathcal{V}$, we can add C to both sides. Using the fact that C is a cone we know $C + C = C$ and hence $C + \mathcal{W} \subseteq C + \mathcal{V}$. As $\mathcal{V} \subseteq \mathcal{W}$, the reverse inclusion is immediate, thus proving $C + \mathcal{V} = C + \mathcal{W}$.

(4) ⇒(2): Using $C + \mathcal{V} = C + \mathcal{W}$, we can intersect both sides with $\mathcal{W}$. Clearly $(C + \mathcal{W}) \cap \mathcal{W} = \mathcal{W}$. Thus proving that $(C + \mathcal{V}) \cap \mathcal{W} = \mathcal{W}$. ■

We move on to a characterization of the property that the difference of two convex cones is equal to a subspace.

Lemma 3.2. Let $\mathcal{C}, \mathcal{K} \subseteq \mathbb{R}^n$ be convex cones. The set $\mathcal{C} - \mathcal{K}$ is a subspace if and only if $\text{ri}(\mathcal{C}) \cap \text{ri}(\mathcal{K}) \neq \varnothing$.

Proof. By [26, Prop. 1.3.7] we see that $\text{ri}(\mathcal{C} - \mathcal{K}) = \text{ri}(\mathcal{K}) - \text{ri}(\mathcal{K})$. As such, $\text{ri}(\mathcal{C}) \cap \text{ri}(\mathcal{K}) \neq \varnothing$ if and only if $0 \in \text{ri}(\mathcal{C} - \mathcal{K})$. In turn, since $\mathcal{C} - \mathcal{K}$ is a cone, this holds if and only if $\mathcal{C} - \mathcal{K}$ is a subspace. ■

The following gives a condition for when the difference of two convex cones is a closed set.

Lemma 3.3. Let $\mathcal{C}, \mathcal{K}$ be closed convex cones such that $\mathcal{C} \cap \mathcal{K}$ is a subspace. Then $\mathcal{C} - \mathcal{K}$ is closed.

Proof. Let $A = \begin{bmatrix} I & -I \end{bmatrix}$ and $\mathcal{S} = \mathcal{C} \times \mathcal{K}$. As a consequence of [142, Thm. 9.1], if every $z \in \mathcal{S}$ such that $Az = 0$ belongs to $\text{lin}\,\mathcal{S}$, then $A\mathcal{S}$ is closed. Note that $z \in \mathcal{S}$ with $Az = 0$ if and only if $z = \begin{bmatrix} y \\ y \end{bmatrix}$ where $y \in \mathcal{C} \cap \mathcal{K}$, proving the lemma. ■

Lastly, we prove a useful property of finitely generated cones that are the union of a nested sequence of convex cones.

Lemma 3.4. Let $\mathcal{C}_\ell$ for $\ell \in \mathbb{N}$ be convex cones with $\mathcal{C}_\ell \subseteq \mathcal{C}_{\ell+1}$. If $\bigcup_{\ell=0}^{\infty} \mathcal{C}_\ell$ is finitely generated then there exists $q \geqslant 0$ such that $\mathcal{C}_{q+\ell} = \mathcal{C}_q$ for all $\ell \geqslant 0$ and $\bigcup_{\ell=0}^{\infty} \mathcal{C}_\ell = \mathcal{C}_q$.

Proof. Let $\mathcal{C} = \bigcup_{\ell=0}^{\infty} \mathcal{C}_\ell = \text{cone}(\mathcal{S})$ where $\mathcal{S}$ is a finite set of vectors. Then, we know that for each element $s \in \mathcal{S}$ there is a number q_s such that $s \in \mathcal{C}_{q_s}$. Since the set $\mathcal{S}$ is finite, there exists $q \geqslant 0$ such that $q \geqslant q_s$ for all $s \in \mathcal{S}$. As $\mathcal{C}_q$ is a convex cone containing $\mathcal{S}$, we have $\mathcal{C} \subseteq \mathcal{C}_q$. By definition $\mathcal{C}_q \subseteq \mathcal{C}$. As such, we obtain $\mathcal{C} = \mathcal{C}_q$, which proves the lemma. ■

3.3.2 Convex and linear processes

Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex process. Clearly, $H(0)$ is a convex cone. For all $x, y \in \mathbb{R}^n$ and $\rho > 0$, we have

$$H(\rho x) = \rho H(x), \tag{3.10}$$
$$H(x) + H(y) \subseteq H(x+y), \tag{3.11}$$
$$H(x) = H(x) + H(0), \tag{3.12}$$

and $H(x)$ is convex.

We define its *domain*, *image* and *kernel* by

$$\begin{aligned} \operatorname{dom}(H) &:= \{x \in \mathbb{R}^n \mid H(x) \neq \varnothing\}, \\ \operatorname{im}(H) &:= \{y \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^n \text{ s.t. } y \in H(x)\}, \\ \ker(H) &:= \{x \in \mathbb{R}^n \mid 0 \in H(x)\}, \end{aligned}$$

respectively. If $\operatorname{dom}(H) = \mathbb{R}^n$, we say that H is *strict*.

The *inverse* of H is defined by

$$H^{-1}(y) := \{x \in \mathbb{R}^n \mid y \in H(x)\}.$$

Clearly, $\operatorname{dom}(H^{-1}) = \operatorname{im}(H)$, $\operatorname{im}(H^{-1}) = \operatorname{dom}(H)$, and

$$\operatorname{gr}(H^{-1}) = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \operatorname{gr}(H). \tag{3.13}$$

For $\lambda \in \mathbb{R}$, we define the set-valued map $H - \lambda I$ by $(H - \lambda I)(x) := \{y - \lambda x \mid y \in H(x)\}$. Then, we have

$$\operatorname{gr}(H - \lambda I) = \begin{bmatrix} I_n & 0 \\ -\lambda I_n & I_n \end{bmatrix} \operatorname{gr}(H).$$

We denote the image of a set $\mathcal{S}$ under H by $H(\mathcal{S}) = \{y \in \mathbb{R}^n \mid \exists x \in \mathcal{S} \text{ s.t. } y \in H(x)\}$. Powers of H are defined as follows. By convention, H^0 is the identity map, that is $H^0(x) := x$ for all $x \in \mathbb{R}^n$. For $q \geqslant 1$, we define

$$H^{q+1}(x) := H(H^q(x)) \quad \forall x \in \mathbb{R}^n.$$

Clearly, $H(0)$, $\operatorname{dom}(H)$, $\ker(H)$, and $\operatorname{im}(H)$ are all convex cones. In addition, the inverse H^{-1}, H^q, and $H - \lambda I$ are convex processes for all $q \geqslant 0$ and $\lambda \in \mathbb{R}$.

Similarly, for a linear process $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we have that $L(0)$, $\mathrm{dom}(L)$, $\ker(L)$, and $\mathrm{im}(L)$ are all subspaces and the set-valued maps L^{-1}, $L - \lambda I$, L^q are all linear processes. Furthermore, for all $x, y \in \mathbb{R}^n$ and nonzero $\rho \in \mathbb{R}$, we have

$$L(\rho x) = \rho L(x), \tag{3.14}$$

$$L(x) + L(y) = L(x + y), \tag{3.15}$$

$$L(x) = L(x) + L(0), \tag{3.16}$$

and $L(x)$ is an affine set.

3.3.3 Eigenvalues/vectors of convex processes

A real number λ and nonzero vector $\xi \in \mathbb{R}^n$ form an *eigenpair* of H if $\lambda\xi \in H(\xi)$. In this case λ is called an *eigenvalue* and ξ is called an *eigenvector* of H corresponding to (the eigenvalue) λ.

For each real number λ, the convex cone $\ker(H - \lambda I)$ contains all eigenvectors corresponding to λ and the origin. This set is called the *eigencone* of H corresponding to λ. This means that λ is an eigenvalue of H if and only if $\ker(H - \lambda I) \neq \{0\}$. Given a set $\mathcal{K}$, we define the *spectrum of* H *with respect to* $\mathcal{K}$ by:

$$\sigma(H, \mathcal{K}) := \{\lambda \in \mathbb{R} \mid \exists \xi \in \mathcal{K} \setminus \{0\} \text{ such that } \lambda\xi \in H(\xi)\}.$$

and the *spectrum* by $\sigma(H) := \sigma(H, \mathbb{R}^n)$.

3.3.4 Dual processes

For a nonempty set $\mathcal{C} \subseteq \mathbb{R}^n$, we define the *negative* and *positive polar cone* by

$$\begin{aligned} \mathcal{C}^- &:= \{y \in \mathbb{R}^n \mid \langle x, y\rangle \leqslant 0 \quad \forall x \in \mathcal{C}\}, \\ \mathcal{C}^+ &:= \{y \in \mathbb{R}^n \mid \langle x, y\rangle \geqslant 0 \quad \forall x \in \mathcal{C}\}, \end{aligned}$$

respectively.

Both the negative and positive polar cones of a set are always closed convex cones. In addition, $\mathcal{C}$, its closure, convex hull and conic hull have the same polar cones. Furthermore, if $\mathcal{C}$ is a closed convex cone, then $(\mathcal{C}^-)^- = \mathcal{C}$. We also point out that for sets $\mathcal{C}$ and $\mathcal{S}$:

$$(\mathcal{C} + \mathcal{S})^- = \mathcal{C}^- \cap \mathcal{S}^-, \quad (\mathcal{C} \cap \mathcal{S})^- = \mathrm{cl}(\mathcal{C}^- + \mathcal{S}^-). \tag{3.17}$$

We define the *negative* and *positive dual* processes H^- and H^+ of H by:

$$p \in H^-(q) \iff \langle p, x\rangle \geqslant \langle q, y\rangle, \ \forall x \in \mathbb{R}^n, \ \forall y \in H(x), \tag{3.18a}$$

$$p \in H^+(q) \iff \langle p, x\rangle \leqslant \langle q, y\rangle, \ \forall x \in \mathbb{R}^n, \ \forall y \in H(x), \tag{3.18b}$$

respectively. The graphs of the dual processes are related to that of H as follows:

$$\operatorname{gr}(H^-) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \big(\operatorname{gr}(H)\big)^- \qquad \operatorname{gr}(H^+) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \big(\operatorname{gr}(H)\big)^+. \tag{3.19}$$

The following lemma collects properties of the dual processes that will be used later.

The following properties of the dual processes follow from the definitions.

Lemma 3.5. Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a closed convex process. Then, we have

1. $H(0) = \operatorname{dom}(H^+)^+ = (\operatorname{dom}(H^-))^-$.
2. $(H^{-1})^- = (H^+)^{-1}$.
3. $\operatorname{gr}(H^-) = -\operatorname{gr}(H^+)$.
4. $\operatorname{gr}\big((H^+)^+\big) = \operatorname{gr}\big((H^-)^-\big) = -\operatorname{gr}(H)$.
5. $\operatorname{gr}\big((H^-)^+\big) = \operatorname{gr}\big((H^+)^-\big) = \operatorname{gr}(H)$.
6. $\big(\operatorname{dom}(H)\big)^- = -H^-(0) = H^+(0)$.
7. $\big(\operatorname{im}(H - \lambda I)\big)^- = \ker(H^- - \lambda I)$ for all $\lambda \in \mathbb{R}$.

3.3.5 Minimal and maximal linear process

For a cone $\mathcal{C}$ we define $\operatorname{lin}(\mathcal{C}) := -\mathcal{C} \cap \mathcal{C}$ and $\operatorname{Lin}(\mathcal{C}) = \mathcal{C} - \mathcal{C}$. Note that both $\operatorname{lin}(\mathcal{C})$ and $\operatorname{Lin}(\mathcal{C})$ are subspaces and, in particular $\operatorname{lin}(\mathcal{C})$ is the largest subspace contained in $\mathcal{C}$ whereas $\operatorname{Lin}(\mathcal{C})$ is the smallest subspace that contains $\mathcal{C}$.

Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex process. Associated with H, we define two linear processes L_- and L_+ by

$$\operatorname{gr}(L_-) := \operatorname{lin}\big(\operatorname{gr}(H)\big) \quad \text{and} \quad \operatorname{gr}(L_+) := \operatorname{Lin}\big(\operatorname{gr}(H)\big). \tag{3.20}$$

Further, L_- and L_+ are, respectively, the largest and smallest (with respect to the graph inclusion) linear processes satisfying

$$\operatorname{gr}(L_-) \subseteq \operatorname{gr}(H) \subseteq \operatorname{gr}(L_+). \tag{3.21}$$

We call L_- and L_+, respectively, the minimal and maximal linear processes associated with H. If H is not clear from context, we will specify it by writing $L_-(H)$ and $L_+(H)$.

The domains of the minimal/maximal linear processes are related to that of H:

$$\operatorname{dom}(L_-) \subseteq \operatorname{lin}\big(\operatorname{dom}(H)\big) \quad \text{and} \quad \operatorname{dom}(L_+) = \operatorname{Lin}\big(\operatorname{dom}(H)\big). \tag{3.22}$$

The reverse inclusion for the former does not hold in general.

Inverses of minimal/maximal linear processes can be characterized in terms of the inverse of H:

$$L_-(H^{-1}) = L_-^{-1}(H) \quad \text{and} \quad L_+(H^{-1}) = L_+^{-1}(H). \tag{3.23}$$

Note that for a linear process L, the positive and negative duals coincide. Therefore, we denote it by $L^\perp := L^- = L^+$. The minimal and maximal linear processes associated with a convex process enjoy the following properties that immediately follow from the definitions:

$$L_-(H^-) = L_+(H)^\perp \quad \text{and} \quad L_+(H^-) = L_-(H)^\perp. \tag{3.24}$$

Example 3.1. Consider the following linear system with constraints (3.8), and its corresponding convex process H, defined in (3.9). It is easy to check that the following hold:

- If the set $\{u \mid Bu = 0, Du \in \mathcal{Y}\}$ is a subspace, then
$$L_- = \{Ax + Bu \mid Cx + Du \in \operatorname{lin}(\mathcal{Y})\}.$$
- If $\operatorname{im}\begin{bmatrix} C & D \end{bmatrix} + \mathcal{Y}$ is a subspace, then
$$L_+ = \{Ax + Bu \mid Cx + Du \in \operatorname{Lin}(\mathcal{Y})\}.$$
- If $\operatorname{im}\begin{bmatrix} C & D \end{bmatrix} + \mathcal{Y}$ is a subspace, then
$$H^-(x) = \left\{A^\top x + C^\top u \mid u \in \mathcal{Y}^+, B^\top x + D^\top u = 0\right\}.$$

3.4 MAIN RESULTS

In this section we will state the main results whose proofs can be found in Section 3.6.

For linear processes, it can easily be verified that both the feasible and reachable sets can be computed in finitely many steps. For later use, we state this fact below and omit its rather elementary proof.

Lemma 3.6. Let $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a linear process. Then, $\mathcal{F}(L) = \operatorname{dom}(L^n) = L^{-n}(\mathbb{R}^n)$ and $\mathcal{R}(L) = L^n(0)$ are subspaces.

Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex process. In the rest of the chapter, we will use the following shorthand notational conventions:

$$\mathcal{R}_- := \mathcal{R}\big(L_-(H)\big) \quad \text{and} \quad \mathcal{R}_+ := \mathcal{R}\big(L_+(H)\big). \tag{3.25}$$

Both $\mathcal{R}_-$ and $\mathcal{R}_+$ are subspaces that can be computed in finitely many steps as stated in Lemma 3.6.

Our main results will rely on the following *domain condition*:

$$\operatorname{dom}(H) + \mathcal{R}_- \text{ is a subspace} \quad \text{and} \quad \operatorname{dom}(H) + \mathcal{R}_- = \operatorname{dom}(H) + \mathcal{R}_+. \tag{DC}$$

Note that the domain condition (DC) readily holds whenever H is strict, or H is a linear process.

Since $\operatorname{dom}(H)$ is a convex cone and $\mathcal{R}_-, \mathcal{R}_+$ are subspaces with $\mathcal{R}_- \subseteq \mathcal{R}_+$, it follows from Lemmas 3.1 and 3.2 that the domain condition (DC) is equivalent to

$$\operatorname{ri}\big(\operatorname{dom}(H)\big) \cap \mathcal{R}_- \neq \varnothing \quad \text{and} \quad \mathcal{R}_+ \subseteq \operatorname{dom}(H) + \mathcal{R}_-.$$

Next, we will introduce two convex processes associated with H that capture the behavior of H inside and outside $\mathcal{R}_+$, respectively.

We define the the *inner process* $H_{\text{in}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$\operatorname{gr}(H_{\text{in}}) := \operatorname{gr}(H) \cap (\mathcal{R}_+ \times \mathcal{R}_+)$$

and the *outer process* $H_{\text{out}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$\operatorname{gr}(H_{\text{out}}) := \Big(\operatorname{gr}(H) + \big(\mathcal{R}_+ \times \mathcal{R}_+\big)\Big) \cap \big(\mathcal{V} \times \mathcal{V}\big),$$

where $\mathcal{V} \subseteq \mathbb{R}^n$ is a subspace such that $\mathcal{R}_+ \oplus \mathcal{V} = \mathbb{R}^n$.

We will study these processes in detail in Section 3.5. For the moment, we mention only the following result that will be needed to state our main results.

Lemma 3.7. Let H be a convex process satisfying the domain condition (DC). Then, the outer process H_{out} is a single valued linear process, $\mathcal{F}(H_{\text{out}}) = \big(\mathcal{F}(H) + \mathcal{R}_+\big) \cap \mathcal{V}$ is a subspace and $H_{\text{out}}(\mathcal{F}(H_{\text{out}})) \subseteq \mathcal{F}(H_{\text{out}})$.

We denote the restriction of H_{out} to $\mathcal{F}(H_{\text{out}})$ by $H_{\text{out}}|_{\mathcal{F}(H_{\text{out}})}$.

Example 3.2. Consider the linear system

$$x_{k+1} = Ax_k + Bu_k. \tag{3.26}$$

It is well known that a linear system admits a Kalman decomposition, that is, the system is similar to a system of the form:

$$x_{k+1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} x_k + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u_k,$$

where (A_{11}, B_1) is reachable. As such, the system (3.26) can be said to consist of a reachable part and an autonomous part. Clearly, (A, B) is reachable if and only if the autonomous part is trivial.

Now consider the linear process corresponding to this system, that is,

$$L(x) = \{Ax + \operatorname{im} B\}.$$

Then L_{in} corresponds to the reachable part of the Kalman decomposition and L_{out} to the autonomous part.

Now, we are in a position to state the main results of the chapter. We begin with reachability.

Theorem 3.1. Let H be a convex process satisfying the domain condition (DC). Then, the following statements are equivalent:

(i) H is reachable.

(ii) Both H_{in} and H_{out} are reachable.

(iii) All eigenvectors of H_{in}^- corresponding to eigenvalues in $[0, \infty)$ belong to $\mathcal{R}_+^\perp$ and $\mathcal{F}(H_{\text{out}}) = \{0\}$.

Moreover, if H is reachable, then $\mathcal{R}(H) = \mathcal{R}_+$ and $\mathcal{R}(H)$ is finitely determined.

Theorem 3.1 captures all existing reachability results for convex processes in the literature as special cases.

The well-known reachability characterization for *strict* convex processes [132, Thm. 3.1] (see also [14, Thm. 0.4] for the continuous-time counterpart) follows from Theorem 3.1. To see this, note that if $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is strict, that is $\operatorname{dom}(H) = \mathbb{R}^n$, then the domain condition readily holds and $\mathcal{F}(H) = \mathbb{R}^n$. In view of Lemma 3.7, this means that $\mathcal{F}(H_{\text{out}}) = \mathcal{V}$. Therefore, Theorem 3.1

boils down to H is reachable if and only if $\mathcal{R}_+ = \mathbb{R}^n$ and H_{in}^- does not have any nonnegative eigenvalues. Note that $\mathcal{R}_+ = \mathbb{R}^n$ implies $H_{\text{in}} = H$. Hence, Theorem 3.1, for the strict case, states that $\mathcal{R}_+ = \mathbb{R}^n$ and H^- does not have any nonnegative eigenvalues. As what [132] calls the *rank condition* is equivalent to $\mathcal{R}_+ = \mathbb{R}^n$, [132, Thm. 3.1] is a special case of Theorem 3.1.

Reachability of convex processes of the form (3.9) have been studied in [76] and [86]. While [76, Thm. V.3] assumes that $\operatorname{im} D + C\mathcal{T}^* = \mathbb{R}^n$, [86, Thm. 6.3] assumes that $\operatorname{im} D + C\mathcal{T}^* + \mathcal{Y} = \mathbb{R}^n$. Here $\mathcal{T}^*$ is the so-called strongly reachable subspace associated with the linear system (3.8). These assumptions imply $\operatorname{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$ which, in turn, implies the domain condition (DC). Hence both [76, Thm. V.3] and [86, Thm. 6.3] are special cases of Theorem 3.1. In addition, [45, Thm. 1] is a special case of Theorem 3.1 since it works under the stronger domain condition $\operatorname{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$ as well.

Another noteworthy point is that the results [132, Thm. 3.1] and [76, Thm. V.3] require closedness of the convex processes that they deal with whereas closedness is not assumed by Theorem 3.1.

Last but not the least, none of the existing results [132, Thm. 3.1], [76, Thm. V.3], [86, Thm. 6.3], and [45, Thm. 1] can directly be applied to nonstrict linear processes. In case H is a linear process, we have $H = L_- = L_+$. Together with Lemma 3.6, this implies that the domain condition (DC) is readily satisfied for linear processes. Moreover, H_{in} is reachable whenever H is linear. Therefore, Theorem 3.1 asserts that a linear process H is reachable if and only if $\mathcal{F}(H_{\text{out}}) = \{0\}$.

The domain condition (DC) proves itself useful also in the context of stabilizability. Even though the next result has a very much parallel statement to that of reachability, its proof is substantially more involved as we will see later. This is mainly because of the different nature of the sets $\mathcal{R}(H)$ and $\mathcal{S}(H)$. Indeed, as will be discussed in detail in Section 3.5, $\mathcal{R}(H)$ turns out to be a strongly H invariant set whereas $\mathcal{S}(H)$ is a weakly H invariant set.

Theorem 3.2. Let H be a convex process satisfying the domain condition (DC). Then, the following statements are equivalent:

(i) H is stabilizable.

(ii) H is exponentially stabilizable.

(iii) Both H_{in} and H_{out} are exponentially stabilizable.

(iv) All eigenvectors of H_{in}^- corresponding to eigenvalues in $[1, \infty)$ belong to $\mathcal{R}_+^\perp$ and all eigenvalues of the linear map $H_{\text{out}}|_{\mathcal{F}(H_{\text{out}})}$ are in the open unit disc.

To the best of our knowledge, spectral conditions for stabilizability as stated above have appeared only in [159, Prob. 8.6.4] (see also [159, Thm. 8.10] for the continuous-time counterpart). Since [159, Prob. 8.6.4] deals with strict closed convex processes, the domain condition (DC) is automatically satisfied. As such, [159, Prob. 8.6.4] can be recovered as a special case from Theorem 3.2. In [64, Thm. 5.1], it is shown that a sufficient condition for stabilizability is that the domain of a convex process admits a particular representation via certain types of eigenvectors of the process itself. Since our stabilizability result stated in terms of the eigenvectors of the dual process instead, it is difficult to compare our result with [64, Thm. 5.1]. Nevertheless, it should be remarked that [64, Thm. 5.1] could be applied only if $\mathcal{F}(H) = \mathrm{dom}(H)$ whereas our result does not require this assumption.

Example 3.3. Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^{n\times 1}$ and define

$$H(x) = \{Ax + bu \mid u \geqslant 0\}.$$

Clearly, $L_+(x) = \{Ax + \mathrm{im}\, b\}$. Therefore $\mathcal{F}(H_{\mathrm{out}}) = \{0\}$ if and only if $\mathcal{R}_+ = \mathbb{R}^n$, that is, if and only if (A, b) is reachable. Furthermore, if $\mathcal{R}_+ = \mathbb{R}^n$, then

$$H_{\mathrm{in}}^-(x) = \{A^\top x \mid b^\top x \leqslant 0\}.$$

As such, H is reachable if and only if (A, b) is reachable and $A^\top$ has no real eigenvalues $\lambda \geqslant 0$. This is the well-known result of [52, Thm. 1].

Next, we turn our attention to null-controllability. It is a well-known fact that reachability implies null-controllability for discrete-time linear systems which correspond to linear processes in the framework of this chapter. As shown in the following example, however, reachability does not imply null-controllability in general for convex processes.

Example 3.4. Let H be the convex process given by:

$$H(x) := \begin{cases} [x, \infty) & x \geqslant 0, \\ \varnothing & x < 0. \end{cases}$$

Then clearly $\mathcal{R}(H) = \mathcal{F}(H) = \{x \mid x \geqslant 0\}$, but $\mathcal{N}(H) = \{0\}$. Thus H is reachable but not null-controllable.

Interestingly, this implication does always hold under the domain condition (DC).

Theorem 3.3. Let H be a convex process satisfying the domain condition (DC). If H is reachable, then it is null-controllable. In particular, this means that H is reachable if and only if H is controllable.

One may think that the results on reachability and stabilizability can be extended to null-controllability in the same way. However, the following example reveals why the domain condition (DC) is not enough to formulate spectral conditions for null-controllability. Nevertheless, it is still possible to give a spectral characterization by assuming the following *image condition*:

$$H(\mathcal{R}_+) - \big(\mathcal{N}(L_-(H)) \cap \mathcal{R}_+\big) = \mathcal{R}_+. \tag{IC}$$

Example 3.5. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be the convex process defined by:

$$H(x) = \begin{cases} [0, \infty) & \text{if } x = 0, \\ (0, \infty) & \text{if } x \neq 0. \end{cases}$$

Clearly, H is strict, and therefore the domain condition (DC) holds. However, H is not null-controllable since $\mathcal{N}(H) = \{0\}$.

Let $\bar{H}$ be the closure of H, that is, $\bar{H}(x) = [0, \infty)$ for any $x \in \mathbb{R}$. Then the process $\bar{H}$ is also strict. Since $0 \in \bar{H}(x)$ for every $x \in \mathbb{R}$, the convex process $\bar{H}$ is null-controllable.

Even though H and $\bar{H}$ have the same dual H^-, the latter is null-controllable whereas the former is not. This reveals the role played by the assumption on the image. Indeed, in both cases $\mathcal{R}_+ = \mathbb{R}$, but we have $\operatorname{im} H - \mathcal{N}_- = [0, \infty)$ whereas $\operatorname{im} \bar{H} - \mathcal{N}(L_-(\bar{H})) = \mathbb{R}$.

We will state our main result on null-controllability

Theorem 3.4. Let H be a convex process satisfying the domain condition (DC) and the image condition (IC). Then, the following statements are equivalent:

(i) H is null-controllable.

(ii) Both H_{in} and H_{out} are null-controllable.

(iii) All eigenvectors of H_{in}^- corresponding to eigenvalues in $(0, \infty)$ belong to $\mathcal{R}_+^\perp$ and the linear map $H_{\text{out}}|_{\mathcal{F}(H_{\text{out}})}$ is nilpotent.

Unlike reachability, null-controllability for convex processes has not been extensively studied in the literature. In [132, Thm. 3.2], the authors assume that both H and H^{-1} are strict. In that case, both the domain condition (DC) and the image condition (IC) are trivially satisfied. As such, [132, Thm. 3.2] is a particular case of Theorem 3.4. Yet another particular case is [45, Thm. 2] which works under the stronger domain condition $\mathrm{dom}(H) + \mathcal{R}_- = \mathcal{R}_+ = \mathbb{R}^n$ as well as the stronger image condition $\mathrm{im}(H) + \mathcal{N}(L_-(H)) = \mathbb{R}^n$.

Example 3.6. Let $c_2 \neq 0$ and consider the linear system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k,$$

which is constrained to the half-space given by:

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} x_k \geqslant 0.$$

We are interested in characterizing all c_1, c_2 with $c_2 \neq 0$ for which the system is reachable, stabilizable or null-controllable. For this, we first write the system as a convex process, by letting:

$$H(x) := \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \{0\} \times \mathbb{R} & \begin{bmatrix} c_1 & c_2 \end{bmatrix} x \geqslant 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

It is straightforward to conclude that L_- and L_+ are given by

$$L_-(x) = \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \{0\} \times \mathbb{R} & \begin{bmatrix} c_1 & c_2 \end{bmatrix} x = 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

$$L_+(x) = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \{0\} \times \mathbb{R} \right\}.$$

Note that, since $c_2 \neq 0$, we have that $\mathrm{dom}\, H + \mathcal{R}_- = \mathbb{R}^2$. Furthermore, L_+ is reachable and $\mathcal{R}_+ = \mathbb{R}^2$. Therefore $\mathcal{F}(H_{\mathrm{out}}) = \{0\}$. Calculating the dual of H, we get:

$$H^-(x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \mathbb{R}_+ & \begin{bmatrix} 0 & 1 \end{bmatrix} x = 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

Suppose that $\lambda\xi \in H^-(\xi)$, then

$$\begin{pmatrix} \lambda\xi_1 \\ \lambda\xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 + c_1 \nu \\ \xi_1 + \xi_2 + c_2 \nu \end{pmatrix}, \quad \xi_2 = 0, \quad \text{and} \quad \nu \geqslant 0.$$

We see that $\lambda\xi \in H^-(\xi)$ if and only if $\lambda = \left(1 - \frac{c_1}{c_2}\right)$. Therefore:

- H is stabilizable if and only $\frac{c_1}{c_2} \geqslant 0$.
- H is null-controllable if and only if $\frac{c_1}{c_2} \geqslant 1$.
- H is reachable if and only if $\frac{c_1}{c_2} > 1$.

3.5 TOWARDS THE PROOFS

In this section we will introduce the notions and tools that will be used in the proofs of the main results.

3.5.1 Strong and weak invariance

In the rest of the chapter the following invariance notions will play a key role.

Definition 3.1. Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex process and $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex cone. We say that $\mathcal{C}$ is

(i) *weakly* H *invariant* if $H(x) \cap \mathcal{C} \neq \varnothing$ for all $x \in \mathcal{C}$.

(ii) *strongly* H *invariant* if $H(x) \subseteq \mathcal{C}$ for all $x \in \mathcal{C}$.

From these definitions the following facts immediately follow.

Lemma 3.8. Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex process. Then, the following statements hold:

(i) A cone $\mathcal{W}$ is weakly H invariant if and only if $\mathcal{W} \subseteq H^{-1}(\mathcal{W})$. A cone $\mathcal{S}$ is strongly H invariant if and only if $H(\mathcal{S}) \subseteq \mathcal{S}$.

(ii) If $\mathcal{S}$ is strongly H invariant, then it is also weakly invariant if and only if $\mathcal{S} \subseteq \operatorname{dom}(H)$.

These two notions of invariance enjoy the following properties

Lemma 3.9. Let H be a convex process. If $\mathcal{W}$ and $\mathcal{S}$ are, respectively, weakly and strongly H invariant, then $\mathcal{W} \cap \mathcal{S}$ and $\mathcal{S} - \mathcal{W}$ are, respectively, weakly and strongly H invariant.

Proof. To prove the first part of the statement, let $x \in \mathcal{W} \cap \mathcal{S}$. Since $\mathcal{W}$ is weakly invariant, there exists $y \in H(x) \cap \mathcal{W} \neq \varnothing$. In view of strong invariance of $\mathcal{S}$, we have $H(x) \subseteq \mathcal{S}$. Therefore, $y \in H(x) \cap \mathcal{S} \cap \mathcal{W} \neq \varnothing$ and hence $\mathcal{W} \cap \mathcal{S}$ is weakly H invariant.

For the second part, let $x \in \mathcal{S} - \mathcal{W}$. Then, there exists $s \in \mathcal{S}$ and $w \in \mathcal{W}$ such that $x = s - w$. If $x \notin \operatorname{dom}(H)$, we have $H(x) = \varnothing \subseteq \mathcal{S} - \mathcal{W}$. Suppose that $x \in \operatorname{dom}(H)$. Since $\mathcal{W}$ is weakly invariant, $w \in \operatorname{dom}(H)$. As $\operatorname{dom}(H)$ is a convex cone, we see that $s = x + w \in \operatorname{dom}(H)$. Note that $H(x) + H(w) \subseteq H(s) \subseteq \mathcal{S}$ since H is a convex process and $\mathcal{S}$ is strongly invariant. Weak invariance of $\mathcal{W}$ implies that there exists $z \in H(w) \cap \mathcal{W} \neq \varnothing$. Then, we have $H(x) + \{z\} \subseteq \mathcal{S}$ which implies that $H(x) \subseteq \mathcal{S} - \{z\} \subseteq \mathcal{S} - \mathcal{W}$. Consequently, $\mathcal{S} - \mathcal{W}$ is strongly H invariant. ∎

Next, we will investigate invariance properties of the feasible, reachable, (exponentially) stabilizable and null-controllable sets.

Lemma 3.10. For a convex process H, the feasible set $\mathcal{F}(H)$ is the largest weakly H invariant convex cone. Moreover, $\mathcal{F}(H) = H^{-1}(\mathcal{F}(H))$.

Proof. Clearly, $\mathcal{F}(H)$ is a convex cone. By definition, we can see that the feasible set of H is weakly H invariant: If a trajectory exists from x_0, there also exists one from any corresponding x_1. Therefore $x_1 \in H(x_0) \cap \mathcal{F}(H)$ and hence, by Definition 3.1 the set $\mathcal{F}(H)$ is weakly H invariant. As any weakly H invariant set naturally allows a trajectory, we can see that $\mathcal{F}(H)$ is the largest weakly H invariant cone.

For the second part we know by Lemma 3.8 that $\mathcal{F}(H) \subseteq H^{-1}(\mathcal{F}(H))$. It thus suffices to prove the reverse. Applying H^{-1} on both sides, we know $H^{-1}(\mathcal{F}(H)) \subseteq H^{-1}\big(H^{-1}(\mathcal{F}(H))\big)$ and therefore $H^{-1}(\mathcal{F}(H))$ is weakly H invariant. As $\mathcal{F}(H)$ is the largest of such, we have proven the statement. ∎

Any feasible state is contained in the domain of H^{ℓ} for any ℓ, hence

$$\mathcal{F}(H) \subseteq \bigcap_{\ell \in \mathbb{N}} H^{-\ell}(\mathbb{R}^n) = \bigcap_{\ell \in \mathbb{N}} \operatorname{dom}(H^{\ell}). \tag{3.27}$$

A case where (3.27) holds as equality is when $\mathcal{F}(H) = \operatorname{dom}(H^q)$ for some q. In this case we say that $\mathcal{F}(H)$ is *finitely determined*.

Lemma 3.11. The feasible set $\mathcal{F}(H)$ is finitely determined if and only if $\operatorname{dom}(H^q) = \operatorname{dom}(H^{q+1})$ for some q.

Proof. As $\mathcal{F}(H) \subseteq \operatorname{dom}(H^{q+1}) \subseteq \operatorname{dom}(H^q)$ for all $q \geqslant 0$, necessity is clear. For sufficiency, let q be such that $\operatorname{dom}(H^q) = \operatorname{dom}(H^{q+1})$ and let $x \in \operatorname{dom}(H^q)$. As $\operatorname{dom}(H^q) = \operatorname{dom}(H^{q+1})$, clearly, there exists $y \in H(x)$ such that $y \in \operatorname{dom}(H^q)$.

Thus $y \in H(x) \cap \operatorname{dom}(H^q)$, proving that $\operatorname{dom}(H^q)$ is weakly H invariant. As $\mathcal{F}(H)$ is the largest of such sets, we see that $\mathcal{F}(H) = \operatorname{dom}(H^q)$, proving the lemma. ∎

For the reachable set, we can prove analogous results. To do so, first note that

$$\mathcal{R}(H) = \bigcup_{q=0}^{\infty} H^q(0). \tag{3.28}$$

Lemma 3.12. For a convex process H, the reachable set $\mathcal{R}(H)$ is the smallest strongly H invariant convex cone. Moreover, $\mathcal{R}(H) = H(\mathcal{R}(H))$.

Proof. Clearly, $\mathcal{R}(H)$ is a convex cone. Let $\xi \in \mathcal{R}(H)$. From (3.28), we see that $\xi \in H^q(0)$ for some $q \geqslant 0$. Let $\eta \in H(\xi)$. Then, we have $\eta \in H^{q+1}(0)$. This means that $\eta \in \mathcal{R}(H)$. Therefore, $\mathcal{R}(H)$ is strongly H invariant. Let $\mathcal{R}'$ be a strongly H invariant cone. As $0 \in \mathcal{R}'$, we have $H^{\ell}(0) \subseteq \mathcal{R}'$ for all $\ell \geqslant 0$. Therefore, $\mathcal{R} \subseteq \mathcal{R}'$ and hence $\mathcal{R}(H)$ is the smallest strongly H invariant convex cone.

For the second part, note that $H(\mathcal{R}(H)) \subseteq \mathcal{R}(H)$. Applying H on both sides, we see that $H(\mathcal{R}(H))$ is strongly H invariant. Since $\mathcal{R}(H)$ is the smallest of such cones, we see that $\mathcal{R}(H) \subseteq H(\mathcal{R}(H))$. Therefore, we have $H(\mathcal{R}(H)) = \mathcal{R}(H)$. ∎

We say $\mathcal{R}(H)$ is *finitely determined* if $\mathcal{R}(H) = H^q(0)$ for some q. Similar to Lemma 3.11, we can state the following.

Lemma 3.13. The reachable set $\mathcal{R}(H)$ is finitely determined if and only if $H^q(0) = H^{q+1}(0)$ for some q.

Proof. Since $H^q(0) \subseteq H^{q+1}(0) \subseteq \mathcal{R}(H)$, necessity is clear. As $H(H^q(0)) = H^{q+1}(0)$, we see that $H^q(0)$ is a strongly H invariant cone. As $\mathcal{R}(H)$ is the smallest of such cones, we see that $\mathcal{R}(H) = H^q(0)$. ∎

Unlike for linear systems, the reachable set is not always finitely determined, as shown by the following example.

Example 3.7. Let the convex process $H : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by

$$\operatorname{gr}(H) := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \;\middle|\; \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \geqslant 0 \right\}.$$

Clearly, this is a convex cone. For any $x \in \mathbb{R}^2$ we have

$$H(x) = \{y \mid x_1 \leqslant y_1, \quad 0 \leqslant y_2 \leqslant x_1 + x_2\}.$$

Note that $H^k(0) = \{x \mid 0 \leqslant x_1, \quad 0 \leqslant x_2 \leqslant kx_1\}$. As such,

$$\mathcal{R}(H) = \{0\} \cup \{x \mid x_1 > 0, x_2 \geqslant 0\}.$$

This means that, even though H is closed, $\mathcal{R}(H)$ is not a closed convex cone. Furthermore, we see that $\mathcal{R}(H)$ is not finitely determined, that is, $\mathcal{R}(H) \neq H^k(0)$ for any $k \geqslant 0$.

Even if $\mathcal{R}(H)$ is finitely determined, it is, in general not possible to give an upper bound on the number of steps required.

Example 3.8. Let $\theta \in [0, 2\pi)$, and define the strict convex process $H_\theta : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$H_\theta(x) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x + \mathbb{R}_+ \times \{0\}.$$

Then it is straightforward to check that $\mathcal{R}(H_0) = H_0(0) = \mathbb{R}_+ \times \{0\}$ and $\mathcal{R}(H_\pi) = H_\pi^2(0) = \mathbb{R} \times \{0\}$. If $\theta < \pi$ then $\mathcal{R}(H_\theta) = \mathbb{R}^2$ and $\mathcal{R}(H_\theta) = H_\theta^q(0)$ for q such that $q\theta > \pi$. This means that, for small θ, certain points in the state space can only be reached in a large number of steps.

Similar to feasible and reachable sets, stabilizable sets also enjoy certain invariance properties.

Lemma 3.14. Let H be a convex process. Then, $\mathcal{S}_e(H) \subseteq \mathcal{S}(H) \subseteq \mathcal{F}(H)$. In addition, the sets $\mathcal{S}(H)$ and $\mathcal{S}_e(H)$ are both weakly H invariant convex cones satisfying $\mathcal{S}(H) = H^{-1}(\mathcal{S}(H))$ and $\mathcal{S}_e(H) = H^{-1}(\mathcal{S}_e(H))$.

Proof. The inclusions are immediate from the definitions. Now let $\xi \in \mathcal{S}(H)$, then there exists a stable trajectory $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H)$ such that $x_0 = \xi$. Now clearly $x_1 \in H(\xi) \cap \mathcal{S}(H)$ and thus $\mathcal{S}(H)$ is weakly H invariant.

Now let $\xi \in \mathcal{S}_e(H)$. Therefore there exists $\alpha > 0, \mu \in [0, 1)$ and a trajectory $(x_k)_{k\in\mathbb{N}} \in \mathcal{B}(H)$ such that $x_0 = \xi$ and $|x_k| \leqslant \alpha\mu^k|\xi|$ for all $k \geqslant 0$. If $x_1 = 0$, then we have $x_1 \in H(\xi) \cap \mathcal{S}_e(H)$. If $x_1 \neq 0$, let $y_k = x_{k+1}$ for all $k \geqslant 0$. Clearly, $(y_k)_{k\in\mathbb{N}} \in \mathcal{B}(H)$ and $y_0 = x_1$. Note that $|y_k| = |x_{k+1}| \leqslant \alpha\mu^{k+1}|\xi| = \beta\mu^k|y_0|$ for all $k \geqslant 0$ where $\beta = \alpha\mu|\xi|/|y_0|$. Therefore $y_0 = x_1 \in H(\xi) \cap \mathcal{S}_e(H)$ and thus $\mathcal{S}_e(H)$ is weakly H invariant.

Due to Lemma 3.8.(i), we already know that $\mathcal{S}(H) \subseteq H^{-1}(\mathcal{S}(H))$ and $\mathcal{S}_e(H) \subseteq H^{-1}(\mathcal{S}_e(H))$. What remains to be shown are the reverse inclusions. To do so, let first $\eta \in H^{-1}(\mathcal{S}(H))$. Then, there must exist $\xi \in \mathcal{S}(H)$ such that $\eta \in H^{-1}(\xi)$. Since $\xi \in \mathcal{S}(H)$, there exists a stable trajectory $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H)$ with $x_0 = \xi$. Now, define $y_0 = \eta$ and $y_k = x_{k-1}$ for $k \geqslant 1$. Clearly, $(y_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H)$ is a stable trajectory. Therefore, $y_0 = \eta \in \mathcal{S}(H)$. Consequently, we can conclude that $H^{-1}(\mathcal{S}(H)) \subseteq \mathcal{S}(H)$. The same argument is still valid if one replaces stability by exponential stability. As such, we also have that $H^{-1}(\mathcal{S}_e(H)) \subseteq \mathcal{S}_e(H)$. ∎

For the exponentially stabilizable set, we have the following property.

Lemma 3.15. Let H be a convex process. Then, the exponentially stabilizable set $\mathcal{S}_e(H)$ is strongly $(H-\mu I)^{-1}$ invariant for all $\mu \in [0,1)$.

Proof. Let $\mu \in [0,1)$ and $\xi \in \mathcal{S}_e(H)$. If $\xi \notin \operatorname{dom}(H-\mu I)^{-1}$, then we have $\varnothing = (H-\mu I)^{-1}(\xi) \subseteq \mathcal{S}_e(H)$. If $\xi \in \operatorname{dom}(H-\mu I)^{-1}$, there exists $\eta \in (H-\mu I)^{-1}(\xi)$. Since $\xi \in \mathcal{S}_e(H)$, we know that there exists an exponentially stable trajectory $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H)$ with $x_0 = \xi$. Define $y_0 = \eta$ and $y_k = \mu^k\eta + \sum_{\ell=0}^{k-1} \mu^{k-1-\ell}x_\ell$ for $k \geqslant 1$. It can be verified that $y_{k+1} \in H(y_k)$ for all $k \in \mathbb{N}$. Since $\mu \in [0,1)$ and $(x_k)_{k\in\mathbb{N}}$ is exponentially stable, so is the trajectory $(y_k)_{k\in\mathbb{N}}$. Therefore, $y_0 = \eta \in \mathcal{S}_e(H)$. Consequently, $\mathcal{S}_e(H)$ is strongly $(H-\mu I)^{-1}$ invariant. ∎

For the null-controllable set, we can prove the following invariance properties in a similar fashion to Lemma 3.14.

Lemma 3.16. Let H be a convex process. Then, $\mathcal{N}(H) \subseteq \mathcal{F}(H)$. In addition, $\mathcal{N}(H)$ is weakly H invariant and $\mathcal{N}(H) = H^{-1}(\mathcal{N}(H))$.

In addition to the reachable set of H, we will also consider the reachable sets of L_- and L_+. One relation between these respective reachable sets, is given in the following lemma.

Lemma 3.17. Let H be a convex process and denote $\mathcal{R} = \mathcal{R}(H)$ and $\mathcal{R}_+ = \mathcal{R}(L_+)$. If $\operatorname{dom}(H) - \mathcal{R}$ is a subspace, then $\mathcal{R} - \mathcal{R} = \mathcal{R}_+$ and $\mathcal{R}_+$.

Proof. It follows from (3.21) that $H^\ell(0) \subseteq L_+^\ell(0)$ for all $\ell \in \mathbb{N}$. Then, we get $\mathcal{R} - \mathcal{R} \subseteq \mathcal{R}_+ - \mathcal{R}_+ = \mathcal{R}_+$ where the last equality follows from the fact that $\mathcal{R}_+$ is a subspace. As $\mathcal{R}_+$ is the smallest strongly L_+ invariant cone, the reverse inclusion $\mathcal{R}_+ \subseteq \mathcal{R} - \mathcal{R}$ would follow if $\mathcal{R} - \mathcal{R}$ is strongly L_+ invariant. Therefore, it suffices to show that $L_+(\mathcal{R} - \mathcal{R}) \subseteq \mathcal{R} - \mathcal{R}$. Let $x \in \mathcal{R} - \mathcal{R}$. Then, there exist $r_1, r_2 \in \mathcal{R}$ such that $x = r_1 - r_2$. If $x \notin \operatorname{dom}(L_+) = \operatorname{dom}(H) - \operatorname{dom}(H)$, we readily have $L_+(x) = \varnothing \subseteq \mathcal{R} - \mathcal{R}$. Suppose that $x \in \operatorname{dom}(H) - \operatorname{dom}(H)$ and $y \in L_+(x)$. From the definition of L_+ (3.20), it follows that there exist $x_1, x_2 \in \operatorname{dom}(H)$ such that $x = r_1 - r_2 = x_1 - x_2$ and $y \in H(x_1) - H(x_2)$. Note that

$r_1 - x_1 = r_2 - x_2 \in \mathcal{R} - \mathrm{dom}(H)$. Since $\mathrm{dom}(H) - \mathcal{R}$ is a subspace, we have $\mathcal{R} - \mathrm{dom}(H) = \mathrm{dom}(H) - \mathcal{R}$. Therefore, there exist $\bar{x} \in \mathrm{dom}(H)$ and $\bar{r} \in \mathcal{R}$ such that $r_1 - x_1 = r_2 - x_2 = \bar{x} - \bar{r}$. Since H is a convex process, we have $H(\bar{x}) + H(x_i) \subseteq H(\bar{x} + x_i) = H(r_i + \bar{r})$ for $i = 1, 2$. This leads to $H(x_1) - H(x_2) \subseteq H(r_1 + \bar{r}) - H(r_2 + \bar{r}) + H(\bar{x}) - H(\bar{x})$ since $\bar{x} \in \mathrm{dom}(H)$. From $H(\mathcal{R}) \subseteq \mathcal{R}$, we know that $H(r_1 + \bar{r}) - H(r_2 + \bar{r}) \subseteq \mathcal{R} - \mathcal{R}$ since $r_1, r_2, \bar{r} \in \mathcal{R}$. Thus, it suffices to show that $H(\bar{x}) - H(\bar{x}) \in \mathcal{R} - \mathcal{R}$ for all $\bar{x} \in \mathrm{dom}(H)$. Let $\bar{x} \in \mathrm{dom}(H)$. As $0 \in \mathcal{R}$, we have $\bar{x} \in \mathrm{dom}(H) - \mathcal{R}$. Since $\mathrm{dom}(H) - \mathcal{R}$ is a subspace, $-\bar{x} \in \mathrm{dom}(H) - \mathcal{R}$ and hence there exist $\xi \in \mathrm{dom}(H)$ and $\eta \in \mathcal{R}$ such that $-\bar{x} = \xi - \eta$. This means that $H(\bar{x}) + H(\xi) \subseteq H(\bar{x} + \xi) = H(\eta)$ as H is a convex process. Since $\xi \in \mathrm{dom}(H)$, we get $-H(\bar{x}) \subseteq H(\xi) - H(\eta)$. Then, we obtain $H(\bar{x}) - H(\bar{x}) \subseteq H(\bar{x}) + H(\xi) - H(\eta) \subseteq H(\bar{x} + \xi) - H(\eta) = H(\eta) - H(\eta)$. As $H(\eta) \subseteq \mathcal{R}$, we finally arrive at $H(\bar{x}) - H(\bar{x}) \subseteq \mathcal{R} - \mathcal{R}$ since $\eta \in \mathcal{R}$. ■

Next, we investigate some consequences of the domain condition (DC).

Lemma 3.18. Let H be a convex process and let $\mathcal{V}$ be a strongly H invariant subspace with $\mathcal{R}_- \subseteq \mathcal{V}$. If $\mathrm{dom}(H) + \mathcal{R}_- = \mathrm{dom}(H) + \mathcal{V}$ then $\mathrm{dom}(H^k) + \mathcal{R}_- = \mathrm{dom}(H^k) + \mathcal{V}$ for all $k \geqslant 1$.

Proof. We can use Lemma 3.1 to give an equivalent statement of the implication in the lemma as:

$$\mathcal{V} \subseteq \mathrm{dom}(H) + \mathcal{R}_- \implies \mathcal{V} \subseteq \mathrm{dom}(H^k) + \mathcal{R}_- \quad \forall k \geqslant 1$$

To prove this, we begin with defining the convex process $H_e : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, by

$$\mathrm{gr}(H_e) = \mathrm{gr}(H) + \mathcal{R}_- \times \{0\}.$$

We will start by proving that for all $k \geqslant 1$, we have $\mathrm{dom}(H_e^k) = \mathrm{dom}(H^k) + \mathcal{R}_-$. Clearly $\mathcal{R}_- \subseteq \mathrm{dom}(H_e^k)$ and $\mathrm{dom}(H^k) \subseteq \mathrm{dom}(H_e^k)$. Therefore $\mathrm{dom}(H^k) + \mathcal{R}_- \subseteq \mathrm{dom}(H_e^k)$.

It remains to show that the reverse inclusion holds. This will be achieved by induction on k. For $k = 1$, note that clearly $\mathrm{dom}(H_e) \subseteq \mathrm{dom}(H) + \mathcal{R}_-$. For the induction step assume that $\mathrm{dom}(H_e^k) \subseteq \mathrm{dom}(H^k) + \mathcal{R}_-$ for some $k \geqslant 1$. Let $\xi \in \mathrm{dom}(H_e^{k+1})$. Therefore, there exists $\zeta \in \mathrm{dom}(H_e^k)$ such that $\zeta \in H_e(\xi)$. By the induction hypothesis, we see that $\zeta = \zeta_1 - \zeta_2$ where $\zeta_1 \in \mathrm{dom}(H^k)$ and $-\zeta_2 \in \mathcal{R}_-$. Hence, we obtain $\zeta_1 \in H(\xi) + \zeta_2$.

As $\mathcal{R}_-$ is the reachable set of L_-, we know there exists $\eta \in \mathcal{R}_-$ such that $\zeta_2 \in L_-(\eta)$. This yields $\zeta_1 \in H(\xi) + L_-(\eta)$. Then, Lemma 2.1 implies that $\zeta_1 \in H(\xi + \eta)$. Since $\zeta_1 \in \mathrm{dom}(H^k)$, we can conclude that $\xi + \eta \in \mathrm{dom}(H^{k+1})$

and hence $\xi \in \operatorname{dom}(H^{k+1}) + \mathcal{R}_-$. This proves that for all $k \geqslant 1$, we have $\operatorname{dom}(H_e^k) = \operatorname{dom}(H^k) + \mathcal{R}_-$.

To prove the lemma, recall that it suffices to show that $\mathcal{V} \subseteq \operatorname{dom}(H^k) + \mathcal{R}_-$. By the hypothesis, we have $\mathcal{V} \subseteq \operatorname{dom}(H) + \mathcal{R}_- = \operatorname{dom}(H_e)$. Next, we claim that $\mathcal{V}$ is strongly H_e invariant. To see this, let $x \in \mathcal{V}$ and $y \in H_e(x)$. Then, there must exist $x' \in \operatorname{dom}(H)$ and $z \in \mathcal{R}_-$ such that $(x,y) = (x',y) - (z,0)$ due to the definition of H_e. As such, we have $y \in H(x+z) \subseteq H(\mathcal{R}_+) \subseteq \mathcal{V}$ where the first inclusion follows from the fact that $x+z \in \mathcal{V} + \mathcal{R}_- = \mathcal{V}$ and the second from the fact that $\mathcal{V}$ is strongly H invariant. Therefore, we have $H_e(\mathcal{V}) \subseteq \mathcal{V}$, in other words, $\mathcal{V}$ is strongly H_e invariant. Since $\mathcal{V} \subseteq \operatorname{dom}(H_e)$, we can use Lemma 3.8.(ii) to conclude that $\mathcal{V}$ is also weakly H_e invariant. Therefore, we get $\mathcal{V} \subseteq \mathcal{F}(H_e) \subseteq \operatorname{dom}(H_e^k) = \operatorname{dom}(H^k) + \mathcal{R}_-$ for all $k \geqslant 1$ where the first inclusion follows from the fact that $\mathcal{F}(H_e)$ is the largest weakly H_e invariant cone, the second from (3.27), and finally the third from the first part of this proof. ∎

The image of a convex cone under a convex process enjoys the following duality relation. Note also [12, Thm. 2.5.7] which additionally assumes closedness.

Proposition 3.1. Let H be a convex process and K be a convex cone such that $K - \operatorname{dom}(H)$ is a subspace. Then,

$$\big(H(K)\big)^- = (H^-)^{-1}(K^-).$$

Proof. We can write the left-hand side in terms of the graph by:

$$(H(K))^- = \left(\begin{bmatrix} 0 & I \end{bmatrix} (\operatorname{gr}(H) \cap K \times \mathbb{R}^n)\right)^- .$$

Using [142, Cor. 16.3.2], this means that:

$$\begin{aligned}
(H(K))^- &= \begin{bmatrix} 0 \\ I \end{bmatrix}^{-1} (\operatorname{gr}(H) \cap K \times \mathbb{R}^n)^- \\
&= \begin{bmatrix} 0 \\ I \end{bmatrix}^{-1} \operatorname{cl}\left(\operatorname{gr}(H)^- + K^- \times \{0\}\right) \\
&= \begin{bmatrix} 0 \\ I \end{bmatrix}^{-1} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}^{-1} \operatorname{cl}\left(\operatorname{gr}(H^-) + \{0\} \times K^+\right).
\end{aligned}$$

By Lemma 3.5 we have

$$[\operatorname{dom}(H) - K]^+ = H^-(0) \cap K^-.$$

Therefore, from our assumption follows that $\operatorname{gr}(H^-) \cap \{0\} \times K^-$ is a subspace, which allows us to use Lemma 3.3 to reveal that $\operatorname{gr}(H^-) + \{0\} \times K^+$ is closed. Therefore we can drop the closure from our derivation, and find:

$$(H(K))^- = \begin{bmatrix} I \\ 0 \end{bmatrix}^{-1} \big(\operatorname{gr}(H^-) + \{0\} \times K^+\big) = (H^-)^{-1}(K^-).$$

Thus proving the lemma. ■

Strong and weak invariance become dual notions under certain conditions.

Theorem 3.5. Let H be a convex process and K be a convex cone such that $K - \operatorname{dom}(H)$ is a subspace. Then, K^- is weakly H^- invariant if K is strongly H invariant. Conversely, K is strongly H invariant if K is closed and K^- is weakly H^- invariant.

Proof. Suppose that K is strongly H invariant. Then, $H(K) \subseteq K$ in view of Lemma 3.8. Hence, $K^- \subseteq [H(K)]^-$. From Proposition 3.1, we have $K^- \subseteq (H^-)^{-1}(K^-)$. Therefore, K^- is weakly H^- invariant due to Lemma 3.8.

Now suppose that K is closed and K^- is weakly H^- invariant. Then, $K^- \subseteq (H^-)^{-1}(K^-)$. Proposition 3.1 implies that $\big([H(K)]^-\big)^- \subseteq (K^-)^-$. This means that $\operatorname{cl}(H(K)) \subseteq K$ since K is closed. Hence, $H(K) \subseteq K$. In other words, K is strongly H invariant due to Lemma 3.8. ■

3.5.2 Eigenvalues of convex processes

For a detailed study of eigenvalues and eigenvectors of convex processes, we refer the reader to Chapter 2.

Next, we relate nonnegative eigenvalues of H^- to the reachable set of H.

Lemma 3.19. Let H be a convex process and $\lambda \geqslant 0$. Then, $\ker(H^- - \lambda I) \subseteq \mathcal{R}(H)^-$.

Proof. For this, let ξ be an eigenvector of H^- corresponding to a nonnegative eigenvalue λ, that is, $\lambda\xi \in H^-(\xi)$. Clearly this means that $(\lambda^j\xi, \lambda^{j+1}\xi) \in \operatorname{gr} H^-$ for any $j \geqslant 0$. Now take $\eta \in \mathcal{R}(H)$, i.e. $\eta \in H^q(0)$ for some q. Then there exists a (finite) sequence $(x_k)_{k=0}^q$ such that $x_0 = 0$, $x_q = \eta$ and $(x_k, x_{k+1}) \in \operatorname{gr} H$ for $k = 0, \dots, q-1$. By the definition of the dual process in (3.18), we know:

$$\langle \lambda^{j+1}\xi, x_k \rangle \geqslant \langle \lambda^j \xi, x_{k+1} \rangle$$

for any $j \geqslant 0$ and $k = 0, \dots, q-1$. In particular we can conclude that:

$$0 = \langle \lambda^q \xi, x_0 \rangle \geqslant \langle \lambda^{q-1} \xi, x_1 \rangle \geqslant \cdots \geqslant \langle \lambda \xi, x_{q-1} \rangle \geqslant \langle \xi, x_q \rangle = \langle \xi, \eta \rangle.$$

This allows us to conclude that $\xi \in \mathcal{R}(H)^-$. ■

3.5.3 Inner and outer processes

In this section, we will study inner and outer processes as defined in Section 3.4. We begin with recalling the definition of the inner process $H_{in} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$:

$$\operatorname{gr}(H_{in}) := \operatorname{gr}(H) \cap (\mathcal{R}_+ \times \mathcal{R}_+). \tag{3.29}$$

The following observations readily follow from (3.29):

$$\operatorname{dom}(H_{in}^k) = \operatorname{dom}(H^k) \cap \mathcal{R}_+ \quad \forall\, k \geqslant 1 \tag{3.30a}$$
$$\mathcal{F}(H_{in}) = \mathcal{F}(H) \cap \mathcal{R}_+, \tag{3.30b}$$
$$\mathcal{R}(H_{in}) = \mathcal{R}(H), \tag{3.30c}$$
$$\mathcal{S}(H_{in}) = \mathcal{S}(H) \cap \mathcal{R}_+, \tag{3.30d}$$
$$\mathcal{S}_e(H_{in}) = \mathcal{S}_e(H) \cap \mathcal{R}_+, \tag{3.30e}$$
$$\mathcal{N}(H_{in}) = \mathcal{N}(H) \cap \mathcal{R}_+. \tag{3.30f}$$

The subsequent results play an instrumental role in studying reachability, stabilizability and null-controllability.

Lemma 3.20. Let H be a convex process satisfying the domain condition (DC). Then, $\mathcal{F}(H_{in}) = \operatorname{dom}(H_{in}^n)$ and $\mathcal{F}(H_{in}) + \mathcal{R}_- = \mathcal{R}_+$. Moreover, if $\mathcal{W}$ is a weakly H_{in} invariant convex cone, then the following statements are equivalent:

(i) $\mathcal{F}(H_{in}) \subseteq \mathcal{R}(H_{in}) - \mathcal{W}$

(ii) $\mathcal{R}(H_{in}) - \mathcal{W} = \mathcal{R}_+$

(iii) H_{in}^- has no eigenvector in $\big(\mathcal{R}(H_{in}) - \mathcal{W}\big)^- \setminus \mathcal{R}_+^\perp$ that corresponds to a nonnegative eigenvalue.

If, in addition, $H_{in}^{-1}(\mathcal{W}) \subseteq \mathcal{W}$ then the statements are above equivalent to:

(iv) $\mathcal{F}(H_{in}) = \mathcal{W}$

Proof. To show that $\mathcal{F}(H_{\text{in}}) = \text{dom}(H_{\text{in}}^n)$, we first claim that $\mathcal{R}(L_-(H_{\text{in}})) = \mathcal{R}_-$. Since $\text{gr}(H_{\text{in}}) \subseteq \text{gr}(H)$, we readily have that $\mathcal{R}(L_-(H_{\text{in}})) \subseteq \mathcal{R}_-$. For the reverse inclusion, let $\xi \in \mathcal{R}_-$. Then, there exists $q \geqslant 0$ and $(x_k)_{k=0}^q \in \mathfrak{B}_q(L_-(H))$ such that $x_0 = 0$ and $x_q = \xi$. Note that $x_k \in \mathcal{R}_-$ for all $k \in \{0, 1, \ldots, q\}$. Since $\mathcal{R}_- \subseteq \mathcal{R}_+$ and $\text{gr}(L_-(H_{\text{in}})) = \text{gr}(L_-(H)) \cap (\mathcal{R}_+ \times \mathcal{R}_+)$, we see that $(x_k)_{k=0}^q \in \mathfrak{B}_q(L_-(H_{\text{in}}))$ and hence $\xi \in \mathcal{R}(L_-(H_{\text{in}}))$. This proves that

$$\mathcal{R}(L_-(H_{\text{in}})) = \mathcal{R}_-. \tag{3.31}$$

Next, note that $(\text{dom}(H) \cap \mathcal{R}_+) + \mathcal{R}_- = \mathcal{R}_+$ due to $\mathcal{R}_- \subseteq \mathcal{R}_+$ and Lemma 3.1. In view of (3.30a), we see that

$$\text{dom}(H_{\text{in}}) + \mathcal{R}_- = \mathcal{R}_+. \tag{3.32}$$

Now, we claim that $\text{dom}(H_{\text{in}}^n) = \text{dom}(H_{\text{in}}^{n+1})$. To see this, let $\eta \in \text{dom}(H_{\text{in}}^n)$ and $\zeta \in H_{\text{in}}^n(\eta)$. Since $\text{gr}(H_{\text{in}}) \subseteq \mathcal{R}_+ \times \mathcal{R}_+$, we see that $\zeta \in \mathcal{R}_+$. From (3.32), it follows that $\zeta = \zeta_1 - \zeta_2$ where $\zeta_1 \in \text{dom}(H_{\text{in}})$ and $\zeta_2 \in \mathcal{R}_-$. As $\mathcal{R}_- \subseteq H_{\text{in}}^n(0)$ due to (3.31) and Lemma 3.6, we have that $\zeta_2 \in H_{\text{in}}^n(0)$. Therefore, $\zeta_1 = \zeta + \zeta_2 \in H_{\text{in}}^n(\eta) + H_{\text{in}}^n(0) = H_{\text{in}}^n(\eta)$. Since $\zeta_1 \in \text{dom}(H_{\text{in}})$, we can conclude that $\eta \in \text{dom}(H_{\text{in}}^{n+1})$. This means that $\text{dom}(H_{\text{in}}^n) \subseteq \text{dom}(H_{\text{in}}^{n+1})$. As the reverse inclusion is obvious, we obtain $\text{dom}(H_{\text{in}}^n) = \text{dom}(H_{\text{in}}^{n+1})$. Now, it follows from Lemma 3.11 that $\mathcal{F}(H_{\text{in}}) = \text{dom}(H_{\text{in}}^n)$.

To show that $\mathcal{F}(H_{\text{in}}) + \mathcal{R}_- = \mathcal{R}_+$, note first that the domain condition (DC) and Lemma 3.18 imply $\text{dom}(H^n) + \mathcal{R}_- = \text{dom}(H^n) + \mathcal{R}_+$. Since $\mathcal{R}_- \subseteq \mathcal{R}_+$, Lemma 3.1 implies that $(\text{dom}(H^n) \cap \mathcal{R}_+) + \mathcal{R}_- = \mathcal{R}_+$. Then, it follows from (3.30a) that $\text{dom}(H_{\text{in}}^n) + \mathcal{R}_- = \mathcal{R}_+$. Since $\mathcal{F}(H_{\text{in}}) = \text{dom}(H_{\text{in}}^n)$, we see that $\mathcal{F}(H_{\text{in}}) + \mathcal{R}_- = \mathcal{R}_+$.

For the rest, we will prove the implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (i), (ii) $\Leftrightarrow$ (iii), (iv) $\Rightarrow$ (ii), and finally (ii) $\Rightarrow$ (iv) under the extra hypothesis $H_{\text{in}}^{-1}(\mathcal{W}) \subseteq \mathcal{W}$.

(i) $\Rightarrow$ (ii): Since $\mathcal{R}_- \subseteq \mathcal{R}(H) = \mathcal{R}(H_{\text{in}})$, we see that $\mathcal{F}(H_{\text{in}}) + \mathcal{R}_- \subseteq \mathcal{R}(H_{\text{in}}) - \mathcal{W}$. Then, we have $\mathcal{R}_+ \subseteq \mathcal{R}(H_{\text{in}}) - \mathcal{W}$ as $\mathcal{F}(H_{\text{in}}) + \mathcal{R}_- = \mathcal{R}_+$. The reverse inclusion readily holds since $\mathcal{W} \subseteq \text{dom}(H_{\text{in}}) \subseteq \mathcal{R}_+$. Therefore, we can conclude that $\mathcal{R}(H_{\text{in}}) - \mathcal{W} = \mathcal{R}_+$.

(ii) $\Rightarrow$ (i): This readily follows from the fact that $\mathcal{F}(H_{\text{in}}) \subseteq \mathcal{R}_+$.

(ii) $\Leftrightarrow$ (iii): For this part of the proof, we want to apply Theorem 2.3 to the closed convex cone $\mathcal{K} := \big(\mathcal{R}(H_{\text{in}}) - \mathcal{W}\big)^- = \mathcal{R}(H_{\text{in}})^- \cap \mathcal{W}^+$. To do this, we need to show that the following hypotheses are satisfied:

(a) $\mathcal{K}$ is weakly H_{in}^- invariant.

(b) $H_{\text{in}}^-(0) \cap \mathcal{K}$ is a subspace.

(c) $\mathrm{lin}(\mathcal{K})$ is weakly $L_-(H_{\mathrm{in}}^-)$ invariant.

(d) $\mathrm{lin}(\mathcal{K}) \subseteq (L_-(H_{\mathrm{in}}^-) - \lambda I)\,\mathrm{lin}(\mathcal{K})$ for all $\lambda \geqslant 0$.

To verify these hypotheses, note first that we have

$$\begin{aligned} \mathrm{dom}(H_{\mathrm{in}}) + \mathcal{R}_- \subseteq \mathrm{dom}(H_{\mathrm{in}}) - \mathcal{R}(H) \subseteq \mathrm{dom}(H_{\mathrm{in}}) + \mathcal{R}_+, \\ \mathcal{R}_+ \subseteq \mathrm{dom}(H_{\mathrm{in}}) - \mathcal{R}(H) \subseteq \mathcal{R}_+, \end{aligned}$$

where the first line follows from $\mathcal{R}_- \subseteq \mathcal{R}(H) \subseteq \mathcal{R}_+$ and the second from (3.32) and (3.30a). Therefore, we see that

$$\mathrm{dom}(H_{\mathrm{in}}) - \mathcal{R}(H) = \mathcal{R}_+.$$

Since $\mathcal{W}$ is weakly H_{in} invariant, we have

$$\mathcal{W} \subseteq \mathrm{dom}(H_{\mathrm{in}}) \subseteq \mathcal{R}_+. \tag{3.33}$$

This means that

$$\mathrm{dom}(H_{\mathrm{in}}) - \big(\mathcal{R}(H) - \mathcal{W}\big) = \mathcal{R}_+. \tag{3.34}$$

Since $\mathcal{R}(H) = \mathcal{R}(H_{\mathrm{in}})$ is strongly H_{in} invariant and $\mathcal{W}$ is weakly H_{in} invariant, we see from Lemma 3.9 that $\mathcal{R}(H) - \mathcal{W}$ is strongly H_{in} invariant. Then, it follows from (3.34) and Theorem 3.5 that $\mathcal{K}$ is weakly H_{in}^- invariant. This proves (a).

Let G be the convex process given by $\mathrm{gr}(G) = \mathrm{cl}\big(\mathrm{gr}(H_{\mathrm{in}})\big)$. Note that we have $\mathrm{ri}\big(\mathrm{dom}(G)\big) = \mathrm{ri}\big(\mathrm{dom}(H_{\mathrm{in}})\big)$. Since G is closed and $G^- = H_{\mathrm{in}}^-$, Lemma 3.5 yields that $H_{\mathrm{in}}^-(0) = \big(\mathrm{dom}(H_{\mathrm{in}})\big)^+$. Then, we see from (3.34) that $H_{\mathrm{in}}^-(0) \cap \mathcal{K} = \mathcal{R}_+^\perp$ is a subspace. This proves (b).

From Lemma 3.17, we know that $\mathcal{R}(H) - \mathcal{R}(H) = \mathcal{R}_+$. Since $\mathcal{R}(H_{\mathrm{in}}) = \mathcal{R}(H)$, it follows from (3.33) that $\mathrm{Lin}\,\big(\mathcal{R}(H) - \mathcal{W}\big) = \mathcal{R}_+$. This results in

$$\mathrm{lin}(\mathcal{K}) = \mathcal{R}_+^\perp. \tag{3.35}$$

From the definition of H_{in} in (3.29), we have that

$$\mathrm{gr}(H_{\mathrm{in}}^-) = \mathrm{gr}(H) + (\mathcal{R}_+^\perp \times \mathcal{R}_+^\perp). \tag{3.36}$$

In particular, we have $\mathcal{R}_+^\perp \subseteq H_{\mathrm{in}}^-(\xi)$ for all $\xi \in \mathcal{R}_+^\perp$. This implies that $\mathcal{R}_+^\perp \subseteq (H_{\mathrm{in}}^-)^{-1}(\mathcal{R}_+^\perp)$. As such, $\mathcal{R}_+^\perp$ is weakly H_{in}^- invariant. Together with (3.35), this proves (c).

Note that $L_-(H_{\mathrm{in}}^-)(0) = \mathrm{lin}\,\big(H_{\mathrm{in}}^-(0)\big)$. Then, we see that $\mathcal{R}_+^\perp \subseteq L_-(H_{\mathrm{in}}^-)(0)$ from (3.36). Therefore, we have

$$\mathcal{R}_+^\perp \subseteq L_-(H_{\mathrm{in}}^-)(0) = \big(L_-(H_{\mathrm{in}}^-) - \lambda I\big)(0) \subseteq \big(L_-(H_{\mathrm{in}}^-) - \lambda I\big)(\mathcal{R}_+^\perp),$$

for all $\lambda \geqslant 0$. Together with (3.35), this proves (d).

Since H_{in}^- and $\mathcal{K}$ satisfy the hypotheses (a)-(d), Theorem 2.3 and (3.35) imply that (ii) holds if and only if (iii) holds.

(iv) $\Rightarrow$ (ii): As $\mathcal{F}(H_{\mathrm{in}}) + \mathcal{R}_- = \mathcal{R}_+$, we see that $\mathcal{R}_+ \subseteq \mathcal{F}(H_{\mathrm{in}}) - \mathcal{R}(H_{\mathrm{in}}) = \mathcal{W} - \mathcal{R}(H_{\mathrm{in}})$. Since the reverse inclusion is evident, we se that $\mathcal{R}(H_{\mathrm{in}}) - \mathcal{W} = \mathcal{R}_+$.

$H_{\mathrm{in}}^{-1}(\mathcal{W}) \subseteq \mathcal{W}$ *and (ii) $\Rightarrow$ (iv)*: Note that $\mathcal{R}(H_{\mathrm{in}}) - \mathcal{W} = \cup_{\ell \geqslant 0} H_{\mathrm{in}}^{\ell}(0) - \mathcal{W}$. Since $\mathcal{R}(H_{\mathrm{in}}) - \mathcal{W} = \mathcal{R}_+$, we see that $\cup_{\ell \geqslant 0} H_{\mathrm{in}}^{\ell}(0) - \mathcal{W}$ is finitely generated. Then, it follows from Lemma 3.4 that $H_{\mathrm{in}}^{q}(0) - \mathcal{W} = \mathcal{R}_+$ for some $q \geqslant 0$. Let $\xi \in \mathcal{F}(H_{\mathrm{in}})$. Therefore, there exists a trajectory $(x_k) \in \mathfrak{B}(H_{\mathrm{in}})$ such that $x_0 = \xi$. Clearly, $x_q \in H_{\mathrm{in}}^{q}(\xi) \in \mathcal{R}_+$. Therefore, $x_q = \zeta - \eta$ where $\zeta \in \mathcal{W}$ and $\eta \in H_{\mathrm{in}}^{q}(0)$. This means that $\zeta \in H_{\mathrm{in}}^{q}(\xi)$. Thus, we see that $\xi \in H_{\mathrm{in}}^{-q}(\mathcal{W})$. Since $H_{\mathrm{in}}^{-1}(\mathcal{W}) \subseteq \mathcal{W}$, we further see that $\xi \in \mathcal{W}$. Therefore, we proved $\mathcal{F}(H_{\mathrm{in}}) \subseteq \mathcal{W}$. The reverse inclusion readily holds since $\mathcal{W}$ is weakly H_{in} invariant and $\mathcal{F}(H_{\mathrm{in}})$ is the largest weakly H_{in} invariant set. Therefore, we can conclude that $\mathcal{F}(H_{\mathrm{in}}) = \mathcal{W}$. ∎

Lemma 3.20 leads to the following results for H_{in}.

Lemma 3.21. Let H be a convex process satisfying the domain condition (DC). Then, the following statements hold:

(i) H_{in} is reachable if and only if all eigenvectors of H_{in}^- corresponding to eigenvalues in $[0, \infty)$ belong to $\mathcal{R}_+^{\perp}$.

(ii) H_{in} is exponentially stabilizable if and only if all eigenvectors of H_{in}^- corresponding to eigenvalues in $[1, \infty)$ belong to $\mathcal{R}_+^{\perp}$.

(iii) If H_{in} is null-controllable, then all eigenvectors of H_{in}^- corresponding to eigenvalues in $(0, \infty)$ belong to $\mathcal{R}_+^{\perp}$.

(iv) Suppose that H satisfies in addition the image condition (IC). If all eigenvectors of H_{in}^- corresponding to eigenvalues in $(0, \infty)$ belong to $\mathcal{R}_+^{\perp}$, then H_{in} is null-controllable.

Proof. *(i)*: By applying Lemma 3.20 with the choice $\mathcal{W} = \{0\}$, we see that H_{in} is reachable if and only if H_{in}^- has no eigenvector in $\mathcal{R}(H_{\mathrm{in}})^- \setminus \mathcal{R}_+^{\perp}$ that correspond to a nonnegative eigenvalue. Since all eigenvectors of H_{in}^- corresponding to nonnegative eigenvalues necessarily belong to $\mathcal{R}(H_{\mathrm{in}})^-$ due to Lemma 3.19, we can conclude that H_{in} is reachable if and only if all eigenvectors of H_{in}^- corresponding to eigenvalues in $[0, \infty)$ belong to $\mathcal{R}_+^{\perp}$.

(ii): To prove the 'only if' part, let $\lambda \geqslant 1$ and ξ be such that $\lambda\xi \in H_{\mathrm{in}}^-(\xi)$. Also let $\bar{x} \in \mathcal{F}(H_{\mathrm{in}})$. Since H_{in} is exponentially stabilizable, there exists an

exponentially stable trajectory $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H_{in})$ with $x_0 = \bar{x}$. Note that we have $\langle \xi, x_{k+1}\rangle \leqslant \lambda\langle \xi, x_k\rangle$ for all $k \geqslant 0$. In particular, this means that

$$\frac{1}{\lambda^k}\langle \xi, x_k\rangle \leqslant \langle \xi, x_0\rangle$$

for all $k \geqslant 0$. By taking the limit as k tends to infinity, we see that $\xi \in \big(\mathcal{F}(H_{in})\big)^+ = \big(\mathcal{S}_e(H_{in})\big)^+$. Together with Lemma 3.19, this implies that $\xi \in \big(\mathcal{R}(H_{in}) - \mathcal{S}_e(H_{in})\big)^-$. From Lemma 3.14, we know that $H_{in}^{-1}(\mathcal{S}_e(H_{in})) \subseteq \mathcal{S}_e(H_{in})$. Since H_{in} is exponentially stabilizable, it is clear that Lemma 3.20 implies that $\mathcal{R}(H_{in}) - \mathcal{S}_e(H_{in}) = \mathcal{R}_+$. Therefore, we see that $\xi \in \mathcal{R}_+^\perp$.

Finally, what remains to be proven is the 'if' part. Since $H_{in}^{-1}(\mathcal{S}_e(H_{in})) \subseteq \mathcal{S}_e(H_{in})$ due to Lemma 3.14, the claim would follow from Lemma 3.20 if all eigenvectors of H_{in}^- within $\big(\mathcal{R}(H_{in}) - \mathcal{S}_e(H_{in})\big)^-$ corresponding to nonnegative eigenvalues belong to $\mathcal{R}_+^\perp$. To show this, suppose, on the contrary, that there exist $\lambda \geqslant 0$ and

$$\xi \in \big(\mathcal{R}(H_{in}) - \mathcal{S}_e(H_{in})\big)^- \setminus \mathcal{R}_+^\perp \quad \text{such that} \quad \lambda\xi \in H_{in}^-(\xi). \tag{3.37}$$

Clearly, we have

$$\sigma(H_{in}^-, (\mathcal{R}(H_{in}) - \mathcal{S}_e(H_{in}))^- \setminus \mathcal{R}_+^\perp) \subseteq \sigma(H_{in}^-, \mathcal{R}(H_{in})^- \setminus \mathcal{R}_+^\perp).$$

Since ξ belongs to the set on the left hand side, both sets are nonempty. In addition, we know from Theorem 2.2.1 that both sets are closed and bounded above. Therefore, there exists $\hat{\lambda} \in \sigma(H_{in}^-, \mathcal{R}(H_{in})^- \setminus \mathcal{R}_+^\perp)$ such that $\lambda' \leqslant \hat{\lambda}$ for all $\lambda' \in\in \sigma(H_{in}^-, \mathcal{R}(H_{in})^- \setminus \mathcal{R}_+^\perp)$. As all eigenvectors of H_{in}^- corresponding to eigenvalues in $[1, \infty)$ belong to $\mathcal{R}_+^\perp$, we know that $\hat{\lambda} < 1$. Now take μ such that $\hat{\lambda} < \mu < 1$. From (3.36), we see that $\mathcal{R}_+^\perp \subseteq \ker(H_{in}^- - \mu I)$. Since $\mu > \hat{\lambda}$, we also see that $\ker(H_{in}^- - \mu I) \subseteq \mathcal{R}_+^\perp$. Hence, we have

$$\ker(H_{in}^- - \mu I) = \mathcal{R}_+^\perp. \tag{3.38}$$

From Lemma 3.15, we know that

$$(H_{in}^- - \mu I)^{-1}(\mathcal{S}_e(H_{in})) \subseteq \mathcal{S}_e(H_{in}). \tag{3.39}$$

Now, we claim that $\operatorname{dom}\big((H_{in}^- - \mu I)^{-1}\big) - \mathcal{S}_e(H_{in}) = \mathcal{R}_+$. To see this, let G be the convex process given by $\operatorname{gr}(G) = \operatorname{cl}\big(\operatorname{gr}(H_{in})\big)$. Since G is closed, it follows from Lemma 3.5 that $\big(\operatorname{im}(G - \mu I)\big)^- = \ker(G^- - \mu I)$. Note that $\big(\operatorname{im}(G - \mu I)\big)^- = \big(\operatorname{im}(H_{in} - \mu I)\big)^-$ and $G^- = H_{in}^-$. Therefore, we see that $\big(\operatorname{im}(H_{in} - \mu I)\big)^- = \ker(H_{in}^- - \mu I)$. From (3.38) and the fact that $\operatorname{im}(H_{in} - \mu I) = \operatorname{dom}\big((H_{in}^- - \mu I)^{-1}\big)$, we see that $\operatorname{dom}\big((H_{in}^- - \mu I)^{-1}\big) = \mathcal{R}_+$. Since $\mathcal{S}_e(H_{in}) \subseteq \mathcal{R}_+$, we can conclude that $\operatorname{dom}\big((H_{in}^- - \mu I)^{-1}\big) - \mathcal{S}_e(H_{in}) = \mathcal{R}_+$. Then, the application of Proposition 3.1 to (3.39) results in

$$\mathcal{S}_e(H_{in})^+ \subseteq (H_{in}^- - \mu I)\mathcal{S}_e(H_{in})^+. \tag{3.40}$$

Recall that $\mu > \lambda \geqslant 0$ and $\xi \in (\mathcal{R}(H_{in}) - \mathcal{S}_e(H_{in}))^- \setminus \mathcal{R}_+^\perp$. As $(\mathcal{R}(H_{in}) - \mathcal{S}_e(H_{in}))^- = \mathcal{R}(H_{in})^- \cap \mathcal{S}_e(H_{in})^+$, we see from (3.40) that there exists $\eta \in \mathcal{S}_e(H_{in})^+$ such that $\xi + \mu\eta \in H_{in}^-(\eta)$. Together with $\lambda\xi \in H_{in}^-(\xi)$, this yields

$$\mu\big(\xi + (\mu - \lambda)\eta\big) \in H_{in}^-(\xi + (\mu - \lambda)\eta).$$

Since $\ker(H_{in}^- - \mu I)$ is a subspace, we see that $-\xi - (\mu - \lambda)\eta \in \ker(H_{in}^- - \mu I)$. By using the fact that $\xi + \mu\eta \in H_{in}(\eta)$, we obtain

$$-\lambda\xi \in H_{in}^-(-\xi).$$

Then, Lemma 3.19 implies that $-\xi \in \mathcal{R}(H_{in})^-$. Thus, we see that $\xi \in \mathcal{R}(H_{in})^- \cap \mathcal{R}(H_{in})^+ = \mathcal{R}_+^\perp$. This contradicts with (3.37).

(iii): Let $\lambda > 0$ and ξ be such that $\lambda\xi \in H_{in}^-(\xi)$. Also let $\bar{x} \in \mathcal{F}(H_{in})$. Since H_{in} is null-controllable, there exists a trajectory $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H_{in})$ with $x_0 = \bar{x}$ and $x_q = 0$ for some $q \geqslant 0$. Note that we have $\langle \xi, x_{k+1}\rangle \leqslant \lambda\langle \xi, x_k\rangle$ for all $k \geqslant 0$. In particular, this means that $\langle \xi, x_k\rangle \leqslant \lambda^k\langle \xi, \bar{x}\rangle$ for all $k \geqslant 0$. Since $x_q = 0$ and $\lambda > 0$, we see that $\langle \xi, \bar{x}\rangle \geqslant 0$. As H_{in} is null-controllable, we see that $\xi \in \mathcal{N}(H_{in})^+$. From Lemma 3.19, we know that $\xi \in \mathcal{R}(H_{in})^-$. As such, we have that $\xi \in \big(\mathcal{R}(H_{in}) - \mathcal{N}(H_{in})\big)^-$. It follows from Lemma 3.16 that $H_{in}^{-1}(\mathcal{N}(H_{in})) \subseteq \mathcal{N}(H_{in})$. By taking $\mathcal{W} = \mathcal{N}(H_{in})$ and applying Lemma 3.20, we see that $\mathcal{R}(H_{in}) - \mathcal{N}(H_{in}) = \mathcal{R}_+$. Therefore, we see that $\xi \in \mathcal{R}_+^\perp$.

(iv): Since $H_{in}^{-1}(\mathcal{N}(H_{in})) \subseteq \mathcal{N}(H_{in})$ due to Lemma 3.16, the claim would follow from Lemma 3.20 if all eigenvectors of H_{in}^- within $\big(\mathcal{R}(H_{in}) - \mathcal{N}(H_{in})\big)^-$ corresponding to nonnegative eigenvalues belong to $\mathcal{R}_+^\perp$. Since all eigenvectors of H_{in}^- corresponding to eigenvalues in $(0, \infty)$ already belong to $\mathcal{R}_+^\perp$, it remains to prove that $\xi \in \mathcal{R}_+^\perp$ whenever $0 \in H_{in}^-(\xi)$ and $\xi \in \big(\mathcal{R}(H_{in}) - \mathcal{N}(H_{in})\big)^-$. To see this, note first that $0 \in H_{in}^-(\xi)$ implies that $\xi \in (\operatorname{im}(H_{in}))^-$. Since $\mathcal{R}(H_{in}) \subseteq \operatorname{im}(H_{in})$, we see that $\xi \in \big(\operatorname{im}(H_{in}) - \mathcal{N}(H_{in})\big)^-$. Note that $\operatorname{im}(H_{in}) = H(\mathcal{R}_+)$ due to (3.29) and $\mathcal{N}(H_{in}) = \mathcal{N}(H) \cap \mathcal{R}_+$ due to (3.30f). Since $\mathcal{N}(L_-(H)) \subseteq \mathcal{N}(H)$, we see that the image condition (IC) implies that $\mathcal{R}_+ \subseteq \operatorname{im}(H_{in}) - \mathcal{N}(H_{in})$. Therefore, $\xi \in \mathcal{R}_+^\perp$. ∎

The next result will be employed for exponential stabilizability.

Lemma 3.22. Let $\mathcal{P} \subseteq \mathcal{F}(H)$ be a bounded polyhedron. Suppose that there exist q and ρ with $q \geqslant 1$ and $\rho \in (0,1)$ such that $H^q(x) \cap \rho\mathcal{P} \neq \varnothing$ for all $x \in \mathcal{P}$. Then $\mathcal{P} \subseteq \mathcal{S}_e(H)$.

Proof. Let $\mathcal{P} = \operatorname{conv}\{x^i \mid i = 1, \dots, r\}$. Since $H^q(x) \cap \rho\mathcal{P} \neq \varnothing$ for all $x \in \mathcal{P}$, there exist $x^i_0, \dots, x^i_q$ such that

$$x^i_0 = x^i, \tag{3.41}$$

$$x^i_q \in \rho\mathcal{P}, \tag{3.42}$$

$$x^i_{k+1} \in H(x^i_k) \quad \forall k = 0, \dots, q-1. \tag{3.43}$$

From (3.42), we see that $x^i_q = \rho \sum_j a_{ji} x^j$ where $a_{ji} \geqslant 0$ and $\sum_j a_{ji} = 1$. For $k = 0, 1, \dots, q$, define

$$X_k := \begin{bmatrix} x^1_k & x^2_k & \cdots & x^r_k \end{bmatrix} \in \mathbb{R}^{n \times r}.$$

Also, define the matrix $A = (a_{ji}) \in \mathbb{R}^{r \times r}$. Then, $X_q = \rho X_0 A$.

Let $\xi \in \mathcal{P}$. If $\xi = 0$, then clearly $\xi \in \mathcal{S}_e(H)$. Suppose that $\xi \neq 0$. Then, $\xi = \sum_i b_i x^i$ where $b_i \geqslant 0$ and $\sum_i b_i = 1$. Alternatively, we can write $\xi = X_0 b$. Now, we construct a trajectory $(x_k)_{k\in\mathbb{N}}$ as follows:

$$x_{qm+\ell} = \rho^m X_\ell A^m b \quad \text{for} \quad m \in \mathbb{N} \quad \text{and} \quad \ell = 0, \dots, q-1.$$

Recall that all entries of A and b are nonnegative. Thus, so are the entries of $A^m b$ for all $m \in \mathbb{N}$. This implies that

$$x_{qm+\ell+1} = \rho^m X_{\ell+1} A^m b \in H(\rho^m X_\ell A^m b) = H(x_{qm+\ell})$$

for any $m \in \mathbb{N}$ and $\ell = 0, \dots, q-2$. Further, it follows from $X_q = \rho X_0 A$ that

$$x_{q(m+1)} = \rho^{m+1} X_0 A^{m+1} b = \rho^m X_q A^m b \in H(\rho^m X_{q-1} A^m b) = H(x_{qm+(q-1)}).$$

Therefore $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H)$ with $x_0 = \xi$. It remains to prove that this sequence is exponentially stable. For this, let μ and α be real numbers such that

$$\mu^q = \rho, \quad \text{and} \quad \alpha = \frac{1}{\rho|\xi|} \max_{\substack{1 \leqslant i \leqslant r \\ 0 \leqslant \ell \leqslant q-1}} |x^i_\ell|.$$

Then, we see that

$$|x_{qm+\ell}| = |\rho^m X_\ell A^m b| \leqslant \mu^{mq} \max_{\substack{1 \leqslant i \leqslant r \\ 0 \leqslant \ell \leqslant q-1}} |x^i_\ell| \leqslant \alpha \mu^{qm+\ell} |\xi|,$$

since $\rho \leqslant \mu^\ell$ for all ℓ with $0 \leqslant \ell \leqslant q-1$. This proves that $\xi \in \mathcal{S}_e(H)$ and therefore concludes the proof. ∎

Now, we turn our attention to the outer process. Recall that outer process $H_{\text{out}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by

$$\operatorname{gr}(H_{\text{out}}) := \Big(\operatorname{gr}(H) + (\mathcal{R}_+ \times \mathcal{R}_+)\Big) \cap (\mathcal{V} \times \mathcal{V}), \tag{3.44}$$

where $\mathcal{V} \subseteq \mathbb{R}^n$ is a subspace such that

$$\mathcal{R}_+ \oplus \mathcal{V} = \mathbb{R}^n. \tag{3.45}$$

Even though the subspace $\mathcal{V}$ is not unique in general, the subsequent theory will work regardless of the choice made.

In addition to H_{out}, we define $L_{+,\text{out}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ in a similar fashion:

$$\operatorname{gr}(L_{+,\text{out}}) := \Big(\operatorname{gr}(L_+) + (\mathcal{R}_+ \times \mathcal{R}_+)\Big) \cap (\mathcal{V} \times \mathcal{V}). \tag{3.46}$$

The following lemma will collect essential properties of the outer process.

Lemma 3.23. Let H be a convex process. Then, the following statements hold:

(i) $\operatorname{dom}(H_{\text{out}}) = \big(\operatorname{dom}(H) + \mathcal{R}_+\big) \cap \mathcal{V}$.

(ii) $L_{+,\text{out}}$ is a single valued linear process, i.e. a linear map.

If, in addition, H satisfies (DC), then we have:

(iii) $H_{\text{out}} = L_{+,\text{out}}$.

(iv) $\operatorname{dom}(H^k_{\text{out}}) = \big(\operatorname{dom}(H^k) + \mathcal{R}_+\big) \cap \mathcal{V}$ for all $k \geqslant 1$.

(v) $\mathcal{F}(H) = \operatorname{dom}(H^n)$.

(vi) $\mathcal{F}(H_{\text{out}}) = \big(\mathcal{F}(H) + \mathcal{R}_+\big) \cap \mathcal{V}$ is a subspace.

(vii) $\mathcal{F}(H) + \mathcal{R}_+$ is a subspace.

Proof. (i): Clearly, we have $\operatorname{dom}(H_{\text{out}}) \subseteq \big(\operatorname{dom}(H) + \mathcal{R}_+\big) \cap \mathcal{V}$. As such, it suffices to prove the reverse inclusion $\big(\operatorname{dom}(H) + \mathcal{R}_+\big) \cap \mathcal{V} \subseteq \operatorname{dom}(H_{\text{out}})$. Let $\xi \in \big(\operatorname{dom}(H) + \mathcal{R}_+\big) \cap \mathcal{V}$. Then, $\xi = \xi_1 + \xi_2$ where $\xi_1 \in \operatorname{dom}(H)$ and $\xi_2 \in \mathcal{R}_+$. Let $\zeta_1 \in H(\xi_1)$. From the direct sum (3.45), we see that $\zeta_1 = \zeta - \zeta_2$ for some $\zeta \in \mathcal{V}$ and $\zeta_2 \in \mathcal{R}_+$. Note that $(\xi, \zeta) = (\xi_1, \zeta_1) + (\xi_2, \zeta_2) \in \operatorname{gr}(H_{\text{out}})$. This means that $\xi \in \operatorname{dom}(H_{\text{out}})$, proving the statement.

(ii): For a linear process to be single valued, it is enough to show that $L_{+,\mathrm{out}}(0) = \{0\}$. To do so, let $y \in L_{+,\mathrm{out}}(0)$. This means that $y \in \mathcal{V}$ and there exist $x' \in \mathrm{dom}(L_+) \cap \mathcal{R}_+$, $y' \in L_+(x')$, and $z' \in \mathcal{R}_+$ such that $(0, y) = (x', y') + (-x', z')$. Since $\mathcal{R}_+$ is strongly invariant, we know $L_+(\mathcal{R}_+) \subseteq \mathcal{R}_+$ and thus we see that $y' \in \mathcal{R}_+$. This means that $y = y' + z' \in \mathcal{R}_+$. Recall that $y \in \mathcal{V}$ as well, and hence $y = 0$.

(iii): It follows from (i) that $\mathrm{dom}(L_{+,\mathrm{out}}) = \big(\mathrm{dom}(L_+) + \mathcal{R}_+\big) \cap \mathcal{V}$ by replacing H and H_{out}, respectively, by L_+ and $L_{+,\mathrm{out}}$. Since $\mathrm{dom}(H) + \mathcal{R}_+$ is a subspace and $\mathrm{dom}(L_+) = \mathrm{Lin}(\mathrm{dom}(H))$ due to (3.22), we know $\mathrm{dom}(L_+) + \mathcal{R}_+ = \mathrm{Lin}(\mathrm{dom}(H)) + \mathcal{R}_+ = \mathrm{dom}(H) + \mathcal{R}_+$. As $L_{+,\mathrm{out}}$ is single valued and its graph is larger than that of H_{out}, we see that H_{out} and $L_{+,\mathrm{out}}$ coincide.

(iv): First, we claim that $\big(\mathrm{dom}(H^k) + \mathcal{R}_+\big) \cap \mathcal{V} \subseteq \mathrm{dom}(H^k_{\mathrm{out}})$ for all $k \geqslant 1$. To see this, let $k \geqslant 1$ and $x_0 \in \big(\mathrm{dom}(H^k) + \mathcal{R}_+\big) \cap \mathcal{V}$. Then, there exist $y_0 \in \mathrm{dom}(H^k)$ and $z_0 \in \mathcal{R}_+$ such that $x_0 = y_0 + z_0$. Since $y_0 \in \mathrm{dom}(H^k)$, there exist $y_1, y_2, \ldots, y_k$ such that $y_{\ell+1} \in H(y_\ell)$ with $\ell = 0, 1, \ldots, k-1$. From the direct sum (3.45), we see that there exist $z_1, z_2, \ldots, z_k \in \mathcal{R}_+$ and $x_1, x_2, \ldots, x_k \in \mathcal{V}$ such that $y_\ell = x_\ell - z_\ell$ for all $\ell = 1, 2, \ldots, k$. Note that $(x_\ell, x_{\ell+1}) = (y_\ell, y_{\ell+1}) + (z_\ell, z_{\ell+1}) \in \mathrm{gr}(H_{\mathrm{out}})$ for all $\ell = 0, 1, \ldots, k-1$ since $(y_\ell, y_{\ell+1}) \in \mathrm{gr}(H)$ and $(z_\ell, z_{\ell+1}) \in \mathcal{R}_+ \times \mathcal{R}_+$. Therefore, we have $x_{\ell+1} \in H_{\mathrm{out}}(x_\ell)$ for all $\ell = 0, 1, \ldots, k-1$ and hence $x_k \in H^k_{\mathrm{out}}(x_0)$. In other words, $x_0 \in \mathrm{dom}(H^k_{\mathrm{out}})$. Consequently, we obtain $\big(\mathrm{dom}(H^k) + \mathcal{R}_+\big) \cap \mathcal{V} \subseteq \mathrm{dom}(H^k_{\mathrm{out}})$.

Therefore, it remains to show that $\mathrm{dom}(H^k_{\mathrm{out}}) \subseteq \big(\mathrm{dom}(H^k) + \mathcal{R}_+\big) \cap \mathcal{V}$ for all $k \geqslant 1$. We will prove this by induction on k. Note that $\mathrm{dom}(H_{\mathrm{out}}) = \big(\mathrm{dom}(H) + \mathcal{R}_+\big) \cap \mathcal{V}$ due to (i). As the induction hypothesis, we assume that $\mathrm{dom}(H^k_{\mathrm{out}}) = \big(\mathrm{dom}(H^k) + \mathcal{R}_+\big) \cap \mathcal{V}$ for some $k \geqslant 1$.

Now let $x \in \mathrm{dom}(H^{k+1}_{\mathrm{out}})$. In particular, we have $x \in \mathrm{dom}(H_{\mathrm{out}})$. Due to (i) and the domain condition, we see that $x_0 \in \big(\mathrm{dom}(H) + \mathcal{R}_-\big) \cap \mathcal{V}$. Therefore, there exist $x_1 \in \mathrm{dom}(H)$ and $x_2 \in \mathcal{R}_-$ such that $x = x_1 + x_2$. Let $y_1 \in H(x_1)$. In view of the direct sum (3.45), $y_1 = y - y_2$ where $y \in \mathcal{V}$ and $y_2 \in \mathcal{R}_+$. Then, we have $(x, y) = (x_1, y_1) + (x_2, y_2) \in \mathrm{gr}(H_{\mathrm{out}})$ since $(x_1, y_1) \in \mathrm{gr}(H)$ and $(x_2, y_2) \in \mathcal{R}_- \times \mathcal{R}_+ \subseteq \mathcal{R}_+ \times \mathcal{R}_+$.

Recall that H_{out} is single valued due to (ii) and (iii). As $x \in \mathrm{dom}(H^{k+1}_{\mathrm{out}})$, this means that $y \in \mathrm{dom}(H^k_{\mathrm{out}})$. From the induction hypothesis, we then have $y \in \big(\mathrm{dom}(H^k) + \mathcal{R}_+\big) \cap \mathcal{V}$. Note that $y_1 = y - y_2 \in \mathrm{dom}(H^k) + \mathcal{R}_+$. Since $\mathrm{dom}(H^k) + \mathcal{R}_+ = \mathrm{dom}(H^k) + \mathcal{R}_-$ due to Lemma 3.18, there exist $\xi \in \mathrm{dom}(H^k)$ and $\eta \in \mathcal{R}_-$ such that $y_1 = \xi - \eta$. As $\mathcal{R}_- = L^n_-(0)$ there exists $\zeta \in \mathcal{R}_-$ such that $\eta \in L_-(\zeta)$. Therefore, we have $\xi \in H(x_1) + \eta \subseteq H(x_1) + L_-(\zeta) = H(x_1 + \zeta)$. Since $\xi \in \mathrm{dom}(H^k)$, we get $x_1 + \zeta \in \mathrm{dom}(H^{k+1})$. This yields $x_1 \in \mathrm{dom}(H^{k+1}) + \mathcal{R}_-$ since $\zeta \in \mathcal{R}_-$. Note that $x = x_1 + x_2$ where $x_2 \in \mathcal{R}_-$. As such, we can conclude that $x \in \big(\operatorname{dom}(H^{k+1}) + \mathcal{R}_-\big) \cap \mathcal{V}$. Finally, Lemma 3.18 implies that $x \in \big(\operatorname{dom}(H^{k+1}) + \mathcal{R}_+\big) \cap \mathcal{V}$ which proves that $\operatorname{dom}(H_{\text{out}}^{k+1}) \subseteq \big(\operatorname{dom}(H^{k+1}) + \mathcal{R}_+\big) \cap \mathcal{V}$.

(v): From the statement (iii), we know that H_{out} is a linear process. Therefore, we have

$$\mathcal{F}(H_{\text{out}}) = \operatorname{dom}(H_{\text{out}}^n) \tag{3.47}$$

due to Lemma 3.6. Clearly, $\mathcal{F}(H) \subseteq \operatorname{dom}(H^n)$. The reverse inclusion would follow from Lemma 3.11 if $\operatorname{dom}(H^n) \subseteq \operatorname{dom}(H^{n+1})$.

Let $\xi \in \operatorname{dom}(H^n)$. As such, we see that there exist $x_0, x_1, \ldots, x_n$ such that $\xi = x_0$ and $x_{k+1} \in H(x_k)$ for all k with $0 \leqslant k \leqslant n-1$. From the direct sum (3.45), there exist $y_0, y_1, \ldots, y_n \in \mathcal{R}_+$ and $z_0, z_1, \ldots, z_n \in \mathcal{V}$ such that $x_k = y_k + z_k$ for all $0 \leqslant k \leqslant n-1$. Note that $(z_k, z_{k+1}) = (x_k, x_{k+1}) - (y_k, y_{k+1}) \in \operatorname{gr}(H_{\text{out}})$ since $(x_k, x_{k+1}) \in \operatorname{gr}(H)$ and $(y_k, y_{k+1}) \in \mathcal{R}_+ \times \mathcal{R}_+$.

As H_{out} is single valued, and $z_0 \in \mathcal{F}(H_{\text{out}})$, we know that after n steps, we are still inside the feasible set: $z_n \in \mathcal{F}(H_{\text{out}}) \subseteq \operatorname{dom}(H_{\text{out}})$. The domain condition and $\operatorname{dom}(H) + \mathcal{R}_- = \operatorname{dom}(H) + \mathcal{R}_+$ and (i) imply that $\operatorname{dom}(H_{\text{out}}) = \big(\operatorname{dom}(H) + \mathcal{R}_-\big) \cap \mathcal{V}$. Therefore there exist $\bar{z} \in \operatorname{dom}(H)$ and $\tilde{z} \in \mathcal{R}_-$ such that $z_n = \bar{z} + \tilde{z}$. Due to the domain condition and Lemma 3.1, we have $\mathcal{R}_+ \subseteq \operatorname{dom}(H) + \mathcal{R}_-$. Then, there exist $\bar{y} \in \operatorname{dom}(H)$ and $\tilde{y} \in \mathcal{R}_-$ such that $y_n = \bar{y} + \tilde{y}$. We know that $\mathcal{R}_- = L_-^n(0) \subseteq H^n(0)$ where the last inclusion follows from (3.21). This means that $\bar{y} + \bar{z} = x_n - \tilde{y} - \tilde{z} \in H^n(x_0)$. Since $\bar{y} + \bar{z} \in \operatorname{dom}(H)$, we see that $\xi = x_0 \in \operatorname{dom}(H^{n+1})$. Therefore, we see that $\operatorname{dom}(H^n) \subseteq \operatorname{dom}(H^{n+1})$, which proves the statement.

(vi): From (3.47), (iv) and (v), we see that $\mathcal{F}(H_{\text{out}}) = \operatorname{dom}(H_{\text{out}}^n) = \big(\mathcal{F}(H) + \mathcal{R}_+\big) \cap \mathcal{V}$. Since H_{out} is a linear process due to (iii), $\mathcal{F}(H_{\text{out}})$ is a subspace.

(vii): Since $\mathcal{F}(H) + \mathcal{R}_+$ is a convex cone, it is enough to show that $\xi \in \mathcal{F}(H) + \mathcal{R}_+$ implies $-\xi \in \mathcal{F}(H) + \mathcal{R}_+$. Let $\xi \in \mathcal{F}(H) + \mathcal{R}_+$. Then, $\xi = \zeta + \eta$ where $\zeta \in \mathcal{F}(H)$ and $\eta \in \mathcal{R}_+$. We also know from (3.45) that $\xi = \xi_1 + \xi_2$ where $\xi_1 \in \mathcal{R}_+$ and $\xi_2 \in \mathcal{V}$. It now follows from (vi) that $\xi_2 = \zeta + \eta - \xi_1 \in \mathcal{F}(H_{\text{out}})$. Since $\mathcal{F}(H_{\text{out}})$ is a subspace, we have that $-\xi_2 = \mathcal{F}(H) + \mathcal{R}_+$. Then, $-\xi = -\xi_1 - \xi_2 \in \mathcal{F}(H) + \mathcal{R}_+$. Therefore, $\mathcal{F}(H) + \mathcal{R}_+$ is a subspace. ■

It turns out that the trajectories of the difference inclusion (3.2) can be decomposed according to (3.45) by using the outer process H_{out} as stated next.

Lemma 3.24. Let H be a convex process and $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H)$. Then, there exist sequences $(\xi_k)_{k\in\mathbb{N}} \subset \mathcal{F}(H_{\text{out}})$ and $(\eta_k)_{k\in\mathbb{N}} \subset \mathcal{R}_+$ such that $x_k = \xi_k + \eta_k$ and $\xi_{k+1} = H_{\text{out}}|_{\mathcal{F}(H_{\text{out}})}(\xi_k)$ for all $k \in \mathbb{N}$.

Proof. Existence of sequences $(\xi_k)_{k\in\mathbb{N}} \subset \mathcal{V}$ and $(\eta_k)_{k\in\mathbb{N}} \subset \mathcal{R}_+$ such that $x_k = \xi_k + \eta_k$ for all $k \in \mathbb{N}$ follows from the direct sum (3.45). From (3.44), we see that $\xi_{k+1} \in H_{\mathrm{out}}(\xi_k)$ and hence $\xi_k \in \mathcal{F}(H_{\mathrm{out}})$ for all $k \in \mathbb{N}$. Since H_{out} is a linear map due to Lemma 3.23.(ii)-(iii), we further see that $\xi_{k+1} = H_{\mathrm{out}}|_{\mathcal{F}(H_{\mathrm{out}})}(\xi_k)$ for all $k \in \mathbb{N}$. ■

3.6 PROOFS

This section will use the derived framework, and previously proven results to prove the main results of this chapter.

3.6.1 Proof of Lemma 3.7

It follows from Lemma 3.23 that H_{out} is a single valued linear process, i.e. a linear map and $\mathcal{F}(H_{\mathrm{out}}) = \big(\mathcal{F}(H) + \mathcal{R}_+\big) \cap \mathcal{V}$ is a subspace. Since $\mathcal{F}(H_{\mathrm{out}})$ is weakly H_{out} invariant due Lemma 3.10, single valuedness of H_{out} readily imply that $H_{\mathrm{out}}(\mathcal{F}(H_{\mathrm{out}})) \subseteq \mathcal{F}(H_{\mathrm{out}})$.

3.6.2 Proof of Theorem 3.1

We will prove the implications (i) ⇒ (ii), (ii) ⇒ (iii), and (iii) ⇒ (i).

(i) ⇒ (ii): From (3.30b), we have that $\mathcal{F}(H_{\mathrm{in}}) = \mathcal{F}(H) \cap \mathcal{R}_+$. Since H is reachable, this implies that $\mathcal{F}(H_{\mathrm{in}}) \subseteq \mathcal{R}(H)$. In view of (3.30c), we have that $\mathcal{F}(H_{\mathrm{in}}) \subseteq \mathcal{R}(H_{\mathrm{in}})$. Therefore, H_{in} is reachable. To show that H_{out} is reachable, note that $\mathcal{F}(H_{\mathrm{out}}) = \big(\mathcal{F}(H) + \mathcal{R}_+\big) \cap \mathcal{V}$ due to Lemma 3.23.(vi). Since H is reachable, we see that $\mathcal{F}(H_{\mathrm{out}}) \subseteq \mathcal{R}_+ \cap \mathcal{V} = \{0\}$. As such, H_{out} is reachable.

(ii) ⇒ (iii): From Lemma 3.21.(i), reachability of H_{in} implies that all eigenvectors of H_{in}^- corresponding to eigenvalues in $[0,\infty)$ belong to $\mathcal{R}_+^\perp$. Note that H_{out} is single valued due to Lemma 3.23.(ii) and (iii). This implies that $H_{\mathrm{out}}(0) = \{0\}$ and hence that $\mathcal{R}(H_{\mathrm{out}}) = \{0\}$. Therefore, reachability of H_{out} implies that $\mathcal{F}(H_{\mathrm{out}}) = \{0\}$.

(iii) ⇒ (i): Since $\mathcal{F}(H_{\mathrm{out}}) = \{0\}$, it follows from (3.45) and Lemma 3.23.(vi) that $\mathcal{F}(H) \subseteq \mathcal{R}_+$. Then, we have that $\mathcal{F}(H) = \mathcal{F}(H_{\mathrm{in}})$ in view of (3.30b). From Lemma 3.21.(i), we see that H_{in} is reachable since all eigenvectors of H_{in}^- corresponding to eigenvalues in $[0,\infty)$ belong to $\mathcal{R}_+^\perp$. Therefore, we have $\mathcal{F}(H) = \mathcal{F}(H_{\mathrm{in}}) \subseteq \mathcal{R}(H_{\mathrm{in}})$. Since $\mathcal{R}(H_{\mathrm{in}}) = \mathcal{R}(H)$ due to (3.30c), H is reachable.

For the last claim, suppose that H is reachable. From Theorem 3.1.(ii), we know that H_{in} is reachable. In turn, Lemma 3.20, with the choice of $\mathcal{W} = \{0\}$, implies that $\mathcal{R}(H_{\text{in}}) = \mathcal{R}_+$. Since it readily holds that $\mathcal{R}(H_{\text{in}}) \subseteq \mathcal{R}(H) \subseteq \mathcal{R}_+$, we see that $\mathcal{R}(H) = \mathcal{R}_+$. By taking $\mathcal{C}_\ell = H^\ell(0)$ and applying Lemma 3.4, we obtain $\mathcal{R}(H) = H^q(0)$ for some $q \geqslant 0$.

3.6.3 Proof of Theorem 3.2

The implication *(ii)* $\Rightarrow$ *(i)* is evident. In what follows, we will prove the implications *(i)* $\Rightarrow$ *(iv)*, *(iv)* $\Rightarrow$ *(iii)*, and *(iii)* $\Rightarrow$ *(ii)*.

(i) $\Rightarrow$ *(iv)*: Since H is stabilizable, it follows from Lemma 3.24 that all eigenvalues of the linear map $H_{\text{out}}|_{\mathcal{F}(H_{\text{out}})}$ are in the open unit disc. To prove the rest, we first observe that H_{in} is stabilizable whenever so is H. Now, let $\lambda \geqslant 1$ and ξ be such that $\lambda\xi \in H_{\text{in}}^-(\xi)$. The arguments used in the proof of Lemma 3.21.(ii) result in $\xi \in \big(\mathcal{R}(H_{\text{in}}) - \mathcal{S}(H_{\text{in}})\big)^-$. From Lemma 3.14, we know that $H_{\text{in}}^{-1}(\mathcal{S}(H_{\text{in}})) \subseteq \mathcal{S}(H_{\text{in}})$. By taking $\mathcal{W} = \mathcal{S}(H_{\text{in}})$ and applying Lemma 3.20, we see that $\mathcal{R}(H_{\text{in}}) - \mathcal{S}(H_{\text{in}}) = \mathcal{R}_+$. Therefore, we see that $\xi \in \mathcal{R}_+^\perp$.

(iv) $\Rightarrow$ *(iii)*: This implication follows from Lemma 3.24 and Lemma 3.21.(ii).

(iii) $\Rightarrow$ *(ii)*: To prove this implication, we will first construct a bounded polyhedron $\mathcal{P}$ as in Lemma 3.22 and then show that every feasible point can be steered to $\mathcal{P}$ in finitely many steps.

To construct $\mathcal{P}$, we need some preparation. Let $\mathcal{W}$ be a subspace such that

$$\mathcal{R}_+ = (\operatorname{Lin}(\mathcal{F}(H)) \cap \mathcal{R}_+) \oplus \mathcal{W}. \tag{3.48}$$

For $\xi \in \mathcal{F}(H_{\text{out}})$, define

$$G(\xi) = \Big(\big((\mathcal{F}(H) - \xi) \cap \mathcal{R}_+\big) + \big(\operatorname{Lin}(\mathcal{F}(H)) \cap \mathcal{R}_+\big)\Big) \cap \mathcal{W}. \tag{3.49}$$

It can be easily verified that G is a convex process. From Lemma 3.23.(vi), we know that for every $\xi \in \mathcal{F}(H_{\text{out}})$ there exists $\eta \in \mathcal{R}_+$ such that $\xi + \eta \in \mathcal{F}(H)$. Further, the decomposition (3.48) implies that there exist $\eta_1 \in \operatorname{Lin}(\mathcal{F}(H)) \cap \mathcal{R}_+$ and $\eta_2 \in \mathcal{W}$ such that $\eta = \eta_1 + \eta_2$. Then, we see that $\eta_2 \in G(\xi)$. Therefore, we have that $\operatorname{dom} G = \mathcal{F}(H_{\text{out}})$.

Now, we claim that G is single valued. For this, take $\eta, \zeta \in G(\xi)$. This means that there exist $\eta_1, \zeta_1 \in \mathcal{R}_+$ and $\eta_2, \zeta_2 \in \operatorname{Lin}(\mathcal{F}(H)) \cap \mathcal{R}_+$ such that

$$\xi + \eta_1 \in \mathcal{F}(H), \quad \xi + \zeta_1 \in \mathcal{F}(H), \quad \eta_1 = \eta_2 + \eta, \quad \zeta_1 = \zeta_2 + \zeta.$$

By using the first two relations, we obtain $\eta_1 - \zeta_1 \in \operatorname{Lin}(\mathcal{F}(H))$. This implies that $\eta_1 - \zeta_1 \in \operatorname{Lin}(\mathcal{F}(H)) \cap \mathcal{R}_+$. Note that $\eta - \zeta = (\eta_1 - \zeta_1) - (\eta_2 - \zeta_2)$. Therefore, we see that $\eta - \zeta \in \operatorname{Lin}(\mathcal{F}(H)) \cap \mathcal{R}_+$. Since $\eta - \zeta \in \mathcal{W}$, it follows from (3.48) that $\eta = \zeta$, i.e. G is single valued. Since its domain is a subspace, we further see that G is a linear map.

Since $\mathcal{F}(H_{out})$ is a subspace due to Lemma 3.23.(ii)-(iii), we can find a bounded polyhedron $\mathcal{Q} \subseteq \mathcal{F}(H_{out})$ containing the unit ball relative to $\mathcal{F}(H_{out})$, i.e.

$$\{\xi \in \mathcal{F}(H_{out}) \mid |\xi| \leqslant 1\} \subset \mathcal{Q}. \tag{3.50}$$

From (3.49), we see that

$$(I + G)\mathcal{Q} \subseteq \mathrm{Lin}(\mathcal{F}(H)). \tag{3.51}$$

Moreover, $(I + G)\mathcal{Q}$ is a bounded polyhedron since G is a linear map.

From Lemma 3.17, we know that $\mathcal{R}(H) - \mathcal{R}(H) = \mathcal{R}_+$. Then, Lemma 3.4 implies that there exists $q_1 \geqslant 0$ such that

$$H^q(0) - H^q(0) = \mathcal{R}_+ \tag{3.52}$$

for all $q \geqslant q_1$. Since H_{in} is exponentially stabilizable, we have $\mathcal{S}_e(H_{in}) - \mathcal{R}(H) = \mathcal{R}_+$ due to Lemma 3.20 and the fact that $\mathcal{R}(H_{in}) = \mathcal{R}(H)$. By applying Lemma 3.4, we see that there exists $q_2 \geqslant 0$ such that $\mathcal{S}_e(H_{in}) - H^q(0) = \mathcal{R}_+$ for $q \geqslant q_2$.

Let $q_3 = \max\{q_1, q_2\}$. Since $\mathcal{S}_e(H_{in}) - H^{q_3}(0)$ is a subspace, it follows from Lemma 3.2 that there exists y such that

$$y \in \mathrm{ri}(\mathcal{S}_e(H_{in})) \cap \mathrm{ri}(H^{q_3}(0)). \tag{3.53}$$

As $\mathrm{ri}(\mathrm{cone}(y)) \cap \mathrm{ri}(H^{q_3}(0)) \neq \varnothing$, we know $\mathrm{cone}(y) - H^{q_3}(0)$ is a subspace from Lemma 3.2. This means that $\mathcal{R}_+ = \mathrm{Lin}(H^{q_3}(0)) \subseteq \mathrm{cone}(y) - H^{q_3}(0) \subseteq \mathcal{R}_+$. Thus, we see that

$$\mathrm{cone}(y) - H^{q_3}(0) = \mathcal{R}_+. \tag{3.54}$$

As $H^{q_3}(0) \subseteq H^q(0)$ for all $q \geqslant q_3$ and $q_3 \geqslant q_1$, (3.52) implies that

$$\mathrm{ri}(H^{q_3}(0)) \subseteq \mathrm{ri}(H^q(0)) \tag{3.55}$$

for all $q \geqslant q_3$. Therefore, we see that

$$y \in \mathrm{ri}(\mathcal{S}_e(H_{in})) \cap \mathrm{ri}(H^q(0)), \tag{3.56}$$

$$\mathrm{cone}(y) - H^q(0) = \mathcal{R}_+ \tag{3.57}$$

for all $q \geqslant q_3$.

Now, let $\lambda \in (0, 1)$. Let $\mathcal{B} = \{x \in \mathcal{R}_+ \mid |x| \leqslant 1\}$ denote the unit ball in $\mathcal{R}_+$. From (3.55) and (3.56), we see that there exists $\varepsilon > 0$ such that $y + \varepsilon\mathcal{B} \subseteq H^q(0)$ for all $q \geqslant q_3$. Since $y \in \mathcal{S}_e(H_{in})$, there exists $q_4 \geqslant 0$ such that for all $q \geqslant q_4$

there exists $y_q \in H^q(y)$ with $|y_q| \leqslant \varepsilon\lambda$. As $y \in \mathcal{R}_+$ and $\mathcal{R}_+$ is strongly H invariant, $y_q \in \mathcal{R}_+$. Then, $\lambda y - y_q \in H^q(0)$ for all $q \geqslant q_4$. As such, we can conclude that

$$\lambda y = y_q - y_q + \lambda y \in H^q(y) + H^q(0) \subseteq H^q(y) \tag{3.58}$$

for all $q \geqslant q_4$.

Since H_{in} is exponentially stabilizable, $\mathcal{F}(H_{\text{in}}) = \mathcal{S}_e(H_{\text{in}}))$. In view of (3.56), this means that $y \in \text{ri}(\mathcal{F}(H_{\text{in}}))$. From Lemma 3.23.(vi), we have that $\mathcal{F}(H) + \mathcal{R}_+$ is a subspace. According to Lemma 3.2, we have that $\text{ri}(\mathcal{F}(H)) \cap \mathcal{R}_+ \neq \varnothing$. Together with [142, Thm. 6.5], this implies that $\text{ri}(\mathcal{F}(H) \cap \mathcal{R}_+) = \text{ri}(\mathcal{F}(H)) \cap \mathcal{R}_+$. Thus, we see that

$$y \in \text{ri}(\mathcal{F}(H_{\text{in}})) = \text{ri}(\mathcal{F}(H) \cap \mathcal{R}_+) = \text{ri}(\mathcal{F}(H)) \cap \mathcal{R}_+.$$

Now, let $\mathcal{Q} = \text{conv}\{\xi_0^i \mid i = 1, \dots, r\}$. Since $y \in \text{ri}(\mathcal{F}(H))$, it follows from (3.51) that there exists $\beta_1 > 0$ such that

$$(I + G)\xi_0^i + \beta y \in \mathcal{F}(H) \tag{3.59}$$

for all i and $\beta \geqslant \beta_1$. Let $\xi_q^i \in H_{\text{out}}^q(\xi_0^i)$. Take $\rho \in (\lambda, 1)$. As H_{out} is single valued and exponentially stabilizable, there exists $q_5 \geqslant 0$ such that $|\xi_q^i| < \rho$ for all i and $q \geqslant q_5$.

From (3.59), we see that there exists $(z_k^i)_{k \in \mathbb{N}} \in \mathfrak{B}(H)$ with $z_0^i = \xi_0^i + G(\xi_0^i) + \beta y$. In view of (3.45), (3.48), and Lemma 3.24, we have that

$$z_k^i = \xi_k^i + G(\xi_k^i) + \eta_k^i$$

where $\xi_k^i \in H_{\text{out}}^k(\xi_0^i)$ and $\eta_k^i \in \mathcal{R}_+$ for all $k \in \mathbb{N}$. Now, let $q = \max\{q_3, q_4, q_5\}$. Then, we see from (3.57) that $\eta_q^i = \alpha^i y - \zeta^i$, where $\alpha^i \geqslant 0$ and $\zeta^i \in H^q(0)$. Then, we have that

$$\xi_q^i + G(\xi_q^i) + \alpha^i y \in H^q\big(\xi_0^i + G(\xi_0^i) + \beta y\big).$$

From (3.56) and (3.58) and , we know that $y \in H^q(0)$ and $\lambda y \in H^q(y)$. Therefore, we have

$$\xi_q^i + G(\xi_q^i) + (\alpha^i + \gamma\lambda + \delta^i)y \in H^q\big(\xi_0^i + G(\xi_0^i) + (\beta + \gamma)y\big) \tag{3.60}$$

for all $\gamma \geqslant 0$ and $\delta^i \geqslant 0$. Note that we can take γ and δ^i such that $\alpha^i + \gamma\lambda + \delta^i = \rho(\beta + \gamma)$ for all i since $\lambda < \rho$. Then, (3.60) boils down to

$$\xi_q^i + G(\xi_q^i) + \rho(\beta + \gamma)y \in H^q\big(\xi_0^i + G(\xi_0^i) + (\beta + \gamma)y\big).$$

Now, take $\mathcal{P} = (I+G)\mathcal{Q} + \bar{\beta}y$ where $\bar{\beta} = \beta + \gamma$. Let $x \in \mathcal{P}$. Then, $x = \xi_0 + G(\xi_0) + \bar{\beta}y$ where $\xi_0 \in \mathcal{Q}$. This means that $\xi_0 = \sum_i a_i \xi_0^i$ where $a_i \geqslant 0$ and $\sum_i a_i = 1$. Thus, we can conclude that

$$\xi + G(\xi) + \rho\bar{\beta}y \in H^q\big(\xi_0 + G(\xi_0) + \bar{\beta}y\big) = H^q(x),$$

where $\xi = \sum_i a_i \xi_q^i$. Since $|\xi_q^i| \leqslant \rho$, we have $|\xi| \leqslant \rho$. As such, we see from (3.50) that $\xi \in \rho\mathcal{Q}$. This proves that $H^q(x) \cap \rho\mathcal{P} \neq \varnothing$. By applying Lemma 3.22, we can conclude that $\mathcal{P} \subseteq \mathcal{S}_e(H)$.

Now, let $\bar{x} \in \mathcal{F}(H)$. If $\bar{x} = 0$, then we clearly have that $\bar{x} \in \mathcal{S}_e(H)$. Suppose that $\bar{x} \neq 0$. Let $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H)$ be a trajectory with $x_0 = \bar{x}$. From (3.45), (3.48), and Lemma 3.24, we see that $x_k = \xi_k + G(\xi_k) + \eta_k$ where $\xi_{k+1} \in H_{\text{out}}|_{\mathcal{F}(H_{\text{out}})}(\xi_k)$ and $(\eta_k)_{k\in\mathbb{N}} \subset \mathcal{R}_+$. Since H_{out} is exponentially stabilizable, there exists $\bar{q} \geqslant \max\{q_3, q_4, q_5\}$ such that $|\xi_{\bar{q}}| \leqslant 1$. From (3.56) and (3.57), we see that there exists $\bar{\alpha}$ such that

$$\xi_{\bar{q}} + G(\xi_{\bar{q}}) + \alpha y \in H^{\bar{q}}(\bar{x}) \tag{3.61}$$

for all $\alpha \geqslant \bar{\alpha}$. Due to (3.50), we have that $\xi_{\bar{q}} + G(\xi_{\bar{q}}) \in (I+G)\mathcal{Q}$. As such, $\xi_{\bar{q}} + G(\xi_{\bar{q}}) + \bar{\beta}y \in \mathcal{P} \subseteq \mathcal{S}_e(H)$. By taking $\alpha' = \max(\bar{\alpha}, \bar{\beta})$, we see that $\xi_{\bar{q}} + G(\xi_{\bar{q}}) + \alpha' y \in \mathcal{S}_e(H)$ since $y \in \mathcal{S}_e(H)$ due to (3.56). From (3.61), we have that $\bar{x} \in H^{-\bar{q}}(\xi_{\bar{q}} + G(\xi_{\bar{q}}) + \alpha' y)$. Therefore, it follows from Lemma 3.14 that $\bar{x} \in \mathcal{S}_e(H)$. This proves that $\mathcal{F}(H) \subseteq \mathcal{S}_e(H)$ and hence H is exponentially stabilizable. ∎

3.6.4 Proof of Theorem 3.3

Clearly, H is controllable if and only if H is both reachable and null-controllable. Therefore, what needs to be proven is that reachability of H implies its null-controllability. From Theorem 3.1 we obtain $\mathcal{R}(H) = H^q(0)$ for some $q \geqslant 0$. Let $\xi \in \mathcal{F}(H) \subseteq \mathcal{R}(H)$. Then, there exists a trajectory $(x_k)_{k\in\mathbb{N}} \in \mathcal{B}(H)$ with $x_0 = \xi$. Clearly, we have $x_q \in H^q(\xi)$. As $\mathcal{R}(H)$ is strongly H invariant, we know that $x_q \in \mathcal{R}(H) = \mathcal{R}_+$. Hence, we see that $-x_q \in \mathcal{R}(H) = H^q(0)$. It then follow from (3.12) that $0 = x_q - x_q \in H^q(\xi) + H^q(0) = H^q(\xi)$. Consequently, $\mathcal{F}(H) \subseteq \mathcal{N}(H)$, that is H is null-controllable.

3.6.5 Proof of Theorem 3.4

We will prove the implications *(i)* $\Rightarrow$ *(iii)*, *(iii)* $\Rightarrow$ *(ii)*, and *(iii)* $\Rightarrow$ *(i)*.

(i) $\Rightarrow$ *(iii)*: Since H is null-controllable, it follows from Lemma 3.24 that all eigenvalues of the linear map $H_{\text{out}}|_{\mathcal{F}(H_{\text{out}})}$ are zero and thus $H_{\text{out}}|_{\mathcal{F}(H_{\text{out}})}$ is

nilpotent. For the rest, we first observe that H_{in} is null-controllable whenever so is H due to (3.30b) and (3.30f). Then, it follows from Lemma 3.21.(iii) that all eigenvectors of H_{in}^{-} corresponding to eigenvalues in $(0,\infty)$ belong to $\mathcal{R}_{+}^{\perp}$.

(iii) $\Rightarrow$ *(ii)*: This implication follows from Lemma 3.24 and Lemma 3.21.(iv).

(ii) $\Rightarrow$ *(i)*: Let $\bar{x} \in \mathcal{F}(H)$. Then, there exists a trajectory $(x_k)_{k\in\mathbb{N}} \in \mathfrak{B}(H)$ such that $x_0 = \bar{x}$. From Lemma 3.24, we have that $x_k = x_k = \xi_k + \eta_k$ where $\eta_k \in \mathcal{R}_+$ and $\xi_{k+1} \in H_{\text{out}}|_{\mathcal{F}(H_{\text{out}})}(\xi_k)$ for all $k \in \mathbb{N}$. Since H_{out} is null-controllable, there must exist $q \geqslant 0$ such that $\xi_q = 0$. This means that $\eta_q \in \mathcal{F}(H) \cap \mathcal{R}_+ = \mathcal{F}(H_{\text{in}})$ since $x_q \in \mathcal{F}(H)$. As such, we have that $x_q \in \mathcal{N}(H_{\text{in}}) \subseteq \mathcal{N}(H)$ since H_{in} is null-controllable. This means that $\bar{x} \in H^{-q}(x_q) \subseteq H^{-q}(\mathcal{N}(H))$. Then, it follows from Lemma 3.16 that $\bar{x} \in \mathcal{N}(H)$. This proves that $\mathcal{F}(H) \subseteq \mathcal{N}(H)$ and hence H is null-controllable.

3.7 CONCLUSION

In this chapter, we have developed a framework for analysis of convex processes. Central concepts in this are weakly and strongly invariant cones, the minimal and maximal linear processes and duality. It was shown that these concepts naturally have a central role in the analysis of convex processes.

Within this, we developed Hautus-type spectral tests for reachability, stabilizability and null-controllability of nonstrict convex processes. In essence we have shown that, under a condition on the domain, we can investigate the properties of a convex process by considering its *inner* and *outer* processes separately. This result is akin to the so-called Kalman decomposition for linear systems. After this, spectral characterizations for either of these processes were developed. In particular for the inner process, this required additional developments in duality of convex processes. Moreover, we have proven that, under the domain condition, reachability and controllability are equivalent.

It was shown that these main results generalize all previously known characterizations. In particular, the known results for (strict) convex processes and for linear processes were unified.

Future work

As noted, the framework of this chapter can be applied to many different problems regarding convex processes. The results in duality will prove useful for different stabilizability problems. Indeed, it was shown in [46], that a stricter domain condition plays a role in the study of (duality of) Lyapunov functions for convex processes.

Another avenue of extensions would be to consider control of convex processes. While in the case of linear systems stabilizability is equivalent to the existence of a static state feedback controller, for convex processes such results do not hold. An interesting problem is to develop a theory of control for this class of systems.

In this chapter, we considered only convex processes, with a motivation of developing results for conically constrained linear systems. A logical extension would be to investigate properties of more general set-valued maps. In particular set-valued maps with a convex graph would prove interesting. Parallel to the work of this chapter, an investigation of properties of set-valued maps with affine graphs would prove another relevant intermediate point towards the general convex case. The first steps in this regard were made in [87].

4 LYAPUNOV FUNCTIONS FOR CONVEX PROCESSES

In the previous chapter, we have characterized, among others, stabilizability for nonstrict convex processes. However, knowing whether a convex process is stabilizable is not equivalent to knowing how to find a stabilizing trajectory from a given point. This problem motivates an investigation of Lyapunov functions for convex processes.

4.1 INTRODUCTION

In this chapter we study Lyapunov functions for systems given by difference inclusions of the form

$$x_{k+1} \in H(x_k),$$

where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex process: a set-valued map whose graph is a convex cone. Since being introduced in [141, 142], these maps have attracted attention for a few reasons.

Chief among these is the fact that any conically constrained linear system can be rewritten as one these systems. Conic constraints are ubiquitous in real-life applications: From economic growth models [110] and cable-suspended robots [86, 126, 127] to chemical reaction networks [7]. In addition, any linear positive system, that is, a system whose state is constrained to the nonnegative orthant, can be written as a convex process. For more details on positive systems, we refer to [54], and the references therein. Lastly, as shown in e.g. [11,61], difference inclusions of convex processes can be used as approximations of more complex set-valued maps. As such, local properties of more general difference inclusions can be described in terms of properties of an approximating convex process.

As such, developing tools for the analysis of systems described by convex processes is an interesting and relevant problem. This line of research started in [14] and [132], where the controllability problem is resolved for strict (nonempty everywhere) convex processes in continuous-time and discrete-time, respectively. After this, the works [159] and [131] characterize stabilizability for strict convex processes in a similar fashion. An important ingredient of these results is the use of duality: Controllability and stabilizability of a convex process can be characterized in terms of spectral properties of the dual (or adjoint) convex process. However, the assumption of strictness is rather restrictive in practice. To be precise, strict convex processes correspond to linear systems without constraints on the state of the system. Recently, the works [44, 45, 86] provide conditions under which the previously mentioned results can be generalized towards nonstrict convex processes.

As mentioned, this chapter considers Lyapunov functions for convex processes. It is well known that (quadratic) Lyapunov functions are natural and useful tools in stability analysis and control for linear systems. From a control-oriented standpoint, investigation of different classes of candidate Lyapunov functions has proven very practical with regard to constrained systems. In particular, in [17, 25, 179] (see for a more modern approach [155, 164]), it was shown that linear positive systems are stabilizable if and only if they admit a *diagonal* quadratic Lyaponov function. As such, the amount of variables in the design of a Lyapunov function is linear, instead of quadratic, in the dimension of the state. Indeed, in for example [137] it is shown that this allows for well-scaling control schemes. For Lyapunov functions regarding systems with more general types of constraints, a recent development is the one of control barrier functions. For this, we refer to [6] and the references therein.

More general difference inclusions also admit a useful Lyapunov theory, as evidenced by e.g. [7, 68, 96, 97]. For the more specific class of convex processes, [159] has already employed Lyapunov functions in proving a relation between stabilizability of a primal system and stability of its dual. This relation was generalized in the extensive treatise on Lyapunov functions given in [69]. A result of particular interest is Theorem 2.2 of [69]. This theorem reveals for strict convex processes that the convex conjugate of a weak Lyapunov function for the primal system is a (strong) Lyapunov function for the dual.

Extending on the aforementioned work, this chapter will investigate the duality between different notions of stability and their corresponding types of Lyapunov functions. The contribution of this chapter is as follows.

1. We use extended real-valued functions to modify the definitions of weak and strong Lyapunov functions. These will better reflect the stability properties of nonstrict convex processes. To be precise we will show that the notion of *uniform exponential stabilizability* precisely corresponds admitting a Lyapunov function in a given class of extended real-valued functions.

2. We will prove results generalizing those of, among others, Theorem 2.2 of [69]. To be precise, we show that we can obtain a strong Lyapunov function for the dual process by taking a weak Lyapunov function of the primal process and applying a restriction and the convex conjugate. The domain conditions required to obtain this result can be shown to be close to those required for the existence of such a strong Lyapunov function.

3. Lastly, we will bring together a number of different theorems relating stability and duality. We will show that for convex processes with a polyhedral graph the concepts of stabilizability and strong stability are related by duality. Similarly, this result will relate weak and strong Lyapunov functions.

This chapter starts with some preliminary knowledge on convex processes and extended real-valued functions in Section 4.2. There, we will also define weak and strong Lyapunov functions and motivate this definition. Then, in Section 4.3 we will introduce the notions of *uniform exponential stabilizability* and *uniform exponential strong stability*. These will be used to prove converse Lyapunov results. After this, we turn our attention to duality, and give preliminaries for this in Section 4.4. The main results of the chapter will then be presented in Section 4.5. Lastly we provide conclusions in Section 4.6.

4.2 LYAPUNOV FUNCTIONS FOR CONVEX PROCESSES

Before defining Lyapunov functions for convex processes, we require some preliminaries with regard to convex analysis.

Given a convex set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote its closure by $\operatorname{cl}\mathcal{S}$ and its relative interior by $\operatorname{ri}\mathcal{S}$. Given another convex set $\mathcal{T} \subseteq \mathbb{R}^n$ and scalar $\rho \in \mathbb{R}$ we define the sum and scalar product of sets as:

$$\mathcal{S} + \mathcal{T} = \{s + t \mid s \in \mathcal{S}, t \in \mathcal{T}\}, \quad \rho\mathcal{S} = \{\rho s \mid s \in \mathcal{S}\}.$$

A convex set $\mathcal{C}$ is a *convex cone* if $\rho\mathcal{C} \subseteq \mathcal{C}$ for all $\rho \geqslant 0$. This means that $\mathcal{C}$ is closed under *conic combinations*: If $c_1, ..., c_n \in \mathcal{C}$ then

$$\sum_{i=1}^{n} \alpha_i c_i \in \mathcal{C} \quad \forall \alpha_i \geqslant 0.$$

The set of all conic combinations of a set $\mathcal{S}$ is called the *conic hull* and is denoted by $\operatorname{cone}\mathcal{S}$. If there exists a finite set $\mathcal{S} \subseteq \mathcal{C}$ such that $\mathcal{C} = \operatorname{cone}\mathcal{S}$, we say that $\mathcal{C}$ is *finitely generated* or *polyhedral*.

A set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called a *convex process*, a *linear process*, *closed* if its graph

$$\mathrm{gr}(H) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y \in H(x)\}$$

is a convex cone, a subspace, closed, respectively.

The *domain* and *image* of H are defined as $\mathrm{dom}(H) = \{x \in \mathbb{R}^n \mid H(x) \neq \varnothing\}$ and $\mathrm{im}(H) = \{y \in \mathbb{R}^n : \exists\, x \text{ s.t. } y \in H(x)\}$. If $\mathrm{dom}(H) = \mathbb{R}^n$, we say that H is *strict*.

In this chapter, we consider systems described by a *difference inclusion* of the form:

$$x_{k+1} \in H(x_k) \quad k \geqslant 0, \tag{4.1}$$

where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex process. By a *trajectory* of (3.1), we mean a sequence $(x_k)_{k\in\mathbb{N}}$ such that (3.1) holds for all $k \geqslant 0$. The *behavior* (see e.g. [180]) is the set of all trajectories:

$$\mathfrak{B}(H) := \left\{ (x_k)_{k\in\mathbb{N}} \in (\mathbb{R}^n)^{\mathbb{N}} \mid (x_k)_{k\in\mathbb{N}} \text{ is a trajectory of (3.1)} \right\}.$$

The *feasible set* of the difference inclusion (3.1) is the set of states from which a complete trajectory emanates:

$$\mathcal{F}(H) := \{\xi \mid \exists (x_k) \in \mathfrak{B}(H) \text{ with } x_0 = \xi\}.$$

The *stabilizable set* is the set of states from which a stable trajectory exists:

$$\mathcal{S}(H) := \{\xi \mid \exists (x_k) \in \mathfrak{B}(H) \text{ with } x_0 = \xi, \lim_{k\to\infty} x_k = 0\}.$$

In addition, we define the *strongly stable set* as the set of states from which all trajectories are stable:

$$\bar{\mathcal{S}}(H) := \{\xi \mid \forall (x_k) \in \mathfrak{B}(H) \text{ with } x_0 = \xi, \lim_{k\to\infty} x_k = 0\}.$$

It is straightforward to show that if H is a convex process, the sets $\mathcal{F}(H)$ and $\mathcal{S}(H)$ are convex cones. The set $\bar{\mathcal{S}}(H)$ is a convex cone if it is not empty.

We say the system (4.1) (or the convex process H) is *stabilizable* if every feasible state is stabilizable, that is, $\mathcal{F}(H) \subseteq \mathcal{S}(H)$. Similarly, we say it is *strongly stable* if $\mathcal{F}(H) \subseteq \bar{\mathcal{S}}(H)$.

Remark 4.1. If H is not a strict convex processes, we have that $\mathcal{F}(H) \neq \mathbb{R}^n$. Therefore, the notions on (uniform exponential) stabilizability do not require the existence of a stable trajectory from all points in $\mathbb{R}^n$. This stands in contrast with the notions of stabilizability employed in e.g. [131] and [64], where the authors consider weak asymptotic stabilizability and weak asymptotic stability, respectively. These papers require all points in $\mathbb{R}^n$ to admit a stable trajectory and as such strictness is *required* for the corresponding notions. However, as argued before, strictness is a very restrictive assumption, especially when applied to constrained linear systems.

Similarly, for strong stabilizability we require that all trajectories are stable. This coincides with the notion of pre-attractivity used in [69]. In that paper, a distinction is made between pre-attractivity and attractivity, where the latter requires in our notation that $\operatorname{dom}(H) = \mathcal{F}(H)$.

In this chapter, we will consider *Lyapunov functions* corresponding to the class of systems given by difference inclusions with convex processes and to the notions of stabilizability and strong stability. For the class of linear systems, Lyapunov functions are often chosen (without loss of generality) to be quadratic functions of the state. However, for nonstrict convex processes the feasible set is not the entire state space. As such, it is restrictive to assume that a Lyapunov function is defined outside of the feasible set. To formalize this, we will work with *extended real-valued* Lyapunov functions.

Let f be an extended real-valued function, i.e. a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. We define the *epigraph* of f by

$$\operatorname{epi} f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geqslant f(x)\}.$$

A function is said to be convex or closed if its epigraph is convex or closed as a set. The *closure* and *convex hull* of f, denoted $\operatorname{cl} f$ and $\operatorname{conv} f$ respectively, are the extended real valued functions whose epigraphs are the closure and convex hull of $\operatorname{epi} f$. The *(effective) domain* of the function is the set $\operatorname{dom} f = \{x \in \mathbb{R} \mid f(x) < \infty\}$. The function f is *proper* if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\operatorname{dom} f \neq \varnothing$.

The function f is called *positively homogeneous of degree* 2 if $f(\lambda x) = \lambda^2 f(x)$ for all $\lambda \geqslant 0$ and $x \in \mathbb{R}^n$. If f is positively homogeneous degree 2, it is straightforward to check that the domain of f is a cone.

A function f is *positive semi-definite* if $f(0) = 0$ and $f(x) \geqslant 0$ for all $x \in \mathbb{R}^n$. Clearly, any such function is proper.

Definition 4.1. Let $\mathcal{V}$ denote the set of all extended real-valued functions that are convex, positive semi-definite and positively homogeneous of degree 2.

Definition 4.2. Let $\mathcal{C}$ be a convex cone. We say a function $f \in \mathcal{V}$ is *positive definite with respect to* $\mathcal{C}$ if there exist $0 < \alpha \leqslant \beta < \infty$ such that for all $x \in \mathcal{C}$:

$$\alpha\|x\|^2 \leqslant f(x) \leqslant \beta\|x\|^2.$$

In particular, this implies that if f is positive definite with respect to $\mathcal{C}$, then $0 < f(x) < \infty$ for all $x \in \mathcal{C} \setminus \{0\}$. As a result, this also implies that $\mathcal{C} \subseteq \operatorname{dom} f$.

These preliminaries lead us to a definition of Lyapunov functions for convex processes.

Definition 4.3. Let H be a convex process. A function $V \in \mathcal{V}$ is a *weak Lyapunov function for* H if V is positive definite with respect to $\mathcal{F}(H)$ and there exists $\gamma \in (0,1)$ such that

$$\forall x \in \mathcal{F}(H), \ \ \exists y \in \mathcal{F}(H) \cap H(x) \text{ s.t. } V(y) \leqslant \gamma V(x). \tag{4.2}$$

A function $V \in \mathcal{V}$ is a *strong Lyapunov function for* H if V is positive definite with respect to $\mathcal{F}(H)$ and there exists $\gamma \in (0,1)$ such that

$$\forall x \in \mathcal{F}(H), \ \ \forall y \in \mathcal{F}(H) \cap H(x) \quad V(y) \leqslant \gamma V(x). \tag{4.3}$$

The following example shows that, unlike for linear systems, it is not sufficient to consider only quadratic functions as Lyapunov function candidates.

Example 4.1. Let $0 < \eta < 1$ and define the convex process $H : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by

$$\operatorname{gr}(H) := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \middle| \begin{bmatrix} -\eta & 0 & 1 & 0 \\ -\eta & 0 & 0 & 1 \\ 0 & -\eta & 1 & 0 \\ 0 & -\eta & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \geqslant 0 \right\}.$$

Since this is a convex cone, H is a convex process. Furthermore, H is strict and for any $x \in \mathbb{R}^2$ we have

$$H(x) = \{y \mid y_1, y_2 \geqslant \eta \max\{x_1, x_2\}\}.$$

Let $\xi_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2$ and $m := \max\{\alpha, \beta\}$. Then, for $k \geqslant 1$, define

$$\xi_k := \eta^k \begin{pmatrix} m \\ m \end{pmatrix}.$$

Note that $\xi_{k+1} \in H(\xi_k)$. As such, we see that H is uniformly exponentially stabilizable. In fact, we can find a weak Lyapunov function V by taking

$$V(x) := \max\{x_1^2, x_2^2\}.$$

Note that if η is large enough, there does not exists a weak Lyapunov function of the form $f(x) = x^\top A x$.

We will now compare this definition to earlier notions of Lyapunov functions for nonstrict convex processes. In terms of the notation of this chapter, the definition for a weak Lyapunov function used in [69] requires $V \in \mathcal{V}$ to be positive definite with respect to $\mathbb{R}^n$ and the existence of $\gamma \in (0,1)$ such that

$$\forall x \in \operatorname{dom}(H), \ \ \exists y \in H(x) \text{ s.t. } V(y) \leqslant \gamma V(x). \tag{4.4}$$

Similarly, (strong) Lyapunov functions are defined in [69] as functions $V \in \mathcal{V}$ that are positive definite with respect to $\mathbb{R}^n$ such that there exists $\gamma \in (0,1)$ satisfying

$$\forall x \in \operatorname{dom}(H), \ \ \forall y \in H(x) \quad V(y) \leqslant \gamma V(x). \tag{4.5}$$

Note that for strict convex processes $\operatorname{dom}(H) = \mathcal{F}(H) = \mathbb{R}^n$. This makes these definitions coincide with Definition 4.3. On the other hand, for nonstrict convex processes, important differences arise. Using the following two examples, we will argue that Definition 4.3 is more natural for both weak and strong Lyapunov functions. First, we consider weak Lyapunov functions. The following is an example of a convex process and a function that is a weak Lyapunov function in the sense of (4.4), which fails to be stabilizable.

Example 4.2. Let $H : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be the convex process given by:

$$H(x) = \begin{cases} \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} x + (\mathbb{R} \times \{0\}) & \text{if } \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} x \geqslant 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

Here '$\geqslant$' is understood to hold element-wise. It is straightforward to check that $\mathcal{F}(H) = \operatorname{dom}(H)$. Let $V(x) = \frac{1}{2}\|x\|^2$, then for each $x \in \operatorname{dom}(H)$, there exists $y \in H(x)$ such that $V(y) \leqslant \frac{1}{4}\|x\|^2$. On the other hand, it is straightforward to check that H is not stabilizable. Indeed, for $x \in \mathcal{F}(H)$, we have that $y \in \mathcal{F}(H) \cap H(x)$ implies that $V(y) \geqslant V(x)$.

The following is an example of a convex process which is strongly stable, but for which there does not exist any strong Lyapunov function in the sense of (4.5).

Example 4.3. Let $H : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be the convex process given by:

$$H(x) = \begin{cases} -\frac{1}{2}x + \{0\} \times \mathbb{R}_- & \text{if } \begin{bmatrix} 0 & 1 \end{bmatrix} x \geqslant 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

and empty for other x. It is straightforward to see that $\mathcal{F}(H) = \{x \mid \begin{bmatrix} 0 & 1 \end{bmatrix} x = 0\}$, and that $H(x) \cap \mathcal{F}(H)$ contains only $-\frac{1}{2}x$ for any $x \in \mathcal{F}(H)$. Therefore H is stable.

However, for the point $x = \binom{0}{1}$, we have that $x \in \operatorname{dom}(H)$ and $H(x) = \{\binom{0}{\alpha} \mid \alpha \leqslant -\frac{1}{2}\}$. Clearly there does not exist a function $V \in \mathcal{V}$ that is positive definite with respect to $\mathbb{R}^2$ and $\gamma \in (0,1)$ for which we have that $V(y) \leqslant \gamma V(x)$ for each $y \in H(x)$.

4.3 (CONVERSE) LYAPUNOV RESULTS

In this section we will investigate the links between the existence of weak (strong) Lyapunov functions and stabilizability (resp. strong stability).

We say the system H is *uniformly exponentially stabilizable* if there exists $\nu \in \mathbb{R}$ and $\mu \in [0,1)$ such that for all $\bar{x} \in \mathcal{F}(H)$

$$\exists (x_k) \in \mathfrak{B}(H) \text{ with } x_0 = \bar{x}, \|x_k\| \leqslant \nu\mu^k \|x_0\|.$$

In a similar fashion H is *uniformly exponentially strongly stable* if there exists $\nu \geqslant 1$ and $\mu \in [0,1)$ such that for all $(x_k) \in \mathfrak{B}(H)$ we have that $\|x_k\| \leqslant \nu\mu^k\|x_0\|$. Clearly, H is stabilizable (strongly stable) if it is uniformly exponentially stabilizable (resp. strongly stable).

It is well known that for linear systems, stabilizability and (uniform) exponential stabilizability are equivalent, a result generalized in the following lemma, which is a restatement of [69, Prop. 3.1] in our terminology.

Lemma 4.1. Let H be a convex process such that $\mathcal{F}(H)$ is a polyhedral cone. Then H is stabilizable if and only if it is uniformly exponentially stabilizable.

The following lemma makes the relation between weak and strong Lyapunov functions and stabilizability and strong stability explicit.

Lemma 4.2. Let H be a convex process. If H admits a weak Lyapunov function, then H is uniformly exponentially stabilizable. If H admits a strong Lyapunov function, then H is uniformly exponentially strongly stable.

Proof. We will prove only the first part as the second part follows in a similar fashion. Let V be a weak Lyapunov function for H with corresponding $\gamma \in (0,1)$. Let $\bar{x} \in \mathcal{F}(H)$. Then there exists a sequence $(x_k) \in \mathfrak{B}(H)$ such that $x_0 = \bar{x}$ and $V(x_k) \leqslant \gamma V(x_{k-1})$ for all $k \geqslant 1$. Clearly, this implies that for each $k \geqslant 1$ we have that $V(x_k) \leqslant \gamma^k V(x_0)$. Using the fact that V is positive definite with respect to $\mathcal{F}(H)$, we see that there exists $0 < \alpha \leqslant \beta < \infty$ such that:

$$\alpha\|x_k\|^2 \leqslant V(x_k) \leqslant \gamma^k V(x_0) \leqslant \beta\gamma^k\|x_0\|^2.$$

By dividing each side by α and taking the square root, we see that H is uniformly exponentially stabilizable. ■

This result allows us to conclude stabilizability from the existence of a weak Lyapunov function, and strong stability from the existence of a strong Lyapunov function. We now turn our attention to converse Lyapunov results, that is: We will prove that the existence of a weak Lyapunov function is also necessary for uniform exponential stabilizability.

Given a convex process H, define the extended real-valued function V_w by:

$$V_w(x) := \inf_{\substack{(x_k)\in\mathfrak{B}(H)\\ x_0=x}} \left\{ \sum_{k=0}^{\infty} \|x_k\|^2 \right\}.$$

A few observations can be made immediately. Because $V_w(0) = 0$ and

$$V_w(x) \geqslant \|x\|^2 \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{4.6}$$

we see that V_w is positive semi-definite. In addition $V_w(\lambda x) = \lambda^2 V_w(x)$ for all $\lambda \geqslant 0$ since $\mathfrak{B}(H)$ is a cone. In other words, V_w is positively homogeneous of degree 2. The following lemma will prove that in fact V_w is convex and hence belongs to $\mathcal{V}$.

Lemma 4.3. Let H be a convex process. Then $V_w(x)$ is a convex function.

Proof. Let $x, y \in \mathcal{F}(H)$ and $\varepsilon > 0$. By definition, there exist $(x_k), (y_k) \in \mathfrak{B}(H)$ such that $x_0 = x$, $y_0 = y$ and

$$\sum_{k=0}^{\infty} \|x_k\|^2 \leqslant V_w(x) + \varepsilon, \qquad \sum_{k=0}^{\infty} \|y_k\|^2 \leqslant V_w(y) + \varepsilon.$$

Because H is convex, the convex combination of two trajectories of H is a trajectory. Thus, if we let $0 \leqslant \lambda \leqslant 1$, then:

$$\begin{aligned} V_w(\lambda x + (1-\lambda)y) &\leqslant \sum_{k=0}^{\infty} \|\lambda x_k + (1-\lambda) y_k\|^2 \\ &\leqslant \lambda^2 (V_w(x) + \varepsilon) + (1-\lambda)^2 (V_w(y) + \varepsilon) \\ &\leqslant \lambda V_w(x) + (1-\lambda) V_w(y) + \varepsilon. \end{aligned}$$

As this inequality holds for any $\varepsilon > 0$, this proves that V is convex. ■

Next, we deal with the positive definiteness of V_w.

Lemma 4.4. If H is uniformly exponentially stabilizable, then V_w is positive definite with respect to $\mathcal{F}(H)$.

Proof. Recall that $V_w(x) \geqslant \|x\|^2$ by definition. Therefore it suffices to prove that there exists $\beta \geqslant 1$ such that $V(x) \leqslant \beta\|x\|^2$ for all $x \in \mathcal{F}(H)$. As H is uniformly exponentially stabilizable, there exist $\nu \in \mathbb{R}$ and $\mu \in [0,1)$ such that for all $x \in \mathcal{F}(H)$:

$$\exists(x_k) \in \mathfrak{B}(H) \text{ with } x_0 = x, \|x_k\| \leqslant \nu\mu^k\|x\|. \tag{4.7}$$

Let $x \in \mathcal{F}(H)$ and let (x_k) be the trajectory from (4.7). Then:

$$V_w(x) \leqslant \sum_{k=0}^{\infty} \|x_k\|^2 \leqslant \nu^2 \left(\sum_{k=0}^{\infty} \mu^{2k}\right) \|x\|^2 = \frac{\nu^2}{1-\mu^2}\|x\|^2.$$

Since $\nu \geqslant 1$ and $\mu \in [0,1)$, this proves the lemma. ∎

We can now combine the previous parts to obtain a converse Lyapunov result for convex processes.

Theorem 4.1. Let H be a convex process. Then H is uniformly exponentially stabilizable if and only if V_w is a weak Lyapunov function for H.

Proof. In view of Lemma 4.2, it suffices to prove necessity. Recall that $V_w \in \mathcal{V}$ and that V_w is positive definite with respect to $\mathcal{F}(H)$. Therefore, we are required to prove that (4.2) holds for some $\gamma \in (0,1)$.

As a first observation, we see that $0 \in \mathcal{F}(H) \cap H(0)$ for any convex process, and therefore it suffices to prove that (4.2) holds for all $0 \neq \bar{x} \in \mathcal{F}(H)$.

Let $\bar{x} \in \mathcal{F}(H)$ and let $\varepsilon > 0$. By definition, there exists $(x_k) \in \mathfrak{B}(H)$ such that $x_0 = \bar{x}$ and

$$\sum_{k=0}^{\infty} \|x_k\|^2 \leqslant V_w(\bar{x}) + \varepsilon.$$

Now note that $x_1 \in \mathcal{F}(H) \cap H(\bar{x})$, and that by definition:

$$V_w(x_1) \leqslant \sum_{k=1}^{\infty} \|x_k\|^2 \leqslant V_w(\bar{x}) - \|\bar{x}\|^2 + \varepsilon. \tag{4.8}$$

By Lemma 4.4 and (4.6), we know that for V_w there exist $1 \leqslant \alpha \leqslant \beta < \infty$ such that

$$\tfrac{1}{\beta}V_w(x) \leqslant \|x\|^2 \leqslant \tfrac{1}{\alpha}V_w(x) \quad \text{for all } x \in \mathcal{F}(H) \tag{4.9}$$

Combining (4.8) and (4.9) we see that if $0 \neq \bar{x} \in \mathcal{F}(H)$, then for any $\varepsilon > 0$ there exists $\bar{y} \in \mathcal{F}(H) \cap H(\bar{x})$ for which

$$V_w(\bar{y}) \leqslant \left(1 - \frac{1}{\beta}\right) V_w(\bar{x}) + \varepsilon.$$

Let γ be such that $1 - \frac{1}{\beta} < \gamma < 1$. As $\bar{x} \neq 0$, we can now pick $\varepsilon = (\gamma + \frac{1}{\beta} - 1)V_w(\bar{x}) > 0$, which leads to

$$V_w(\bar{y}) \leqslant \gamma V_w(\bar{x}).$$

This proves that (4.2) holds for all $x \in \mathcal{F}(H)$. ■

Remark 4.2. The previous was proven for strict convex processes in [69, Prop. 4.3].

We now turn our attention to converse Lyapunov results for uniform exponential strong stability. For this, we will first define an intermediate function:

$$W(x) := \sup_{\substack{(x_k) \in \mathfrak{B}(H) \\ x_0 = x}} \left\{ \sum_{k=0}^{\infty} \|x_k\|^2 \right\}.$$

Let H be uniformly exponentially strongly stable. Similar to V_w, it is straightforward to prove that W is positive semi-definite and positively homogeneous of degree 2. Furthermore, analogously to the proof of Lemma 4.4, we can derive that

$$\|x\|^2 \leqslant W(x) \leqslant \frac{\nu^2}{1 - \mu^2} \|x\|^2 \tag{4.10}$$

for all $x \in \mathcal{F}(H)$. Therefore, W is positive definite with respect to $\mathcal{F}(H)$.

Lastly, by the definition of the supremum, we know that

$$W(x) \geqslant W(y) + \|x\|^2 \geqslant W(y) + \frac{1 - \mu^2}{\nu^2} W(x) \quad \text{for all } y \in \mathcal{F}(H) \cap H(x).$$

This results in

$$W(y) \leqslant \left(1 - \frac{1 - \mu^2}{\nu^2}\right) W(x). \tag{4.11}$$

Remark 4.3. Note that W is not necessarily convex but does satisfy the other properties required to be a strong Lyapunov function. This is similar to the result in [69, Lem. 4.5]

The following theorem will give conditions under which W leads to a strong Lyapunov function.

Theorem 4.2. Let H be a convex process. Suppose that H is strongly uniformly exponentially stabilizable with parameters ν and μ such that $\nu^2 < 2 - 2\mu^2$. Then there exists a strong Lyapunov function for H.

Proof. Let $V_s = \operatorname{conv} W$. Then, we have

$$V_s(x) := \inf_{\substack{x=\sum_i \lambda_i x^i \\ 0 \leqslant \lambda_i \leqslant 1}} \sum_i \lambda_i \left(\sup_{\substack{(x_k) \in \mathfrak{B}(H) \\ x_0 = x^i}} \left\{ \sum_{k=0}^{\infty} \|x_k\|^2 \right\} \right).$$

We consider V_s as a strong Lyapunov function candidate. Again, V_s is obviously positive semi-definite and positively homogeneous of degree 2.

Let $x \in \mathcal{F}(H)$. Note that V_s is the largest convex function that is majorized by W. Therefore, taking the convex hull of the functions in (4.10), we obtain

$$\|x\|^2 \leqslant V_s(x) \leqslant W(x) \leqslant \frac{\nu^2}{1-\mu^2} \|x\|^2. \tag{4.12}$$

As such, V_s is positive definite with respect to $\mathcal{F}(H)$. A consequence of (4.12) is that:

$$W(x) \leqslant \frac{\nu^2}{1-\mu^2} V_s(x). \tag{4.13}$$

We can now apply (4.13) to the inequality in (4.11). This shows that

$$V_s(y) \leqslant W(y) \leqslant \left(\frac{\nu^2}{1-\mu^2} - 1 \right) V_s(x),$$

whenever $x \in \mathcal{F}(H)$ and $y \in \mathcal{F}(H) \cap H(x)$. Recall that by assumption $\nu^2 < 2 - 2\mu^2$. Now let $\gamma := \frac{\nu^2}{1-\mu^2} - 1$. Then $\gamma < 1$ and

$$V_s(y) \leqslant \gamma V_s(x).$$

This concludes the proof. ■

In Section 4.5 we will revisit the converse Lyapunov problem for strong stability. There, we will present different conditions under which we can explicitly construct a strong Lyapunov function.

4.4 PRELIMINARIES ON DUALITY

4.4.1 Duality of convex processes

We denote the image of a set $\mathcal{S}$ under H by $H(\mathcal{S}) = \{y \in \mathbb{R}^n \mid \exists x \in \mathcal{S} \text{ s.t. } y \in H(x)\}$. This shorthand notation allows us to define powers of convex processes, by taking $H^1 = H$ and

$$H^{q+1}(x) := H(H^q(x)) \quad \forall x \in \mathbb{R}^n \quad \text{for all } q \geqslant 1.$$

We can define the inverse of a convex process by $H^{-1}(y) = \{x \mid y \in H(x)\}$. Note that this is always defined as a set-valued map. For negative powers of H we write $H^{-n}(x) = (H^{-1})^n(x)$. We define the set of q-*step trajectories* as

$$\mathfrak{B}_q(H) = \left\{(x_k)_{k=0}^q \in (\mathbb{R}^n)^{q+1} \mid (x_k) \text{ satisfies } (4.1)\right\}.$$

Using this, we say that a point $\xi \in \mathbb{R}^n$ is *reachable* if there exists a q-step trajectory from the origin to ξ. The set of such points is the reachable set:

$$\mathcal{R}(H) = \left\{\xi \mid \exists q, (x_k)_{k=0}^q \in \mathfrak{B}_q(H) \text{ s.t. } x_0 = 0, x_q = \xi\right\}.$$

For a convex cone $\mathcal{C} \subseteq \mathbb{R}^n$, we define $\text{lin}(\mathcal{C}) = -\mathcal{C} \cap \mathcal{C}$ and $\text{Lin}(\mathcal{C}) = \mathcal{C} - \mathcal{C}$. It is clear that $\text{lin}(\mathcal{C})$ is the largest subspace contained in $\mathcal{C}$ whereas $\text{Lin}(\mathcal{C})$ is the smallest subspace that contains $\mathcal{C}$.

Let H be a convex process. Associated with H, we define the two linear processes L_- and L_+ by

$$\text{gr}(L_-) = \text{lin}\big(\text{gr}(H)\big) \text{ and } \text{gr}(L_+) = \text{Lin}\big(\text{gr}(H)\big). \tag{4.14}$$

By definition, we therefore have

$$\text{gr}(L_-) \subseteq \text{gr}(H) \subseteq \text{gr}(L_+). \tag{4.15}$$

It is clear that L_- and L_+ are, respectively, the largest and the smallest (with respect to the graph inclusion) linear processes satisfying (4.15). We call L_- and L_+, respectively, the minimal and maximal linear processes associated with H. If H is not clear from context, we write $L_-(H)$ and $L_+(H)$ in order to avoid confusion. For the respective reachable sets, we write:

$$\mathcal{R}_- = \mathcal{R}(L_-), \quad \mathcal{R}_+ = \mathcal{R}(L_+).$$

For a nonempty set $\mathcal{C} \subseteq \mathbb{R}^n$, we define the *negative* and *positive polar cone*, respectively,

$$\begin{aligned}\mathcal{C}^- &= \{y \in \mathbb{R}^n \mid \langle x, y\rangle \leqslant 0 \quad \forall x \in \mathcal{C}\},\\ \mathcal{C}^+ &= \{y \in \mathbb{R}^n \mid \langle x, y\rangle \geqslant 0 \quad \forall x \in \mathcal{C}\}.\end{aligned}$$

For a subspace $\mathcal{S}$, we have that $\mathcal{S}^- = \mathcal{S}^+ = \mathcal{S}^\perp$, where $\mathcal{S}^\perp$ denotes the orthogonal complement of $\mathcal{S}$. Given sets $\mathcal{C}$ and $\mathcal{S}$, we have the following:

$$(\mathcal{C} + \mathcal{S})^- = \mathcal{C}^- \cap \mathcal{S}^-, \quad (\mathcal{C} \cap \mathcal{S})^- = \mathrm{cl}(\mathcal{C}^- + \mathcal{S}^-). \tag{4.16}$$

The same equations hold for the positive polar cone, since $\mathcal{C}^+ = -\mathcal{C}^-$.

Based on the definition of the negative and positive polar cones, we define *negative* and *positive dual* processes H^- and H^+ of H as follows:

$$p \in H^-(q) \iff \langle p, x\rangle \geqslant \langle q, y\rangle \quad \forall\,(x, y) \in \mathrm{gr}(H), \tag{4.17a}$$

$$p \in H^+(q) \iff \langle p, x\rangle \leqslant \langle q, y\rangle \quad \forall\,(x, y) \in \mathrm{gr}(H). \tag{4.17b}$$

Remark 4.4. The positive dual process is alternatively called the *adjoint* or *(left-) transpose* in the literature.

Note that $H^+(q) = -H^-(-q)$ for all q. If H is closed, we know that $(H^+)^- = H$ and

$$H(0) = (\mathrm{dom}(H^+))^+ = (\mathrm{dom}(H^-))^-. \tag{4.18}$$

If L is a linear process it is clear that its negative and positive dual processes coincide, in which case we denote it by $L^\perp := L^- = L^+$. The reachable and feasible sets of a linear process L can be determined in a finite number of steps. To be precise $\mathcal{F}(L) = L^{-n}(\mathbb{R}^n)$ and $\mathcal{R}(L) = L^n(0)$. The feasible and reachable set of a linear process and its dual are related by:

$$\mathcal{F}(L^\perp) = \mathcal{R}(L)^\perp, \tag{4.19a}$$

$$\mathcal{R}(L^\perp) = \mathcal{F}(L)^\perp. \tag{4.19b}$$

In addition, the minimal and maximal linear processes associated with a convex process enjoy the following additional properties:

$$L_-(H^-) = L_-(H^+) = L_+^\perp, \tag{4.20a}$$

$$L_+(H^-) = L_+(H^+) = L_-^\perp. \tag{4.20b}$$

In order to characterize stabilizability for linear processes, we will consider the outer process next. Let H be a convex process and let $\mathcal{W} \subseteq \mathbb{R}^n$ be a subspace such that $\mathcal{R}_+ \oplus \mathcal{W} = \mathbb{R}^n$. Then the *outer process* of H is defined by

$$\operatorname{gr}(H_{\text{out}}) := \Big(\operatorname{gr}(H) + (\mathcal{R}_+ \times \mathcal{R}_+)\Big) \cap (\mathcal{W} \times \mathcal{W}).$$

In the specific case of linear processes, the following is a direct consequence of Lemma 3.7 and Theorem 3.2.

Lemma 4.5. Let L be a linear process. Then L_{out} is a single-valued linear process, that is, a linear map. Furthermore, L is stabilizable if and only if all eigenvalues of $L_{\text{out}}|_{\mathcal{F}(L_{\text{out}})}$ are in the open unit disc.

In the following, we will assume that $\operatorname{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$. This assumption has a few important consequences. First of all, Lemma 3.23.(v) shows that

$$\operatorname{dom}(H) + \mathcal{R}_- = \mathbb{R}^n \implies \mathcal{F}(H) = \operatorname{dom}(H^n). \tag{4.21}$$

Furthermore, Lemma 3.18 shows that

$$\operatorname{dom}(H) + \mathcal{R}_- = \mathbb{R}^n \implies \operatorname{dom}(H^k) + \mathcal{R}_- = \mathbb{R}^n \quad \forall k \geqslant 1. \tag{4.22}$$

These two facts can be combined to obtain the following, which will be used later.

Lemma 4.6. Let H be a convex process such that $\operatorname{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$. Then, $\mathcal{F}(H)^- \cap \operatorname{cl}(\mathcal{F}(H^+)) = \{0\}$ and $\mathcal{F}(H)^- \cap \operatorname{cl}(\mathcal{F}(H^-)) = \{0\}$.

Proof. Combining (4.21) and (4.22), we get that:

$$\mathbb{R}^n = \operatorname{dom}(H) + \mathcal{R}_- = \mathcal{F}(H) + \mathcal{R}_-. \tag{4.23}$$

By (4.20), we have that $L_+(H^+) = L_-^\perp$. Therefore, by (4.19), we have

$$\mathcal{F}(H^+) \subseteq \mathcal{F}(L_+(H^+)) = \mathcal{F}(L_-^\perp) = \mathcal{R}_-^\perp. \tag{4.24}$$

As the right-hand side is closed by definition, we have that $\operatorname{cl}(\mathcal{F}(H^+)) \subseteq \mathcal{R}_-^\perp$.

Taking the negative polar of (4.23), we have by (4.16) that $\mathcal{F}(H)^- \cap \mathcal{R}_-^\perp = \{0\}$. By the previous, we therefore have that $\mathcal{F}(H)^- \cap \operatorname{cl}(\mathcal{F}(H^+)) = \{0\}$. One can repeat the same arguments for $\mathcal{F}(H)^- \cap \operatorname{cl} \mathcal{F}(H^-) = \{0\}$. ∎

4.4.2 Duality of extended real-valued functions

Given an extended real-valued function f, we define its *convex conjugate* by

$$f^\star(y) := \sup_{x \in \mathbb{R}^n} \{y \cdot x - f(x)\}.$$

It is well known (see e.g. [142, Thm. 12.2]) that the convex conjugate of proper convex function is a proper closed convex function. In addition, if f is a convex function, then $f^{\star\star} = \operatorname{cl} f$. As taking the convex conjugate also preserves positive semi-definiteness and positive homogeneity of degree 2, we see that $f \in \mathcal{V}$ if and only if $f^\star \in \mathcal{V}$.

It is straightforward to show that if $f_1(x) \geqslant f_2(x)$ for all $x \in \mathbb{R}^n$, then $f_1^\star(y) \leqslant f_2^\star(y)$ for all $y \in \mathbb{R}^n$.

Let $\mathcal{C} \subseteq \mathbb{R}^n$. The *indicator function* of $\mathcal{C}$ is the function $\delta(\cdot \mid \mathcal{C})$ given by

$$\delta(x \mid \mathcal{C}) := \begin{cases} 0 & x \in \mathcal{C} \\ \infty & x \notin \mathcal{C}. \end{cases}$$

It is straightforward to check that $\operatorname{dom} \delta(\cdot \mid \mathcal{C}) = \mathcal{C}$ and that the indicator function is a convex (closed) function if and only if $\mathcal{C}$ is closed (resp. convex).

We define the *restriction of* f *to* $\mathcal{C}$ by

$$f|_{\mathcal{C}}(x) = f(x) + \delta(x \mid \mathcal{C}).$$

If $f \in \mathcal{V}$ then we have that $f|_{\mathcal{C}} \in \mathcal{V}$ for any convex cone $\mathcal{C}$.

Given proper convex functions f and g, we define their *infimal convolution* $f \,\square\, g$ by the relation $\operatorname{epi}(f \,\square\, g) := \operatorname{epi} f + \operatorname{epi} g$, or equivalently:

$$(f \,\square\, g)(x) := \inf_{y \in \mathbb{R}^n} f(x-y) + g(y).$$

A consequence of [142, Thm. 16.4] is the following.

Corollary 4.1. Let f and g be proper convex functions, then $(f \,\square\, g)^\star = f^\star + g^\star$ and $(\operatorname{cl}(f) + \operatorname{cl}(g))^\star = \operatorname{cl}(f^\star \,\square\, g^\star)$. If in addition $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq \varnothing$, then $(f+g)^\star = f^\star \,\square\, g^\star$.

We will now consider a number of simple examples, which will be of use in the remainder of this chapter.

Example 4.4. Let $\mathcal{C}$ be a convex cone. Consider the functions $f(x) = \frac{1}{2}\|x\|^2$ and $g(x) = \delta(x \mid \mathcal{C})$. Then $f^\star(y) = \frac{1}{2}\|y\|^2$ and $g^\star(y) = \delta(y \mid \mathcal{C}^-)$. Denote the squared *distance function* to $\mathcal{C}$ as

$$h(x) := \inf_{y \in \mathcal{C}} \tfrac{1}{2}\|x-y\|^2 = (f \,\square\, g)(x).$$

Corollary 4.1 implies that:

$$h^\star(y) = \tfrac{1}{2}\|y\|^2 + \delta(y \mid \mathcal{C}^-) = f^\star|_{\mathcal{C}^-}.$$

Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a convex set such that $0 \in \mathcal{S}$. We denote the *squared gauge* of $\mathcal{S}$ by

$$g(x \mid \mathcal{S}) := \inf_{\lambda \geqslant 0} \{\lambda^2 \mid x \in \lambda\mathcal{S}\}.$$

It is straightforward to show that if $\mathcal{S} \subseteq \mathcal{T}$ then $g(x \mid \mathcal{S}) \geqslant g(x \mid \mathcal{T})$ for all $x \in \mathbb{R}^n$.

For $\alpha \geqslant 0$, the *α-sublevel set* of a function f is the set $\mathcal{S}_\alpha(f) := \{x \mid f(x) \leqslant \alpha\}$. The squared gauge is related to the sublevel sets by the following lemma.

Lemma 4.7. Let $f \in \mathcal{V}$. Then $f(x) = g(x \mid \mathcal{S}_1(f))$ for all $x \in \mathbb{R}^n$.

Proof. Let $x \in \mathbb{R}^n$. Note that

$$\begin{aligned} g(x \mid \mathcal{S}_1(f)) &= \inf_{\lambda \geqslant 0} \left\{\lambda^2 \mid x \in \lambda\{y \mid f(y) \leqslant 1\}\right\} \\ &= \inf_{\lambda \geqslant 0} \{\lambda^2 \mid f(x) \leqslant \lambda^2\} \\ &= f(x). \end{aligned}$$

Thus proving the lemma. ■

It is well known that if f is positive definite with respect to $\mathbb{R}^n$, then so is $f^\star$. This result is generalized in the following theorem.

Theorem 4.3. Let $f \in \mathcal{V}$ and let $\mathcal{C}, \mathcal{D} \subseteq \mathbb{R}^n$ be convex cones such that f is positive definite with respect to $\mathcal{C}$ and $\mathcal{C}^- \cap \operatorname{cl}(\mathcal{D}) = \{0\}$. Then, $(f|_\mathcal{C})^\star \in \mathcal{V}$ and $(f|_\mathcal{C})^\star$ is positive definite with respect to $\mathcal{D}$.

Proof. Recall that if f is a function in $\mathcal{V}$, then its restriction to a convex cone and its convex conjugate are also in $\mathcal{V}$. Therefore $(f|_\mathcal{C})^\star \in \mathcal{V}$.

From Definition 4.2, we know that there exists $0 < \alpha < \infty$ such that

$$\alpha\|x\|^2 + \delta(x \mid \mathcal{C}) \leqslant f|_\mathcal{C}(x).$$

Recall that $f|_\mathcal{C}(x) \geqslant h(x)$ for all $x \in \mathbb{R}^n$ implies that $(f|_\mathcal{C})^\star(y) \leqslant h^\star(y)$ for all $y \in \mathbb{R}^n$. By using Corollary 4.1 and Example 4.4, we therefore have that

$$(f|_\mathcal{C})^\star(y) \leqslant \frac{1}{4\alpha} \inf_{x \in \mathcal{C}^-} \|y - x\|^2 \quad \text{for all } y \in \mathbb{R}^n.$$

Note that $0 \in \mathcal{C}^-$. Therefore, we have

$$(f|_{\mathcal{C}})^\star(y) \leqslant \frac{1}{4\alpha}\|y\|^2. \tag{4.25}$$

Before we are able to prove that $(f|_{\mathcal{C}})^\star$ is positive definite with respect to $\mathcal{D}$, we require a preliminary result. Given $\mathcal{D}$, we define the function h by

$$h(y) := \inf_{x \in \mathcal{C}^-} \|y - x\|^2 + \delta(y \mid \mathcal{D}).$$

Taking $\mathcal{S} = \{y \mid \|y\|^2 = 1\}$, we have that

$$\begin{aligned}\inf_{y \in \mathcal{S}} h(y) &= \inf_{y \in \mathcal{S} \cap \mathcal{D}} \inf_{x \in \mathcal{C}^-} \|y - x\|^2 \\ &\geqslant \inf_{y \in \mathcal{S} \cap \mathrm{cl}(\mathcal{D})} \inf_{x \in \mathcal{C}^-} \|y - x\|^2.\end{aligned}$$

We will now prove by contradiction that the last term of the above inequality is positive. Assume the contrary, i.e. that there exist sequences $(x_k)_{k=0}^\infty, (y_k)_{k=0}^\infty$ where $x_k \in \mathcal{C}^-$ and $y_k \in \mathcal{S} \cap \mathrm{cl}(\mathcal{D})$ for all $k \geqslant 0$, such that $\lim_{k\to\infty} \|y_k - x_k\| = 0$.

Note that $\mathcal{S} \cap \mathrm{cl}(\mathcal{D})$ is both closed and bounded. Therefore there exists a subsequence $(y_{k_\ell})_{\ell=0}^\infty$ that converges to $\bar{y} \in \mathcal{S} \cap \mathrm{cl}(\mathcal{D})$. We can now use the triangle inequality to show that

$$\lim_{\ell\to\infty} \|\bar{y} - x_{k_\ell}\| \leqslant \lim_{\ell\to\infty} \|\bar{y} - y_{k_\ell}\| + \lim_{\ell\to\infty} \|y_{k_\ell} - x_{k_\ell}\| = 0.$$

As $\mathcal{C}^-$ is closed, this implies that $\bar{y} \in \mathcal{C}^- \cap (\mathcal{S} \cap \mathrm{cl}(\mathcal{D}))$. Recall that by assumption $\mathcal{C}^- \cap \mathrm{cl}(\mathcal{D}) = \{0\}$ and $0 \notin \mathcal{S}$, thus leading to a contradiction.

Following the previous, we know that there exists $\gamma > 0$ such that $\inf_{y \in \mathcal{S}} h(y) \geqslant \gamma$. As h is positively homogeneous of degree 2, we can conclude that $h(y) \geqslant \gamma\|y\|^2$ for all $y \in \mathbb{R}^n$.

Similar to the first part of this proof, there exists $0 < \beta < \infty$ such that we have

$$\frac{1}{4\beta} \inf_{x \in \mathcal{C}^-} \|y - x\|^2 \leqslant (f|_{\mathcal{C}})^\star(y) \quad \text{for each} y \in \mathbb{R}^n.$$

Using our previous results, we therefore have that

$$(f|_{\mathcal{C}})^\star(y) + \delta(y \mid \mathcal{D}) \geqslant \frac{1}{4\beta} h(y) \geqslant \frac{\gamma}{4\beta}\|y\|^2.$$

Combining this with (4.25) shows that $(f|_{\mathcal{C}})^\star$ is positive definite with respect to $\mathcal{D}$, thus proving the theorem. ∎

Remark 4.5. Indeed, if f is positive definite with respect to $\mathbb{R}^n$, we can take $\mathcal{D} = \mathbb{R}^n$ and see that $f^\star = (f|_{\mathbb{R}^n})^\star \in \mathcal{V}$ is positive definite with respect to $\mathbb{R}^n$.

4.5 DUALITY THEOREMS

This section will focus on the links between duality and the different stability properties under consideration. The following example shows that the fact that stabilizability and strong stability are in some sense dual to each other is already well known, albeit not in the notation we employ.

Example 4.5. Let L be a strict linear process. It is easy to see (see e.g. Lemma 2.4) that we can find matrices A and B such that for all x we have that $L(x) := Ax + \operatorname{im} B$. Consider the linear system given by

$$x_{k+1} = Ax_k + Bu_k$$

Note that the matrix pair (A, B) is stabilizable if and only if L is stabilizable. Taking the dual of L, we see that

$$\operatorname{gr}(L^{\perp}) = \ker \begin{bmatrix} A^{\top} & -I \\ B^{\top} & 0 \end{bmatrix}.$$

Note that therefore $y_{k+1} \in L^{\perp}(y_k)$ if and only if:

$$y_{k+1} = A^{\top} y_k \quad \text{and} \quad B^{\top} y_k = 0.$$

This means that $L^{\perp}$ is strongly stable if and only if the matrix pair $(B^{\top}, A^{\top})$ is detectable. Recall from linear systems theory that (A, B) is stabilizable if and $(B^{\top}, A^{\top})$ is detectable, and therefore L is stabilizable if and only if $L^{\perp}$ is strongly stable.

Another iteration of such duality is found in spectral characterizations of stabilizability. We say λ is an eigenvalue of H if there exists $\xi \neq 0$ such that $\lambda\xi \in H(\xi)$. A special case of Theorem 3.2 gives the following result with regard to stabilizability.

Corollary 4.2. Let H be a convex process such that $\operatorname{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$. Then H is stabilizable if and only if L_+ is stabilizable and H^- has no eigenvalues $\lambda \geqslant 1$.

Together, Example 4.5 and Corollary 4.2 suggest a link between stabilizability of a convex process H and strong stability of the dual H^-. In this section, we will prove that under certain conditions we can transform a weak Lyapunov function of H into a strong Lyapunov function for H^+. A special case of this result, for strict H, was already proven in [69, Thm. 2.2]. This result will then be combined with a previously known characterization of stabilizability to obtain a result relating stabilizability of H to strong stability of its dual process.

Remark 4.6. We will focus our theorems on the positive dual H^+. As for the negative dual, it might be good to note that (x_k) is a trajectory of H^- if and only if $(-x_k)$ is a trajectory of H^+. Therefore, we can interchangeably look at stability properties of H^- and H^+. This means that if V is a weak (respectively strong) Lyapunov function for H^+, then $V(-\cdot)$ is a weak (respectively strong) Lyapunov function for H^-. As such, the choice to focus on H^+ is not restrictive.

4.5.1 Duality for Lyapunov functions

As we are interested in constructing strong Lyapunov functions for the dual process, we will first investigate necessary conditions for (uniform exponential) strong stability of the dual process. If H^+ is strongly stable, then in particular all trajectories from the origin are stable. In view of (4.18) we can state the following result.

Corollary 4.3. Let H be a convex process such that H^+ is strongly stable. Then, $\mathcal{F}(H^+) \cap H^+(0) = \{0\}$.

Example 4.6. Let L be a linear process such that $L(0) = \{0\}$, then it is clear that L is single valued. We also see that, if L is strongly stable, then $L(0) \cap \mathcal{F}(H) = \{0\}$. As such, if L is strongly stable, only a single trajectory emanates from any feasible point. Now let H be the convex process defined by

$$H(x) := \begin{cases} \left[\frac{1}{3}x, \frac{1}{2}x\right] & x \geqslant 0, \\ \varnothing & x < 0. \end{cases}$$

Note that indeed $H(0) = \{0\}$, but that H is not single valued. Furthermore, we see that H is (uniformly exponentially) strongly stable, whilst admitting different trajectories from nonzero feasible points.

Clearly, Corollary 4.3 implies that H^+ admits a strong Lyapunov function only if $\mathcal{F}(H^+) \cap (\mathrm{dom}(H))^- = \{0\}$. In fact, we will work under a slightly stronger assumption, namely that $\mathcal{F}(H)^- \cap \mathrm{cl}(\mathcal{F}(H^+)) = \{0\}$. Note that by Lemma 4.6 we can guarantee that this last condition holds if $\mathrm{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$.

We can now state and prove the result relating strong and weak Lyapunov functions.

Theorem 4.4. Let H be a convex process such that $\mathcal{F}(H)^- \cap \mathrm{cl}(\mathcal{F}(H^+)) = \{0\}$. If $V \in \mathcal{V}$ is a weak Lyapunov function for H, then $W := (V|_{\mathcal{F}(H)})^\star$ is a strong Lyapunov function for H^+.

Proof. We can use Theorem 4.3 to conclude that $W \in \mathcal{V}$ and W is positive definite with respect to $\mathcal{F}(H^+)$. What remains is to prove that there exists $\gamma \in (0,1)$ such that (4.3) holds for H^+ and W. Let $q \in \mathcal{F}(H^+)$ and $p \in \mathcal{F}(H^+) \cap H^+(q)$, then

$$\begin{aligned} W(p) &= (V + \delta(\cdot \mid \mathcal{F}(H)))^\star (p) \\ &= \sup_{x \in \mathcal{F}(H)} \{p \cdot x - V(x)\}. \end{aligned}$$

Note that V is a weak Lyapunov function for H. Therefore, if $x \in \mathcal{F}(H)$, we know that $V(x) \geqslant \frac{1}{\gamma} V(y)$ for some $y \in \mathcal{F}(H) \cap H(x)$, allowing us to write

$$\begin{aligned} W(p) &\leqslant \sup_{x \in \mathcal{F}(H)} \{p \cdot x - \frac{1}{\gamma} \inf_{y \in \mathcal{F}(H) \cap H(x)} V(y)\} \\ &= \sup_{x \in \mathcal{F}(H)} \sup_{y \in \mathcal{F}(H) \cap H(x)} \{p \cdot x - \frac{1}{\gamma} V(y)\}. \end{aligned}$$

Then, following the definition of H^+, we see that $p \cdot x \leqslant q \cdot y$ and hence

$$W(p) \leqslant \sup_{x \in \mathcal{F}(H)} \sup_{y \in \mathcal{F}(H) \cap H(x)} \{q \cdot y - \frac{1}{\gamma} V(y)\}.$$

Note that for each $x \in \mathcal{F}(H)$ the set $\mathcal{F}(H) \cap H(x)$ is nonempty by definition, and therefore

$$\begin{aligned} W(p) &\leqslant \frac{1}{\gamma} \sup_{y \in \mathcal{F}(H)} \{\gamma q \cdot y - V(y)\} \\ &= \frac{1}{\gamma} W(\gamma q) = \gamma W(q). \end{aligned}$$

Thus proving that (4.3) holds, which proves the theorem. ∎

Remark 4.7. Note that if H is a strict convex process, the condition $\mathrm{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$ holds immediately. This means that the previous theorem is a generalization of [69, Thm. 2.2].

Example 4.7. Let H and V be as in Example 4.1. Recall that V is a weak Lyapunov function for H. Calculating the dual of H results in

$$\mathrm{gr}(H^+) := \left\{ \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \in \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \eta & \eta & 0 & 0 \\ 0 & 0 & \eta & \eta \end{bmatrix} \mathbb{R}^4_- \right\}.$$

This implies that

$$H^+(q) = \begin{cases} \{p \mid p_1 + p_2 = \eta(q_1 + q_2), y \leqslant 0\} & q \leqslant 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

In addition,

$$V^\star(y) = \tfrac{1}{4}(|y_1| + |y_2|)^2.$$

Clearly, this is a strong Lyapunov function for H^+.

Combining Theorem 4.1 and Theorem 4.4 implies that if $\mathrm{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$ and H is uniformly exponentially stabilizable, then we have that

$$\bar{V}_s(y) := \sup_{x \in \mathcal{F}(H)} \left\{ y \cdot x - \inf_{\substack{(x_k) \in \mathfrak{B}(H) \\ x_0 = x}} \left\{ \sum_{k=0}^{\infty} \|x_k\|^2 \right\} \right\}$$

is a strong Lyapunov function for H^+. Since $(H^+)^- = \mathrm{cl}(H)$, we obtain the following.

Corollary 4.4. Let H be a closed convex process such that H^- is uniformly exponentially stabilizable and $\mathrm{dom}(H^-) + \mathcal{R}(L_-(H^-)) = \mathbb{R}^n$. Then

$$\sup_{x \in \mathcal{F}(H^-)} \left\{ y \cdot x - \inf_{\substack{(x_k) \in \mathfrak{B}(H^-) \\ x_0 = x}} \left\{ \sum_{k=0}^{\infty} \|x_k\|^2 \right\} \right\}$$

is a strong Lyapunov function for H.

Remark 4.8. Note that $\mathrm{dom}(H^-) + \mathcal{R}(L_-(H^-)) = \mathbb{R}^n$ if and only if $H(0)^- \cap \mathcal{F}(L_+) = \{0\}$.

The following theorem, which is based on [69, Thm. 2.4], provides conditions under which a strong Lyapunov function for a convex process H can be transformed into a weak Lyapunov function for another convex process.

Theorem 4.5. Let H and G be convex processes such that $\mathcal{F}(H)^- \cap \mathrm{cl}(\mathcal{F}(G)) = \{0\}$ and for all $x \in \mathcal{F}(H)$ and $q \in \mathcal{F}(G)$:

$$\inf_{p \in \mathcal{F}(G) \cap G(q)} p \cdot x \leqslant \sup_{y \in \mathcal{F}(H) \cap H(x)} y \cdot q. \tag{4.26}$$

If $V \in \mathcal{V}$ is a strong Lyapunov function for H, then $W := (V|_{\mathcal{F}(H)})^\star$ is a weak Lyapunov function for G.

Proof. We can follow the proof of Theorem 4.4, to see that $W \in \mathcal{V}$ and W is positive definite with respect to $\mathcal{F}(G)$.

Note that to prove the theorem it suffices to prove that for every $q \in \mathcal{F}(G)$, the following holds:

$$\inf_{p \in \mathcal{F}(G) \cap G(q)} W(p) \leqslant \gamma W(q).$$

Let $q \in \mathcal{F}(G)$. Then, we have

$$\inf_{p \in \mathcal{F}(G) \cap G(q)} W(p) = \inf_{p \in \mathcal{F}(G) \cap G(q)} \sup_{x \in \mathcal{F}(H)} \{p \cdot x - V(x)\}.$$

As $V|_{\mathcal{F}(H)}$ is coercive, we can swap the infimum and supremum using [142, Thm. 37.3], leading to:

$$\inf_{p \in \mathcal{F}(G) \cap G(q)} W(p) = \sup_{x \in \mathcal{F}(H)} \inf_{p \in \mathcal{F}(G) \cap G(q)} \{p \cdot x - V(x)\}.$$

Then, we can apply the inequality of (4.26) to obtain:

$$\inf_{p \in \mathcal{F}(G) \cap G(q)} W(p) \leqslant \sup_{x \in \mathcal{F}(H)} \sup_{y \in \mathcal{F}(H) \cap H(x)} \{y \cdot q - V(x)\}.$$

As V is a strong Lyapunov function for H, we know that $V(x) \geqslant \frac{1}{\gamma} V(y)$ for all $y \in \mathcal{F}(H) \cap H(x)$. Therefore we have

$$\begin{aligned} \inf_{p \in \mathcal{F}(G) \cap G(q)} W(p) &\leqslant \sup_{x \in \mathcal{F}(H)} \sup_{y \in \mathcal{F}(H) \cap H(x)} \{y \cdot q - \tfrac{1}{\gamma} V(y)\} \\ &\leqslant \sup_{y \in \mathcal{F}(H)} \{y \cdot q - \tfrac{1}{\gamma} V(y)\} = \gamma W(q). \end{aligned}$$

This proves the theorem. ■

So far, checking the conditions under which the previous theorem works might seem daunting. However, for a few choices of G, we can check this condition easily.

Example 4.8. If $G = H^+$, we see that the assumption that (4.26) is satisfied for all $x \in \mathcal{F}(H)$ and $q \in \mathcal{F}(H^+)$ holds immediately. As any strong Lyapunov function is a weak Lyapunov function, this reduces Theorem 4.5 to a specific case of Theorem 4.4.

Example 4.9. If $G = H^-$, the assumption that (4.26) is satisfied for all $x \in \mathcal{F}(H)$ and $q \in \mathcal{F}(H^-)$ is equivalent to the assumption that for all $x \in \mathcal{F}(H)$ and $q \in \mathcal{F}(H^+)$.

$$\sup_{p \in \mathcal{F}(H^+) \cap H^+(q)} p \cdot x \leqslant \inf_{y \in \mathcal{F}(H) \cap H(x)} y \cdot q.$$

In turn, we can use [159, Thm. 2.9] to show that this follows if for example $\mathcal{F}(H) \subseteq \mathrm{ri}(\mathrm{dom}(H))$. As an example, for linear processes this condition is always satisfied.

4.5.2 Duality and stability

We now move towards a generalization of the result in Example 4.5.

Lemma 4.8. Let L be a linear process. Then $L^{\perp}$ is strongly stable if and only if L is stabilizable and $\mathrm{dom}(L) + \mathcal{R}(L) = \mathbb{R}^n$.

Proof. From Corollary 4.3, we know that if $L^{\perp}$ is strongly stable, then $\mathcal{F}(L^{\perp}) \cap L^{\perp}(0) = \{0\}$. Taking the orthogonal complement and using (4.18) and (4.19), we obtain that $\mathrm{dom}(L) + \mathcal{R}(L) = \mathbb{R}^n$.

In the remainder of the proof, we will make use of the linear process defined by:

$$\mathrm{gr}(\bar{L}) := \mathrm{gr}(L) + \mathcal{R}(L) \times \mathcal{R}(L).$$

To be precise, we will prove that, if $\mathrm{dom}(L) + \mathcal{R}(L) = \mathbb{R}^n$:

(a) L is stabilizable if and only if $\bar{L}$ is stabilizable.

(b) $\bar{L}$ is stabilizable if and only if $\bar{L}^{\perp}$ is strongly stable.

(c) $L^{\perp}$ is strongly stable if and only if $\bar{L}^{\perp}$ is strongly stable.

Note that proving these claims is sufficient to prove that L is stabilizable if and only if $L^{\perp}$ is strongly stable.

(a): Note that $L_{\mathrm{out}} = \bar{L}_{\mathrm{out}}$. Therefore this statement follows immediately from Lemma 4.5.

(b): Note that $\mathrm{dom}(\bar{L}) = \mathrm{dom}(L) + \mathcal{R}(L)$. Therefore by assumption $\bar{L}$ is strict. We can now apply the result in Example 4.5 to conclude the claim.

(c): It is straightforward to show that:

$$\mathrm{gr}(\bar{L}^{\perp}) = \mathrm{gr}(L^{\perp}) \cap \mathcal{F}(L^{\perp}) \times \mathcal{F}(L^{\perp}).$$

Therefore any trajectory of $L^{\perp}$ is one of $\bar{L}^{\perp}$ and vice-versa. As such, the claim holds. ∎

Having proven the previous, we can formulate the following result that summarizes the duality between H and H^{+}.

Theorem 4.6. Let H be a convex process such that $\mathrm{dom}(H) + \mathcal{R}_{-} = \mathbb{R}^n$ and $\mathcal{F}(H)$ is polyhedral. Then the following are equivalent:

1. H is stabilizable.
2. H is uniformly exponentially stabilizable.
3. H admits a weak Lyapunov function.
4. H^+ admits a strong Lyapunov function.
5. H^+ is uniformly exponentially strongly stable.
6. H^+ is strongly stable.
7. $L_-(H^+)$ is strongly stable and H^+ has no eigenvalues $\lambda \geqslant 1$.
8. L_+ is stabilizable and H^+ has no eigenvalues $\lambda \geqslant 1$.

Remark 4.9. By Lemma 3.23.(v) we know that if $\mathrm{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$, then by (4.21) we have $\mathcal{F}(H) = \mathrm{dom}(H^n)$. Recall in addition that $\mathcal{R}_-$ is finitely determined. This means that we can test for both conditions with algorithms that halt in a finite number of steps.

Proof. We will prove the theorem by proving that each statement implies the next one and finally that the last implies the first.

The first four implications are proven as, respectively, Lemma 4.1, Theorem 4.1, Theorem 4.4, and Lemma 4.2. Furthermore, (5) $\Rightarrow$ (6) is immediate.

(6) $\Rightarrow$ (7): Note that if H^+ is strongly stable, then H^+ clearly has no eigenvalues $\lambda \geqslant 1$. Furthermore, by definition any trajectory of H^+ is also one of $L_-(H^+)$. Therefore it is clear that $L_-(H^+)$ is also strongly stable.

(7) $\Rightarrow$ (8): Note that $L_-(H^+) = L_+(H)^\perp$ for any H. Applying Lemma 4.8, we get that L_+ is stabilizable.

What rests is proving that (8) $\Rightarrow$ (1). Recall from Remark 4.6 that any eigenvalue of H^+ is also an eigenvalue of H^-. This implies that H^- has no eigenvalues $\lambda \geqslant 1$. We can now use Corollary 4.2, which states that H is stabilizable if and only if L_+ is stabilizable and H^- has no eigenvalues $\lambda \geqslant 1$. This finalizes the proof. ∎

This theorem has a few noteworthy consequences, which we will discuss next. The first of these formalizes the duality between stabilizability and strong stability.

Corollary 4.5. Let H be a convex process such that $\mathrm{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$ and $\mathcal{F}(H)$ is polyhedral. Then H is stabilizable if and only if H^+ is strongly stable.

Note that this property does *not* hold for any H; we require a condition on the domain. As noted in Corollary 4.3, we know for instance that $H^+(0) \cap \mathcal{F}(H^+) = \{0\}$ is necessary for H^+ to be strongly stable. As a simple example, consider $\mathrm{gr}(H) := \{0\} \times \{0\}$. Clearly, H is stabilizable, but its dual, $\mathrm{gr}(H^+) = \mathbb{R} \times \mathbb{R}$, is not strongly stable.

Now, we note the following analogue of Lemma 4.1 with regard to uniform exponential strong stability.

Corollary 4.6. Let H be a convex process such that $\mathrm{dom}(H) + \mathcal{R}_- = \mathbb{R}^n$ and $\mathcal{F}(H)$ is polyhedral. Then H^+ is strongly stable if and only if H^+ is uniformly exponentially strongly stable.

One aspect of this result that is perhaps surprising is that, while we assume $\mathcal{F}(H)$ is poyhedral, we do *not* assume $\mathcal{F}(H^+)$ is polyhedral. This means that strong stability and uniform exponential strong stability are equivalent, without explicit assumptions on the feasible set.

4.6 CONCLUSIONS

In this chapter, we have provided a new definition for Lyapunov functions for difference inclusions of nonstrict convex processes. As shown in a few examples, this definition better captures the stabilizability properties of these systems. Indeed, we prove that a convex process is uniformly exponentially stable if and only if a there exists a weak Lyapunov function within our framework. Building on this definition, we have shown that under certain conditions, a weak Lyapunov function for a convex process can naturally be transformed to a strong Lyapunov function for its dual. In addition we reveal conditions under which a strong Lyapunov function can be transformed to a weak Lyapunov function for another associated convex process. These results generalize known results and the conditions required on the domain are, in some sense, close to being necessary. Lastly, we have combined this result with some earlier duality results. This proved that under weak assumptions, stabilizability and strong stability are dual notions.

Part II

INFORMATIVITY

5 DATA INFORMATIVITY

In this part of the thesis, we shift our focus to problems involving data. Instead of performing analysis on a given model, we begin with measured data. If the data is such that we can find a unique model, we can simply find this model and use a suitable model-based method. However, it may be the case that there is no unique model that is consistent with the data, and yet we still want to test for certain properties. This, in essence, is what we will investigate in the following chapter.

5.1 INTRODUCTION

One of the main paradigms in the field of systems and control is that of *model-based* control. Indeed, many control design techniques rely on a system model, represented by e.g. a state-space system or transfer function. In practice, system models are rarely known a priori and have to be identified from measured data using system identification methods such as prediction error [108] or subspace identification [167]. As a consequence, the use of model-based control techniques inherently leads to a two-step control procedure consisting of system identification followed by control design.

In contrast, *data-driven* control aims to bypass this two-step procedure by constructing controllers directly from data, without (explicitly) identifying a system model. This direct approach is not only attractive from a conceptual point of view but can also be useful in situations where system identification is difficult or even impossible because the data do not give sufficient information.

The first contribution to data-driven control is often attributed to Ziegler and Nichols for their work on tuning PID controllers [189]. Adaptive control [9], iterative feedback tuning [80, 81] and unfalsified control [147] can also be regarded as classical data-driven control techniques. More recently, the problem of finding optimal controllers from data has received considerable attention [1, 2, 16, 35, 58, 63, 71, 111, 121, 129, 154, 158]. The proposed solutions to this problem are quite varied, ranging from the use of batch-form Riccati equations [158] to approaches that apply reinforcement learning [35]. Additional noteworthy data-driven control problems include predictive control [41,57,148], model reference control [39,60] and (intelligent) PID control [59,95]. For more references and classifications of data-driven control techniques, we refer to the survey [82].

In addition to control problems, also *analysis* problems have been studied within a data-based framework. The authors of [130] analyze the stability of an input/output system using time series data. The papers [106,125,178,188] deal with data-based controllability and observability analysis. Moreover, the problem of verifying dissipativity on the basis of measured system trajectories has been studied in [22,113,143,144].

A result that is becoming increasingly popular in the study of data-driven problems is the so-called *fundamental lemma* by Willems and coworkers [182]. This result roughly states that all possible trajectories of a linear time-invariant system can be obtained from any given trajectory whose input component is persistently exciting. The fundamental lemma has clear implications for system identification. Indeed, it provides criteria under which the data are sufficiently informative to uniquely identify the system model within a given model class. In addition, the result has also been applied to data-driven control problems. The idea is that control laws can be obtained directly from data, with the underlying mechanism that the system is represented implicitly by the so-called Hankel matrix of a measured trajectory. This framework has led to several interesting control strategies, first in a behavioral setting [111,112,114], and more recently in the context of state-space systems [22,23,41,42,83,143].

The above approaches all use persistently exciting data in the control design, meaning that one could (hypothetically) identify the system model from the same data. An intriguing question is therefore the following: is it possible to obtain a controller from data that are *not* informative enough to uniquely identify the system? An affirmative answer would be remarkable, since it would highlight situations in which data-driven control is more powerful than the combination of system identification and model-based control. On the other hand, a negative answer would also be significant, as it would give a theoretic justification for the use of persistently exciting data for data-driven analysis and control.

To address the above question, this chapter introduces a general framework to study data informativity problems for data-driven analysis and control. Specifically, our contributions are the following:

1. Inspired by the concept of data informativity in system identification [66,67,108], we introduce a general notion of informativity for data-driven analysis and control.

2. We study the data-driven analysis of several system theoretic properties like stability, stabilizability and controllability. For each of these problems, we provide necessary and sufficient conditions under which the data are informative for this property, i.e., conditions required to ascertain the system's property from data.

3. We study data-driven control problems such as stabilization by state feedback, stabilization by dynamic measurement feedback, deadbeat control and linear quadratic regulation. In each of the cases, we give conditions under which the data are informative for controller design.

4. For each of the studied control problems, we develop methods to compute a controller from data, assuming that the informativity conditions are satisfied.

Our work has multiple noteworthy implications. First of all, we show that for problems like stabilization by state feedback, the corresponding informativity conditions on the data are *weaker* than those for system identification. This implies that a stabilizing feedback can be obtained from data that are not sufficiently informative to uniquely identify the system.

Moreover, for problems such as linear quadratic regulation (LQR), we show that the informativity conditions are essentially the same as for system identification. Therefore, our results provide a theoretic justification for imposing the strong persistency of excitation conditions in prior work on the LQR problem, such as [111] and [42].

The chapter is organized as follows. In Section 5.2 we introduce the problem at a conceptual level. Subsequently, in Section 5.3 we provide data informativity conditions for controllability and stabilizability. Section 5.4 deals with data-driven control problems with input/state data. Next, Section 5.5 discusses control problems where ouput data plays a role. Finally, Section 5.6 contains our conclusions and suggestions for future work.

5.2 PROBLEM FORMULATION

In this section we will first introduce the *informativity framework* for data-driven analysis and control in a fairly abstract manner.

Let Σ be a model class, i.e. a given set of systems containing the 'true' system denoted by $\mathcal{S}$. We assume that the 'true' system $\mathcal{S}$ is not known but that we have access to a set of data, $\mathcal{D}$, which are generated by this system. In this chapter we are interested in assessing system-theoretic properties of $\mathcal{S}$ and designing control laws for it from the data $\mathcal{D}$.

Given the data $\mathcal{D}$, we define $\Sigma_{\mathcal{D}} \subseteq \Sigma$ to be the set of all systems that are consistent with the data $\mathcal{D}$, i.e. that could also have generated these data.

We first focus on data-driven analysis. Let $\mathcal{P}$ be a system-theoretic property. We will denote the set of all systems within Σ having this property by $\Sigma_{\mathcal{P}}$.

Now suppose we are interested in the question whether our 'true' system $\mathcal{S}$ has the property $\mathcal{P}$. As the only information we have to base our answer on are the data $\mathcal{D}$ obtained from the system, we can only conclude that the 'true' system has property $\mathcal{P}$ if *all* systems consistent with the data $\mathcal{D}$ have the property $\mathcal{P}$. This leads to the following definition:

Definition 5.1 (Informativity)**.** We say that the data $\mathcal{D}$ are *informative* for property $\mathcal{P}$ if $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}}$.

Next, we illustrate the above abstract setup by an example.

Example 5.1. For given n and m, let Σ be the set of all discrete-time linear input/state systems of the form

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

where $\mathbf{x}$ is the n-dimensional state and $\mathbf{u}$ is the m-dimensional input. Let the 'true' system $\mathcal{S}$ be represented by the matrices (A_s, B_s).

An example of a data set $\mathcal{D}$ arises when considering data-driven problems on the basis of input and state measurements. Suppose that we collect input/state data on q time intervals $\{0, 1, \dots, T_i\}$ for $i = 1, 2, \dots, q$. Let

$$U_-^i := \begin{bmatrix} u^i(0) & u^i(1) & \cdots & u^i(T_i - 1) \end{bmatrix}, \tag{5.1a}$$

$$X^i := \begin{bmatrix} x^i(0) & x^i(1) & \cdots & x^i(T_i) \end{bmatrix} \tag{5.1b}$$

denote the input and state data on the i-th interval. By defining

$$X_-^i := \begin{bmatrix} x^i(0) & x^i(1) & \cdots & x^i(T_i - 1) \end{bmatrix}, \tag{5.2a}$$

$$X_+^i := \begin{bmatrix} x^i(1) & x^i(2) & \cdots & x^i(T_i) \end{bmatrix}, \tag{5.2b}$$

we clearly have $X_+^i = A_s X_-^i + B_s U_-^i$ for each i because the 'true' system is assumed to generate the data. Now, introduce the notation

$$U_- := \begin{bmatrix} U_-^1 & \cdots & U_-^q \end{bmatrix}, \quad X := \begin{bmatrix} X^1 & \cdots & X^q \end{bmatrix}, \tag{5.3a}$$

$$X_- := \begin{bmatrix} X_-^1 & \cdots & X_-^q \end{bmatrix}, \quad X_+ := \begin{bmatrix} X_+^1 & \cdots & X_+^q \end{bmatrix}. \tag{5.3b}$$

We then define the data as $\mathcal{D} := (U_-, X)$. In this case, the set $\Sigma_{\mathcal{D}}$ is equal to $\Sigma_{(U_-,X)}$ defined by

$$\Sigma_{(U_-,X)} := \left\{ (A, B) \mid X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{5.4}$$

Clearly, we have $(A_s, B_s) \in \Sigma_{\mathcal{D}}$.

Suppose that we are interested in the system-theoretic property $\mathcal{P}$ of *stabilizability*. The corresponding set $\Sigma_{\mathcal{P}}$ is then equal to Σ_{stab} defined by

$$\Sigma_{\text{stab}} := \{(A, B) \mid (A, B) \text{ is stabilizable}\}.$$

Then, the data (U_-, X) are informative for stabilizability if $\Sigma_{(U_-,X)} \subseteq \Sigma_{\text{stab}}$. That is, if all systems consistent with the input/state measurements are stabilizable.

In general, if the 'true' system $\mathcal{S}$ can be uniquely determined from the data $\mathcal{D}$, that is $\Sigma_{\mathcal{D}} = \{\mathcal{S}\}$ and $\mathcal{S}$ has the property $\mathcal{P}$, then it is evident that the data $\mathcal{D}$ are informative for $\mathcal{P}$. However, the converse may not be true: $\Sigma_{\mathcal{D}}$ might contain many systems, all of which have property $\mathcal{P}$. In this chapter, we are interested in necessary *and* sufficient conditions for informativity of the data. Such conditions reveal the minimal amount of information required to assess the property $\mathcal{P}$. A natural problem statement is therefore the following:

Problem 5.1 (Informativity problem)**.** Provide necessary and sufficient conditions on $\mathcal{D}$ under which the data are informative for property $\mathcal{P}$.

The above gives us a general framework to deal with data-driven analysis problems. Such analysis problems will be the main focus of Section 5.3.

This chapter also deals with data-driven control problems. The objective in such problems is the data-based design of controllers such that the closed loop system, obtained from the interconnection of the 'true' system $\mathcal{S}$ and the controller, has a specified property.

As for the analysis problem, we have only the information from the data to base our design on. Therefore, we can only guarantee our control objective if the designed controller imposes the specified property when interconnected with *any* system from the set $\Sigma_{\mathcal{D}}$.

For the framework to allow for data-driven control problems, we will consider a system-theoretic property $\mathcal{P}(\mathcal{K})$ that depends on a given controller $\mathcal{K}$. For properties such as these, we have the following variant of informativity:

Definition 5.2 (Informativity for control)**.** We say that the data $\mathcal{D}$ are *informative* for the property $\mathcal{P}(\cdot)$ if there exists a controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}(\mathcal{K})}$.

Example 5.2. For systems and data like in Example 5.1, we can take the controller $\mathcal{K} = K \in \mathbb{R}^{m \times n}$ and the property $\mathcal{P}(\mathcal{K})$: 'interconnection with the state feedback K yields a stable closed loop system'. The corresponding set of systems $\Sigma_{\mathcal{P}(\mathcal{K})}$ is equal to Σ_K defined by

$$\Sigma_K = \{(A, B) \mid A + BK \text{ is stable}^1\}.$$

The first step in any data-driven control problem is to determine whether it is possible to obtain a suitable controller from given data. This leads to the following informativity problem:

Problem 5.2 (Informativity problem for control)**.** Provide necessary and sufficient conditions on $\mathcal{D}$ under which there exists a controller $\mathcal{K}$ such that the data are informative for property $\mathcal{P}(\mathcal{K})$.

The second step of data-driven control involves the design of a suitable controller. In terms of our framework, this can be stated as:

Problem 5.3 (Control design problem)**.** Under the assumption that the data $\mathcal{D}$ are informative for property $\mathcal{P}(\cdot)$, find a controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}(\mathcal{K})}$.

As stated in the introduction, we will highlight the strength of this framework by solving multiple problems. We stress that throughout the chapter it is assumed that the data are *given* and are *not corrupted by noise*.

5.3 DATA-DRIVEN ANALYSIS

In this section, we will study data-driven analysis of controllability and stabilizability given input and state measurements. As in Example 5.1, consider the discrete-time linear system

$$\mathbf{x}(t+1) = A_s \mathbf{x}(t) + B_s \mathbf{u}(t). \tag{5.5}$$

We will consider data consisting of input and state measurements. We define the matrices U_- and X as in (5.3a) and define X_- and X_+ as in (5.3b). The set of all systems compatible with these data was introduced in (5.4). In order to stress that we deal with input/state data, we rename it here as

$$\Sigma_{i/s} := \left\{ (A, B) \mid X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{5.6}$$

1 We say that a matrix is *stable* if all its eigenvalues are contained in the open unit disk.

Note that the defining equation of (5.6) is a system of linear equations in the unknowns A and B. The solution space of the corresponding homogeneous equations is denoted by $\Sigma^0_{i/s}$ and is equal to

$$\Sigma^0_{i/s} := \left\{ (A_0, B_0) \mid 0 = \begin{bmatrix} A_0 & B_0 \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{5.7}$$

We consider the problem of data-driven analysis for systems of the form (5.5). If (A_s, B_s) is the only system that explains the data, data-driven analysis could be performed by first identifying this system and then analyzing its properties. It is therefore of interest to know under which conditions there is only one system that explains the data.

Definition 5.3. We say that the data (U_-, X) are *informative for system identification* if $\Sigma_{i/s} = \{(A_s, B_s)\}$.

It is straightforward to derive the following result:

Proposition 5.1. The data (U_-, X) are informative for system identification if and only if

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m. \tag{5.8}$$

Furthermore, if (5.8) holds, there exists a right inverse[2] $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ such that

$$\begin{bmatrix} X_- \\ U_- \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{5.9}$$

and for any such right inverse $A_s = X_+ V_1$ and $B_s = X_+ V_2$.

As we will show in this section, the condition (5.8) is not necessary for data-driven analysis in general. We now proceed by studying data-driven analysis of controllability and stabilizability. Recall the Hautus test [165, Theorem 3.13] for controllability: a system (A, B) is controllable if and only if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \tag{5.10}$$

for all $\lambda \in \mathbb{C}$. For stabilizability, we require that (5.10) holds for all λ outside the open unit disc.

Now, we introduce the following sets of systems:

$$\Sigma_{\text{cont}} := \{(A, B) \mid (A, B) \text{ is controllable}\}$$
$$\Sigma_{\text{stab}} := \{(A, B) \mid (A, B) \text{ is stabilizable}\}.$$

2 Note that $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ is not unique whenever $T > n + m$.

Using Definition 5.1, we obtain the notions of *informativity for controllability* and *stabilizability*. To be precise:

Definition 5.4. We say that the data (U_-, X) are *informative for controllability* if $\Sigma_{i/s} \subseteq \Sigma_{\text{cont}}$ and *informative for stabilizability* if $\Sigma_{i/s} \subseteq \Sigma_{\text{stab}}$.

In the following theorem, we give necessary and sufficient conditions for the above notions of informativity. The result is remarkable as only data matrices are used to assess controllability and stabilizability.

Theorem 5.1 (Data-driven Hautus tests)**.** The data (U_-, X) are informative for controllability if and only if

$$\operatorname{rank}(X_+ - \lambda X_-) = n \quad \forall \lambda \in \mathbb{C}. \tag{5.11}$$

Similarly, the data (U_-, X) are informative for stabilizability if and only if

$$\operatorname{rank}(X_+ - \lambda X_-) = n \quad \forall \lambda \in \mathbb{C} \text{ with } |\lambda| \geqslant 1. \tag{5.12}$$

Before proving the theorem, we will discuss some of its implications. We begin with computational issues.

Remark 5.1. Similar to the classical Hautus test, (5.11) and (5.12) can be verified by checking the rank for finitely many complex numbers λ. Indeed, (5.11) is equivalent to $\operatorname{rank}(X_+) = n$ and

$$\operatorname{rank}(X_+ - \lambda X_-) = n$$

for all $\lambda \neq 0$ with $\lambda^{-1} \in \sigma(X_- X_+^\dagger)$, where $X_+^\dagger$ is any right inverse of X_+. Here, $\sigma(M)$ denotes the spectrum, i.e. set of eigenvalues of the matrix M. Similarly, (5.12) is equivalent to $\operatorname{rank}(X_+ - X_-) = n$ and

$$\operatorname{rank}(X_+ - \lambda X_-) = n$$

for all $\lambda \neq 1$ with $(\lambda - 1)^{-1} \in \sigma(X_-(X_+ - X_-)^\dagger)$, where $(X_+ - X_-)^\dagger$ is any right inverse of $X_+ - X_-$.

A noteworthy point to mention is that there are situations in which we can conclude controllability/stabilizability from the data without being able to identify the 'true' system uniquely, as illustrated next.

Example 5.3. Suppose that $n = 2$, $m = 1$, $q = 1$, $T_1 = 2$ and we obtain the data

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } U_- = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

This implies that

$$X_+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } X_- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly, by Theorem 5.1 we see that these data are informative for controllability, as

$$\operatorname{rank} \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} = 2 \quad \forall \lambda \in \mathbb{C}.$$

As therefore all systems explaining the data are controllable, we conclude that the 'true' system is controllable. It is worthwhile to note that the data are not informative for system identification, as

$$\Sigma_{i/s} = \left\{ \left(\begin{bmatrix} 0 & a_1 \\ 1 & a_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \mid a_1, a_2 \in \mathbb{R} \right\}. \tag{5.13}$$

Proof of Theorem 5.1. We will only prove the characterization of informativity for controllability. The proof for stabilizability uses very similar arguments, and is hence omitted.

Note that the condition (5.11) is equivalent to the implication:

$$z \in \mathbb{C}^n, \lambda \in \mathbb{C} \text{ and } z^* X_+ = \lambda z^* X_- \implies z = 0. \tag{5.14}$$

Suppose that the implication (5.14) holds. Let $(A, B) \in \Sigma_{i/s}$ and suppose that $z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0$. We want to prove that $z = 0$. Note that $z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0$ implies that

$$z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = 0,$$

or equivalently $z^* X_+ = \lambda z^* X_-$. This means that $z = 0$ by (5.14). We conclude that (A, B) is controllable, i.e., (U_-, X) are informative for controllability.

Conversely, suppose that (U_-, X) are informative for controllability. Let $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ be such that $z^* X_+ = \lambda z^* X_-$. This implies that for all $(A, B) \in \Sigma_{i/s}$, we have $z^* \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = \lambda z^* X_-$. In other words,

$$z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = 0. \tag{5.15}$$

We now distinguish two cases, namely the case that λ is real, and the case that λ is complex. First suppose that λ is real. Without loss of generality, z is real. We want to prove that $z = 0$. Suppose on the contrary that $z \neq 0$ and $z^\top z = 1$. We define the (real) matrices

$$\bar{A} := A - zz^\top (A - \lambda I) \text{ and } \bar{B} := B - zz^\top B.$$

In view of (5.15), we find that $(\bar{A}, \bar{B}) \in \Sigma_{i/s}$. Moreover,

$$z^\top \bar{A} = z^\top A - z^\top (A - \lambda I) = \lambda z^\top$$

and

$$z^\top \bar{B} = z^\top B - z^\top B = 0.$$

This means that

$$z^\top \begin{bmatrix} \bar{A} - \lambda I & \bar{B} \end{bmatrix} = 0.$$

However, this is a contradiction as $(\bar{A}, \bar{B})$ is controllable by the hypothesis that (U_-, X) are informative for controllability. We conclude that $z = 0$ which shows that (5.14) holds for the case that λ is real.

Secondly, consider the case that λ is complex. We write z as $z = p + iq$, where $p, q \in \mathbb{R}^n$ and i denotes the imaginary unit. If p and q are linearly dependent, then $p = \alpha q$ or $q = \beta p$ for $\alpha, \beta \in \mathbb{R}$. If $p = \alpha q$ then substitution of $z = (\alpha + i)q$ into $z^* X_+ = \lambda z^* X_-$ yields

$$(\alpha - i) q^\top X_+ = \lambda (\alpha - i) q^\top X_-,$$

that is, $q^\top X_+ = \lambda q^\top X_-$. As $q^\top X_+$ is real and λ is complex, we must have $q^\top X_+ = 0$ and $q^\top X_- = 0$. This means that $z^* X_+ = z^* X_- = 0$, hence $z^* X_+ = \mu z^* X_-$ for any real μ, which means that $z = 0$ by case 1. Using the same arguments, we can show that $z = 0$ if $q = \beta p$.

It suffices to prove now that p and q are linearly dependent. Suppose on the contrary that p and q are linearly independent. Since λ is complex, $n \geqslant 2$. Therefore, by linear independence of p and q there exist $\eta, \zeta \in \mathbb{R}^n$ such that

$$\begin{bmatrix} p^\top \\ q^\top \end{bmatrix} \begin{bmatrix} \eta & \zeta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We now define the real matrices $\bar{A}$ and $\bar{B}$ as

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} := \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} \eta & \zeta \end{bmatrix} \begin{bmatrix} \operatorname{Re}\left(z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) \\ \operatorname{Im}\left(z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) \end{bmatrix}.$$

By (5.15) we have $(\bar{A}, \bar{B}) \in \Sigma_{i/s}$. Next, we compute

$$\begin{aligned} z^* \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} &= z^* \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} \operatorname{Re}\left(z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) \\ \operatorname{Im}\left(z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) \end{bmatrix} \\ &= z^* \begin{bmatrix} A & B \end{bmatrix} - z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} \\ &= z^* \begin{bmatrix} \lambda I & 0 \end{bmatrix}. \end{aligned}$$

This implies that $z^* \begin{bmatrix} \bar{A} - \lambda I & \bar{B} \end{bmatrix} = 0$. Using the fact that $(\bar{A}, \bar{B})$ is controllable, we conclude that $z = 0$. This is a contradiction with the fact that p and q are linearly independent. Thus p and q are linearly dependent and therefore implication (5.14) holds. This proves the theorem. ■

In addition to controllability and stabilizability, we can also study the *stability* of an autonomous system of the form

$$x(t+1) = A_s x(t). \tag{5.16}$$

To this end, let X denote the matrix of state measurements obtained from (5.16), as defined in (5.3a). The set of all autonomous systems compatible with these data is

$$\Sigma_s := \{A \mid X_+ = AX_-\}.$$

Then, we say the data X are *informative for stability* if any matrix $A \in \Sigma_s$ is stable, i.e. Schur. Using Theorem 5.1 we can show that stability can only be concluded if the 'true' system can be uniquely identified.

Corollary 5.1. The data X are informative for stability if and only if X_- has full row rank and $X_+X_-^\dagger$ is stable for any right inverse $X_-^\dagger$, equivalently $\Sigma_s = \{A_s\}$ and $A_s = X_+X_-^\dagger$ is stable.

Proof. Since the 'if' part is evident, we only prove the 'only if' part. By taking $B = 0$, it follows from Theorem 5.1 that the data X are informative for stability if and only if

$$\operatorname{rank}(X_+ - \lambda X_-) = n \quad \forall \lambda \in \mathbb{C} \text{ with } |\lambda| \geqslant 1. \tag{5.17}$$

Let z be such that $z^\top X_- = 0$. Take $A \in \Sigma_s$ and $\lambda > 1$ such that λ is not an eigenvalue of A. Note that

$$z^\top (A - \lambda I)^{-1} (X_+ - \lambda X_-) = z^\top X_- = 0.$$

Since $\operatorname{rank}(X_+ - \lambda X_-) = n$, we may conclude that $z = 0$. Hence, X_- has full row rank. Therefore, $\Sigma_s = \{A_s\}$ where $A_s = X_+X_-^\dagger$ for any right inverse $X_-^\dagger$ and A_s is stable. ■

Note that there is a subtle but important difference between the characterizations (5.12) and (5.17). For the first the data X are assumed to be generated by a system with inputs, whereas the data for the second characterization are generated by an autonomous system.

5.4 CONTROL USING INPUT AND STATE DATA

In this section we will consider various state feedback control problems on the basis of input/state measurements. First, we will consider the problem of data-driven stabilization by static state feedback, where the data consist of input and state measurements. As described in the problem statement we will look at the informativity and design problems separately as special cases of Problem 5.2 and Problem 5.3. We will then use similar techniques to obtain a result for *deadbeat control*.

After this, we will shift towards the linear quadratic regulator problem, where we wish to find a stabilizing feedback that additionally minimizes a specified quadratic cost.

5.4.1 Stabilization by state feedback

In what follows, we will consider the problem of finding a stabilizing controller for the system (5.5), using only the data (U_-, X). To this end, we define the set of systems (A, B) that are stabilized by a given K:

$$\Sigma_K := \{(A, B) \mid A + BK \text{ is stable}\}.$$

In addition, recall the set $\Sigma_{i/s}$ as defined in (5.6) and $\Sigma^0_{i/s}$ from (5.7). In line with Definition 5.2 we obtain the following notion of informativity for stabilization by state feedback.

Definition 5.5. We say that the data (U_-, X) are *informative for stabilization by state feedback* if there exists a feedback gain K such that $\Sigma_{i/s} \subseteq \Sigma_K$.

Remark 5.2. At this point, one may wonder about the relation between informativity for stabilizability (as in Section 5.3) and informativity for stabilization. It is clear that (U_-, X) are informative for stabilizability if (U_-, X) are informative for stabilization by state feedback. However, the reverse statement does not hold in general. This is due to the fact that all systems (A, B) in $\Sigma_{i/s}$ may be stabilizable, but there may not be a *common* feedback gain K such that $A + BK$ is stable for all of these systems. Note that the existence of a common stabilizing K for all systems in $\Sigma_{i/s}$ is essential, since there is no way to distinguish between the systems in $\Sigma_{i/s}$ based on the given data (U_-, X).

The following example further illustrates the difference between informativity for stabilizability and informativity for stabilization.

Example 5.4. Consider the scalar system

$$\mathbf{x}(t+1) = \mathbf{u}(t),$$

where $\mathbf{x}, \mathbf{u} \in \mathbb{R}$. Suppose that $q = 1$, $T_1 = 1$ and $x(0) = 0$, $u(0) = 1$ and $x(1) = 1$. This means that $U_- = \begin{bmatrix} 1 \end{bmatrix}$ and $X = \begin{bmatrix} 0 & 1 \end{bmatrix}$. It can be shown that $\Sigma_{i/s} = \{(a, 1) \mid a \in \mathbb{R}\}$. Clearly, all systems in $\Sigma_{i/s}$ are stabilizable, i.e., $\Sigma_{i/s} \subseteq \Sigma_{\text{stab}}$. Nonetheless, the data are not informative for *stabilization*. This is because the systems $(-1, 1)$ and $(1, 1)$ in $\Sigma_{i/s}$ cannot be stabilized by the *same* controller of the form $u(t) = Kx(t)$. We conclude that informativity of the data for stabilizability does not imply informativity for stabilization by state feedback.

The notion of informativity for stabilization by state feedback is a specific example of informativity for control. As described in Problem 5.2, we will first find necessary and sufficient conditions for informativity for stabilization by state feedback. After this, we will design a corresponding controller, as described in Problem 5.3.

In order to be able to characterize informativity for stabilization, we first state the following lemma.

Lemma 5.1. Suppose that the data (U_-, X) are informative for stabilization by state feedback, and let K be a feedback gain such that $\Sigma_{i/s} \subseteq \Sigma_K$. Then $A_0 + B_0K = 0$ for all $(A_0, B_0) \in \Sigma^0_{i/s}$. Equivalently,

$$\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

Proof. We first prove that $A_0 + B_0K$ is *nilpotent* for all $(A_0, B_0) \in \Sigma^0_{i/s}$. By hypothesis, $A + BK$ is stable for all $(A, B) \in \Sigma_{i/s}$. Let $(A, B) \in \Sigma_{i/s}$ and $(A_0, B_0) \in \Sigma^0_{i/s}$ and define the matrices $F := A + BK$ and $F_0 := A_0 + B_0K$. Then, the matrix $F + \alpha F_0$ is stable for all $\alpha \geqslant 0$. By dividing by α, it follows that, for all $\alpha \geqslant 1$, the spectral radius of the matrix

$$M_\alpha := \frac{1}{\alpha} F + F_0$$

is smaller than $1/\alpha$. From the continuity of the spectral radius by taking the limit as α tends to infinity, we see that $F_0 = A_0 + B_0K$ is nilpotent for all $(A_0, B_0) \in \Sigma^0_{i/s}$. Note that we have

$$((A_0 + B_0K)^T A_0, (A_0 + B_0K)^T B_0) \in \Sigma^0_{i/s}$$

whenever $(A_0, B_0) \in \Sigma^0_{i/s}$. This means that $(A_0 + B_0K)^T(A_0 + B_0K)$ is nilpotent. Since the only symmetric nilpotent matrix is the zero matrix, we see that $A_0 + B_0K = 0$ for all $(A_0, B_0) \in \Sigma^0_{i/s}$. This is equivalent to

$$\ker \begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix} \subseteq \ker \begin{bmatrix} I & K^\top \end{bmatrix}$$

which is equivalent to $\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$. ∎

The previous lemma is instrumental in proving the following theorem that gives necessary and sufficient conditions for informativity for stabilization by state feedback.

Theorem 5.2. The data (U_-, X) are informative for stabilization by state feedback if and only if the matrix X_- has full row rank and there exists a right inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is stable.

Moreover, K is such that $\Sigma_{i/s} \subseteq \Sigma_K$ if and only if $K = U_-X_-^\dagger$, where $X_-^\dagger$ satisfies the above properties.

Proof. To prove the 'if' part of the first statement, suppose that X_- has full row rank and there exists a right inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is stable. We define $K := U_-X_-^\dagger$. Next, we see that

$$X_+X_-^\dagger = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} X_-^\dagger = A + BK, \tag{5.18}$$

for all $(A, B) \in \Sigma_{i/s}$. Therefore, $A + BK$ is stable for all $(A, B) \in \Sigma_{i/s}$, i.e., $\Sigma_{i/s} \subseteq \Sigma_K$. We conclude that the data (U_-, X) are informative for stabilization by state feedback, proving the 'if' part of the first statement. Since $K = U_-X_-^\dagger$ is such that $\Sigma_{i/s} \subseteq \Sigma_K$, we have also proven the 'if' part of the second statement as a byproduct.

Next, to prove the 'only if' part of the first statement, suppose that the data (U_-, X) are informative for stabilization by state feedback. Let K be such that $A + BK$ is stable for all $(A, B) \in \Sigma_{i/s}$. By Lemma 5.1 we know that

$$\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

This implies that X_- has full row rank and there exists a right inverse $X_-^\dagger$ such that

$$\begin{bmatrix} I \\ K \end{bmatrix} = \begin{bmatrix} X_- \\ U_- \end{bmatrix} X_-^\dagger. \tag{5.19}$$

By (5.18), we obtain $A + BK = X_+X_-^\dagger$, which shows that $X_+X_-^\dagger$ is stable. This proves the 'only if' part of the first statement. Finally, by (5.19), the stabilizing feedback gain K is indeed of the form $K = U_-X_-^\dagger$, which also proves the 'only if' part of the second statement. ∎

Theorem 5.2 gives a characterization of all data that are informative for stabilization by state feedback and provides a stabilizing controller. Nonetheless, the procedure to compute this controller might not be entirely satisfactory since it is not clear how to find a right inverse of X_- that makes $X_+X_-^\dagger$ stable. In general, X_- has many right inverses, and $X_+X_-^\dagger$ can be stable or unstable depending on the particular right inverse $X_-^\dagger$. To deal with this problem and to solve the design problem, we give a characterization of informativity for stabilization in terms of linear matrix inequalities (LMI's). The feasibility of such LMI's can be verified using standard methods.

Theorem 5.3. The data (U_-, X) are informative for stabilization by state feedback if and only if there exists a matrix $\Theta \in \mathbb{R}^{T\times n}$ satisfying

$$X_-\Theta = (X_-\Theta)^\top \quad \text{and} \quad \begin{bmatrix} X_-\Theta & X_+\Theta \\ \Theta^\top X_+^\top & X_-\Theta \end{bmatrix} > 0. \tag{5.20}$$

Moreover, K satisfies $\Sigma_{i/s} \subseteq \Sigma_K$ if and only if $K = U_-\Theta(X_-\Theta)^{-1}$ for some matrix Θ satisfying (5.20).

Remark 5.3. To the best of our knowledge, LMI conditions for data-driven stabilization were first studied in [42]. In fact, the linear matrix inequality (5.20) is the same as that of [42, Theorem 3]. However, an important difference is that the results in [42] assume that the input u is persistently exciting of sufficiently high order. In contrast, Theorem 5.3, as well as Theorem 5.2, do not require such conditions. The characterization (5.20) provides the minimal conditions on the data under which it is possible to obtain a stabilizing controller.

Example 5.5. Consider an unstable system of the form (5.5), where A_s and B_s are given by

$$A_s = \begin{bmatrix} 1.5 & 0 \\ 1 & 0.5 \end{bmatrix}, \quad B_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We collect data from this system on a single time interval from $t = 0$ until $t = 2$, which results in the data matrices

$$X = \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0 & 1 & 1 \end{bmatrix}, \quad U_- = \begin{bmatrix} -1 & -1 \end{bmatrix}.$$

Clearly, the matrix X_- is square and invertible, and it can be verified that

$$X_+X_-^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ 1 & 0.5 \end{bmatrix}$$

is stable, since its eigenvalues are $\frac{1}{2}(1 \pm \sqrt{2}i)$. We conclude by Theorem 5.2 that the data (U_-, X) are informative for stabilization by state feedback. The same conclusion can be drawn from Theorem 5.3 since

$$\Theta = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

solves (5.20). Next, we can conclude from either Theorem 5.2 or Theorem 5.3 that the stabilizing feedback gain in this example is unique, and given by $K = U_-X_-^{-1} = \begin{bmatrix} -1 & -0.5 \end{bmatrix}$. Finally, it is worth noting that the data are not informative for system identification. In fact, $(A, B) \in \Sigma_{i/s}$ if and only if

$$A = \begin{bmatrix} 1.5 + a_1 & 0.5a_1 \\ 1 + a_2 & 0.5 + 0.5a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 + a_1 \\ a_2 \end{bmatrix}$$

for some $a_1, a_2 \in \mathbb{R}$.

Proof of Theorem 5.3. To prove the 'if' part of the first statement, suppose that there exists a Θ satisfying (5.20). In particular, this implies that $X_-\Theta$ is symmetric positive definite. Therefore, X_- has full row rank. By taking a Schur complement and multiplying by -1, we obtain

$$X_+\Theta(X_-\Theta)^{-1}(X_-\Theta)(X_-\Theta)^{-1}\Theta^\top X_+^\top - X_-\Theta < 0.$$

Since $X_-\Theta$ is positive definite, this implies that $X_+\Theta(X_-\Theta)^{-1}$ is stable. In other words, there exists a right inverse $X_-^\dagger := \Theta(X_-\Theta)^{-1}$ of X_- such that $X_+X_-^\dagger$ is stable. By Theorem 5.2, we conclude that (U_-, X) are informative for stabilization by state feedback, proving the 'if' part of the first statement. Using Theorem 5.2 once more, we see that $K := U_-\Theta(X_-\Theta)^{-1}$ stabilizes all systems in $\Sigma_{i/s}$, which in turn proves the 'if' part of the second statement.

Subsequently, to prove the 'only if' part of the first statement, suppose that the data (U_-, X) are informative for stabilization by state feedback. Let K be any feedback gain such that $\Sigma_{i/s} \subseteq \Sigma_K$. By Theorem 5.2, X_- has full row rank and K is of the form $K = U_-X_-^\dagger$, where $X_-^\dagger$ is a right inverse of X_- such that $X_+X_-^\dagger$ is stable. The stability of $X_+X_-^\dagger$ implies the existence of a symmetric positive definite matrix P such that

$$(X_+X_-^\dagger)P(X_+X_-^\dagger)^\top - P < 0.$$

Next, we define $\Theta := X_-^\dagger P$ and note that

$$X_+\Theta P^{-1}(X_+\Theta)^\top - P < 0.$$

Via the Schur complement we conclude that

$$\begin{bmatrix} P & X_+\Theta \\ \Theta^\top X_+^\top & P \end{bmatrix} > 0.$$

Since $X_- X_-^\dagger = I$, we see that $P = X_-\Theta$, which proves the 'only if' part of the first statement. Finally, by definition of Θ, we have $X_-^\dagger = \Theta P^{-1} = \Theta(X_-\Theta)^{-1}$. Recall that $K = U_- X_-^\dagger$, which shows that K is of the form $K = U_-\Theta(X_-\Theta)^{-1}$ for Θ satisfying (5.20). This proves the 'only if' part of the second statement and hence the proof is complete. ■

In addition to the stabilizing controllers discussed in Theorems 5.2 and 5.3, we may also look for a controller of the form $\mathbf{u}(t) = K\mathbf{x}(t)$ that stabilizes the system in *finite time*. Such a controller is called a *deadbeat controller* and is characterized by the property that $(A_s + B_s K)^t x_0 = 0$ for all $t \geqslant n$ and all $x_0 \in \mathbb{R}^n$. Thus, K is a deadbeat controller if and only if $A_s + B_s K$ is nilpotent. Now, for a given matrix K define

$$\Sigma_K^{\text{nil}} := \{(A, B) \mid A + BK \text{ is nilpotent}\}.$$

Then, analogous to the definition of informativity for stabilization by state feedback, we have the following definition of informativity for deadbeat control.

Definition 5.6. We say that the data (U_-, X) are *informative for deadbeat control* if there exists a feedback gain K such that $\Sigma_{i/s} \subseteq \Sigma_K^{\text{nil}}$.

Similarly to Theorem 5.2, we obtain the following necessary and sufficient conditions for informativity for deadbeat control.

Theorem 5.4. The data (U_-, X) are informative for deadbeat control if and only if the matrix X_- has full row rank and there exists a right inverse $X_-^\dagger$ of X_- such that $X_+ X_-^\dagger$ is nilpotent.

Moreover, if this condition is satisfied then the feedback gain $K := U_- X_-^\dagger$ yields a deadbeat controller, that is, $\Sigma_{i/s} \subseteq \Sigma_K^{\text{nil}}$.

Remark 5.4. In order to compute a suitable right inverse $X_-^\dagger$ such that $X_+ X_-^\dagger$ is nilpotent, we can proceed as follows. Since X_- has full row rank, we have $T \geqslant n$. We now distinguish two cases: $T = n$ and $T > n$. In the former case,

X_- is nonsingular and hence $X_+X_-^{-1}$ is nilpotent. In the latter case, there exist matrices $F \in \mathbb{R}^{T\times n}$ and $G \in \mathbb{R}^{T\times(T-n)}$ such that $\begin{bmatrix} F & G \end{bmatrix}$ is nonsingular and $X_- \begin{bmatrix} F & G \end{bmatrix} = \begin{bmatrix} I_n & 0_{n\times(T-n)} \end{bmatrix}$. Note that $X_-^\dagger$ is a right inverse of X_- if and only if $X_-^\dagger = F + GH$ for some $H \in \mathbb{R}^{(T-n)\times n}$. Finding a right inverse $X_-^\dagger$ such that $X_+X_-^\dagger$ is nilpotent, therefore, amounts to finding H such that $X_+F + X_+GH$ is nilpotent, i.e. has only zero eigenvalues. Such a matrix H can be computed by invoking [165, Thm. 3.29 and Thm. 3.32] for the pair (X_+F, X_+G) and the stability domain $\mathbb{C}_g = \{0\}$.

5.4.2 Informativity for linear quadratic regulation

Consider the discrete-time linear system (5.5). Let $x_{x_0,u}(\cdot)$ be the state sequence of (5.5) resulting from the input $u(\cdot)$ and initial condition $x(0) = x_0$. We omit the subscript and simply write $x(\cdot)$ whenever the dependence on x_0 and u is clear from the context.

Associated to system (5.5), we define the quadratic cost functional

$$J(x_0, u) = \sum_{t=0}^{\infty} x^\top(t)Qx(t) + u^\top(t)Ru(t), \tag{5.21}$$

where $Q = Q^\top$ is positive semidefinite and $R = R^\top$ is positive definite. Then, the linear quadratic regulator (LQR) problem is the following:

Problem 5.4 (LQR)**.** Determine for every initial condition x_0 an input u^*, such that $\lim_{t\to\infty} x_{x_0,u^*}(t) = 0$, and the cost functional $J(x_0, u)$ is minimized under this constraint.

Such an input u^* is called optimal for the given x_0. Of course, an optimal input does not necessarily exist for all x_0. We say that the linear quadratic regulator problem is *solvable* for (A, B, Q, R) if for every x_0 there exists an input u^* such that

1. The cost $J(x_0, u^*)$ is finite.
2. The limit $\lim_{t\to\infty} x_{x_0,u^*}(t) = 0$.
3. The input u^* minimizes the cost functional, i.e.,

   $$J(x_0, u^*) \leqslant J(x_0, \bar{u})$$

   for all $\bar{u}$ such that $\lim_{t\to\infty} x_{x_0,\bar{u}}(t) = 0$.

In the sequel, we will require the notion of observable eigenvalues. Recall from e.g. [165, Section 3.5] that an eigenvalue λ of A is (Q, A)-observable if

$$\operatorname{rank}\begin{pmatrix} A - \lambda I \\ Q \end{pmatrix} = n.$$

The following theorem provides necessary and sufficient conditions for the solvability of the linear quadratic regulator problem for (A, B, Q, R). This theorem is the discrete-time analogue to the continuous-time case stated in [165, Theorem 10.18].

Theorem 5.5. Let $Q = Q^\top$ be positive semidefinite and $R = R^\top$ be positive definite. Then the following statements hold:

i If (A, B) is stabilizable, there exists a unique largest real symmetric solution P^+ to the discrete-time algebraic Riccati equation (DARE)

$$P = A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q, \tag{5.22}$$

in the sense that $P^+ \geqslant P$ for every real symmetric P satisfying (5.22). The matrix P^+ is positive semidefinite.

ii If, in addition to stabilizability of (A, B), every eigenvalue of A on the unit circle is (Q, A)-observable then for every x_0 a unique optimal input u^* exists. Furthermore, this input sequence is generated by the feedback law $\mathbf{u} = K\mathbf{x}$, where

$$K := -(R + B^\top P^+ B)^{-1} B^\top P^+ A. \tag{5.23}$$

Moreover, the matrix $A + BK$ is stable.

iii In fact, the linear quadratic regulator problem is solvable for (A, B, Q, R) if and only if (A, B) is stabilizable and every eigenvalue of A on the unit circle is (Q, A)-observable.

If the LQR problem is solvable for (A, B, Q, R), we say that K given by (5.23) is the optimal feedback gain for (A, B, Q, R).

Now, for any given K we define $\Sigma_K^{Q,R}$ as the set of all systems of the form (5.5) for which K is the optimal feedback gain corresponding to Q and R, that is,

$$\Sigma_K^{Q,R} := \{(A, B) \mid K \text{ is the optimal gain for } (A, B, Q, R)\}.$$

This gives rise to another notion of informativity in line with Definition 5.2. Again, let $\Sigma_{i/s}$ be given by (5.6).

Definition 5.7. Given matrices Q and R, we say that the data (U_-, X) are *informative for linear quadratic regulation* if there exists K such that $\Sigma_{i/s} \subseteq \Sigma_K^{Q,R}$.

In order to provide necessary and sufficient conditions for the corresponding informativity problem, we need the following auxiliary lemma.

Lemma 5.2. Let $Q = Q^\top$ be positive semidefinite and $R = R^\top$ be positive definite. Suppose the data (U_-, X) are informative for linear quadratic regulation. Let K be such that $\Sigma_{i/s} \subseteq \Sigma_K^{Q,R}$. Then, there exist a square matrix M and a symmetric positive semidefinite matrix P^+ such that for all $(A, B) \in \Sigma_{i/s}$

$$M = A + BK, \tag{5.24}$$

$$P^+ = A^\top P^+ A - A^\top P^+ B(R + B^\top P^+ B)^{-1} B^\top P^+ A + Q, \tag{5.25}$$

$$P^+ - M^\top P^+ M = K^\top RK + Q, \tag{5.26}$$

$$K = -(R + B^\top P^+ B)^{-1} B^\top P^+ A. \tag{5.27}$$

Proof. Since the data (U_-, X) are informative for linear quadratic regulation, $A + BK$ is stable for every $(A, B) \in \Sigma_{i/s}$. By Lemma 5.1, this implies that $A_0 + B_0 K = 0$ for all $(A_0, B_0) \in \Sigma_{i/s}^0$. Thus, there exists M such that $M = A + BK$ for all $(A, B) \in \Sigma_{i/s}$. For the rest, note that Theorem 5.5 implies that for every $(A, B) \in \Sigma_{i/s}$ there exists $P^+_{(A,B)}$ satisfying the DARE

$$P^+_{(A,B)} = A^\top P^+_{(A,B)} A - A^\top P^+_{(A,B)} B(R + B^\top P^+_{(A,B)} B)^{-1} B^\top P^+_{(A,B)} A + Q \tag{5.28}$$

such that

$$K = -(R + B^\top P^+_{(A,B)} B)^{-1} B^\top P^+_{(A,B)} A. \tag{5.29}$$

It is important to note that, although K is independent of the choice of (A, B), the matrix $P^+_{(A,B)}$ might depend on (A, B). We will, however, show that also $P^+_{(A,B)}$ is independent of the choice of (A, B).

By rewriting (5.28), we see that

$$P^+_{(A,B)} - M^\top P^+_{(A,B)} M = K^\top RK + Q. \tag{5.30}$$

Since M is stable, $P^+_{(A,B)}$ is the unique solution to the discrete-time Lyapunov equation (5.30), see e.g. [157, Section 6]. Moreover, since M and K do not depend on the choice of $(A, B) \in \Sigma_{i/s}$, it indeed follows that $P^+_{(A,B)}$ does not depend on (A, B). It follows from (5.28)–(5.30) that $P^+ := P^+_{(A,B)}$ satisfies (5.25)–(5.27). ∎

The following theorem solves the informativity problem for linear quadratic regulation.

Theorem 5.6. Let $Q = Q^\top$ be positive semidefinite and $R = R^\top$ be positive definite. Then, the data (U_-, X) are informative for linear quadratic regulation if and only if at least one of the following two conditions hold:

i The data (U_-, X) are informative for system identification, that is, $\Sigma_{i/s} = \{(A_s, B_s)\}$, and the linear quadratic regulator problem is solvable for the matrices (A_s, B_s, Q, R). In this case, the optimal feedback gain K is of the form (5.23) where P^+ is the largest real symmetric solution to (5.22).

ii For all $(A, B) \in \Sigma_{i/s}$ we have $A = A_s$. Moreover, A_s is stable, $QA_s = 0$, and the optimal feedback gain is given by $K = 0$.

Remark 5.5. Condition (ii) of Theorem 5.6 is a pathological case in which A is stable and $QA = 0$ for all matrices A that are compatible with the data. Since $x(t) \in \operatorname{im} A$ for all $t > 0$, we have $Qx(t) = 0$ for all $t > 0$ if the input function is chosen as $u = 0$. Additionally, since A is stable, this shows that the optimal input is equal to $u^* = 0$. If we set aside condition (ii), the implication of Theorem 5.6 is the following: if the data are informative for linear quadratic regulation they are also informative for system identification.

At first sight, this might seem like a negative result in the sense that data-driven LQR is only possible with data that are also informative enough to uniquely identify the system. However, at the same time, Theorem 5.6 can be viewed as a positive result in the sense that it provides fundamental justification for the data conditions imposed in e.g. [42]. Indeed, in [42] the data-driven infinite horizon LQR problem[3] is solved using input/state data under the assumption that the input is persistently exciting of sufficiently high order. Under the latter assumption, the input/state data are informative for system identification, i.e., the matrices A_s and B_s can be uniquely determined from data. Theorem 5.6 justifies such a strong assumption on the richness of data in data-driven linear quadratic regulation.

The data-driven *finite* horizon LQR problem was solved under a persistency of excitation assumption in [111]. Our results suggest that also in this case informativity for system identification is necessary for data-driven LQR, although further analysis is required to prove this claim.

Proof of Theorem 5.6. We first prove the 'if' part. Sufficiency of the condition (i) readily follows from Theorem 5.5. To prove the sufficiency of the condition (ii), assume that the matrix A is stable and $QA = 0$ for all $(A, B) \in \Sigma_{i/s}$. By the discussion following Theorem 5.6, this implies that $u^* = 0$ for all $(A, B) \in \Sigma_{i/s}$.

3 Note that the authors of [42] formulate this problem as the minimization of the H_2-norm of a certain transfer matrix.

Hence, for $K = 0$ we have $\Sigma_{i/s} \subseteq \Sigma_K^{Q,R}$, i.e., the data are informative for linear quadratic regulation.

To prove the 'only if' part, suppose that the data (U_-, X) are informative for linear quadratic regulation. From Lemma 5.2, we know that there exist M and P^+ satisfying (5.24)–(5.27) for all $(A, B) \in \Sigma_{i/s}$. By substituting (5.27) into (5.25) and using (5.24), we obtain

$$A^\top P^+ M = P^+ - Q. \tag{5.31}$$

In addition, it follows from (5.27) that $-(R + B^\top P^+ B)K = B^\top P^+ A$. By using (5.24), we have

$$B^\top P^+ M = -RK. \tag{5.32}$$

Since (5.31) and (5.32) hold for all $(A, B) \in \Sigma_{i/s}$, we have that

$$\begin{bmatrix} A_0^\top \\ B_0^\top \end{bmatrix} P^+ M = 0$$

for all $(A_0, B_0) \in \Sigma_{i/s}^0$. Note that $(FA_0, FB_0) \in \Sigma_{i/s}^0$ for all $F \in \mathbb{R}^{n\times n}$ whenever $(A_0, B_0) \in \Sigma_{i/s}^0$. This means that

$$\begin{bmatrix} A_0^\top \\ B_0^\top \end{bmatrix} F^\top P^+ M = 0$$

for all $F \in \mathbb{R}^{n\times n}$. Therefore, either $\begin{bmatrix} A_0 & B_0 \end{bmatrix} = 0$ for all $(A_0, B_0) \in \Sigma_{i/s}^0$ or $P^+ M = 0$. The former is equivalent to $\Sigma_{i/s}^0 = \{0\}$. In this case, we see that the data (U_-, X) are informative for system identification, equivalently $\Sigma_{i/s} = \{(A_s, B_s)\}$, and the LQR problem is solvable for (A_s, B_s, Q, R). Therefore, condition (i) holds. On the other hand, if $P^+ M = 0$ then we have

$$\begin{aligned} 0 = P^+ M &= P^+(A + BK) \\ &= P^+\big(A - B(R + B^\top P^+ B)^{-1} B^\top P^+ A\big) \\ &= \big(I - P^+ B(R + B^\top P^+ B)^{-1} B^\top\big) P^+ A. \end{aligned}$$

for all $(A, B) \in \Sigma_{i/s}$. From the identity

$$(I + P^+ B R^{-1} B^\top)^{-1} = I - P^+ B(R + B^\top P^+ B)^{-1} B^\top,$$

we see that $P^+ A = 0$ for all $(A, B) \in \Sigma_{i/s}$. Then, it follows from (5.27) that $K = 0$. Since $A_0 + B_0 K = 0$ for all $(A_0, B_0) \in \Sigma_{i/s}^0$ due to Lemma 5.1, we see that A_0 must be zero. Hence, we have $A = A_s$ for all $(A, B) \in \Sigma_{i/s}$ and A_s is stable.

Moreover, it follows from (5.31) that $P^+ = Q$. Therefore, $QA_s = 0$. In other words, condition (ii) is satisfied, which proves the theorem. ∎

Theorem 5.6 gives necessary and sufficient conditions under which the data are informative for linear quadratic regulation. However, it might not be directly clear how these conditions can be verified given input/state data. Therefore, in what follows we rephrase the conditions of Theorem 5.6 in terms of the data matrices X and U_-.

Theorem 5.7. Let $Q = Q^\top$ be positive semidefinite and $R = R^\top$ be positive definite. Then, the data (U_-, X) are informative for linear quadratic regulation if and only if at least one of the following two conditions hold:

i The data (U_-, X) are informative for system identification. Equivalently, there exists $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ such that (5.9) holds. Moreover, the linear quadratic regulator problem is solvable for (A_s, B_s, Q, R), where $A_s = X_+V_1$ and $B_s = X_+V_2$.

ii There exists $\Theta \in \mathbb{R}^{T\times n}$ such that $X_-\Theta = (X_-\Theta)^\top$, $U_-\Theta = 0$,
$$\begin{bmatrix} X_-\Theta & X_+\Theta \\ \Theta^\top X_+^\top & X_-\Theta \end{bmatrix} > 0. \tag{5.33}$$
and $QX_+\Theta = 0$.

Proof. The equivalence of condition (i) of Theorem 5.6 and condition (i) of Theorem 5.7 is obvious. It remains to be shown that condition (ii) of Theorem 5.6 and condition (ii) of Theorem 5.7 are equivalent as well. To this end, suppose that there exists a matrix $\Theta \in \mathbb{R}^{T\times n}$ such that the conditions of (ii) holds. By Theorem 5.3, we have $\Sigma_{i/s} \subseteq \Sigma_K$ for $K = 0$, that is, A is stable for all $(A, B) \in \Sigma_{i/s}$. In addition, note that

$$QX_+\Theta(X_-\Theta)^{-1} = Q\begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} X_- \\ U_- \end{bmatrix}\Theta(X_-\Theta)^{-1} = QA \tag{5.34}$$

for all $(A, B) \in \Sigma_{i/s}$. This shows that $QA = 0$ and therefore that condition (ii) of Theorem 5.6 holds. Conversely, suppose that A is stable and $QA = 0$ for all $(A, B) \in \Sigma_{i/s}$. This implies that $K = 0$ is a stabilizing controller for all $(A, B) \in \Sigma_{i/s}$. By Theorem 5.3, there exists a matrix $\Theta \in \mathbb{R}^{T\times n}$ satisfying the first three conditions of (ii). Finally, it follows from $QA = 0$ and (5.34) that Θ also satisfies the fourth equation of (ii). This proves the theorem. ∎

5.4.3 From data to LQ gain

In this section our goal is to devise a method in order to compute the optimal feedback gain K directly from the data. For this, we will employ ideas from the study of Riccati inequalities (see e.g [136]).

The following theorem asserts that P^+ as in Lemma 5.2 can be found as the unique solution to an optimization problem involving only the data. Furthermore, the optimal feedback gain K can subsequently be found by solving a set of linear equations.

Theorem 5.8. Let $Q = Q^\top \geqslant 0$ and $R = R^\top > 0$. Suppose that the data (U_-, X) are informative for linear quadratic regulation. Consider the linear operator $P \mapsto \mathcal{L}(P)$ defined by

$$\mathcal{L}(P) := X_-^\top P X_- - X_+^\top P X_+ - X_-^\top Q X_- - U_-^\top R U_-.$$

Let P^+ be as in Lemma 5.2. The following statements hold:

i The matrix P^+ is equal to the unique solution to the optimization problem

$$\begin{aligned} &\text{maximize} \quad \operatorname{tr} P \\ &\text{subject to} \quad P = P^\top \geqslant 0 \quad \text{and} \quad \mathcal{L}(P) \leqslant 0. \end{aligned}$$

ii There exists a right inverse $X_-^\dagger$ of X_- such that

$$\mathcal{L}(P^+) X_-^\dagger = 0. \tag{5.35}$$

Moreover, if $X_-^\dagger$ satisfies (5.35), then the optimal feedback gain is given by $K = U_- X_-^\dagger$.

Remark 5.6. From a design viewpoint, the optimal feedback gain K can be found in the following way. First solve the semidefinite program in Theorem 5.8(i). Subsequently, compute a solution $X_-^\dagger$ to the linear equations $X_- X_-^\dagger = I$ and (5.35). Then, the optimal feedback gain is given by $K = U_- X_-^\dagger$.

Remark 5.7. The data-driven LQR problem was first solved using semidefinite programming in [42, Theorem 4]. There, the optimal feedback gain was found by minimizing the trace of a weighted sum of two matrix variables, subject to two LMI constraints. The semidefinite program in Theorem 5.8 is attractive since the dimension of the unknown P is (only) $n \times n$. In comparison, the dimensions of the two unknowns in [42, Theorem 4] are $T \times n$ and $m \times m$, respectively. In general, the number of samples T is much larger[4] than n. An additional attractive feature of Theorem 5.8 is that P^+ is obtained from the data. This is useful since the minimal cost associated to any initial condition x_0 can be computed as $x_0^\top P^+ x_0$.

The data-driven LQR approach in [71] is quite different from Theorem 5.8 since the solution to the Riccati equation is approximated using a batch-form solution to the *Riccati difference equation*. A similar approach was used in [1, 63, 154, 158] for the *finite horizon* data-driven LQR/LQG problem. In the setup of [71], the approximate solution to the Riccati equation is exact only if the number of data points tends to infinity. The main difference between our approach and the one in [71] is hence that the solution P^+ to the Riccati equation can be obtained exactly from *finite* data via Theorem 5.8.

Proof of Theorem 5.8. We begin with proving the first statement. Note that

$$\mathcal{L}(P) = \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \begin{bmatrix} P - A^\top PA - Q & -A^\top PB \\ -B^\top PA & -(R + B^\top PB) \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$$

for all $(A, B) \in \Sigma_{i/s}$. We claim that the following implication holds:

$$P = P^\top \geqslant 0 \text{ and } \mathcal{L}(P) \leqslant 0 \implies P^+ \geqslant P. \tag{5.36}$$

To prove this claim, let P be such that $P = P^\top \geqslant 0$ and $\mathcal{L}(P) \leqslant 0$. Since the data are informative for linear quadratic regulation, they are also informative for stabilization by state feedback. Therefore, the optimal feedback gain K satisfies

$$\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$$

due to Lemma 5.1. Therefore, the above expression for $\mathcal{L}(P)$ implies that

$$\begin{bmatrix} I \\ K \end{bmatrix}^\top \begin{bmatrix} P - A^\top PA - Q & -A^\top PB \\ -B^\top PA & -(R + B^\top PB) \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \leqslant 0$$

for all $(A, B) \in \Sigma_{i/s}$. This yields

$$P - M^\top PM \leqslant K^\top RK + Q$$

where M is as in Lemma 5.2. By subtracting this from (5.26), we obtain

$$(P^+ - P) - M^\top (P^+ - P)M \geqslant 0.$$

4 In fact, this is always the case under the persistency of excitation conditions imposed in [42] as such conditions can only be satisfied provided that $T \geqslant nm + n + m$.

Since M is stable, this discrete-time Lyapunov inequality implies that $P^+ - P \geqslant 0$ and hence $P^+ \geqslant P$. This proves the claim (5.36).

Note that $R + B^\top P^+ B$ is positive definite. Then, it follows from (5.25) that

$$\begin{bmatrix} P^+ - A^\top P^+ A - Q & -A^\top P^+ B \\ -B^\top P^+ A & -(R + B^\top P^+ B) \end{bmatrix} \leqslant 0$$

via a Schur complement argument. Therefore, $\mathcal{L}(P^+) \leqslant 0$. Since $P^+ \geqslant P$, we have $\operatorname{tr} P^+ \geqslant \operatorname{tr} P$. Together with (5.36), this shows that P^+ is a solution to the optimization problem stated in the theorem.

Next, we prove uniqueness. Let $\bar{P}$ be another solution of the optimization problem. Then, we have that $\bar{P} = \bar{P}^\top \geqslant 0$, $\mathcal{L}(\bar{P}) \leqslant 0$, and $\operatorname{tr} \bar{P} = \operatorname{tr} P^+$. From (5.36), we see that $P^+ \geqslant \bar{P}$. In particular, this implies that $(P^+)_{ii} \geqslant \bar{P}_{ii}$ for all i. Together with $\operatorname{tr} \bar{P} = \operatorname{tr} P^+$, this implies that $(P^+)_{ii} = \bar{P}_{ii}$ for all i. Now, for any i and j, we have

$$\begin{aligned} &(e_i - e_j)^\top P^+ (e_i - e_j) \geqslant (e_i - e_j)^\top \bar{P} (e_i - e_j) \text{ and} \\ &(e_i + e_j)^\top P^+ (e_i + e_j) \geqslant (e_i + e_j)^\top \bar{P} (e_i + e_j), \end{aligned}$$

where e_i denotes the i-th standard basis vector. This leads to $(P^+)_{ij} \leqslant \bar{P}_{ij}$ and $(P^+)_{ij} \geqslant \bar{P}_{ij}$, respectively. We conclude that $(P^+)_{ij} = \bar{P}_{ij}$ for all i, j. This proves uniqueness.

Finally, we prove the second statement. It follows from (5.25) and (5.27) that

$$\mathcal{L}(P^+) = -\left(U_- - KX_-\right)^\top \left(R + B^\top P^+ B\right)\left(U_- - KX_-\right). \tag{5.37}$$

The optimal feedback K is stabilizing, therefore it follows from Theorem 5.2 that K can be written as $K = U_- \Gamma$, where Γ is some right inverse of X_-. Note that this implies the existence of a right inverse $X_-^\dagger$ of X_- satisfying (5.35). Indeed, $X_-^\dagger := \Gamma$ is such a matrix by (5.37). Moreover, if $X_-^\dagger$ is a right inverse of X_- satisfying (5.35) then $(U_- - KX_-)X_-^\dagger = 0$ by (5.37) and positive definiteness of R. We conclude that the optimal feedback gain is equal to $K = U_- X_-^\dagger$, which proves the second statement. ■

5.5 CONTROL USING INPUT AND OUTPUT DATA

In this section, we will consider problems where the output *does* play a role. In particular, we will consider the problem of stabilization by dynamic measurement feedback. We will first consider this problem based on input, state and output measurements. Subsequently, we will turn our attention to the case of input/output data.

Consider the 'true' system

$$x(t+1) = A_s x(t) + B_s u(t) \tag{5.38a}$$
$$y(t) = C_s x(t) + D_s u(t). \tag{5.38b}$$

We want to design a stabilizing dynamic controller of the form

$$w(t+1) = Kw(t) + Ly(t) \tag{5.39a}$$
$$u(t) = Mw(t) \tag{5.39b}$$

such that the closed-loop system, given by

$$\begin{bmatrix} x(t+1) \\ w(t+1) \end{bmatrix} = \begin{bmatrix} A_s & B_s M \\ LC_s & K + LD_s M \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix},$$

is stable. This is equivalent to the condition that

$$\begin{bmatrix} A_s & B_s M \\ LC_s & K + LD_s M \end{bmatrix} \tag{5.40}$$

is a stable matrix.

5.5.1 Stabilization using input, state and output data

Suppose that we collect input/state/output data on ℓ time intervals $\{0, 1, \dots, T_i\}$ for $i = 1, 2, \dots, q$. Let U_-, X, X_-, and X_+ be defined as in (5.3) and let Y_- be defined in a similar way as U_-. Then, we have

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \tag{5.41}$$

relating the data and the 'true' system (5.38). The set of all systems that are consistent with these data is then given by:

$$\Sigma_{i/s/o} := \left\{ (A, B, C, D) \mid \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{5.42}$$

In addition, for given K, L and M, we define the set of systems that are stabilized by the dynamic controller (5.39) by

$$\Sigma_{K,L,M} := \left\{ (A, B, C, D) \mid \begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix} \text{ is stable} \right\}.$$

Subsequently, in line with Definition 5.2, we consider the following notion of informativity:

Definition 5.8. We say the data (U_-, X, Y_-) are *informative for stabilization by dynamic measurement feedback* if there exist matrices K, L and M such that $\Sigma_{i/s/o} \subseteq \Sigma_{K,L,M}$.

As in the general case of informativity for control, we consider two consequent problems: First, to characterize informativity for stabilization in terms of necessary and sufficient conditions on the data and next to design a controller based on these data. To aid in solving these problems, we will first investigate the case where U_- does not have full row rank. In this case, we will show that the problem can be 'reduced' to the full row rank case.

For this, we start with the observation that any $U_- \in \mathbb{R}^{m\times T}$ of row rank $k < m$ can be decomposed as $U_- = S\hat{U}_-$, where S has full column rank and $\hat{U}_- \in \mathbb{R}^{k\times T}$ has full row rank. We now have the following lemma:

Lemma 5.3. Consider the data (U_-, X, Y_-) and the corresponding set $\Sigma_{i/s/o}$. Let S be a matrix of full column rank such that $U_- = S\hat{U}_-$ with $\hat{U}_-$ a matrix of full row rank. Let $S^\dagger$ be a left inverse of S.

Then the data (U_-, X, Y_-) are informative for stabilization by dynamic measurement feedback if and only if the data $(\hat{U}_-, X, Y_-)$ are informative for stabilization by dynamic measurement feedback.

In particular, if we let $\hat{\Sigma}_{i/s/o}$ be the set of systems consistent with the 'reduced' data set $(\hat{U}_-, X, Y_-)$, and if $\hat{K}$ $\hat{L}$ and $\hat{M}$ are real matrices of appropriate dimensions, then:

$$\Sigma_{i/s/o} \subseteq \Sigma_{K,L,M} \implies \hat{\Sigma}_{i/s/o} \subseteq \Sigma_{K,L,S^\dagger M}, \tag{5.43}$$

$$\hat{\Sigma}_{i/s/o} \subseteq \Sigma_{\hat{K},\hat{L},\hat{M}} \implies \Sigma_{i/s/o} \subseteq \Sigma_{\hat{K},\hat{L},S\hat{M}}. \tag{5.44}$$

Proof. First note that

$$\hat{\Sigma}_{i/s/o} = \left\{ (\hat{A}, \hat{B}, \hat{C}, \hat{D}) \mid \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} X_- \\ \hat{U}_- \end{bmatrix} \right\}.$$

We will start by proving the following two implications:

$$(A, B, C, D) \in \Sigma_{i/s/o} \implies (A, BS, C, DS) \in \hat{\Sigma}_{i/s/o}, \tag{5.45}$$

$$(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \hat{\Sigma}_{i/s/o} \implies (\hat{A}, \hat{B}S^\dagger, \hat{C}, \hat{D}S^\dagger) \in \Sigma_{i/s/o}. \tag{5.46}$$

To prove implication (5.45), assume that $(A, B, C, D) \in \Sigma_{i/s/o}$. Then, by definition

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

From the definition of S, we have $U_- = S\hat{U}_-$. Substitution of this results in

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ S\hat{U}_- \end{bmatrix} = \begin{bmatrix} A & BS \\ C & DS \end{bmatrix} \begin{bmatrix} X_- \\ \hat{U}_- \end{bmatrix}.$$

This implies that $(A, BS, C, DS) \in \hat{\Sigma}_{i/s/o}$. The implication (5.46) can be proven similarly by substitution of $\hat{U}_- = S^\dagger U_-$.

To prove the lemma, suppose that the data (U_-, X, Y_-) are informative for stabilization by dynamic measurement feedback. This means that there exist K, L, and M such that

$$\begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix}$$

is stable for all $(A, B, C, D) \in \Sigma_{i/s/o}$. In particular, if $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \hat{\Sigma}_{i/s/o}$ then $(\hat{A}, \hat{B}S^\dagger, \hat{C}, \hat{D}S^\dagger) \in \Sigma_{i/s/o}$ by (5.46). This means that the matrix

$$\begin{bmatrix} \hat{A} & \hat{B}S^\dagger M \\ L\hat{C} & K + L\hat{D}S^\dagger M \end{bmatrix}$$

is stable for all $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \hat{\Sigma}_{i/s/o}$. In other words, $\hat{\Sigma}_{i/s/o} \subseteq \Sigma_{K,L,S^\dagger M}$ and hence implication (5.43) holds and the data $(\hat{U}_-, X, Y_-)$ are informative for stabilization by dynamic measurement feedback. The proofs of (5.44) and the 'if' part of the theorem are analogous and hence omitted. ∎

We will now solve the informativity and design problems under the condition that U_- has full row rank.

Theorem 5.9. Consider the data (U_-, X, Y_-) and assume that U_- has full row rank. Then (U_-, X, Y_-) are informative for stabilization by dynamic measurement feedback if and only if the following conditions are satisfied:

i We have

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m.$$

Equivalently, there exists $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ such that (5.9) holds. This means that

$$\Sigma_{i/s/o} = \{(X_+V_1, X_+V_2, Y_-V_1, Y_-V_2)\}.$$

ii The pair (X_+V_1, X_+V_2) is stabilizable and (Y_-V_1, X_+V_1) is detectable.

Moreover, if the above conditions are satisfied, a stabilizing controller (K, L, M) can be constructed as follows:

(a) Select a matrix M such that $X_+(V_1 + V_2M)$ is stable.

(b) Choose a matrix L such that $(X_+ - LY_-)V_1$ is stable.

(c) Define $K := (X_+ - LY_-)(V_1 + V_2M)$.

Remark 5.8. Under the condition that U_- has full row rank, Theorem 5.9 asserts that in order to construct a stabilizing dynamic controller, it is necessary that the data are rich enough to identify the system matrices A_s, B_s, C_s and D_s uniquely. The controller proposed in ((a)), ((b)), ((c)) is a so-called *observer-based* controller, see e.g. [165, Section 3.12]. The feedback gains M and L can be computed using standard methods, for example via pole placement or LMI's.

Proof of Theorem 5.9. To prove the 'if' part, suppose that conditions (i) and (ii) are satisfied. This implies the existence of the matrices (K, L, M) as defined in items ((a)), ((b)) and ((c)). We will now show that these matrices indeed constitute a stabilizing controller. Note that by condition (i), $\Sigma_{i/s/o} = \{(A_s, B_s, C_s, D_s)\}$ with

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} X_+V_1 & X_+V_2 \\ Y_-V_1 & Y_-V_2 \end{bmatrix}. \tag{5.47}$$

By definition of K, L and M, the matrices $A_s + B_sM$ and $A_s - LC_s$ are stable and $K = A_s + B_sM - LC_s - LD_sM$. This implies that (5.40) is stable since the matrices

$$\begin{bmatrix} A_s & B_sM \\ LC_s & A_s + B_sM - LC_s \end{bmatrix} \text{ and } \begin{bmatrix} A_s + B_sM & B_sM \\ 0 & A_s - LC_s \end{bmatrix}$$

are similar [165, Section 3.12]. We conclude that (U_-, X, Y_-) are informative for stabilization by dynamic measurement feedback and that the recipe given by ((a)), ((b)) and ((c)) leads to a stabilizing controller (K, L, M).

It remains to prove the 'only if' part. To this end, suppose that the data (U_-, X, Y_-) are informative for stabilization by dynamic measurement feedback. Let (K, L, M) be such that $\Sigma_{i/s/o} \subseteq \Sigma_{K,L,M}$. This means that

$$\begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix}$$

is stable for all $(A, B, C, D) \in \Sigma_{i/s/o}$. Let $\zeta \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ be such that

$$\begin{bmatrix} \zeta^\top & \eta^\top \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = 0.$$

Note that $(A + \zeta\zeta^\top, B + \zeta\eta^\top, C, D) \in \Sigma_{i/s/o}$ if $(A, B, C, D) \in \Sigma_{i/s/o}$. Therefore, the matrix

$$\begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix} + \alpha \begin{bmatrix} \zeta\zeta^\top & \zeta\eta^\top M \\ 0 & 0 \end{bmatrix}$$

is stable for all $\alpha \in \mathbb{R}$. We conclude that the spectral radius of the matrix

$$W_\alpha := \frac{1}{\alpha} \begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix} + \begin{bmatrix} \zeta\zeta^\top & \zeta\eta^\top M \\ 0 & 0 \end{bmatrix}$$

is smaller than $1/\alpha$. By taking the limit as $\alpha \to \infty$, we see that the spectral radius of $\zeta\zeta^\top$ must be zero due to the continuity of spectral radius. Therefore, ζ must be zero. Since U_- has full column rank, we can conclude that η must be zero too. This proves that condition (i) and therefore $\Sigma_{i/s/o} = \{(A_s, B_s, C_s, D_s)\}$. Since the controller (K, L, M) stabilizes (A_s, B_s, C_s, D_s), the pair (A_s, B_s) is stabilizable and (C_s, A_s) is detectable. By (5.47) we conclude that condition (ii) is also satisfied. This proves the theorem. ∎

The following corollary follows from Lemma 5.3 and Theorem 5.9 and gives necessary and sufficient conditions for informativity for stabilization by dynamic measurement feedback. Note that we do not make any a priori assumptions on the rank of U_-.

Corollary 5.2. Let S be any full column rank matrix such that $U_- = S\hat{U}_-$ with $\hat{U}_-$ full row rank k. The data (U_-, X, Y_-) are informative for stabilization by dynamic measurement feedback if and only if the following two conditions are satisfied:

i We have
$$\operatorname{rank} \begin{bmatrix} X_- \\ \hat{U}_- \end{bmatrix} = n + k.$$
Equivalently, there exists a matrix $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ such that
$$\begin{bmatrix} X_- \\ \hat{U}_- \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

ii The pair (X_+V_1, X_+V_2) is stabilizable and (Y_-V_1, X_+V_1) is detectable.

Moreover, if the above conditions are satisfied, a stabilizing controller (K, L, M) is constructed as follows:

(a) Select a matrix $\hat{M}$ such that $X_+(V_1 + V_2\hat{M})$ is stable. Define $M := S\hat{M}$.

(b) Choose a matrix L such that $(X_+ - LY_-)V_1$ is stable.

(c) Define $K := (X_+ - LY_-)(V_1 + V_2\hat{M})$.

Remark 5.9. In the previous corollary it is clear that the system matrices of the data-generating system are related to the data via

$$\begin{bmatrix} A_s & B_s S \\ C_s & D_s S \end{bmatrix} = \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}.$$

Therefore the corollary shows that informativity for stabilization by dynamic measurement feedback requires that A_s and C_s can be identified uniquely from the data. However, this does not hold for B_s and D_s in general.

5.5.2 Stabilization using input and output data

Recall that we consider a system of the form (5.38). When given input, state and output data, any system (A, B, C, D) consistent with these data satisfies

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}. \tag{5.48}$$

In this section, we will consider the situation where we have access to input and output measurements only. Moreover, we assume that the data are collected on a single time interval, i.e. $q = 1$. This means that our data are of the form (U_-, Y_-), where

$$U_- := \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \tag{5.49a}$$

$$Y_- := \begin{bmatrix} y(0) & y(1) & \cdots & y(T-1) \end{bmatrix}. \tag{5.49b}$$

Again, we are interested in informativity of the data, this time given by (U_-, Y_-). Therefore we wish to consider the set of all systems of the form (5.38) with the state space dimension[5] n that admit the same input/output data. This leads to the following set of consistent systems:

$$\Sigma_{i/o} := \left\{ (A, B, C, D) \mid \exists X \in \mathbb{R}^{n \times (T+1)} \text{ s.t. (5.48) holds} \right\}.$$

As in the previous section, we wish to find a controller of the form (5.39) that stabilizes the system. This means that, in line with Definition 5.2, we have the following notion of informativity:

5 The state space dimension of the system may be known a priori. In the case that it is not, it can be computed using subspace identification methods, see e.g. [167, Theorem 2].

Definition 5.9. We say the data (U_-, Y_-) are *informative for stabilization by dynamic measurement feedback* if there exist matrices K, L and M such that $\Sigma_{i/o} \subseteq \Sigma_{K,L,M}$.

In order to obtain conditions under which (U_-, Y_-) are informative for stabilization, it may be tempting to follow the same steps as in Section 5.5.1. In that section we first proved that we can assume without loss of generality that U_- has full row rank. Subsequently, Theorem 5.9 and Corollary 5.2 characterize informativity for stabilization by dynamic measurement feedback based on input, state and output data. It turns out that we can perform the first of these two steps for input/output data as well. Indeed, in line with Lemma 5.3, we can state the following:

Lemma 5.4. Consider the data (U_-, Y_-) and the corresponding set $\Sigma_{i/o}$. Let S be a matrix of full column rank such that $U_- = S\hat{U}_-$ with $\hat{U}_-$ a matrix of full row rank.

Then the data (U_-, Y_-) are informative for stabilization by dynamic measurement feedback if and only if the data $(\hat{U}_-, Y_-)$ are informative for stabilization by dynamic measurement feedback.

The proof of this lemma is analogous to that of Lemma 5.3 and therefore omitted. Lemma 5.4 implies that without loss of generality we can consider the case where U_- has full row rank.

In contrast to the first step, the second step in Section 5.5.1 relies heavily on the affine structure of the considered set $\Sigma_{i/s/o}$. Indeed, the proof of Theorem 5.9 makes use of the fact that $\Sigma^0_{i/s/o}$ is a subspace. However, the set $\Sigma_{i/o}$ is not an affine set. This means that it is not straightforward to extend the results of Corollary 5.2 to the case of input/output measurements.

Nonetheless, under certain conditions on the input/output data it is possible to construct the corresponding state sequence X of (5.38) up to similarity transformation. In fact, state reconstruction is one of the main themes of subspace identification, see e.g. [119, 167]. The construction of a state sequence would allow us to reduce the problem of stabilization using input/output data to that with input, state and output data. The following result gives sufficient conditions on the data (U_-, Y_-) for state construction.

To state the result, we will first require a few standard pieces of notation. First, let $f(0), \dots, f(T-1)$ be a signal and $\ell < T$, then we define the *Hankel matrix of depth* ℓ as

$$\mathcal{H}_\ell(f) = \begin{bmatrix} f(0) & f(1) & \cdots & f(T-\ell) \\ f(1) & f(2) & \cdots & f(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ f(\ell-1) & f(\ell) & \cdots & f(T-1) \end{bmatrix}.$$

Given input and output data of the form (5.49), and k such that $2k < T$ we consider $\mathcal{H}_{2k}(u)$ and $\mathcal{H}_{2k}(y)$. Next, we partition our data into so-called '*past*' and '*future*' data as

$$\mathcal{H}_{2k}(u) = \begin{bmatrix} U_p \\ U_f \end{bmatrix}, \quad \mathcal{H}_{2k}(y) = \begin{bmatrix} Y_p \\ Y_f \end{bmatrix},$$

where U_p, U_f, Y_p and Y_f all have k block rows. Let $x(0), \dots, x(T)$ denote the state trajectory of (5.38) compatible with a given (U_-, Y_-). We now denote

$$\begin{aligned} X_p &= \begin{bmatrix} x(0) & \cdots & x(T-2k) \end{bmatrix}, \\ X_f &= \begin{bmatrix} x(k) & \cdots & x(T-k) \end{bmatrix}. \end{aligned}$$

Lastly, let $\operatorname{rs}(M)$ denote the row space of the matrix M. Now we have the following result, which is a rephrasing of [119, Theorem 3].

Theorem 5.10. Consider the system (5.38) and assume it is minimal. Let the input/output data (U_-, Y_-) be as in (5.49). Assume that k is such that $n < k < \frac{1}{2}T$. If

$$\operatorname{rank} \begin{bmatrix} \mathcal{H}_{2k}(u) \\ \mathcal{H}_{2k}(y) \end{bmatrix} = 2km + n, \tag{5.50}$$

then

$$\operatorname{rs}(X_f) = \operatorname{rs}\left(\begin{bmatrix} U_p \\ Y_p \end{bmatrix} \right) \cap \operatorname{rs}\left(\begin{bmatrix} U_f \\ Y_f \end{bmatrix} \right),$$

and this row space is of dimension n.

Under the conditions of this theorem, we can now find the 'true' state sequence X_f up to similarity transformation. That is, we can find $\bar{X} = SX_f$ for some unknown invertible matrix S. This means that, under these conditions, we obtain an input/state/output trajectory given by the matrices

$$\bar{U}_- = \begin{bmatrix} u(k) & u(k+1) & \cdots & u(T-k-1) \end{bmatrix}, \tag{5.51a}$$

$$\bar{Y}_- = \begin{bmatrix} y(k) & y(k+1) & \cdots & y(T-k-1) \end{bmatrix}, \tag{5.51b}$$

$$\bar{X} = S \begin{bmatrix} x(k) & x(k+1) & \cdots & x(T-k) \end{bmatrix}. \tag{5.51c}$$

We can now state the following sufficient condition for informativity for stabilization with input/output data.

Corollary 5.3. Consider the system (5.38) and assume it is minimal. Let the input/output data (U_-, Y_-) be as in (5.49). Assume that k is such that $n < k < \frac{1}{2}T$. Then the data (U_-, Y_-) are informative for stabilization by dynamic measurement feedback if the following two conditions are satisfied:

i The rank condition (5.50) holds.

ii The data $(\bar{U}_-, \bar{X}, \bar{Y}_-)$, as defined in (5.51), are informative for stabilization by dynamic measurement feedback.

Moreover, if these conditions are satisfied, a stabilizing controller (K, L, M) such that $\Sigma_{i/o} \subseteq \Sigma_{K,L,M}$ can be found by applying Corollary 5.2 ((a)),((b)),((c)) to the data $(\bar{U}_-, \bar{X}, \bar{Y}_-)$.

The conditions provided in Corollary 5.3 are sufficient, but not necessary for informativity for stabilization by dynamic measurement feedback. In addition, it can be shown that data satisfying these conditions are also informative for system identification, in the sense that $\Sigma_{i/o}$ contains only the 'true' system (5.38) and all systems similar to it.

An interesting question is whether the conditions of Corollary 5.3 can be sharpened to necessary and sufficient conditions. In this case it would be of interest to investigate whether such conditions are weaker than those for informativity for system identification.

At this moment, we do not have a conclusive answer to the above question. However, we note that even for subspace identification there are no known necessary and sufficient conditions for data to be informative, although several sufficient conditions exist, e.g. [119, Theorems 3 and 5], [167, Theorem 2] and [175, Theorems 3 and 4].

5.6 CONCLUSIONS AND FUTURE WORK

Results in data-driven control should clearly highlight the differences and possible advantages as compared to system identification paired with model-based control. One clear advantage of data-driven control is its capability of solving problems in the presence of data that are not informative for system identification. Therefore, informativity is a very important concept for data-driven analysis and control.

In this chapter we have introduced a comprehensive framework for studying informativity problems. We have applied this framework to analyze several system-theoretic properties on the basis of data. The same framework was used to solve multiple data-driven control problems.

After solving these problems, we have made the comparison between our data-driven methods, and the 'classical' combination of identification and model-based control. We have shown that for many analysis and control problems, such as controllability analysis and stabilization, the data-driven approach can indeed be performed on data that are not informative for system identification. On the other hand, for data-driven linear quadratic regulation it has been shown that informativity for system identification is a necessary condition. This effectively means that for this data-driven control problem, we have given a theoretic justification for the use of persistently exciting data.

Future work

Due to the generality of the introduced framework, many different problems can be studied in a similar fashion: one could consider different types of data, where more results based on only input and output data would be particularly interesting. Many other system-theoretic properties could be considered as well, for example, analyzing passivity or tackling robust control problems based on data.

It would also be of interest to generalize the model class under consideration. One could, for instance, consider larger classes of systems like differential algebraic or polynomial systems. On the other hand, the class under consideration can also be made smaller by prior knowledge of the system. For example, the system might have an observed network structure, or could in general be parametrized.

A framework similar to ours could be employed in the presence of disturbances, which is a problem of practical interest. A study of data-driven control problems in this situation is particularly interesting, because system identification is less straightforward. We note that data-driven stabilization under measurement noise has been studied in [42] and under unknown disturbances in [23]. Additionally, the data-driven LQR problem is popular in the machine learning community, where it is typically assumed that the system is influenced by (Gaussian) process noise, see e.g. [43].

In this chapter, we have assumed that the data are given. Yet another problem of practical interest is that of *experiment design*, where inputs need to be chosen such that the resulting data are informative. In system identification, this problem led to the notion of persistence of excitation. For example, it is shown in [182] that the rank condition (5.8) can be imposed by injecting an input sequence that is persistently exciting of order $n+1$. However, as we have shown, this rank condition is not necessary for some data-driven control problems, like stabilization by state feedback. The question therefore arises whether we can find tailor-made conditions on the input only, that guarantee informativity for data-driven control.

6 INFORMATIVITY OF NOISY DATA

In the previous chapter, we saw that resolving an informativity problem can be done in essentially two parts: First, we describe the structure of the set of systems compatible with the data. After this, we develop methods to test whether all systems in such a set have the required property. In this chapter consider a situation where the measurements have noise, but where certain system matrices are known. Using the specific affine structure that arises, we can then develop spectral and geometric tests for a number of properties.

6.1 INTRODUCTION

In this chapter we will study the problem of determining whether a given unknown dynamical system has certain structural properties, based on noisy data obtained from this system. One way to approach this problem is to use the data to identify an explicit model representing the system, and apply a suitable, model based, test to this model. In the present chapter we will approach the problem from a different angle, and establish tests, directly on the noisy data, to check structural system properties.

As a major tool, we will use the general framework of informativity of data, recently introduced in [172]. In that paper, it was shown that the data-driven approach can also be useful if the data do not give sufficient information to identify the 'true' unknown system. In that case, a given set of data gives rise to a whole *family of system models*, all of which could have given the same data. On the basis of the data it is then impossible to distinguish between models, and a given system property will hold for the 'true' system model only if its holds for all models compatible with the data. Formalizing this, a set of data is called *informative* for a given system property if the property holds for all systems that could have given this set of data.

In [172], tests were established for checking whether a given set of noiseless data is informative for controllability, stability and stabilizability. In [50], informativity of noiseless data for observability was studied. In the present chapter we will deal with informativity of noisy data. We will establish informativity tests for several relevant structural system properties. More specifically, we will study informativity for observability and detectability, strong observability and detectability, strong controllability and stabilizability, and invertibility of linear systems. These structural properties are relevant in in a wide range of observer, filter and control design problems. For definitions and extensive treatments we refer to [75, 117, 120, 150, 156, 183], and [165] and the references therein.

Analysis of system properties based on data has been studied also in [106,125, 178, 188], which deal with data-based controllability and observability analysis. Whereas in the present chapter general data sets are allowed, these references impose restrictions on the data. The paper [130] deals with the problem of determining stability properties of input-output systems using time series data. More recently, there has also been an increasing interest in the problem of verifying dissipativity on the basis of system data. This problem has, for example, been addressed in [22, 99, 113, 143].

6.2 PROBLEM FORMULATION

In this chapter we will consider the linear discrete-time input-state-output system with noise given by

$$x(t+1) = A_{\text{true}}x(t) + Bu(t) + Ew(t), \tag{6.1a}$$
$$y(t) = Cx(t) + Du(t) + Fw(t), \tag{6.1b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ a control input, $y \in \mathbb{R}^p$ an output, and $w \in \mathbb{R}^r$ is unknown noise. We assume that A_{true} is an unknown matrix, but that the matrices B, C, D and E, F are known. The assumption that these matrices are known is reasonable, for example in networked systems, in which the input and output nodes are given, but the interconnection topology is unknown. Typically, in that context, the matrices B, C and D would be matrices whose columns only contain 0's and 1's, with in each column at most one entry equal to 1. The term $Ew(t)$ represents process noise, whereas $Fw(t)$ represents measurement noise. The special case that $E = 0$ and $F = 0$ is called the *noiseless case*.

We assume that we have input-state-output data concerning this unknown 'true' system in the form of samples of x, u and y on a given finite time interval $\{0, 1 \dots, T\}$. These data are denoted by

$$U_- := \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}, \tag{6.2a}$$

$$X := \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}, \tag{6.2b}$$

$$Y_- := \begin{bmatrix} y(0) & y(1) & \cdots & y(T-1) \end{bmatrix}. \tag{6.2c}$$

It will be assumed that the data (6.2) are 'harvested' from the true system (6.1), meaning that there exists some matrix

$$W_- = \begin{bmatrix} w(0) & w(1) & \cdots & w(T-1) \end{bmatrix}$$

such that

$$X_+ = A_{\text{true}} X_- + BU_- + EW_-, \tag{6.3a}$$

$$Y_- = CX_- + DU_- + FW_-, \tag{6.3b}$$

where we denote

$$X_- := \begin{bmatrix} x(0) & x(2) & \cdots & x(T-1) \end{bmatrix},$$
$$X_+ := \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}.$$

We then say that the data are *compatible* with the true system $(A_{\text{true}}, B, C, D, E, F)$.

The set of all $n \times n$ matrices A such that the data (6.2) are compatible with the system (A, B, C, D, E, F) is denoted by $\mathcal{A}_{\text{dat}}$, i.e.,

$$\mathcal{A}_{\text{dat}} := \Big\{ A \in \mathbb{R}^{n \times n} \mid \exists W_- : \begin{pmatrix} X_+ \\ Y_- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X_- \\ U_- \end{pmatrix} + \begin{pmatrix} E \\ F \end{pmatrix} W_- \Big\}. \tag{6.4}$$

Let $\mathcal{P}$ denote some system theoretic property that might or might not hold for a given linear system. The general problem that we will address in this chapter is to determine from the data harvested from (6.1) whether the property $\mathcal{P}$ holds for the unknown true system $(A_{\text{true}}, B, C, D, E, F)$. Since on the basis of the data we can not distinguish between the true A_{true} and any $A \in \mathcal{A}_{\text{dat}}$, we need to check whether the property holds for *all* systems (A, B, C, D, E, F) with $A \in \mathcal{A}_{\text{dat}}$. Following [172], in that case we call the data *informative for property $\mathcal{P}$.*

Example 6.1. For $\mathcal{P}$ take the property '(A, B) is a controllable pair'. Suppose that on the basis of the data (U_-, X, Y_-) we want to determine whether $\mathcal{P}$ holds for the pair (A_{true}, B) corresponding to the true system. This requires to check whether the data are informative for property $\mathcal{P}$. Using Theorem 8 in [172], it can be shown that in the noiseless case (i.e. the case that $E = 0$ and $F = 0$) the data (U_-, X, Y_-) are informative for $\mathcal{P}$ if and only if $\operatorname{rank} \begin{bmatrix} X_+ - \lambda X_- & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$.

Example 6.2. For $\mathcal{P}$ take the property 'the pair (C, A) is detectable'. In the noiseless case it can be shown that the data (U_-, X, Y_-) are informative for $\mathcal{P}$ if and only if $\ker C \subseteq \operatorname{im} X_-$ and for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$ we have

$$\operatorname{rank} \begin{pmatrix} X_+ - BU_- - \lambda X_- \\ CX_- \end{pmatrix} = \operatorname{rank} X_-.$$

This will be one of the results in this chapter.

Remark 6.1. We note that the case of independent process noise and measurement noise is also covered by the noisy model (6.1) introduced above. The noise matrices should then be taken of the form $E = \begin{pmatrix} E_1 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & F_2 \end{pmatrix}$, while the noise signal is given by the vector $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and likewise $W_- = \begin{pmatrix} W_{1-} \\ W_{2-} \end{pmatrix}$. A special case of this is that only process noise occurs, in which case F_2 is void and $E = E_1$ and $F = 0$. In other words, in the case of independent process and measurement noise we have $A \in \mathcal{A}_{\text{dat}}$ if and only if there exists a matrix W_{1-} such that $X_+ = AX_- + BU_- + E_1 W_{1-}$. The equation $Y_- = CX_- + DU_- + F_2 W_{2-}$ can then be ignored since it does not put any constraint on A.

The purpose of this chapter is to establish necessary and sufficient conditions on the input-state-output data obtained from (6.1) to be informative for a range of system properties $\mathcal{P}$. Throughout, we will restrict ourselves to the situation introduced above, namely, that the state map A_{true} is unknown, but that the matrices B, C and D are known. We will study both the noisy case as well as the noiseless case. In the noisy case it will be assumed that the noise matrices E and F are known.

The outline of the remainder of this chapter is as follows. In Section 6.3, we will state and prove a theorem that will be instrumental in order to obtain our results on informativity in the rest of the chapter. The theorem expresses a rank property of the Rosenbrock system matrix of the unknown system in terms of a polynomial matrix that collects available information about the unknown system. In Section 6.4, this result will be applied to obtain necessary and sufficient conditions for informativity of noisy data for the following system properties:

- strong observability and strong detectability of (A, B, C, D),

- observability and detectability of (C, A),
- strong controllability and strong stabilizability of (A, B, C, D),
- controllability and stabilizability of (A, B).

In Section 6.5, we apply ideas from the geometric approach to linear systems, see [165, 183] to set up a geometric framework for informativity analysis for strong observability and observability. This framework will then be applied to the analysis of informativity for left-invertibility. Finally, in Section 6.6 we close this chapter with concluding remarks.

6.3 A RANK PROPERTY FOR AN AFFINE SET OF SYSTEMS

In this section we will establish a general framework that will enable us characterize informativity of input-state-output data for the properties listed in Section II.

Let $P \in \mathbb{R}^{n\times r}$, $Q \in \mathbb{R}^{\ell\times n}$ and $R \in \mathbb{R}^{\ell\times r}$ be given matrices. Here, r and ℓ are positive integers, and the symbol n has the usual meaning of state space dimension. Using these matrices, we define an affine space of state matrices A by

$$\mathcal{A} := \{A \in \mathbb{R}^{n\times n} \mid R = QAP\}. \tag{6.5}$$

It is easily seen that $\mathcal{A}$ is nonempty if and only if $\operatorname{im} R \subseteq \operatorname{im} Q$ and $\ker P \subseteq \ker R$. Assume this to be the case.

Now let $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ be given, and for each $A \in \mathcal{A}$ consider the system

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{6.6a}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t). \tag{6.6b}$$

The Rosenbrock system matrix associated with the system (6.6) is defined as the polynomial matrix

$$\begin{pmatrix} A - sI & B \\ C & D \end{pmatrix}. \tag{6.7}$$

In addition, we we will consider the polynomial matrix

$$\begin{pmatrix} R - sQP & QB \\ CP & D \end{pmatrix} \tag{6.8}$$

associated with the given matrices (P, Q, R) and (B, C, D). The following theorem expresses a uniform rank property of the set of system matrices (6.7), with A ranging over the affine set $\mathcal{A}$, in terms of a rank property of the single polynomial matrix (6.8).

Theorem 6.1. Let (P, Q, R) and (B, C, D) be given. Then

$$\operatorname{rank}\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} B \\ D \end{pmatrix} \tag{6.9}$$

for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ and

$$\operatorname{rank}\begin{pmatrix} R-\lambda QP & QB \\ CP & D \end{pmatrix} = \operatorname{rank} P + \operatorname{rank}\begin{pmatrix} QB \\ D \end{pmatrix} \tag{6.10}$$

for all $\lambda \in \mathbb{C}$.[1]

In addition, (6.9) holds for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$ if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ and (6.10) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

Proof. To start the proof, first observe that for any $A \in \mathcal{A}$:

$$\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} = \begin{pmatrix} A-\lambda I & I & 0 \\ C & 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \\ 0 & D \end{pmatrix}. \tag{6.11}$$

Note that for any pair of matrices M and N we have $\operatorname{rank} MN = \operatorname{rank} N$ if and only if $\ker MN = \ker N$. By applying this to (6.11), we see that (6.9) is equivalent to

$$\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \implies \begin{pmatrix} I & 0 \\ 0 & B \\ 0 & D \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0.$$

It is straightforward to check that, in turn, this holds if and only if

$$\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \implies \xi = 0. \tag{6.12}$$

Similarly, note that for all $A \in \mathcal{A}$

$$\begin{pmatrix} R-\lambda QP & QB \\ CP & D \end{pmatrix} = \begin{pmatrix} Q(A-\lambda I) & I & 0 \\ C & 0 & I \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & QB \\ 0 & D \end{pmatrix}.$$

1 For a given subspace $\mathcal{L}$ and matrix M we denote by $M^{-1}\mathcal{L}$ the inverse image $\{x \mid Mx \in \mathcal{L}\}$.

This makes (6.10) equivalent to

$$\begin{pmatrix} R-\lambda QP & QB \\ CP & D \end{pmatrix} \begin{pmatrix} \nu \\ \eta \end{pmatrix} = 0 \implies P\nu = 0 \tag{6.13}$$

From here on, we will prove the first statement of the theorem, noting any changes required for the second part.

($\Leftarrow$): Let $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ (resp. $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$). Assume that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ and (6.10) holds for λ. We will prove that (6.12) holds. For this, let ξ and η satisfy

$$\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0.$$

Since $\xi \in C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$, we can write $\xi = P\nu$ for some ν. Now, by pre-multiplying with $\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$ we obtain that

$$\begin{pmatrix} R-\lambda QP & QB \\ CP & D \end{pmatrix} \begin{pmatrix} \nu \\ \eta \end{pmatrix} = 0.$$

We can now apply (6.13) and thereby conclude that $\xi = P\nu = 0$. This proves that (6.12) holds.

($\Rightarrow$): Assume that (6.12) holds for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ (resp. $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$). We will first prove that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$.

Let $\hat{x} \in C^{-1} \operatorname{im} D \setminus \operatorname{im} P$, that is, $\hat{x} \notin \operatorname{im} P$ and there exists a $\hat{u}$ such that $C\hat{x} + D\hat{u} = 0$. Without loss of generality take $\hat{x}$ and $\hat{u}$ as real vectors. Take any $A \in \mathcal{A}$ and $\mu \in \mathbb{R}$ (resp. $\mu \in \mathbb{R}$ such that $|\mu| \geqslant 1$). Let A_0 be any real $n \times n$ matrix such that $QA_0P = 0$ and $A_0\hat{x} = -(A-\mu I)\hat{x} - B\hat{u}$. Note that such matrix exists as $\hat{x} \notin \operatorname{im} P$ and $-(A-\mu I)\hat{x} - B\hat{u}$ is a real vector. Now define $\bar{A} := A + A_0$. Note that $\bar{A} \in \mathcal{A}$ and

$$\begin{pmatrix} \bar{A}-\mu I & B \\ C & D \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{u} \end{pmatrix} = 0.$$

By (6.12), we see that $\hat{x} = 0$, which contradicts with $x \notin \operatorname{im} P$. Therefore $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$.

We now move to proving (6.13). Let $\lambda \in \mathbb{C}$ (resp. $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$), and let ν and η satisfy

$$\begin{pmatrix} R-\lambda QP & QB \\ CP & D \end{pmatrix} \begin{pmatrix} \nu \\ \eta \end{pmatrix} = 0.$$

Denote $\xi = P\nu$, then we see that $C\xi + D\eta = 0$ and $(A-\lambda I)\xi + B\eta \in \ker Q$ for any $A \in \mathcal{A}$.

We will prove (6.13) holds in three separate cases: First, we prove the statement for real λ. For complex λ we consider the cases where the real and complex parts of ξ are linearly dependent and where these are linearly independent.

First suppose that $\lambda \in \mathbb{R}$. Then, without loss of generality, ν and η are real, and as such ξ is real. Suppose that $\xi \neq 0$, and take any $A \in \mathcal{A}$. Let A_0 be any real $n \times n$ matrix such that $A_0\xi = -(A - \lambda I)\xi - B\eta$ and $QA_0P = 0$. Such a matrix exists as $-(A - \lambda I)\xi - B\eta \in \ker Q$ and is a real vector and $\xi \neq 0$. Now take $\bar{A} = A + A_0$. Then it is immediate that $\bar{A} \in \mathcal{A}$ and:

$$\begin{pmatrix} \bar{A} - \lambda I & B \\ C & D \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0.$$

As (6.12) holds for $\bar{A}$ by assumption, we see that $\xi = 0$, which leads to a contradiction. Therefore $\xi = 0$.

Now consider that case where $\lambda \notin \mathbb{R}$. Suppose that the real and complex parts of ξ are linearly dependent. Therefore, there exist real scalars $\alpha, \beta \in \mathbb{R}$ and a real vector r such that $\xi = (\alpha + i\beta)r$. Let $\hat{r} = (\alpha - i\beta)\xi = (\alpha^2 + \beta^2)r$. Let $A \in \mathcal{A}$, then:

$$\begin{pmatrix} Q(A - \lambda I) & QB \\ C & D \end{pmatrix} \begin{pmatrix} \hat{r} \\ (\alpha - i\beta)\eta \end{pmatrix} = 0.$$

Denote $\lambda = a + bi$, where $b \neq 0$, and $(\alpha - i\beta)\eta = \eta_1 + i\eta_2$. Then we see that: $Q(A - aI)\hat{r} + QB\eta_1 = -b\hat{r} + QB\eta_2 = 0$ and $C\hat{r} + D\eta_1 = D\eta_2 = 0$. Let $\mu \in \mathbb{R}$ (resp. $\mu \in \mathbb{R}$ such that $|\mu| \geqslant 1$). Note that

$$\begin{pmatrix} Q(A - \mu I) & QB \\ C & D \end{pmatrix} \begin{pmatrix} b\hat{r} \\ b\eta_1 + (\mu - a)\eta_2 \end{pmatrix} = 0.$$

As μ is real, we can now apply the previous part of the proof to note that $b\hat{r} = 0$, which holds only if $\xi = 0$.

Now suppose that $\xi = Pp + iPq$, where Pp and Pq are linearly independent. If we take any $A \in \mathcal{A}$, we know that $Q(A - \lambda I)\xi + QB\eta = 0$, and that we can denote $(A - \lambda I)\xi + B\eta = \zeta_1 + \zeta_2 i$, where $\zeta_1, \zeta_2 \in \ker Q$. Take A_0 any real map such that $A_0Pp = -\zeta_1$, $A_0Xq = -\zeta_2$ and $QA_0P = 0$. Such a map exists as Pp and Pq are linearly independent. Now take $\bar{A} = A + A_0$, then $\bar{A} \in \mathcal{A}$ and clearly

$$\begin{pmatrix} \bar{A} - \lambda I & B \\ C & D \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0.$$

Using (6.12), this implies that $\xi = 0$. This is a contradiction with the fact that Pp and Pq are linearly independent. ∎

6.4 DATA DRIVEN INFORMATIVITY ANALYSIS

In this section we will apply Theorem 6.1 to obtain necessary and sufficient conditions for informativity of input-state-output data for the system properties listed in Section II. For a given system (6.6) we will denote by $\mathbf{x}(t, x_0, \mathbf{u})$ and $\mathbf{y}(t, x_0, \mathbf{u})$ the state and output sequence corresponding to the initial state $\mathbf{x}(0) = x_0$ and input sequence $\mathbf{u}$.

6.4.1 Informativity for strong observability and detectability

We first briefly review the properties of strong observability and strong detectability (see also [165]).

Definition 6.1. The system (6.6) is called *strongly observable* if for each $x_0 \in \mathbb{R}^n$ and input sequence $\mathbf{u}$ the following holds: $\mathbf{y}(t, x_0, \mathbf{u}) = 0$ for all $t \in \mathbb{Z}_+$ implies that $x_0 = 0$. The system is called *strongly detectable* if for all $x_0 \in \mathbb{R}^n$ and every input sequence $\mathbf{u}$ the following holds: $\mathbf{y}(t, x_0, \mathbf{u}) = 0$ for all $t \in \mathbb{Z}_+$ implies that $\lim_{t\to\infty} \mathbf{x}(t, x_0, \mathbf{u}) = 0$.

For continuous-time systems, necessary and sufficient conditions for strong observability and strong detectability were formulated in [165]. It can be verified that also the discrete-time system (6.6) is strongly observable (strongly detectable) if and only if the pair $(C + DK, A + BK)$ is observable (detectable) for all K. It is also straightforward to verify the following.

Proposition 6.1. The system (6.6) is strongly observable if and only if for all $\lambda \in \mathbb{C}$

$$\operatorname{rank} \begin{pmatrix} A - \lambda I & B \\ C & D \end{pmatrix} = n + \operatorname{rank} \begin{pmatrix} B \\ D \end{pmatrix}. \tag{6.14}$$

The system (6.6) is strongly detectable if and only if (6.14) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

As in Section III, we now consider the situation that only the matrices B, C and D are given, and that the matrix A can be any matrix from the affine set (6.5) with P, Q and R given matrices. By applying Theorem 6.1 we then get the following necessary and sufficient conditions for strong observability and strong detectability of *all* systems (6.6) with A ranging over the affine set $\mathcal{A}$.

Theorem 6.2 (Uniform rank condition)**.** Let (P, Q, R) and (B, C, D) be given matrices. Then (6.6) is strongly observable for all $A \in \mathcal{A}$ if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X$ and for all $\lambda \in \mathbb{C}$ we have

$$\operatorname{rank} \begin{pmatrix} R - \lambda QP & QB \\ CP & D \end{pmatrix} = \operatorname{rank} P + \operatorname{rank} \begin{pmatrix} QB \\ D \end{pmatrix}. \tag{6.15}$$

Similarly, (6.6) is strongly detectable for all $A \in \mathcal{A}$ if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ and (6.15) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

Proof. This follows immediately by combining Proposition 6.1 and Theorem 6.1. ∎

We will now apply the previous result to informativity of input-state-output data. Suppose the data are (U_-, X, Y_-). Recall Definition (6.4) of the affine set $\mathcal{A}_{\text{dat}}$ of all $n \times n$ matrices A such that the data are compatible with the system (A, B, C, D, E, F). We want to obtain conditions under which the data are informative for strong observability and for strong detectability. To this end, let $(M \;\; N)$ be any matrix such that

$$\ker(M \;\; N) = \operatorname{im} \begin{pmatrix} E \\ F \end{pmatrix}. \tag{6.16}$$

Then we have $A \in \mathcal{A}_{\text{dat}}$ if and only if $R = MAX_-$ with

$$R := (M \;\; N) \begin{pmatrix} X_+ - BU_- \\ Y_- - CX_- - DU_- \end{pmatrix}. \tag{6.17}$$

The following then immediately follows from Theorem 6.2.

Theorem 6.3. The data (U_-, X, Y_-) are informative for strong observability if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ and for all $\lambda \in \mathbb{C}$ we have

$$\operatorname{rank} \begin{pmatrix} R - \lambda MX_- & MB \\ CX_- & D \end{pmatrix} = \operatorname{rank} X_- + \operatorname{rank} \begin{pmatrix} MB \\ D \end{pmatrix}, \tag{6.18}$$

where R is given by (6.17).

The data are informative for strong detectability if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ and (6.18) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$.

In the case of independent process and measurement noise (see Remark 6.1), in which $E = (E_1 \;\; 0)$ and $F = (0 \;\; F_2)$, we have $A \in \mathcal{A}_{\text{dat}}$ if and only if there exists a matrix W_{1-} such that $X_+ = AX_- + BU_- + E_1W_{1-}$. Thus, $A \in \mathcal{A}_{\mathrm{dat}}$ if and only if $R = MAX_-$ with

$$R := M(X_+ - BU_-), \tag{6.19}$$

and M such that $\ker M = \operatorname{im} E_1 = \operatorname{im} E$. In this case, the formulation of Theorem 6.3 holds verbatim with this M, and the new R given by (6.19).

Finally, for the special case $E = 0$ (the case with no process noise), we have $A \in \mathcal{A}_{\mathrm{dat}}$ if and only if $R = AX_-$ with

$$R := X_+ - BU_- , \tag{6.20}$$

In that case, Theorem 6.3 holds verbatim with $M = I_n$ and R given by (6.20).

6.4.2 Informativity for observability and detectability

Next, we turn to characterizing informativity of the data for the properties of observability and detectability. Consider the system

$$x(t+1) = Ax(t), \ y(t) = Cx(t). \tag{6.21}$$

The Hautus test states that (6.21) is observable (detectable) if and only if

$$\operatorname{rank} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n$$

for all $\lambda \in \mathbb{C}$ (for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$).

Now, take the situation that only C is known, that matrices P, Q and R are given, and that A can be any matrix from the affine set $\mathcal{A}$ given by (6.5). By applying Theorem 6.7 to the special case $B = 0$ and $D = 0$, we then obtain the following.

Corollary 6.1 (Uniform Hautus test)**.** Let (P, Q, R) and C be given matrices. Then (6.21) is observable for all $A \in \mathcal{A}$ if and only if $\ker C \subseteq \operatorname{im} P$ and for any $\lambda \in \mathbb{C}$ we have

$$\operatorname{rank} \begin{pmatrix} R - \lambda QP \\ CP \end{pmatrix} = \operatorname{rank} P. \tag{6.22}$$

Similarly, (6.21) is detectable for all $A \in \mathcal{A}$ if and only if $\ker C \subseteq \operatorname{im} P$ and (6.22) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

We now apply the previous result to the situation that input-state-output data on the system are available, as explained in Section II. As before, suppose the data are (U_-, X, Y_-) and consider the affine set $\mathcal{A}_{\mathrm{dat}}$ of all $n \times n$ matrices given by (6.4). The next result establishes conditions under which the data are informative for observability and for detectability.

Corollary 6.2. Let (U_-, X, Y_-) be given input-state-output data. Let $(M \;\; N)$ be any matrix such that (6.16) holds. Let R be given by (6.17). The data are informative for observability if and only if $\ker C \subseteq \operatorname{im} X_-$ and for all $\lambda \in \mathbb{C}$ we have

$$\operatorname{rank} \begin{pmatrix} R - \lambda M X_- \\ C X_- \end{pmatrix} = \operatorname{rank} X_-. \tag{6.23}$$

The data are informative for detectability if and only if $\ker C \subseteq \operatorname{im} X_-$ and (6.22) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$.

Again, in the special case that the process noise and measurement noise are independent, Corollary 6.2 holds verbatim with M such that $\ker M = \operatorname{im} E$ and R given by (6.19). For the case that there is no process noise, in the rank test (6.23) we should take $M = I_n$ and R given by (6.20).

Example 6.3. As an example, consider the system (6.1) with

$$A_{\text{true}} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; E = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \; D = 0, \; F = 0.$$

Suppose that the following data are given:

$$U_- = \begin{pmatrix} 1 & 1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \; Y_- = \begin{pmatrix} 0 & 0 \end{pmatrix}. \tag{6.24}$$

These data are indeed compatible with the true system, since (6.3) holds with $W_- = \begin{pmatrix} 0 & 1 \end{pmatrix}$. It is easily verified that

$$\mathcal{A}_{\text{dat}} = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

We will check whether the data are informative for strong detectability. Take $M = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Since $F = 0$ we have $R = M(X_+ - BU_-) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $MX_- = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $CX_- = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $MB = 1$. The condition $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ is satisfied, so informativity for strong detectability holds if and only if

$$\operatorname{rank} \begin{pmatrix} 0 & -\lambda & 1 \\ 0 & 0 & 0 \end{pmatrix} = 2$$

for $|\lambda| \geqslant 1$, which is clearly not the case. We now check informativity for detectability. This requires $\ker C \subseteq \operatorname{im} X_-$ and

$$\operatorname{rank} \begin{pmatrix} 0 & -\lambda \\ 0 & 0 \end{pmatrix} = 1$$

for $|\lambda| \geqslant 1$. Both conditions indeed hold. On the other hand, the data are not informative for observability since the rank condition fails for $\lambda = 0$. If, in the example, we modify C and take $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, and accordingly $Y_- = \begin{pmatrix} 0 & 1 \end{pmatrix}$, then the data are still not informative for strong observability. In that case the rank condition does hold for all $\lambda \in \mathbb{C}$, but the condition $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ is violated.

Remark 6.2. For the noiseless case, without proof we mention that if, apart from A_{true}, also C_{true} is unknown (but B and D are still known), then both for informativity for observability and detectability a necessary condition is that X_- has full row rank. As illustrated in Example 6.3, this is no longer the case if C_{true} is known. Since $X_+ = A_{\text{true}}X_- + BU_-$ and $Y_- = C_{\text{true}}X_- + DU_-$, this implies $A_{\text{true}} = (X_+ - BU_-)X_-^\dagger$ and $C_{\text{true}} = (Y_- - DU_-)X_-^\dagger$ for any right-inverse $X_-^\dagger$ of X_-. Hence, in that case the data are informative for observability (detectability) if and only if X_- has full row rank, and the pair $((Y_- - DU_-)X_-^\dagger, (X_+ - BU_-)X_-^\dagger)$ is observable (detectable). The unknown A_{true} and C_{true} are then uniquely determined by the data.

6.4.3 Informativity for strong controllability and stabilizability

For the system (6.6), the dual properties of strong observability and strong detectability are strong controllability and strong stabilizability. These properties can be defined in terms of trajectories of the system. Here, for brevity, we define (6.6) to be strongly controllable (strongly stabilizable) if the pair $(A + LC, B + LD)$ is controllable (stabilizable) for all L. From this it is immediate that (6.6) is strongly controllable (strongly stabilizable) if and only if the dual system $(A^\top, C^\top, B^\top, D^\top)$ is strongly observable (strongly detectable). As before, assume that B, C and D are given, but that A can be any matrix from the affine set $\mathcal{A} := \{A \in \mathbb{R}^{n \times n} \mid R = QAP\}$, where P, Q and R are given. Obviously, $A \in \mathcal{A}$ if and only if $A^\top$ satisfies $R^\top = P^\top A^\top Q^\top$. The above observations make the following a matter of course.

Corollary 6.3. Let (P, Q, R) and (B, C, D) be given. Then (6.6) is strongly controllable for all $A \in \mathcal{A}$ if and only if $\ker Q \subseteq B \ker D$ and for all $\lambda \in \mathbb{C}$

$$\operatorname{rank} \begin{pmatrix} R - \lambda QP & QB \\ CP & D \end{pmatrix} = \operatorname{rank} Q + \operatorname{rank} \begin{pmatrix} CP & D \end{pmatrix}. \tag{6.25}$$

Similarly, (6.6) is strongly stabilizable for all $A \in \mathcal{A}$ if and only if $\ker Q \subseteq B \ker D$ and (6.25) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

Since a given pair (A, B) is controllable (stabilizable) if and only if the quadruple $(A, B, 0, 0)$ is strongly controllable (strongly stabilizable), the following also follows immediately.

Corollary 6.4 (Uniform Hautus test)**.** Given (P, Q, R) and B, the pair (A, B) is controllable for all $A \in \mathcal{A}$ if and only if $\ker Q \subseteq \operatorname{im} B$ and for any $\lambda \in \mathbb{C}$

$$\operatorname{rank}\begin{pmatrix} R - \lambda QP & QB \end{pmatrix} = \operatorname{rank} Q. \tag{6.26}$$

Furthermore (A, B) is stabilizable if and only if $\ker Q \subseteq \operatorname{im} B$ and (6.26) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

By appying the above in the context of informativity, we immediately obtain the following.

Corollary 6.5. Let $\begin{pmatrix} M & N \end{pmatrix}$ be such that (6.16) holds. Given the data (U_-, X, Y_-), let R be given by (6.17). The data are informative for strong controllability if and only if $\ker M \subseteq \operatorname{im} B$ and for all $\lambda \in \mathbb{C}$ we have

$$\operatorname{rank}\begin{pmatrix} R - \lambda M X_- & MB \\ CX_- & D \end{pmatrix} = \operatorname{rank} M + \operatorname{rank}\begin{pmatrix} CX_- & D \end{pmatrix}. \tag{6.27}$$

The data are informative for strong stabilizability if and only if $\ker M \subseteq \operatorname{im} B$ and (6.27) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$.

Corollary 6.6. Let $\begin{pmatrix} M & N \end{pmatrix}$ be such that (6.16) holds and let R be given by (6.17). The data (U_-, X, Y_-) are informative for controllability if and only if $\ker M \subseteq \operatorname{im} B$ and for all $\lambda \in \mathbb{C}$ we have

$$\operatorname{rank}\begin{pmatrix} R - \lambda M X_- & MB \end{pmatrix} = \operatorname{rank} M. \tag{6.28}$$

The data are informative for stabilizability if and only if $\ker M \subseteq \operatorname{im} B$ and (6.28) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$.

As before, in the special case of independent process and measurement noise, Corollary 6.5 and 6.6 hold verbatim with M such that $\ker M = \operatorname{im} E$ and R given by (6.19). In this special case, the rank test for controllability and stabilizability can be simplified to $\operatorname{rank} M \begin{pmatrix} X_+ - \lambda X_- & B \end{pmatrix} = \operatorname{rank} M$ for all $\lambda \in \mathbb{C}$, and $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$, respectively.

If there is no process noise, in the rank tests (6.27) and (6.28) we should take $M = I_n$ and $R = X_+ - BU_-$. For this special case, the rank test for controllability and stabilizability can even be simplified to

$$\operatorname{rank}\begin{pmatrix} X_+ - \lambda X_- & B \end{pmatrix} = n \tag{6.29}$$

for all $\lambda \in \mathbb{C}$, and $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$, respectively.

Remark 6.3. The rank test (6.29) can also be derived from [172, Theorem 8]. Indeed, that theorem states that all pairs (A, B) that satisfy the linear equation $X_+ = AX_- + BU_-$ are controllable if and only if rank $\big(X_+ - \lambda X_-\big) = n$ for all $\lambda \in \mathbb{C}$. This result can be applied to our set up, where we assume that only A is unknown and that B is given. Indeed, by defining 'new data' by

$$\tilde{X}_+ := \begin{bmatrix} X_+ & B \end{bmatrix}, \tilde{X}_- := \begin{bmatrix} X_- & 0 \end{bmatrix}, \tilde{U}_- := \begin{bmatrix} U_- & I_m \end{bmatrix}, \tag{6.30}$$

we have that a matrix A satisfies $X_+ = AX_- + BU_-$ if and only if (A, B) satisfies $\tilde{X}_+ = A\tilde{X}_- + B\tilde{U}_-$. By applying [172, Theorem 8] to the new data (6.30) we then get that (A, B) is controllable for all A satisfying $X_+ = AX_- + BU_-$ if and only if (6.29) holds.

Example 6.4. Again take as the true system the one specified in Example 6.3. Also, let the data be given by (6.24). Note that the condition $\ker M \subset \operatorname{im} B$ is violated, so the data are neither informative for strong controllability nor for strong stabilizability. They are also not informative for controllability or stabilizability.

6.5 A GEOMETRIC APPROACH TO INFORMATIVITY

It is well known, see for example [165], that observability and strong observability also allow tests in terms of certain subspaces of the state space, more specifically, the unobservable subspace and weakly unobservable subspace. Properties of the weakly unobservable subspace also characterize left-invertibility of the system. In this section we will use these ideas to characterize informativity for strong observability, observability and left-invertibility.

Again consider the system (6.6). We call a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ output-nulling controlled invariant if

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \times \{0\} + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{6.31}$$

(see [117, 165]). Since any finite sum of such subspaces retains this property, there exists a unique largest output-nulling controlled invariant subspace, which will be denoted by $\mathcal{V}(A, B, C, D)$. This subspace is called the *weakly unobservable subspace* of the system (6.6). The system (6.6) is strongly observable if and only if $\mathcal{V}(A, B, C, D) = \{0\}$, see [165, pp. 159-160 and Theorem 7.16].

Now, again consider the situation that the matrices B, C and D are specified, but that A can be any matrix from the affine set $\mathcal{A}$ given by (6.5), where the matrices $P \in \mathbb{R}^{n\times r}$, $Q \in \mathbb{R}^{\ell\times n}$ and $R \in \mathbb{R}^{\ell\times r}$ are given. We consider the set of all subspaces $\mathcal{J} \subseteq \mathbb{R}^r$ that satisfy the following inclusion:

$$\begin{bmatrix} R \\ CP \end{bmatrix} \mathcal{J} \subseteq QP\mathcal{J} \times \{0\} + \operatorname{im} \begin{bmatrix} QB \\ D \end{bmatrix}. \tag{6.32}$$

It is easily verified that any finite sum retains this property, and therefore there exists a largest subspace of $\mathbb{R}^r$ that satisfies the inclusion (6.32). We will denote this subspace by $\mathcal{J}^\star$.

Remark 6.4. It is straightforward to check that $\mathcal{J}^\star$ can be found from B, C, D, P, Q and R in at most r steps by letting $\mathcal{J}_0 = \mathbb{R}^r$, and iterating

$$\mathcal{J}_{t+1} = \begin{bmatrix} R \\ CP \end{bmatrix}^{-1} \left(QP\mathcal{J}_t \times \{0\} + \operatorname{im} \begin{bmatrix} QB \\ D \end{bmatrix} \right). \tag{6.33}$$

The following result will be instrumental in the remainder of this section.

Theorem 6.4. Let (P, Q, R) and (B, C, D) be such that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$. Then the following hold:

1. For all $A \in \mathcal{A}$, we have $\mathcal{V}(A, B, C, D) \subseteq P\mathcal{J}^\star$.
2. There exists $\bar{A} \in \mathcal{A}$ such that $P\mathcal{J}^\star \subseteq \mathcal{V}(\bar{A}, B, C, D)$.

Proof. (1): Assume that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ holds. Let $A \in \mathcal{A}$ and let $\mathcal{V} \subseteq \mathbb{R}^n$ be an output nulling controlled invariant subspace. Note that $C\mathcal{V} \subseteq \operatorname{im} D$, and therefore there exists a subspace $\mathcal{J}$ such that $\mathcal{V} = P\mathcal{J}$. We now see that

$$\begin{bmatrix} R \\ CP \end{bmatrix} \mathcal{J} = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \begin{bmatrix} A \\ C \end{bmatrix} P\mathcal{J} \subseteq QP\mathcal{J} \times \{0\} + \operatorname{im} \begin{bmatrix} QB \\ D \end{bmatrix}.$$

Due to the definition of $\mathcal{J}^\star$, we then obtain $\mathcal{V}(A, B, C, D) \subseteq P\mathcal{J}^\star$.

(2): Let $\mathcal{J}$ satisfy (6.32). Then for any $A \in \mathcal{A}$ and $x \in P\mathcal{J}$ there exists $u \in \mathbb{R}^m$ such that:

$$Cx + Du = 0, \quad \text{and} \quad QAx + QBu \in QP\mathcal{J}.$$

This implies that

$$Ax + Bu \subseteq Q^{-1}Q(Ax + Bu) \subseteq Q^{-1}QP\mathcal{J} = P\mathcal{J} + \ker Q.$$

Now let $\{x_1, ..., x_k\}$ be a basis of the subspace $P\mathcal{J}$. We can use the previous to write $Ax_i + Bu_i = y_i + z_i$, where $Cx_i + Du_i = 0$, $y_i \in P\mathcal{J}$ and $z_i \in \ker Q$. Let A_0 be any real $n \times n$ matrix such that $A_0 x_i = -z_i$ for $i = 1, \dots, k$ and

$QA_0P = 0$. Then, if we define $\bar{A} = A + A_0$, we see that $\bar{A} \in \mathcal{A}$. By definition $\bar{A}x_i + Bu_i = y_i \in P\mathcal{J}$, and therefore, if we write $\mathcal{V} = P\mathcal{J}$, we have:

$$\begin{bmatrix} \bar{A} \\ C \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \times \{0\} + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}.$$

Therefore $P\mathcal{J} \subseteq \mathcal{V}(\bar{A}, B, C, D)$, proving that $P\mathcal{J}^\star \subseteq \mathcal{V}(\bar{A}, B, C, D)$. ■

Using Theorem 6.4 we immediately obtain the following.

Theorem 6.5. Let (B, C, D) and (P, Q, R) be given. Then the system (6.6) is strongly observable for all $A \in \mathcal{A}$ if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ and $\mathcal{J}^\star \subseteq \ker P$.

Proof. From Theorem 6.2 we see that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ is a necessary condition. The rest follows from Theorem 6.4. ■

The procedure can be mimicked in order to characterize observability. For the system (6.21), the unobservable subspace $\mathcal{N}$ is the largest A-invariant subspace contained in $\ker C$, and (6.21) is observable if and only if $\mathcal{N} = \{0\}$. In the situation that only C and matrices (P, Q, R) are given, while A can be any matrix in the affine set $\mathcal{A}$, we should look at the largest subspace $\mathcal{L} \subseteq \mathbb{R}^r$ with the properties that

$$R\mathcal{L} \subseteq QP\mathcal{L} \text{ and } CP\mathcal{L} = \{0\}. \tag{6.34}$$

Denote this subspace by $\mathcal{L}^\star$. Then we have

Corollary 6.7. Given (P, Q, R) and C, then (6.21) is observable for all $A \in \mathcal{A}$ if and only if $\ker C \subseteq \operatorname{im} P$ and $\mathcal{L}^\star \subseteq \ker P$.

The subspace $\mathcal{L}^\star$ is obtained in at most r steps by applying the iteration (6.33) with $B = 0$ and $D = 0$.

We now very briefly put the above in the context of informativity of input-state-output data. As before, let (U_-, X, Y_-) be the noisy data obtained from the system (6.1). Let $(M \;\; N)$ be any matrix such that (6.16) holds. Then, by Theorem 6.4, these data are informative for strong observability of (6.6) if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ and $\mathcal{J}^\star \subseteq \ker X_-$, where $\mathcal{J}^\star$ is the largest subspace satisfying (6.32) with R given by (6.17), $P = X_-$ and $Q = M$. Likewise, informativity for observability holds if and only if $\ker C \subseteq \operatorname{im} X_-$ and $\mathcal{L}^\star \subseteq \ker X_-$.

Obviously, the above can, again, be dualized to obtain alternative tests for informativity for controllability and strong controllability. We omit the details here. Instead, we will turn to informativity for the property of left-invertibility of the system (6.6) now. We briefly recall the definition.

Definition 6.2. The system (6.6) is called *left-invertible* if for each input sequence $\mathbf{u}$ the following holds: $\mathbf{y}(t, 0, \mathbf{u}) = 0$ for all $t \in \mathbb{Z}_+$ implies that $\mathbf{u}(t) = 0$ for all $t \in \mathbb{Z}_+$.

The following characterization of left-invertibility was given in [165, Thm. 8.26].

Proposition 6.2. The following are equivalent:

1. The system (6.6) is left-invertible.
2. $\mathcal{V}(A, B, C, D) \cap B \ker D = \{0\}$ and $\begin{pmatrix} B \\ D \end{pmatrix}$ has full column rank.

The next result then, again, follows from Theorem 6.4.

Theorem 6.6. Let (P, Q, R) and (B, C, D) be given. Assume that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$. Then the system (6.6) is left-invertible for all $A \in \mathcal{A}$ if and only if $P\mathcal{J}^\star \cap B \ker D = \{0\}$ and $\begin{pmatrix} B \\ D \end{pmatrix}$ has full column rank.

As before, this can immediately be applied in the context of informativity. We omit the details.

Remark 6.5. Note that Theorem 6.6 requires $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$, which, unfortunately, for left-invertibility for all $A \in \mathcal{A}$ is not a necessary condition. This can be seen, for example, by taking $D = I$. Then, regardless of our choice of (P, Q, R), B and C, we see that (6.6) is left-invertible for all $A \in \mathcal{A}$. However, in this case $C^{-1} \operatorname{im} D = \mathbb{R}^n$, so the condition $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ is violated if P does not have full row rank.

To conclude this section, we note that Theorem 6.6 can be dualized in a straightforward way to obtain a characterization of right-invertibility for all $A \in \mathcal{A}$, and conditions for informativity of data for right-invertibility. Again, we omit the details.

To illustrate the the theory developed in this section we give the following example.

Example 6.5. Consider the system (6.1) with

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = 0, \quad F = 0.$$

Let data be given by

$$X = \begin{bmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; U_- = \begin{bmatrix} 0 & 0 & 4 \end{bmatrix}, \; Y_- = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Since there is only process noise, we should take M such that $\ker M = \operatorname{im} E$. Define

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad R = M(X_+ - BU_-) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easily verified that

$$\mathcal{A}_{\text{dat}} = \left\{ \begin{bmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & 0 & 1 & 0 \\ a_{31} & 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \mid a_{ij} \in \mathbb{R} \right\}.$$

Note that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$. In this case $\mathcal{J}^\star = \mathbb{R}^3$, and therefore the data are not informative for strong observability. On the other hand, $\mathcal{L}^\star = \{0\}$, proving that we do have informativity for observability.

If we modify our system by taking $B = e_i$, the ith standard basis vector in $\mathbb{R}^4$ ($i = 2, 3, 4$), and adapt the data X accordingly, we get $\mathcal{J}^\star = \mathbb{R}^{4-i} \times \{0\}^{i-1}$. This means that $X_- \mathcal{J}^\star = \{0\}^i \times \mathbb{R}^{4-i}$. Thus, only for $i = 4$, the data are informative for strong observability. For $i = 2, 3, 4$ the data are informative for left-invertibility.

6.6 CONCLUSIONS

In this chapter we have given necessary and sufficient conditions for informativity of noisy data obtained from a given unknown system for a range of system properties. These conditions are in terms of rank tests on polynomial matrices that can be constructed from these noisy data. The main instrument used to obtain these tests was a general theorem that expresses a rank property of the Rosenbrock system matrix of an unknown system in terms of a polynomial matrix that collects available information about that system. We have also set up a geometric framework for informativity analysis. Within that framework we have found geometric tests for informativity of data for strong observability, observability, and left-invertibility.

Within the framework of this chapter, no assumptions are made on the noise samples, and in that sense our noise model is very general. On the other hand, complete knowledge on how the noise influences the system is assumed to be available (via E and F). A drawback of this noise model is that in some situations it may not be possible to draw conclusions on the system on the basis of data. For example, if within our framework $E = I$ and $F = 0$, it is impossible to draw conclusions from data, no matter how many input/state/output samples have been collected. An interesting problem for future research would therefore be to investigate data-driven analysis from noisy data under the assumption that the noise samples W_- are bounded. Relevant noise models with bounded noise samples have, for example, been proposed in [23, 42, 99, 173].

As an extension of the work in this chapter we also see its generalization to the case that, apart from the A-matrix of the unknown system, also (parts of) the matrices B, C and D are unknown. Finally, it would be interesting to apply Theorem 6.1 in the context of structured systems, where specific entries of the system matrices are constrained to satisfy certain linear equations, and the remaining entries are arbitrary.

7 INFORMATIVITY FOR CONVEX PROCESSES

In this chapter, we will bring together the two parts of this thesis. In particular, we will work on informativity conditions for data collected from a convex process. This will highlight a number of appealing properties of both the framework of convex processes and that of informativity.

7.1 INTRODUCTION

This chapter deals with the question: what can be inferred from an unknown constrained linear system on the basis of state measurements? A similar question, for unconstrained systems, has recently led to the development of the *informativity framework* in [172]. The observation at the center of this framework is that we can only conclude that the unknown system has a given property if *all* systems compatible with the measurements have this property. In the context of linear systems this has lead to, among others, results for analysis problems in [49] and control problems in [166, 170]. Parallel to the work performed within this framework, similar analysis problems are addressed in [99], while control problems are addressed in [23, 42].

In contrast to this earlier work, we will be focusing on conically constrained linear systems. Such conic constraints often arise naturally in modeling, taking the form of e.g. nonnegativity constraints on the input or states. Specifically, we will be looking at the class of difference inclusions of the form

$$x_{k+1} \in H(x_k)$$

where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex process, that is, a set-valued map whose graph is a convex cone. It it straightforward to show that any conically constrained linear system can be written as such a system and vice versa. Such difference inclusions arise naturally in many different contexts, including chemical reaction networks [7], von Neumann-Gale economic growth models [110] and cable-suspended robots [86, 126]. Lastly, as shown in e.g. [11, 61], difference inclusions of convex processes can be used as meaningful approximations of more complex set-valued maps.

The many applications of convex processes have led to interest in the analysis of such systems. In particular, this chapter will consider the system-theoretic properties of reachability and null-controllability. For a given convex process, tests for these properties have been developed in terms of spectral conditions. First among these were the characterizations of reachability and null-controllability in [14, 132]. However, the aforementioned characterizations only regard *strict* (nonempty everywhere) convex processes, which limits the applicability for our goals. In [45] both of these results are generalized to work for a class of nonstrict convex processes. In this chapter, the characterizations of [45] will be fundamental in our investigation of informativity. In particular, we will be interested in analyzing whether these system-theoretic properties hold for *all* convex processes compatible with a measured state trajectory.

Apart from the aforementioned work, some results in data-driven analysis and control should be mentioned. With regard to unconstrained linear systems [130] analyzes stability of an input/output system using time series data. The works [106, 125, 178, 188] deal with data-based controllability and observability analysis. Lastly, many methods arising from Model Predictive Control (MPC) are well suited to constrained systems. For an overview of such methods, we refer to [115, 116]. More specifically, MPC has recently been brought into a data-based context in [41, 133].

The contribution of this chapter is threefold:

1. We expand the informativity framework of [172] towards the class of convex processes. This framework will naturally lead to the formulation of a number of problems. In particular, we will illustrate the framework by resolving the problems of informativity for reachability and null-controllability.

2. We develop explicit tools to manipulate and perform analysis on convex processes with a polyhedral graph. Assuming polyhedrality will allow us to represent convex processes and the conditions required for reachability and null-controllability in a convenient way.

3. Lastly, we note the fact that polyhedral convex processes naturally arise from the aforementioned informativity problems with finite measurements. This allows us to combine the previous points to formulate tests *on measured state data* to conclude that all convex processes consistent with the data are reachable or null-controllable.

This chapter is organized as follows: We begin in Section 7.2 with definitions of convex process and reachability and null-controllability. After this, Section 7.3 introduces informativity and formally states the problem we will consider in this chapter. In Section 7.4, we will present some known results regarding the analysis of convex processes, which will be applied in Section 7.5 to our problem. We finalize the chapter with conclusions in Section 7.6.

7.2 CONVEX PROCESSES

Given convex sets $\mathcal{S}, \mathcal{T} \subseteq \mathbb{R}^n$ and scalar $\rho \in \mathbb{R}$ we define the sum and scalar multiplication of sets as:

$$\mathcal{S} + \mathcal{T} := \{s + t \mid s \in \mathcal{S}, t \in \mathcal{T}\}, \quad \rho\mathcal{S} := \{\rho s \mid s \in \mathcal{S}\}.$$

We denote the closure of $\mathcal{S}$ by $\operatorname{cl} \mathcal{S}$. A *convex cone* is a nonempty convex set that is closed under nonnegative scalar multiplication.

A *set-valued map*, denoted $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a map taking elements of $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$. It is called a *convex process*, *closed convex process* or *linear process* if its graph

$$\operatorname{gr}(H) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y \in H(x)\}$$

is a convex cone, closed convex cone or subspace, respectively.

The *domain* and *image* of H are defined as $\operatorname{dom}(H) = \{x \in \mathbb{R}^n \mid H(x) \neq \varnothing\}$ and $\operatorname{im} H = \{y \in \mathbb{R}^n : \exists\, x \text{ s.t. } y \in H(x)\}$. If $\operatorname{dom}(H) = \mathbb{R}^n$, we say that H is *strict*.

In this chapter we consider systems described by a *difference inclusion* of the form

$$x_{k+1} \in H(x_k) \tag{7.1}$$

where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex process. Our main motivation for considering this class of systems is the fact that this class of systems captures the behavior of all linear systems with convex conic constraints. This will be made explicit in the following example.

Example 7.1. Consider states x_k in $\mathbb{R}^n$ and inputs $u_k \in \mathbb{R}^m$. Let A and B be linear maps of appropriate dimensions and let $\mathcal{C} \subseteq \mathbb{R}^{n+m}$ be a convex cone. Consider the linear system with conic constraints given by:

$$x_{k+1} = Ax_k + Bu_k, \quad \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \mathcal{C}. \tag{7.2}$$

Note that this description can be applied to any combination of input, state and output constraints.

We can describe the dynamics of (7.2) by the difference inclusion (7.1) with the convex process H defined by:

$$H(x) := \left\{ Ax + Bu \;\middle|\; \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{C} \right\}.$$

This reveals that we can study the properties of conically constrained linear systems by studying convex processes, without any loss of generality.

Next, we define a number of sets associated with the difference inclusion (7.1). A q-*step trajectory* is a (finite) sequence $x_0, \dots, x_q$ such that (7.1) holds for all $k < q$. We define the q-*step behavior* as:

$$\mathfrak{B}_q(H) := \left\{ (x_k)_{k=0}^q \in (\mathbb{R}^n)^{q+1} \mid (x_k) \text{ satisfies } (7.1) \right\}.$$

Using this, we define the *reachable* and *null-controllable* sets by:

$$\begin{aligned} \mathcal{R}(H) &:= \left\{ \xi \mid \exists q, (x_k)_{k=0}^q \in \mathfrak{B}_q(H) \text{ s.t. } x_0 = 0, x_q = \xi \right\}, \\ \mathcal{N}(H) &:= \left\{ \xi \mid \exists q, (x_k)_{k=0}^q \in \mathfrak{B}_q(H) \text{ s.t. } x_0 = \xi, x_q = 0 \right\}. \end{aligned}$$

We say that a point $\xi \in \mathbb{R}^n$ is *reachable* if $\xi \in \mathcal{R}(H)$. That is, there exists a q-step trajectory from the origin to ξ. Similarly, we say a point $\xi \in \mathbb{R}^n$ is *null-controllable* if $\xi \in \mathcal{N}(H)$.

By a *trajectory* of (7.1), we mean a sequence $(x_k)_{k \in \mathbb{N}}$ such that (7.1) holds for all $k \geqslant 0$. The *behavior* is the set of all trajectories:

$$\mathfrak{B}(H) := \left\{ (x_k) \in (\mathbb{R}^n)^{\mathbb{N}} \mid (x_k) \text{ is a trajectory of } (7.1) \right\}.$$

The set of *feasible* states of the difference inclusion (7.1) is the set of states from which a trajectory emanates:

$$\mathcal{F}(H) := \{ \xi \mid \exists (x_k) \in \mathfrak{B}(H) \text{ with } x_0 = \xi \}.$$

Clearly, if H is a convex process, then $\mathcal{F}(H)$ is a convex cone.

It is important to stress that in general not every point in the state space is feasible: In Example 7.1, if we consider a point x_0 for which no u_0 satisfies the constraints, we have that $H(x_0) = \varnothing$. This means that x_0 is not a feasbile point. As there is no need to reach or control states that violate the constraints, we say the system (7.1) is *reachable* or *null-controllable* if every feasible state is reachable or null-controllable respectively. In terms of the previously defined sets, these can be written as $\mathcal{F}(H) \subseteq \mathcal{R}(H)$ and $\mathcal{F}(H) \subseteq \mathcal{N}(H)$ respectively.

It is important to note that, as is the case for discrete-time linear systems, reachability and null-controllability are not equivalent notions.

7.3 PROBLEM FORMULATION

In this chapter we are interested in analyzing the properties of an unknown system based on measurements performed on it. We will assume that the system under consideration is given by

$$x_{k+1} \in H_s(x_k)$$

where H_s is an unknown convex process. However, we do have access to a number of exact state measurements corresponding to ($q-$step) trajectories of H_s. It is clear to see that we can view a single $q-$step trajectory as q separate $1-$step trajectories. Therefore, without loss of generality, we assume that we measure single steps. That is, we are given a finite number of pairs $(x_k, y_k) \in \mathrm{gr}(H_s)$, with $k = 0, \dots, T$

Suppose that we are interested in characterizing reachability of H_s. As H_s is unknown, it is indistinguishable from all other convex processes that could have generated the measurements. Therefore, we may only conclude that H_s is reachable if *all* convex processes that are compatible with the data are reachable. This motivates the following definition. Let Σ denote the set of all convex processes $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a finite set of measurements. Define the set of all convex processes compatible with these measurements by:

$$\Sigma_{\mathcal{D}} := \{H \in \Sigma \mid \mathcal{D} \subseteq \mathrm{gr}(H)\}. \tag{7.4}$$

Recall that, in order to characterize whether H_s is reachable, we require all convex processes compatible with the measurements to be reachable. As such, we say that the data $\mathcal{D}$ are *informative for reachability* if every $H \in \Sigma_{\mathcal{D}}$ is reachable. In a similar way we define *informativity for null-controllability*.

Note that informativity is fundamentally a property of the data and the system class, but *not* of the system H_s. This leads to the following problem formulation:

Problem 7.1. Provide necessary and sufficient conditions on the data $\mathcal{D}$ under which the data are informative for reachability or null-controllability.

Remark 7.1. Following Example 7.1, it is clear that all convex processes consistent with the data are reachable if and only if all conically constrained linear systems consistent with the data are reachable. As these problems are equivalent we will only focus on formulations in terms of convex processes in the remainder of this chapter.

It should be noted that, in certain cases, the informativity problem can be resolved trivially, as shown by the following example.

Example 7.2. Let $n = 1$, and assume that we measure the 2-step trajectory given by $x_0 = 0$, $x_1 = 1$, and $x_2 = -1$. Then we have $\mathcal{D} = \{(0,1),(1,-1)\}$.

Note that nonnegative scalar multiples of these measurements are also (finite step) trajectories of any convex process in $\Sigma_{\mathcal{D}}$. As such, it is clear that for any $\alpha, \beta \geqslant 0$ we have 2-step trajectories $y_0 = 0$, $y_1 = \alpha$, $y_2 = -\alpha$ and $z_0 = 0$, $z_1 = 0$, $z_2 = \beta$. Furthermore, the sum of two such 2-step trajectories is one as well. Therefore $(0, \alpha, \beta - \alpha) \in \mathfrak{B}_2(H)$ for any H consistent with the data. As such, $\mathcal{R}(H) = \mathbb{R}$ for any $H \in \Sigma_{\mathcal{D}}$.

In general, however, resolving the problem is not this straightforward. To be precise, it is made difficult by two things. First of all, apart from trivial examples, the set $\Sigma_{\mathcal{D}}$ contains infinitely many convex processes. As such, it is usually not possible to take an approach based on identification. In addition, there may *not* exist q for a convex process H such that

$$\mathcal{R}(H) := \left\{ \xi \mid \exists (x_k)_{k=0}^{q} \in \mathfrak{B}_q(H) \text{ s.t. } x_0 = 0, x_q = \xi \right\}.$$

Therefore, testing whether a given convex process is reachable or null-controllable is a nontrivial problem in itself (see e.g. [14, 45]).

7.4 ANALYSIS OF CONVEX PROCESSES

By definition a convex cone $\mathcal{C}$ is closed under *conic combinations*: If $c_1, \ldots, c_\ell \in \mathcal{C}$ then

$$\sum_{i=1}^{\ell} \alpha_i c_i \in \mathcal{C} \quad \forall \alpha_i \geqslant 0.$$

The set of all conic combinations of a set $\mathcal{S}$ is called the *conic hull* and is denoted by $\operatorname{cone} \mathcal{S}$. If there exists a finite set $\mathcal{S}$ such that $\mathcal{C} = \operatorname{cone} \mathcal{S}$ we say that $\mathcal{C}$ is *finitely generated* or *polyhedral*. We denote the set of vectors of length ℓ with nonnegative and nonpositive elements by $\mathbb{R}^\ell_+$ and $\mathbb{R}^\ell_-$ respectively. Then, if $M \in \mathbb{R}^{k \times \ell}$ and $\mathcal{S}$ is the set of columns of M, we have that

$$\operatorname{cone} \mathcal{S} = M\mathbb{R}^\ell_+. \tag{7.5}$$

For a nonempty set $\mathcal{C} \subseteq \mathbb{R}^n$, we define the *negative* and *positive polar cone*, respectively,

$$\begin{aligned} \mathcal{C}^- &:= \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leqslant 0 \quad \forall x \in \mathcal{C}\}, \\ \mathcal{C}^+ &:= \{y \in \mathbb{R}^n \mid \langle x, y \rangle \geqslant 0 \quad \forall x \in \mathcal{C}\}. \end{aligned}$$

Given sets $\mathcal{C}$ and $\mathcal{S}$, we have that $(\mathcal{C}^-)^- = \mathrm{cl}(\mathrm{cone}\,\mathcal{C})$, and:

$$(\mathcal{C}+\mathcal{S})^- = \mathcal{C}^- \cap \mathcal{S}^-, \quad (\mathcal{C}\cap\mathcal{S})^- = \mathrm{cl}(\mathcal{C}^- + \mathcal{S}^-). \tag{7.6}$$

Let A be a linear map and let $\bullet^{-1}$ denotes the inverse image, that is, $A^{-1}\mathcal{C}^- = \{x \mid Ax \in \mathcal{C}^-\}$. Then if $\mathcal{C}$ is a convex cone we have that (see e.g. [12, Theorem 2.4.3]):

$$(A^\top \mathcal{C})^- = A^{-1}\mathcal{C}^-. \tag{7.7}$$

The aforementioned properties also hold for the positive polar cone.

Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex process. We define *negative* and *positive dual* processes H^- and H^+ of H as follows:

$$p \in H^-(q) \iff \langle p, x\rangle \geqslant \langle q, y\rangle \quad \forall\,(x,y) \in \mathrm{gr}(H), \tag{7.8a}$$

$$p \in H^+(q) \iff \langle p, x\rangle \leqslant \langle q, y\rangle \quad \forall\,(x,y) \in \mathrm{gr}(H). \tag{7.8b}$$

Note that $H^+(q) = -H^-(-q)$ for all q. If H is a closed convex process, we know that $(H^+)^- = H$ and

$$H(0) = (\mathrm{dom}(H^+))^+ = (\mathrm{dom}(H^-))^-. \tag{7.9}$$

It is straightforward to check that

$$\mathrm{gr}(H^-) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} (\mathrm{gr}(H))^-, \mathrm{gr}(H^+) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} (\mathrm{gr}(H))^+. \tag{7.10}$$

For a convex cone $\mathcal{C} \subseteq \mathbb{R}^n$, we define $\mathrm{lin}(\mathcal{C}) = -\mathcal{C} \cap \mathcal{C}$ and $\mathrm{Lin}(\mathcal{C}) = \mathcal{C} - \mathcal{C}$. We can now define two linear processes L_- and L_+ associated with H by

$$\mathrm{gr}(L_-) := \mathrm{lin}\big(\mathrm{gr}(H)\big) \text{ and } \mathrm{gr}(L_+) := \mathrm{Lin}\big(\mathrm{gr}(H)\big). \tag{7.11}$$

By definition, we therefore have

$$\mathrm{gr}(L_-) \subseteq \mathrm{gr}(H) \subseteq \mathrm{gr}(L_+). \tag{7.12}$$

It is clear that L_- and L_+ are, respectively, the largest and the smallest (with respect to the graph inclusion) linear processes satisfying (7.12). We call L_- and L_+, respectively, the minimal and maximal linear processes associated with H. If H is not clear from context, we write $L_-(H)$ and $L_+(H)$ in order to avoid confusion.

If L is a linear process it is clear that the negative and positive dual processes are equal, which allows us to denote it by $L^\perp := L^- = L^+$. In fact, the minimal and maximal linear processes associated with a convex process enjoy the following additional properties:

$$L_-(H^-) = L_-(H^+) = L_+^\perp, \tag{7.13a}$$

$$L_+(H^-) = L_+(H^+) = L_-^\perp. \tag{7.13b}$$

For the reachable and null-controllable sets of L_- and L_+ we use the following shorthand notation:

$$\mathcal{R}_- := \mathcal{R}(L_-), \quad \mathcal{R}_+ := \mathcal{R}(L_+).$$
$$\mathcal{N}_- := \mathcal{N}(L_-), \quad \mathcal{N}_+ := \mathcal{N}(L_+).$$

We denote the image of a set $\mathcal{S}$ under a convex process H by $H(\mathcal{S}) := \{y \in \mathbb{R}^n \mid \exists x \in \mathcal{S} \text{ s.t. } y \in H(x)\}$. A direct consequence of this definition is that

$$H(\mathcal{S}) = \begin{bmatrix} 0 & I_n \end{bmatrix} \left(\operatorname{gr}(H) \cap (\mathcal{S} \times \mathbb{R}^n) \right). \tag{7.14}$$

We can define powers of convex processes, by taking H^0 equal to the identity map, and letting for $q \geqslant 0$:

$$H^{q+1}(x) := H(H^q(x)) \quad \forall x \in \mathbb{R}^n.$$

We can define the inverse of a convex process by $H^{-1}(y) = \{x \mid y \in H(x)\}$. Note that this is always defined as a set-valued map. For higher negative powers of H we use the shorthand: $H^{-n}(x) = (H^{-1})^n(x)$.

Let $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a linear process, then we know that $\mathcal{F}(L) = L^{-n}(\mathbb{R}^n)$ and $\mathcal{R}(L) = L^n(0)$. In addition

$$\mathcal{F}(L^\perp) = \mathcal{R}(L)^\perp, \tag{7.15a}$$
$$\mathcal{R}(L^\perp) = \mathcal{F}(L)^\perp. \tag{7.15b}$$

We will characterize reachability in terms of spectral conditions. For this we require one more definition: A real number λ and vector $\xi \in \mathbb{R}^n \setminus \{0\}$ form an *eigenpair* of H if $\lambda\xi \in H(\xi)$. In this case λ is called an *eigenvalue* and ξ is called an *eigenvector* of H.

In the following, we will need the assumption:

$$\operatorname{dom}(H) + \mathcal{R}_- = \mathbb{R}^n. \tag{7.16}$$

As proven in Theorem 3.1, Lemma 3.20 (see also [45, Thm. 1, Lem. 7]), we can characterize reachability in terms of eigenvalues of the dual process.

Theorem 7.1. Let H be a convex process such that (7.16) holds. Then, the following are equivalent:

1. H is reachable.
2. $\mathcal{R}(H) = \mathbb{R}^n$.

3. $\mathcal{R}_+ = \mathbb{R}^n$ and H^- has no nonnegative eigenvalues.

We now move towards null-controllability. It is tempting to think that null-controllability of H is equivalent to reachability of H^{-1}. However, while indeed it is true that $\mathcal{R}(H^{-1}) = \mathcal{N}(H)$, we do not necessarily have that $\mathcal{F}(H^{-1}) = \mathcal{F}(H)$.

As such, we require a characterization of null-controllability. This will be done under slightly more restrictive assumptions than Theorem 7.1. To be precise, we will assume both (7.16) and

$$\mathcal{R}_+ = \operatorname{im} H + \mathcal{N}_- = \mathbb{R}^n. \tag{7.17}$$

The following was proven in Theorem 3.4, Lemma 3.20 (see also [45, Thm. 2, Lem. 9]):

Theorem 7.2. Let H be a convex process such that (7.16) and (7.17) hold. Then, the following are equivalent:

1. H is null-controllable.
2. $\mathcal{N}(H) - \mathcal{R}(H) = \mathbb{R}^n$.
3. H^- has no positive eigenvalues.

The following shows why we require separate tests for these two properties.

Example 7.3. Recall that, as is the case for discrete time linear systems, a convex process can be null-controllable without being reachable. As a simple example consider the convex process given by:

$$\operatorname{gr}(H) := \mathbb{R} \times \{0\}.$$

On the other hand, we know that reachability implies null-controllability for discrete time linear systems. For general convex processes this is not the case. As an example, let:

$$\operatorname{gr}(G) := \{(x, y) \mid 0 \leqslant x \leqslant y\}.$$

Note that $\mathcal{R}(G) = \mathbb{R}_+ = \mathcal{F}(G)$, and therefore G is reachable. As any trajectory of G is a nondecreasing sequence, G is clearly not null-controllable. This means that in general tests for reachability can not be employed to obtain results for null-controllability.

These two theorems allow us to check for reachability and null-controllability without explicitly determining $\mathcal{R}(H)$ or $\mathcal{N}(H)$. This will be central in resolving Problem 7.1 in the next section.

7.5 INFORMATIVITY FOR CONVEX PROCESSES

We turn our attention to the context of informativity. Let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a finite set of measurements. We define the *most powerful unfalsified process*, $H_{\mathcal{D}}$, by:

$$\operatorname{gr}(H_{\mathcal{D}}) := \operatorname{cone} \mathcal{D}.$$

By definition we see that $H_{\mathcal{D}} \in \Sigma_{\mathcal{D}}$ and $\operatorname{gr}(H_{\mathcal{D}}) \subseteq \operatorname{gr}(H)$ if and only if $H \in \Sigma_{\mathcal{D}}$. Our goal is to find conditions on $\mathcal{D}$ under which every $H \in \Sigma_{\mathcal{D}}$ is reachable or null-controllable. we start with the following theorem:

Theorem 7.3. Suppose that (7.16) holds for $H_{\mathcal{D}}$. Then $H_{\mathcal{D}}$ is reachable if and only if every $H \in \Sigma_{\mathcal{D}}$ is reachable.

Proof. Note that $H_{\mathcal{D}} \in \Sigma_{\mathcal{D}}$. Therefore the 'if' part is immediate. For the 'only if' part, assume that $H_{\mathcal{D}}$ is reachable. By Theorem 7.1, we have that $\mathcal{R}(H_{\mathcal{D}}) = \mathbb{R}^n$. Now let H be a convex process such that $\operatorname{gr}(H_{\mathcal{D}}) \subseteq \operatorname{gr}(H)$. As any $q-$step trajectory of $H_{\mathcal{D}}$ is one of H, it is immediate that $\mathcal{R}(H_{\mathcal{D}}) \subseteq \mathcal{R}(H)$. Therefore $\mathcal{R}(H) = \mathbb{R}^n$. This implies that H is reachable. ■

Remark 7.2. It is important to stress that a convex process H is defined to be reachable if $\mathcal{F}(H) \subseteq \mathcal{R}(H)$. Therefore a nonstrict convex process H can be reachable whilst $\mathcal{R}(H) \neq \mathbb{R}^n$. Now let $\operatorname{gr}(H) \subseteq \operatorname{gr}(G)$. Note that we may *not* conclude reachability of G from reachability of H in general. As an example, let $\operatorname{gr}(H) = \{0\}$. This convex process is reachable, and its graph is contained in the graph of any other convex process, which are not necessarily reachable.

Next, we study null-controllability. It is clear that the reasoning of Remark 7.2 also applies to null-controllability. This leads to an important point of contrast between Theorem 7.1 and Theorem 7.2: Under the conditions of the latter the convex process H can be null-controllable even if $\mathcal{N}(H) \neq \mathbb{R}^n$.

Theorem 7.4. Suppose that (7.16) and (7.17) hold for $H_{\mathcal{D}}$. Then, $H_{\mathcal{D}}$ is null-controllable if and only if every $H \in \Sigma_{\mathcal{D}}$ is null-controllable.

Proof. Again the 'if' part is immediate. For the 'only if' part, assume that $H_{\mathcal{D}}$ is null-controllable. Let H be a convex process such that $\operatorname{gr}(H_{\mathcal{D}}) \subseteq \operatorname{gr}(H)$. As in the proof of Theorem 7.3, we see that $\mathcal{R}(H_{\mathcal{D}}) \subseteq \mathcal{R}(H)$ and $\mathcal{N}(H_{\mathcal{D}}) \subseteq \mathcal{N}(H)$. This implies that

$$\mathbb{R}^n = \mathcal{N}(H_{\mathcal{D}}) - \mathcal{R}(H_{\mathcal{D}}) \subseteq \mathcal{N}(H) - \mathcal{R}(H).$$

Note that we also have $\mathrm{gr}(L_-(H_\mathcal{D})) \subseteq \mathrm{gr}(L_-(H))$ and $\mathrm{gr}(L_+(H_\mathcal{D})) \subseteq \mathrm{gr}(L_+(H))$. Therefore, it is clear that (7.16) and (7.17) hold for H. This implies that H is null-controllable. ■

The question rests whether we can provide simple tests for reachability and null-controllability of $H_\mathcal{D}$ in terms of the data $\mathcal{D}$. In order to resolve this, we will begin by giving two equivalent representations of $H_\mathcal{D}$.

Denote $T = |\mathcal{D}|$ and $\mathcal{D} = \{(x_t, y_t) : t = 1, \dots, T\}$. We define the matrices $X, Y \in \mathbb{R}^{n \times T}$ by taking:

$$X := \begin{bmatrix} x_1 & x_2 & \cdots & x_T \end{bmatrix}, \quad Y := \begin{bmatrix} y_1 & y_2 & \cdots & y_T \end{bmatrix}.$$

Since $\operatorname{cone} \mathcal{D}$ is a convex cone, we have that $\mathcal{D}^+ = (\operatorname{cone} \mathcal{D})^+$. As $\mathcal{D}$ is a finite set, we have that $\operatorname{cone} \mathcal{D}$ and $\mathcal{D}^+$ are polyhedral cones. This means that there exists $\ell \in \mathbb{N}$ and $\eta_1, \dots, \eta_\ell \in \mathbb{R}^{2n}$, such that $\mathcal{D}^+ = \operatorname{cone}\{\eta_1, \dots, \eta_\ell\}$. We can now define matrices $Z, W \in \mathbb{R}^{\ell \times n}$ by the following partition:

$$\begin{bmatrix} Z & -W \end{bmatrix} := \begin{bmatrix} \eta_1 & \dots & \eta_\ell \end{bmatrix}^\top.$$

As $\operatorname{cone} \mathcal{D}$ is closed, it is equal to $(\mathcal{D}^+)^+$. Recall that $\mathrm{gr}(H_\mathcal{D}) = \operatorname{cone} \mathcal{D}$. Therefore, we can use (7.5) to represent $H_\mathcal{D}$ in the following ways:

$$\mathrm{gr}(H_\mathcal{D}) = \begin{bmatrix} X \\ Y \end{bmatrix} \mathbb{R}^T_+ = \left\{ (x, y) \mid \begin{bmatrix} Z & -W \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^\ell_+ \right\}. \tag{7.18}$$

Immediately, we see that

$$\mathrm{dom}(H_\mathcal{D}) = X\mathbb{R}^T_+ \quad \text{and} \quad \operatorname{im} H_\mathcal{D} = Y\mathbb{R}^T_+.$$

Using (7.18) we can express the minimal and maximal linear processes of $H_\mathcal{D}$ as follows:

$$\begin{aligned} \mathrm{gr}(L_-(H_\mathcal{D})) &= \ker \begin{bmatrix} Z & -W \end{bmatrix}, \\ \mathrm{gr}(L_+(H_\mathcal{D})) &= \operatorname{im} \begin{bmatrix} X \\ Y \end{bmatrix}. \end{aligned}$$

For the characterizations of reachability and null-controllability in Theorem 7.1 and Theorem 7.2 respectively, we need the reachable and null-controllable sets of L_+ and L_-. In order to characterize these in terms of the data $\mathcal{D}$, we first look at the image of a set under these linear processes. For a given set $\mathcal{S} \subseteq \mathbb{R}^n$ we can apply (7.14) to verify that:

$$\begin{aligned} L_-(H_\mathcal{D})(\mathcal{S}) &= W^{-1} Z \mathcal{S}, \\ L_+(H_\mathcal{D})(\mathcal{S}) &= Y X^{-1} \mathcal{S}. \end{aligned}$$

Recall that for a linear process L the reachable set is *finitely determined* and $\mathcal{R}(L) = L^n(0)$. Combining the above with some slight abuse of notation, we can write:

$$\begin{aligned}\mathcal{R}(L_-(H_{\mathcal{D}})) &= (W^{-1}Z)^n\{0\},\\ \mathcal{R}(L_+(H_{\mathcal{D}})) &= (YX^{-1})^n\{0\}.\end{aligned}$$

This characterizes the reachable sets of $L_-(H_{\mathcal{D}})$ and $L_+(H_{\mathcal{D}})$ using subspace algorithms with at most n steps. Following the same reasoning with negative powers, we obtain that:

$$\begin{aligned}\mathcal{N}(L_-(H_{\mathcal{D}})) &= (Z^{-1}W)^n\{0\},\\ \mathcal{N}(L_+(H_{\mathcal{D}})) &= (XY^{-1})^n\{0\}.\end{aligned}$$

We now shift our focus to the negative dual of $H_{\mathcal{D}}$, and show that it can be represented in terms of X and Y or Z and W as well.

By (7.10) and the first representation of (7.18) we have that:

$$\operatorname{gr}(H_{\mathcal{D}}^-) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \left(\begin{bmatrix} X \\ Y \end{bmatrix} \mathbb{R}_+^T \right)^- .$$

By (7.7) this implies that:

$$\operatorname{gr}(H_{\mathcal{D}}^-) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} X^\top & Y^\top \end{bmatrix}^{-1} \mathbb{R}_-^T = \begin{bmatrix} Y^\top & -X^\top \end{bmatrix}^{-1} \mathbb{R}_-^T.$$

Similarly, we can begin from (7.10) and the second representation in (7.18) instead. As such, we can conclude that the negative dual of $H_{\mathcal{D}}$ satisfies:

$$\operatorname{gr}(H_{\mathcal{D}}^-) = \begin{bmatrix} W^\top \\ Z^\top \end{bmatrix} \mathbb{R}_+^\ell = \left\{ (x,y) \mid \begin{bmatrix} Y^\top & -X^\top \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}_-^T \right\}.$$

Then, we have that λ and ξ form an eigenpair of $H_{\mathcal{D}}^-$ if and only if $\xi \neq 0$ and $\xi^\top(Y - \lambda X) \leqslant 0$.

We can now combine the previous discussion with Theorem 7.1 and Theorem 7.3 to obtain the following characterization of informativity for reachability in terms of data:

Theorem 7.5. Let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a finite set. Suppose that

$$X\mathbb{R}_+^T + (W^{-1}Z)^n\{0\} = \mathbb{R}^n.$$

Then, $\mathcal{D}$ is informative for reachability if and only if $(YX^{-1})^n\{0\} = \mathbb{R}^n$ and for all $\lambda \geqslant 0$:

$$\xi^\top(Y - \lambda X) \leqslant 0 \implies \xi = 0.$$

Remark 7.3. Note that $(YX^{-1})^n\{0\} = \mathbb{R}^n$ implies that $\mathcal{R}(L) = \mathbb{R}^n$ for all *linear processes* L such that $\mathcal{D} \subseteq \text{gr}(L)$. That is, all such linear processes are reachable.

Example 7.4. Let $n = 2$ and suppose that we measure the following 4−step trajectory:

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, x_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

If we define X and Y as before, we get

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

We can use these to find Z and W:

$$\begin{bmatrix} Z & -W \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

First, note that $X\mathbb{R}^4_+ = \mathbb{R}_+ \times \mathbb{R}$ and $(W^{-1}Z)^2\{0\} = \mathbb{R} \times \{0\}$. Therefore, we can now use Theorem 7.5 to check for informativity.

Now, it is straightforward to verify that $(YX^{-1})^2\{0\} = \mathbb{R}^2$. Lastly, let $\lambda \geqslant 0$ and

$$\begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda & 0 & -1 \\ 0 & 1 & -1-\lambda & \lambda \end{bmatrix} \leqslant 0.$$

By direct inspection, it is clear that this implies that

$$\xi_1 \leqslant 0, \quad \xi_2 \leqslant \lambda\xi_1, \quad 0 \leqslant (1+\lambda)\xi_2, \quad \lambda\xi_2 \leqslant \xi_1.$$

These inequalities show that for any $\lambda \geqslant 0$ we have that $\xi_1 = \xi_2 = 0$. This proves that $\mathcal{D}$ is informative for reachability.

In a similar fashion we can apply our discussion to Theorem 7.2 and Theorem 7.4 to obtain a characterization of informativity for null-controllability.

Theorem 7.6. Let $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a finite set. Suppose that

$$X\mathbb{R}^T_+ + (W^{-1}Z)^n\{0\} = \mathbb{R}^n$$

and

$$(YX^{-1})^n\{0\} = Y\mathbb{R}^T_+ + (Z^{-1}W)^n\{0\} = \mathbb{R}^n.$$

Then $\mathcal{D}$ is informative for null-controllability if and only if for all $\lambda > 0$:

$$\xi^\top(Y - \lambda X) \leqslant 0 \implies \xi = 0.$$

Remark 7.4. If H is a convex process whose graph is polyhedral, we can always find a finite set $\mathcal{D}$ such that $H = H_{\mathcal{D}}$. This means that the results of Theorem 7.5 and Theorem 7.6 can be applied to any polyhedral convex process without loss of generality.

7.6 CONCLUSIONS

In this chapter, we have resolved a number of informativity problems for conically constrained linear systems. This means that we have formulated conditions on finite, exact, state measurements under which we can test whether the measured system is reachable or null-controllable. The resulting tests take the convenient form of subspace inclusions and spectral conditions.

Future work includes extending the ideas in this chapter towards the more general class of linear systems with convex constraints. It is easy to see that these systems can be viewed as difference inclusions of convex set-valued maps. Similar to the approach in this chapter, we can define the smallest set-valued map consistent with the data by taking the convex hull instead of the conic hull. As such, a characterization of reachability for such systems will lead to informativity results for this class of systems. Another direction of future work is investigating informativity for the analysis of other properties and for control. Interesting problems are for example dissipativity or feedback stabilization. Resolving such a problem would require formulating characterizations for a given convex process to have the aforementioned properties. Lastly, this chapter considers only exact measurements of the state. However, many realistic scenarios will involve noisy measurements. Incorporating noisy data within this framework will lead to interesting informativity problems.

8 CONCLUSION

8.1 CONTRIBUTION

In Chapter 2 we gave conditions under which a convex process has an eigenvalue in a given convex cone. These conditions were stated in terms of properties of the convex process and its minimal process. Using a realization of this minimal process, we were able to reformulate these in terms of classical geometric concepts. As such, verifying the conditions of the main results was proven to be straightforward.

Chapter 3 contains Hautus-like tests of reachability, stabilizability and null-controllability of convex processes. That is, under certain assumptions we developed tests for these properties in terms of eigenvalues of convex processes. The main result of the previous chapter was shown to be instrumental in this. The results of Chapter 3 were shown to generalize all previously known cases, including the linear case. As an aside it was proven that under the aforementioned conditions reachability implies controllability.

The focus of Chapter 4 was Lyapunov functions. We developed a framework of extended real-valued functions, in order to better capture the intricacies that arise when considering nonstrict convex processes. In particular, we have proven that the notion of uniform exponential stabilizability is precisely captured by weak Lyapunov functions in the specific class of functions we consider. Furthermore, the second part of this chapter discussed the relation between duality and stability notions. In particular, we generalized all known results on duality of weak and strong Lyapunov functions for convex processes. Further duality results were formulated with the help of the results of the previous chapter.

Chapter 5 moved the attention to data-driven analysis and control. The main contribution of this chapter is the introduction of the informativity framework, which provides a unified approach to the problems of analysis and control of unknown systems based on measured data. After the introduction of this framework, we resolved a number of different problems. To be precise, for linear time-invariant input-state systems, we characterized controllability, stabilizability, stabilization by static feedback and linear quadratic regulation based on noiseless data. Furthermore, for systems with outputs we looked at dynamic output feedback controllers.

Extending these results, in Chapter 6 we considered systems with unbounded noise. To compensate for this added uncertainty we assumed that the system maps were known, apart from the state map. Based on data, we formulated the informativity problem of characterizing whether an unknown systems with this structure has the system-theoretic properties of strong controllability, stabilizability, observability and detectability. Instrumental in resolving this result was a rank property for affine sets of matrices, which led to a Hautus-like test. After this, we considered a geometric version of the same problem, which led to a result for common invariant subspaces.

In Chapter 7 we brought together the results of the earlier chapters. To be precise, we used convex processes to obtain informativity conditions for constrained linear systems. This showed that looking at the graphs of set-valued maps is a natural way to look at the sets of systems that arise from data-driven problems. On the way of resolving the informativity problem, we also formulated characterizations specifically for polyhedral convex processes.

8.2 FUTURE WORK RELATING TO PART I

Due to its many applications, developing the theory of analysis of convex processes further is a promising research direction. However, the theory is still very much in the early stages of development. As such, an extension of the ideas presented in the first part of this thesis is in order. Some clear extensions of the work, like considering e.g. strong reachability, can be found in the properties under consideration in Chapter 6.

Of course, most analysis results are used as steps towards control design methods. As such, a logical extension of the work in this thesis would be to consider control problems. Since in the framework of difference inclusions with convex processes the input is implicit, a natural course of action would be to investigate methods in the vein of *control by interconnection*. This would reduce the stabilization problem for convex processes to the following: Find a stable single valued map, whose graph is contained in the graph of the convex process and whose feasible set is equal to that of the convex process.

A different approach would be the investigation of systems with external variables. That is, systems with inputs and outputs, whose dynamics are given by difference inclusions of convex processes. A particularly important problem for this class of systems is stabilization.

After developing Lyapunov theory for convex processes, another natural extension would be the development of *passivity* and *dissipativity* theories for convex processes. Since the inception of the notion of dissipativity, its connection to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control and the study of linear matrix inequalities made dissipativity one of the central concepts in system analysis. A good starting point for this is the discussion on duality and dissipativity for linear processes in [70].

Apart from the continuation of research on convex processes, different generalizations of linear maps are also of great interest. For example, we could broaden the class of systems under consideration further towards just *convex set-valued maps*. Such convex set-valued maps correspond to linear systems with convex constraints, a class of systems with a great number of applications. Properties of such systems can be investigated in a similar fashion to the work in this thesis, with central roles for invariance and duality. Of course, any results for this class of systems should reduce to the results of this thesis as special cases.

Instead of tackling this problem immediately, it is also possible to first consider different generalizations of linear processes. In terms of the graph, certain avenues for generalization are clear. Since the graph of a linear process is closed under linear combinations, set-valued maps with an *affine graph* are of particular interest. Again, linear systems with affine constraints can be viewed as special cases of this class of systems. Such affine constraints often arise in the form of conservation laws: For instance if the sum of certain states is constant. Interestingly, this class of systems can serve, alongside convex processes, as a second intermediate point in the development of results for more general convex set-valued maps.

In this thesis, we have considered only systems in discrete time. Developing results similar to the ones presented in *continuous time* is an interesting research direction. As for linear systems, the study of discrete time and continuous time are similar, but not equal. Clearly, stabilizability and null-controllability would have rather different characterizations.

8.3 FUTURE WORK RELATING TO PART II

By the nature of the informativity framework, the bulk of resolving a data-driven control problem in this manner is comprised of verifying properties for sets of systems. In other words, this requires resolving a certain type of robust control problem. However, in contrast to the usual setup of robust control, there is not a nominal system. Furthermore, instead of describing the 'uncertainty' a priori, it is derived from the data. For example, in Chapter 5, data-driven control and analysis are viewed as robust control and analysis of a class of affine sets of systems. In addition the data-driven analysis results in Chapter 6 hinge on a uniform rank condition for affine sets of systems. Yet another example is given in [170], which considers sets of systems described by quadratic matrix inequalities. From this viewpoint, any development of new data-driven methods is naturally reliant on developments in analysis and control of sets of systems. As such, *developing methods for robust analysis and control of sets of systems* is in order.

More specifically, certain specific types of sets of systems are particularly interesting. For instance, those arising from *combining the use of data with a priori knowledge of structure*. So far, most data-driven methods treat the system under consideration as completely unknown. Naturally, this leads to relatively strict conditions on the data in order to guarantee, for example, stability. In practice, however, it is often clear that certain assumptions can be made on the structure of the unknown system. For instance, if the system admits a parametrization, network structure or can be viewed as an additive/multiplicative perturbation to a nominal system. Now, if we collect measurements on the system the interesting situation can arise where the data nor the structure alone are sufficient to guarantee stability whereas the combination of these is. Formulating necessary and sufficient conditions for informativity in the presence of structure is a wide and open area of research.

In addition to considering different model classes, resolving different control problems is interesting. So far, most works in the literature have focused on the use of relatively simple controllers for data-driven stabilization. Of course, it is well known that an LTI system is stabilizable if and only if there exists a static state feedback law such that the closed loop is stable. However a similar result no longer holds in the context of data-driven control. Indeed, it is shown in Chapter 5 that conditions guaranteeing that all systems consistent with the data are stabilizable are significantly weaker than those guaranteeing stabilization with a single static feedback. As such, an *investigation of dynamic and adaptive controllers* is in order. This line of research will aim at finding classes of controllers such that, if all systems in a given set are stabilizable, there exists a controller stabilizing them.

Another class of problems arises when considering different types of data. For instance, problems based on only input-output measurements or, more generally, measurements on external variables. Of course, investigating e.g. state controllability on the basis of such data is impossible, as the same external behavior can be generated by many different choices of state variables. However, properties such as stability and dissipativity can be investigated based on only such measurements.

Data can also be corrupted by different types of noise. Many recent methods in data-driven control have focused on the situation where any noise is assumed to be bounded. However, this creates a mismatch with the field of system identification, where *stochastic noise* is commonly used. For many applications, such as measurement errors, it is more natural to model the noise using stochastic methods. It would therefore be interesting to extend the notion of informativity towards data corrupted by such stochastic noise. Problems like stabilization and optimal control in particular will form interesting challenges.

Lastly, the methods developed in the first part of this thesis, and the aforementioned topics of future research are particularly well suited to applications in data-driven analysis and control. In particular, we have shown in Chapter 7 that taking conic combinations of measurements led to the definition of the most powerful unfalsified convex process. This concept played a natural role within the study of informativity problems. Clearly, a similar approach can be taken for linear and affine processes and, more generally, convex set-valued maps. Of course, any research in this direction is dependent on developments in the analysis of difference inclusions of such set-valued maps.

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SUMMARY

Different scientific disciplines employ dynamical systems to model (natural) phenomena. In this thesis we specifically consider dynamical systems in discrete time. In mathematical terms, such a system consists of two parts, working together. First of all there is the *state space*, consisting of all variables derived from the considered phenomenon. Some examples are the location and velocity of a moving object, or the voltages and currents in an electrical network. Secondly we consider *discrete-time dynamics*. This means that we describe the system at instances of time, and have a mathematical description of which states can 'follow' a given state. Often these descriptions are *linear* and *time invariant*: we can find the next state by applying a linear map to the current state and adding the effect of an *input*.

As noted, such models have applications in a large number of scientific fields. Therefore, characterizing when dynamical systems exhibit certain quantitative or qualitative behavior is of great interest. Can we, for instance, decide from the mathematical description whether there exists inputs that brings the system to rest? Do there exist inputs such that from a given state we can reach any other point? This type of problems are central in *systems analysis*. In line with this is actually finding the input that achieves the aforementioned properties. In essence, this is the main problem of *control*. Clearly, analysis results are often required in order to resolve control problems.

For linear time-invariant systems the theories of analysis and control are well developed. Of particular interest with respect to this thesis is the geometric approach to systems and control. To illustrate the this approach in a simplified way, we can consider for instance the *reachability problem*: Determine whether for every point of the state space there exists a sequence of inputs such that, starting from zero and applying these inputs, we end up at the prescribed point. However, we might not be able to test each point separately. The geometric approach to this problem is to consider the set of all states that can be 'reached'. If we can show that this *reachable set* has certain mathematical properties, and that the only set that has these properties is the entire state space, then the reachable set must be equal to the entire state space.

In this thesis we consider a number of analysis and control problems in this manner. In part one we consider *linear systems with constraints*. Many phenomena naturally have certain constraints, but these are often disregarded in mod-

eling. To be specific, we work with linear systems with *convex conic constraints*. This will allow us, for example, to consider systems where certain variables are taken to be nonnegative. However, in the development of geometric methods previous work has focused on either linear systems, or very general nonlinear systems. Since considering more general classes of systems inevitably restricts the results we can obtain, we will take an approach that is specifically tailored to the class of linear systems with convex conic constraints.

As such, we will consider the problem in a framework of *difference inclusions* of *convex processes*. These form a mathematical formalism, that can be used to capture the dynamics of all linear systems with such constraints. For this class of systems we develop a theory parallel to the classical theory of geometric control, with a special focus on *invariance* and *duality*. As a consequence we obtain new characterizations of reachability, null-controllability, controllability and stabilizability. Then we show that these generalize all previously known results for both (conically) constrained linear systems and convex processes. Beyond developing these characterizations, we extend the theory of *Lyapunov functions* within this framework.

So far, we have considered situations where we have access to a model of the phenomenon under consideration. However, sometimes we are in the situation where a unique system is unknown, but where we instead have access to measurements. Performing analyis or control based on such measured data is known as *data-driven analysis and control*. In this thesis, we develop the theory of *data-informativity*, that is, we describe conditions on the data under which the measured system is guaranteed to have certain properties. This is based on the following simple observation: We can only conclude whether the measured system is (for instance) controllable if *all systems compatible with the data* are controllable.

The second part of this thesis is set in a context of data-informativity. As a first step, we consider *exact measurements*. For this type of data we provide informativity conditions for a range of different analysis and control problems. After this we consider the situation of data with *structured noise*. This means that the measurements are corrupted by an unknown, but structured disturbance. For this situation we develop a generalization of the earlier, noiseless results. Finally, we bring the two parts of this thesis together and consider data-informativity in a context of convex processes.

SAMENVATTING

Titel: "Een geometrisch raamwerk voor beperkingen en data: van lineaire systemen tot convexe processen."

Verschillende wetenschapsdisciplines gebruiken dynamische systemen om (natuurlijke) fenomenen te modelleren. In dit proefschrift beschouwen we specifiek dynamische systemen in discrete tijd. In wiskundige termen, bestaat een dergelijk systeem uit een tweetal samenwerkende delen. In de eerste plaats is er de *toestandsruimte*, welke bestaat uit variabelen afgeleid van het beschouwde fenomeen. Een aantal voorbeelden zijn de locatie en snelheid van een bewegend object, of the spanning en stroom van een elektrisch netwerk. Ten tweede beschouwen we *discrete tijd dynamica*. Dit betekent dat we het systeem beschrijven op instanties van tijd, en een wiskundige beschrijven hebben van welke toestanden een gegeven toestand 'op kunnen volgen'. Vaak zijn deze omschrijvingen *lineair* en *tijdsinvariant*: We kunnen de volgende toestand bepalen door een lineaire afbeelding toe te passen op de huidige toestand en daar het effect van een *ingangssignaal* bij op te tellen.

Zoals aangegeven hebben dergelijke modellen toepassingen in een groot aantal wetenschappen. Daarom is karakteriseren wanneer een dynamisch systeem bepaalde kwalitatieve en kwantitatieve eigenschappen vertoont een probleem van groot belang. Kunnen we, bijvoorbeeld, op basis van de wiskundige beschrijving bepalen of er ingangssignalen bestaan welke het systeem tot rust brengen? Bestaan er ingangssignalen zodanig dat we vanaf een gegeven toestand een ander punt kunnen bereiken? Dit soort vragen staan centraal in *systeemanalyse*. In het verlengde hiervan ligt het daadwerkelijk bepalen van het ingangssignaal dat de bovenstaande eigenschappen behaalt. Dit is het essentiële probleem in de *regeltechniek*. Het moge duidelijk wezen dat analyseresultaten vaak noodzakelijk zijn voor het oplossen van regelproblemen.

Voor lineaire, tijdsinvariante systemen zijn de theorieën van analyse en regeling goed ontwikkeld. Van bijzonder belang met betrekking tot dit proefschrift is de geometrische benadering van systeem- en regeltechniek. Om deze benadering op een simpele manier te illustreren, kunnen we bijvoorbeeld het *bereikbaarheidsprobleem* bestuderen: Bepaal of voor elk punt in de toestandsruimte of er een ingangssignaal bestaat zodanig dat, als we beginnen in het nulpunt en deze signalen toepassen, we uitkomen bij het voorgeschreven punt.

Echter, het hoeft niet mogelijk te zijn om elk punt apart te testen. De geometrische benadering van dit probleem is om de verzameling van alle 'bereikbare' punten te beschouwen. Als we kunnen laten zien dat deze *bereikbare verzameling* bepaalde eigenschappen heeft, en dat de enige verzameling met deze eigenschappen de volledige toestandsruimte is, dan is de bereikbare verzameling gelijk aan de toestandsruimte.

In dit proefschrift beschouwen we een aantal analyse- en regelproblemen op deze manier. In deel een bekijken we *lineaire systemen met begrenzingen*. Veel fenomenen zijn op een natuurlijke manier begrensd, maar deze worden vaak buiten beschouwing gelaten bij het modelleren. Om specifiek te zijn werken we met lineaire systemen met begrenzingen in de vorm van *convexe kegels*. Dit zorgt ervoor dat we bijvoorbeeld systemen kunnen bekijken waarbij bepaalde variabelen niet-negatief zijn. Echter, bij het ontwikkelen van geometrische methoden lag voor zover de focus ofwel bij lineaire systemen ofwel bij zeer algemene niet-lineaire systemen. Aangezien het beschouwen van algemenere klassen van system er onvermijdelijk voor zorgt dat de resultaten die te behalen zijn beperkt zijn, nemen we een benadering die precies aansluit bij de klasse van lineaire systemen met convexe kegelbegrenzingen.

Met dit doel ontwikkelen we een raamwerk van *differentie-inclusies* van *convexe processen*. Deze vormen een wiskundig formalisme, waarin de dynamica van elk lineair systeem met dergelijke begrenzingen gevat kan worden. Voor deze klasse systemen ontwikkelen we een theorie gestoeld op de klassieke geometrische regeltechniek, met een specifieke focus op *invariantie* end *dualiteit*. Als gevolg hiervan verkrijgen we nieuwe karakterisaties van bereikbaarheid, nulregelbaarheid, regelbaarheid, en stabiliseerbaarheid. Vervolgens laten we zien dat deze alle bekende resultaten generaliseren. Naast het ontwikkelen van deze karakterisaties, bereiden we de theorie van *Lyapunovfuncties* uit naar deze klasse.

Tot nu toe hebben we gekeken naar situaties waarin we een model van het fenomeen in kwestie tot onze beschikking hebben. Echter, soms zijn we in de situatie waarin een uniek systeem onbekend is, maar waarin we in plaats daarvan een aantal metingen hebben. Het uitvoeren van analyse en regeling op basis van dergelijke gemeten data staat bekend als *datagestuurde analyse en regeling*. In dit proefschrift ontwikkelen we de theorie van *data-informativiteit*, dat wil zeggen, we omschrijven voorwaarden voor de data onder welke het gemeten system gegarandeerd is om zekere eigenschappen te hebben. Dit is gebaseerd op de volgende simpele observatie: We kunnen louter concluderen dat het gemeten systeem (bijvoorbeeld) regelbaar is als *alle systemen die verenigbaar zijn met de data* regelbaar zijn.

Het tweede deel van dit proefschrift staat in een context van data-informativiteit. Als eerste stap beschouwen we *exacte data*. Voor dit type data formuleren we informativiteitsvoorwaarden voor een reeks aan verschillende analyse- en regelproblemen. Hierna beschouwen we een situatie met data met *gestructureerde ruis*. Dit betekent dat de metingen gecorrumpeerd zijn met een onbekende, maar gestructureerde verstoring. Voor deze situatie ontwikkelen we een generalisatie van de eerdere, ruisloze resultaten. Ten slotte brengen we de twee delen van dit proefschrift samen en beschouwen we data-informativiteit in een context van convexe processen.