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Published in:
 Indagationes Mathematicae

DOI:
[10.1016/j.indag.2021.03.002](https://doi.org/10.1016/j.indag.2021.03.002)

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Document Version
 Publisher's PDF, also known as Version of record

Publication date:
 2021

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Boer, N. B., & Sterk, A. E. (2021). Generalized Fibonacci numbers and extreme value laws for the Rényi map. *Indagationes Mathematicae*, 32(3), 704-718. <https://doi.org/10.1016/j.indag.2021.03.002>

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Generalized Fibonacci numbers and extreme value laws for the Rényi map

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Received 16 April 2020; received in revised form 11 December 2020; accepted 2 March 2021

Communicated by R. van der Hofstad

Abstract

In this paper we prove an extreme value law for a stochastic process obtained by iterating the Rényi map $x \mapsto \beta x \pmod{1}$, where we assume that $\beta > 1$ is an integer. Haiman (2018) derived a recursion formula for the Lebesgue measure of threshold exceedance sets. We show how this recursion formula is related to a rescaled version of the k -generalized Fibonacci sequence. For the latter sequence we derive a Binet formula which leads to a closed-form expression for the distribution of partial maxima of the stochastic process. The proof of the extreme value law is completed by deriving sharp bounds for the dominant root of the characteristic polynomial associated with the Fibonacci sequence.

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Keywords: Extreme value laws; Fibonacci numbers; Rényi map

1. Introduction

Extreme value theory for a sequence of i.i.d. random variables $(X_i)_{i=0}^{\infty}$ studies the asymptotic distribution of the partial maximum

$$M_n = \max(X_0, \dots, X_{n-1}) \tag{1}$$

as $n \rightarrow \infty$. Since the distribution of M_n has a degenerate limit it is necessary to consider a rescaling. Under appropriate conditions there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$ for which the limiting distribution of $a_n(M_n - b_n)$ is nondegenerate. As an elementary example, assume

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that the variables $X_i \sim U(0, 1)$ are independent. Then with $a_n = n$ and $b_n = 1$ it follows for $\lambda \geq 0$ that

$$\lim_{n \rightarrow \infty} \mathbb{P}(a_n(M_n - b_n) \leq -\lambda) = \lim_{n \rightarrow \infty} \mathbb{P}\left(M_n \leq 1 - \frac{\lambda}{n}\right) = \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}. \tag{2}$$

More generally, it can be proven that extreme value distributions for i.i.d. random variables are either a Weibull, Gumbel, or Fréchet distribution [12,13,27]. For extensions of extreme value theory to dependent random variables, see [21].

In the last twenty years the applicability of extreme value theory has been extended to the setting of deterministic dynamical systems. The pioneering work [6] introduced many ideas that were used in subsequent papers by various authors. A particularly important development was proving the link between hitting and return time statistics on the one hand and extreme value laws on the other hand [10]. Hence, extreme value laws can be proven by using the many results on hitting and return time statistics that are available. The latter have been derived for general classes of dynamical systems [1,16–19,28] and go beyond the context of the piecewise linear maps that will be considered in the present paper. For a detailed account on the subject of extremes in dynamical systems the interested reader is referred to the recent monograph [23] and the extensive list of references therein.

In this paper we consider the Rényi map [26] given by

$$f : [0, 1) \rightarrow [0, 1), \quad f(x) = \beta x \pmod{1},$$

where we restrict to the case where $\beta > 1$ is an integer. This map is an active topic of study within the field of dynamical systems and ergodic theory. In the special case $\beta = 2$ the map f is also known as the doubling map which is an archetypical example of a chaotic dynamical system [4]. Other applications, in which also non-integer values of β are considered, include the study of random number generators [2] and dynamical systems with holes in their state space [20].

The assumption that $\beta > 1$ is an integer implies that the Lebesgue measure is an invariant probability measure of the map f :

Lemma 1.1. *If X is a random variable such that $X \sim U(0, 1)$, then $f(X) \sim U(0, 1)$.*

Proof. For $u \in (0, 1]$ we have that $\mathbb{P}(X \in [0, u)) = u$. This gives

$$\mathbb{P}(f(X) \in [0, u)) = \mathbb{P}(X \in f^{-1}([0, u))) = \sum_{k=1}^{\beta} \mathbb{P}\left(X \in \left[\frac{k-1}{\beta}, \frac{k-1+u}{\beta}\right)\right) = u,$$

which implies that $f(X) \sim U(0, 1)$. \square

Consider the stochastic process $(X_i)_{i=0}^{\infty}$ defined by $X_{i+1} = f(X_i)$, where $X_0 \sim U(0, 1)$. Lemma 1.1 implies that the variables X_i are identically distributed, but they are no longer independent. Let M_n be the partial maximum as defined in (1). Haiman [15] proved the following result:

Theorem 1.2. *For fixed $\lambda > 0$ and the sequence $n_k = \lfloor \beta^k \lambda \rfloor$ it follows that*

$$\lim_{k \rightarrow \infty} \mathbb{P}(M_{n_k} \leq 1 - \beta^{-k}) = e^{-\frac{\beta-1}{\beta} \lambda}.$$

Note that for $\lambda \in \mathbb{N}$ we have $\mathbb{P}(M_{n_k} \leq 1 - \beta^{-k}) = \mathbb{P}(\beta^k \lambda (M_{\beta^k \lambda} - 1) \leq -\lambda)$. Therefore, the result of Theorem 1.2 is in spirit similar to the example in (2), albeit that a subsequence of M_n is considered.

The aim of this paper is to give an alternative proof for [Theorem 1.2](#) which relies on asymptotic properties of a rescaled version of the k -generalized Fibonacci numbers. The restriction that β is an integer is essential for our proof. Indeed, for non-integer values of $\beta > 1$ the invariant measure of the map f is generally different from the Lebesgue measure; see [\[5,26\]](#) for the case $\beta = (\sqrt{5} + 1)/2$. A more general approach to establish an extreme value law would be to show that two mixing conditions are satisfied which guarantee that an extreme value law for a time series generated by a dynamical system can be obtained as if it were an i.i.d. stochastic process. An application of this approach to the tent map process can be found in [\[8\]](#). However, in [Appendix A](#) we show that one of these conditions does not hold the Rényi map process.

The fact that the limit in [Theorem 1.2](#) is not equal to $e^{-\lambda}$ has a particular statistical interpretation. The coefficient $\theta := (\beta - 1)/\beta$ in the exponential is called the extremal index and measures the degree of clustering in extremes arising as a consequence of the dependence between the variables X_i ; see [\[21,23\]](#) for more details. In [Appendix B](#) we show how the extremal index for the Rényi map process can be derived in an elementary way. For more general dynamical systems, conditions for extreme value laws with particular extremal indices are derived in [\[11\]](#).

2. The relation with generalized Fibonacci numbers

In this section we fix the numbers $k \in \mathbb{N}$ and $u = \beta^{-k}$. For any integer $i \geq 0$ we define the set

$$E_i = \{x \in [0, 1) : f^i(x) > 1 - u\},$$

where the dependence on k is suppressed in the notation for convenience. Then

$$\mathbb{P}(M_n \leq 1 - u) = 1 - B_n \quad \text{where} \quad B_n = \text{Leb}\left(\bigcup_{i=0}^{n-1} E_i\right),$$

where Leb denotes the Lebesgue measure. Based on self-similarity arguments [Haiman \[15\]](#) derived the following recursion formula which holds for each fixed $k \in \mathbb{N}$:

$$B_n = (n - 1) \frac{\beta - 1}{\beta} u + u \quad \text{if } 1 \leq n \leq k + 1, \tag{3}$$

$$B_{n+1} = B_n + \frac{\beta - 1}{\beta} u(1 - B_{n-k}) \quad \text{if } n \geq k + 1. \tag{4}$$

The same idea was used earlier by [Haiman](#) to study extreme value laws for the tent map [\[14\]](#).

For $n \in \mathbb{Z}$ we define the following numbers:

$$F_n = \begin{cases} 0 & \text{if } n < 1, \\ 1 & \text{if } n = 1, \\ \frac{B_n - B_{n-1}}{u/\beta^{n-1}} & \text{if } n > 1. \end{cases} \tag{5}$$

These numbers have the following geometric meaning. Note that the sets E_i can be written as a union of β^i intervals:

$$E_i = \bigcup_{j=1}^{\beta^i} \left[\frac{j - u}{\beta^i}, \frac{j}{\beta^i} \right), \quad i \geq 0.$$

For $n \geq 2$ the number F_n equals the number of subintervals of the set E_{n-1} which need to be added to $E_0 \cup \dots \cup E_{n-2}$ in order to obtain $E_0 \cup \dots \cup E_{n-1}$. Fig. 1 illustrates this for the special case $\beta = 2$ and $k = 2$.

Lemma 2.1. For any $k, n \in \mathbb{N}$ it follows that

$$\mathbb{P}(M_n \leq 1 - \beta^{-k}) = \frac{\beta^{1-n-k}}{\beta - 1} F_{n+k+1}.$$

Proof. For $n \geq k + 2$ Eq. (4) gives

$$F_n = \frac{B_n - B_{n-1}}{u/\beta^{n-1}} = (\beta - 1)\beta^{n-2}(1 - B_{n-k-1}),$$

or, equivalently,

$$B_{n-k-1} = 1 - \frac{\beta^{2-n}}{\beta - 1} F_n.$$

The proof is completed by substituting n for $n - k - 1$. \square

The following result provides the connection between the sequence (B_n) and generalizations of the Fibonacci numbers. In particular, for $\beta = 2$ the sequence (F_n) is the well-known k -generalized Fibonacci sequence.

Lemma 2.2. The following statements are equivalent:

- (i) Eqs. (3) and (4) hold;
- (ii) For fixed $k \in \mathbb{N}$, the sequence (F_n) , where $n \in \mathbb{Z}$, defined in (5) satisfies

$$F_n = \begin{cases} 0 & \text{if } n < 1, \\ 1 & \text{if } n = 1, \\ (\beta - 1)(F_{n-1} + F_{n-2} + \dots + F_{n-k}) & \text{if } n \geq 2. \end{cases} \tag{6}$$

In particular, $F_n = (\beta - 1)\beta^{n-2}$ for $2 \leq n \leq k + 1$.

Proof. Assume that statement (i) holds. By definition $F_1 = 1$ and for $2 \leq n \leq k + 1$ Eq. (3) implies that

$$F_n = \frac{B_n - B_{n-1}}{u/\beta^{n-1}} = \frac{\beta^{n-1}}{u} \left[\left((n-1) \frac{\beta - 1}{\beta} u + u \right) - \left((n-2) \frac{\beta - 1}{\beta} u + u \right) \right] = (\beta - 1)\beta^{n-2}.$$

We proceed with induction on n . For any $n \geq k + 1$ Eq. (4) gives

$$F_{n+1} = \frac{B_{n+1} - B_n}{u/\beta^n} = (\beta - 1)\beta^{n-1}(1 - B_{n-k}). \tag{7}$$

In particular, for $n = k + 1$ we have

$$\begin{aligned} F_{k+2} &= (\beta - 1)\beta^k(1 - B_1) \\ &= (\beta - 1)(\beta^k - 1) = (\beta - 1)^2 \sum_{i=1}^k \beta^{k-i} = (\beta - 1) \sum_{i=1}^k F_{k+2-i}. \end{aligned}$$

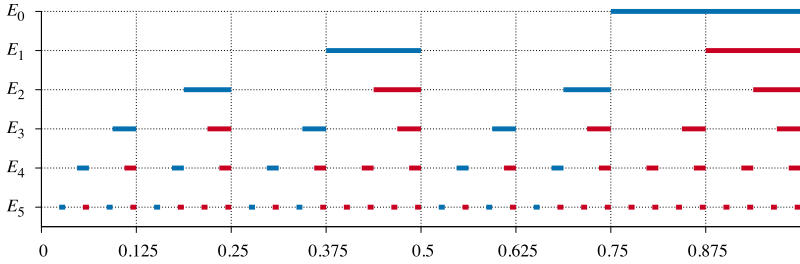


Fig. 1. Illustration of the sets E_0, \dots, E_5 for $\beta = 2$ and $k = 2$. Each set E_n is a union of β^n intervals. Intervals in E_n which are disjoint from (resp. contained in) the intervals comprising E_0, \dots, E_{n-1} are drawn in blue (resp. red). For $n \geq 2$ the number F_n equals the number of subintervals of the set E_{n-1} which need to be added to $E_0 \cup \dots \cup E_{n-2}$ in order to obtain $E_0 \cup \dots \cup E_{n-1}$. The figure clearly shows that $F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$, and $F_6 = 8$ which are the starting numbers of the Fibonacci sequence. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Assume that for some $n \geq k + 1$ it follows that

$$F_{n+1} = (\beta - 1) \sum_{i=1}^k F_{n+1-i}.$$

First using Eq. (5) and then Eq. (4) twice gives

$$\begin{aligned} F_{n+2} &= (\beta - 1)\beta^n(1 - B_{n-k+1}) \\ &= (\beta - 1)\beta^n(1 - B_{n-k}) - (\beta - 1)^2\beta^{n-k-1}(1 - B_{n-2k}) \\ &= \beta F_{n+1} - (\beta - 1)F_{n-k+1}, \end{aligned}$$

where the last equality follows from (7). Finally, the induction hypothesis implies that

$$\begin{aligned} F_{n+2} &= (\beta - 1)F_{n+1} + F_{n+1} - (\beta - 1)F_{n-k+1} \\ &= (\beta - 1)F_{n+1} + (\beta - 1) \sum_{i=1}^k F_{n+1-i} - (\beta - 1)F_{n-k+1} \\ &= (\beta - 1) \sum_{i=1}^k F_{n+2-i}. \end{aligned}$$

Hence, statement (ii) follows.

Conversely, assume that statement (ii) holds. In particular, $F_n = (\beta - 1)\beta^{n-2}$ for $2 \leq n \leq k + 1$ so that by Eq. (5) it follows that

$$B_n = B_{n-1} + \frac{u}{\beta^{n-1}} F_n = B_{n-1} + \frac{\beta - 1}{\beta} u.$$

Eq. (3) now follows by recalling that $B_1 = u$.

We proceed by strong induction on n . We have

$$F_{k+2} = (\beta - 1) \sum_{i=1}^k F_{k+2-i} = (\beta - 1) \sum_{i=1}^k (\beta - 1)\beta^{k-i} = (\beta - 1)(\beta^k - 1).$$

Recalling that $B_1 = u = \beta^{-k}$, Eq. (5) implies that

$$B_{k+2} = B_{k+1} + \frac{u}{\beta^{k+1}} F_{k+2} = B_{k+1} + \frac{u}{\beta^{k+1}} (\beta - 1)(\beta^k - 1) = B_{k+1} + \frac{\beta - 1}{\beta} u(1 - B_1),$$

which shows that Eq. (4) holds for $n = k + 1$. Assume that there exists $m \in \mathbb{N}$ such that (4) holds for all $k + 1 \leq n \leq m$. Observe that

$$\begin{aligned} F_{m+2} &= (\beta - 1) \sum_{i=1}^k F_{m+2-i} \\ &= (\beta - 1) \left(F_{m+1} - F_{m+1-k} + \sum_{i=1}^k F_{m+1-i} \right) \\ &= (\beta - 1) \left(F_{m+1} - F_{m+1-k} + \frac{F_{m+1}}{\beta - 1} \right) \\ &= \beta F_{m+1} - (\beta - 1) F_{m+1-k}. \end{aligned}$$

Therefore,

$$\begin{aligned} B_{m+2} - B_{m+1} &= \frac{u}{\beta^{m+1}} F_{m+2} \\ &= \frac{u}{\beta^{m+1}} (\beta F_{m+1} - (\beta - 1) F_{m+1-k}) \\ &= B_{m+1} - B_m - \frac{\beta - 1}{\beta^{k+1}} (B_{m+1-k} - B_{m-k}). \end{aligned}$$

The induction hypothesis gives

$$\begin{aligned} B_{m+2} - B_{m+1} &= \frac{\beta - 1}{\beta} u(1 - B_{m-k}) - \frac{(\beta - 1)^2}{\beta^{k+2}} u(1 - B_{m-2k}) \\ &= \frac{\beta - 1}{\beta} u \left(1 - \left(B_{m-k} + \frac{\beta - 1}{\beta} u(1 - B_{m-2k}) \right) \right) \\ &= \frac{\beta - 1}{\beta} u(1 - B_{m-k+1}). \end{aligned}$$

Hence, statement (i) follows. \square

3. The Binet formula

Let the sequence (F_n) be as defined in (6), where $\beta \geq 2$ is assumed to be an integer. In this section we will derive a closed-form expression for F_n as a function of n along the lines of Spickerman and Joyner [29] and Dresden and Du [7]. Levesque [22] derived a closed-form expression for sequences of the form (6) in which each term is multiplied with a different factor. Another interesting paper by Wolfram [30] considers explicit formulas for the k -generalized Fibonacci sequence with arbitrary starting values, but we will not pursue those ideas here.

The characteristic polynomial corresponding to the recursion relation (6) is given by

$$p_k(x) = x^k - (\beta - 1) \sum_{i=0}^{k-1} x^i. \tag{8}$$

The following result concerns properties of the roots of this polynomial. The proof closely follows Miller [25]. For alternative proofs for the special case $\beta = 2$, see [24,30].

Lemma 3.1. *Let $k \geq 2$ and $\beta \geq 2$ be integers. Then*

- (i) *the polynomial p_k has a real root $1 < r_{k,1} < \beta$;*
- (ii) *the remaining roots $r_{k,2}, \dots, r_{k,k}$ of p_k lie within the unit circle of the complex plane;*
- (iii) *the roots of p_k are simple.*

Proof. (i) Descartes’ rule of signs implies that p_k has exactly one positive root $r_{k,1}$. Since

$$p_k(1) = 1 - k(\beta - 1) < 0 \quad \text{and} \quad p_k(\beta) = 1$$

the Intermediate Value Theorem implies the existence of a root $1 < r_{k,1} < \beta$.

(ii) Define the polynomial

$$q_k(x) = (x - 1)p_k(x) = x^{k+1} - \beta x^k + \beta - 1,$$

and make the following observations:

(O1) if $x > r_{k,1}$, then $p_k(x) > 0$, and if $0 < x < r_{k,1}$, then $p_k(x) < 0$;

(O2) if $x > r_{k,1}$, then $q_k(x) > 0$, and if $1 < x < r_{k,1}$, then $q_k(x) < 0$.

Note that p_k has no root r such that $|r| > r_{k,1}$. Indeed, if such a root exists, then $p_k(r) = 0$, or, equivalently, $r^k = (\beta - 1) \sum_{i=0}^{k-1} r^i$. The triangle inequality gives $|r|^k \leq (\beta - 1) \sum_{i=0}^{k-1} |r|^i$. Hence, $p_k(|r|) \leq 0$, which contradicts observation (O1).

In addition, p_k has no root r with $1 < |r| < r_{k,1}$. Indeed, if such a root exists, then $q_k(r) = (r - 1)p_k(r) = 0$ so that $\beta r^k = r^{k+1} + \beta - 1$. The triangle inequality implies that $\beta|r|^k \leq |r|^{k+1} + \beta - 1$. Hence, $q_k(|r|) \geq 0$, which contradicts observation (O2).

Finally, p_k has no root r with either $|r| = 1$ or $|r| = r_{k,1}$ but $r \neq r_{k,1}$. Indeed, if such a root exists, then $q_k(r) = (r - 1)p_k(r) = 0$, which implies $\beta r^k = r^{k+1} + \beta - 1$ and

$$\beta|r|^k = |r^{k+1} + \beta - 1| \leq |r|^{k+1} + \beta - 1. \tag{9}$$

If the inequality in (9) is strict, then $q_k(|r|) > 0$. Since $q_k(1) = 0$ and $q_k(r_{k,1}) = 0$ it then follows that $|r| \neq 1$ and $|r| \neq r_{k,1}$. If the inequality in (9) is an equality, then r^{k+1} must be real. Since $q_k(r) = 0$, it follows that $r^k = ((\beta - 1) + r^{k+1})/\beta$ is real as well and hence r itself is real. An application of Descartes’ rule of signs to q_k implies that when k is even p_k has one negative root, and when k is odd p_k has no negative root. If k is even, then $p_k(0) = -(\beta - 1)$ and $p_k(-1) = 1$. By the Intermediate Value Theorem it follows that $-1 < r < 0$. We conclude that no root of p_k , except $r_{k,1}$ itself, has absolute value 1 or $r_{k,1}$.

(iii) If p_k has a multiple root, then so has q_k . In that case, there exists r such that $q_k(r) = q'_k(r) = 0$. Note that $q'_k(r) = 0$ implies that $r = 0$ or $r = \beta k/(k + 1)$. Clearly, $r = 0$ is not a root of q_k . By the Rational Root Theorem it follows that the only rational roots of q_k can be integers that divide $\beta - 1$. Hence, $r = \beta k/(k + 1)$ is not a root of q_k either. We conclude that q_k , and thus p_k , cannot have multiple roots. \square

The proof of the following result closely follows the method of Spickerman and Joyner [29] and then uses a rewriting step as in Dresden and Du [7].

Lemma 3.2. *The sequence (F_n) as defined in (6) is given by the following Binet formula:*

$$F_n = \sum_{j=1}^k \frac{r_{k,j} - 1}{\beta + (k + 1)(r_{k,j} - \beta)} r_{k,j}^{n-1},$$

where $r_{k,1}, \dots, r_{k,k}$ are the roots of the polynomial p_k defined in (8).

Proof. The generating function of the sequence (F_n) is given by

$$G(x) = \sum_{n=0}^{\infty} F_{n+1}x^n.$$

The equation

$$\sum_{n=k}^{\infty} \left(F_{n+1} - (\beta - 1) \sum_{i=1}^k F_{n+1-i} \right) x^n = 0$$

leads to

$$G(x) = \sum_{n=0}^{k-1} F_{n+1}x^n - (\beta - 1) \sum_{i=1}^{k-1} \sum_{n=0}^{k-i-1} F_{n+1}x^n + (\beta - 1)G(x) \sum_{i=1}^k x^i.$$

Finally, using that $F_1 = 1$ and $F_n = (\beta - 1)\beta^{n-2}$ for $2 \leq n \leq k - 1$ implies that

$$G(x) = \frac{1}{1 - (\beta - 1) \sum_{i=1}^k x^i}.$$

Note that $1/r$ is a root of the denominator of G if and only if r is a root of the characteristic polynomial p_k . By Lemma 3.1 part (iii) we can expand the generating function in terms of partial fractions as follows:

$$G(x) = \sum_{j=1}^k \frac{c_j}{x - 1/r_{k,j}},$$

where the coefficients are given by

$$c_j = \lim_{x \rightarrow 1/r_{k,j}} \frac{x - 1/r_{k,j}}{1 - (\beta - 1) \sum_{i=1}^k x^i} = -\frac{1}{(\beta - 1) \sum_{i=1}^k i(1/r_{k,j})^{i-1}}.$$

Observe that

$$\begin{aligned} \left(1 - \frac{1}{r_{k,j}}\right) \sum_{i=1}^k i \left(\frac{1}{r_{k,j}}\right)^{i-1} &= \sum_{i=1}^k \left[i \left(\frac{1}{r_{k,j}}\right)^{i-1} - (i+1) \left(\frac{1}{r_{k,j}}\right)^i \right] + \sum_{i=1}^k \left(\frac{1}{r_{k,j}}\right)^i \\ &= 1 - (k+1) \frac{1}{r_{k,j}^k} + \frac{1}{\beta - 1}. \end{aligned}$$

This results in

$$c_j = -\frac{1 - 1/r_{k,j}}{\beta - (\beta - 1)(k + 1)/r_{k,j}^k}.$$

Since $r_{k,j}^{k+1} - \beta r_{k,j}^k + \beta - 1 = (r_{k,j} - 1)p(r_{k,j}) = 0$ it follows that $r_{k,j} - \beta = (1 - \beta)/r_{k,j}^k$ so that

$$c_j = -\frac{1 - 1/r_{k,j}}{\beta + (k + 1)(r_{k,j} - \beta)}.$$

Finally, we have that

$$G(x) = \sum_{j=1}^k c_j \left(-r_{k,j} \sum_{n=0}^{\infty} r_{k,j}^n x^n \right) = \sum_{n=0}^{\infty} \left(- \sum_{j=1}^k c_j r_{k,j}^{n+1} \right) x^n.$$

Substituting the values for the coefficients completes the proof. \square

For the special case $\beta = 2$ Dresden and Du [7] go one step further and derive the following simplified Binet formula:

$$F_n = \left\lfloor \frac{r_{k,1} - 1}{\beta + (k + 1)(r_{k,1} - \beta)} r_{k,1}^{n-1} + \frac{1}{2} \right\rfloor \quad \text{for } n \geq k - 2,$$

where $r_{k,1}$ is the unique root of p_k for which $1 < r_{k,1} < \beta$; see Lemma 3.1. We expect that this formula can be proven for all integers $\beta > 1$ for n sufficiently large, where the lower bound on n may depend on both β and k . However, we will not pursue this question in this paper.

4. Exponentially growing sequences

In preparation to the proof of Theorem 1.2 we will prove two facts on sequences that exhibit exponential growth. The first result is a variation on a well-known limit:

Lemma 4.1. *If (a_k) is a sequence such that $\lim_{k \rightarrow \infty} ka_k = c$, then*

$$\lim_{k \rightarrow \infty} (1 - a_k)^k = e^{-c}.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Then there exists $N \in \mathbb{N}$ such that $|ka_k - c| \leq \varepsilon$, or, equivalently,

$$\left(1 - \frac{c + \varepsilon}{k}\right)^k \leq (1 - a_k)^k \leq \left(1 - \frac{c - \varepsilon}{k}\right)^k$$

for all $k \geq N$. Hence, we obtain

$$e^{-(c+\varepsilon)} \leq \liminf_{k \rightarrow \infty} (1 - a_k)^k \leq \limsup_{k \rightarrow \infty} (1 - a_k)^k \leq e^{-(c-\varepsilon)}.$$

Since $\varepsilon > 0$ is arbitrary, the result follows. \square

The next result provides sufficient conditions under which the difference of two exponentially increasing sequences grows at a linear rate:

Lemma 4.2. *If $a > 1$ and (b_k) is a positive sequence such that $\lim_{k \rightarrow \infty} a^k b_k = c$, then*

$$\lim_{k \rightarrow \infty} \frac{a^k - (a - b_k)^k}{k} = \frac{c}{a}.$$

Proof. The algebraic identity

$$x^k - y^k = (x - y) \sum_{i=0}^{k-1} x^{k-1-i} y^i$$

leads to

$$\frac{a^k - (a - b_k)^k}{k} = \frac{a^k b_k}{a} \cdot S_k \quad \text{where} \quad S_k = \frac{1}{k} \sum_{i=0}^{k-1} \left(1 - \frac{b_k}{a}\right)^i.$$

It suffices to show that $\lim_{k \rightarrow \infty} S_k = 1$. To that end, note that the assumption implies that $\lim_{k \rightarrow \infty} b_k = 0$ so that $-1 < -b_k/a < 0$ for k sufficiently large. Bernoulli’s inequality gives

$$1 - i \frac{b_k}{a} \leq \left(1 - \frac{b_k}{a}\right)^i < 1,$$

which implies that

$$1 - \frac{k-1}{2} \cdot \frac{b_k}{a} < S_k < 1$$

for k sufficiently large. Moreover, the assumption implies that $\lim_{k \rightarrow \infty} kb_k = 0$. An application of the Squeeze Theorem completes the proof. \square

5. Proof of the extreme value law

Let $\lambda > 0$ and define $n_k = \lfloor \beta^k \lambda \rfloor$. Combining Lemmas 2.1 and 3.2 gives

$$\mathbb{P}(M_{n_k} \leq 1 - \beta^{-k}) = \frac{\beta}{\beta - 1} \sum_{i=1}^k a_i(k) \quad \text{where}$$

$$a_i(k) = \frac{r_{k,i} - 1}{\beta + (k+1)(r_{k,i} - \beta)} \left(\frac{r_{k,i}}{\beta} \right)^{n_k+k},$$

where $r_{k,i}$ are the roots of p_k . Recall that $r_{k,1}$ is the unique root in the interval $(1, \beta)$, and that $|r_{k,i}| < 1$ for $i = 2, \dots, k$. In the remainder of this section Theorem 1.2 will be proven by a careful analysis of the asymptotic behaviour of the dominant root $r_{k,1}$.

We define the following numbers:

$$r_{k,\min} = \beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) \quad \text{and} \quad r_{k,\max} = \beta - \frac{\beta - 1}{\beta^k - 1}.$$

The number $r_{k,\max}$ is obtained by applying a single iteration of Newton’s method to p_k using the starting point $x = \beta$. The number $r_{k,\min}$ is a correction of $r_{k,\max}$ with an exponentially decreasing factor.

Lemma 5.1. *If $\beta \geq 2$ is an integer and $k \in \mathbb{N}$ is sufficiently large, then*

- (i) $p_k(r_{k,\max}) > 0$;
- (ii) $p_k(r_{k,\min}) < 0$;
- (iii) $r_{k,\min} < r_{k,1} < r_{k,\max}$.

Proof. (i) For $x \neq 1$ we have

$$p_k(x) = x^k - (\beta - 1) \sum_{i=0}^{k-1} x^i = x^k - (\beta - 1) \frac{1 - x^k}{1 - x} = \frac{1}{1 - x} ((\beta - x)x^k - (\beta - 1)).$$

In particular, for $k \geq 2$ it follows that

$$p_k(r_{k,\max}) = \frac{1}{\beta^k - 2} \left[\beta^k - \left(\beta - \frac{\beta - 1}{\beta^k - 1} \right)^k - 1 \right].$$

It suffices to show that the expression between brackets is positive for k sufficiently large. Lemma 4.2 gives

$$\lim_{k \rightarrow \infty} \frac{1}{k} \left(\beta^k - \left(\beta - \frac{\beta - 1}{\beta^k - 1} \right)^k \right) = \frac{\beta - 1}{\beta}.$$

Hence, for k sufficiently large it follows that

$$\beta^k - \left(\beta - \frac{\beta - 1}{\beta^k - 1} \right)^k - 1 \geq \frac{\beta - 1}{2\beta} k - 1,$$

and the right-hand side is positive for $k > 2\beta/(\beta - 1)$.

(ii) Similar to the proof of part (i) it follows that

$$p_k(r_{k,\min}) = \frac{1}{2 + \beta^{-k/2} - \beta^k} \left[\left(\beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) \right)^k (1 + \beta^{-k/2}) - \beta^k + 1 \right].$$

It suffices to show that the expression between brackets is positive for k sufficiently large.

Lemma 4.2 gives

$$\lim_{k \rightarrow \infty} \frac{1}{k} \left(\beta^k - \left(\beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) \right)^k \right) = \frac{\beta - 1}{\beta}.$$

Hence, for k sufficiently large it follows that

$$\beta^k - \left(\beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) \right)^k \leq k.$$

This gives

$$\begin{aligned} & \left(\beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) \right)^k (1 + \beta^{-k/2}) - \beta^k + 1 \\ &= \beta^{k/2} + 1 - (1 + \beta^{-k/2}) \left(\beta^k - \left(\beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) \right)^k \right) \\ &\geq \beta^{k/2} + 1 - (1 + \beta^{-k/2})k, \end{aligned}$$

and the right-hand side is positive for k sufficiently large.

(iii) By the Intermediate Value Theorem there exists a point $c \in (r_{k,\min}, r_{k,\max})$ such that $p_k(c) = 0$. Note that $c > 1$ for k sufficiently large. Since $r_{k,1}$ is the only zero of p_k which lies outside the unit circle it follows that $c = r_{k,1}$. \square

In the particular, for $\beta = 2$ the previous result improves the bound $2(1 - 2^{-k}) < r_{1,k} < 2$ derived by Wolfram [30].

Lemma 5.2. *We have that*

$$\lim_{k \rightarrow \infty} a_1(k) = \frac{\beta - 1}{\beta} e^{-\frac{\beta-1}{\beta} \lambda}.$$

Proof. From Lemma 5.1 it follows for sufficiently large k that

$$\beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) < r_{k,1} < \beta - \frac{\beta - 1}{\beta^k - 1}. \tag{10}$$

In particular, this implies

$$\lim_{k \rightarrow \infty} r_{k,1} = \beta \quad \text{and} \quad \lim_{k \rightarrow \infty} (k + 1)(r_{k,1} - \beta) = 0$$

so that

$$\lim_{k \rightarrow \infty} \frac{r_{k,1} - 1}{\beta + (k + 1)(r_{k,1} - \beta)} = \frac{\beta - 1}{\beta}. \tag{11}$$

Define the sequences

$$a_k = \frac{\beta - 1}{\beta^{k+1} - \beta} (1 + \beta^{-k/2}) \quad \text{and} \quad b_k = \frac{\beta - 1}{\beta^{k+1} - \beta}.$$

The inequality $\beta^k \lambda - 1 \leq n_k \leq \beta^k \lambda$ combined with (10) implies that

$$(1 - a_k)^{\beta^k \lambda - 1 + k} \leq \left(\frac{r_{k,1}}{\beta}\right)^{n_k + k} \leq (1 - b_k)^{\beta^k \lambda + k}. \tag{12}$$

By Lemma 4.1 it follows that

$$\lim_{k \rightarrow \infty} (1 - b_k)^{\beta^{k+1}} = e^{-\frac{\beta-1}{\beta}} \quad \text{and} \quad \lim_{k \rightarrow \infty} (1 - b_k)^k = 1,$$

which implies that

$$\lim_{k \rightarrow \infty} (1 - b_k)^{\beta^k \lambda + k} = e^{-\frac{\beta-1}{\beta} \lambda}.$$

A similar result holds for the sequence (a_k) . Hence, (11) together with the Squeeze Theorem applied to (12) completes the proof. \square

Lemma 5.3. *For k sufficiently large we have that*

$$|a_i(k)| < \frac{2}{|\beta + (k + 1)(1 - \beta)|} \cdot \frac{1}{\beta^{n_k + k}} \quad \text{for } i = 2, \dots, k.$$

Proof. Using that $|r_{k,i}| < 1$ for $i = 2, \dots, k$ gives

$$|a_i(k)| = \frac{|r_{k,i} - 1|}{|\beta + (k + 1)(r_{k,i} - \beta)|} \cdot \left(\frac{|r_{k,i}|}{\beta}\right)^{n_k + k} < \frac{2}{|\beta + (k + 1)(r_{k,i} - \beta)|} \cdot \frac{1}{\beta^{n_k + k}}.$$

For $z \in \mathbb{C}$ we consider the function

$$f(z) = \beta + (k + 1)(z - \beta).$$

Writing $z = x + iy$ gives

$$\begin{aligned} |f(z)|^2 &= (\beta + (k + 1)(x - \beta))^2 + (k + 1)^2 y^2 \\ &\geq (\beta + (k + 1)(x - \beta))^2. \end{aligned}$$

The quadratic function in the right-hand side attains its minimum value at $x_k = \beta - \beta/(k + 1)$, and for k sufficiently large it follows that $x_k > 1$. Using that $\text{Re}(r_{k,i}) \in (-1, 1)$ gives

$$|f(r_{k,i})| \geq |\beta + (k + 1)(1 - \beta)|.$$

This completes the proof. \square

From Lemma 5.3 it follows for k sufficiently large that

$$\left| \sum_{i=2}^k a_i(k) \right| \leq \sum_{i=2}^k |a_i(k)| \leq \frac{2(k - 1)}{|\beta + (k + 1)(1 - \beta)|} \cdot \frac{1}{\beta^{n_k + k}},$$

so that Lemma 5.2 implies that

$$\lim_{k \rightarrow \infty} \mathbb{P}(M_{n_k} \leq 1 - \beta^{-k}) = \lim_{k \rightarrow \infty} \frac{\beta}{\beta - 1} \sum_{i=1}^k a_i(k) = \lim_{k \rightarrow \infty} \frac{\beta}{\beta - 1} a_1(k) = e^{-\frac{\beta-1}{\beta} \lambda},$$

whereby Theorem 1.2 has been proven.

Appendix A. The conditions $D(u_n)$ and $D'(u_n)$

A more general approach to study extreme value laws is to determine for which sequences (u_n) , depending on a parameter $\lambda \geq 0$, it follows that

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_n \leq u_n) = e^{-\lambda}.$$

In [21, Theorem 1.5.1] the following equivalence is proven: if $(X_i)_{i=0}^\infty$ is an i.i.d. sequence of random variables and $\lambda \geq 0$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_n \leq u_n) = e^{-\lambda} \iff \lim_{n \rightarrow \infty} n\mathbb{P}(X_0 > u_n) = \lambda. \tag{13}$$

For example, if the variables $X_i \sim U(0, 1)$ are independent, then with $u_n = 1 - \lambda/n$ it clearly follows that $n\mathbb{P}(X_0 > u_n) = \lambda$ for all n and the left hand side of (13) yields precisely the statement in (2).

When the variables X_i are generated by a dynamical system, and therefore dependent, the equivalence (13) need not hold in general and additional conditions need to be satisfied. Let $f : M \rightarrow M$ be a map on a manifold M admitting an invariant Borel probability measure μ . In addition, consider a random variable $X : M \rightarrow \mathbb{R}$ on the probability space (M, \mathcal{B}, μ) , where \mathcal{B} is the Borel σ -algebra on M , with $\mathbb{P}(X \leq u) = \mu(X^{-1}(-\infty, u])$. The sequence $X_i = X \circ f^i$ is identically distributed but not independent. Based on [21] the following two conditions were presented in [9]:

Definition A.1. The condition $D(u_n)$ holds for the sequences $(X_i)_{i=0}^\infty$ and $(u_n)_{n=1}^\infty$ if for any integers $\ell, t, n \geq 1$ we have

$$\left| \mathbb{P}(X_0 > u_n, X_t \leq u_n, \dots, X_{t+\ell-1} \leq u_n) - \mathbb{P}(X_0 > u_n)\mathbb{P}(X_t \leq u_n, \dots, X_{t+\ell-1} \leq u_n) \right| \leq \gamma(n, t),$$

where $\gamma(n, t)$ is non-increasing in t for each n and $n\gamma(n, t_n) \rightarrow 0$ as $n \rightarrow \infty$ for some sequence $t_n = o(n)$ as $t_n \rightarrow \infty$.

Definition A.2. The condition $D'(u_n)$ holds for the sequences $(X_i)_{i=0}^\infty$ and $(u_n)_{n=1}^\infty$ if

$$\lim_{k \rightarrow \infty} \left(\limsup_{n \rightarrow \infty} n \sum_{j=1}^{\lfloor n/k \rfloor} \mathbb{P}(X_0 > u_n, X_j > u_n) \right) = 0.$$

The $D(u_n)$ condition imposes a decay rate on the dependence of specific events concerning threshold exceedances. The $D'(u_n)$ condition restricts the amount of clustering of exceedances over a threshold. Under these two conditions the equivalence in (13) remains true for the process $X_i = X \circ f^i$ [9, Theorem 1]. For the Rényi map process we will now show that $D(u_n)$ is satisfied, but $D'(u_n)$ is not.

It follows from [3, Theorem 8.3.2] that the Rényi map has exponential decay of correlations. This means the following: for all functions $\varphi \in BV([0, 1])$ and $\psi \in L^\infty([0, 1])$ there exist constants $C > 0$ and $0 < r < 1$ such that

$$\left| \int_0^1 \varphi \cdot (\psi \circ f^t) d\mu - \int_0^1 \varphi d\mu \int_0^1 \psi d\mu \right| \leq C \text{Var}(\varphi) \|\psi\|_\infty r^t \quad \text{for all } t \geq 0.$$

By taking the indicator functions $\varphi = 1_{\{X_0 > u_n\}}$ and $\psi = 1_{\{X_0 \leq u_n, \dots, X_{\ell-1} \leq u_n\}}$ it follows that the $D(u_n)$ condition is satisfied with $\gamma(n, t) = 2Cr^t$ and $t_n = n^\alpha$ for any $0 < \alpha < 1$.

Now we show that $D'(u_n)$ does not hold for any sequence u_n that satisfies

$$\lim_{n \rightarrow \infty} n\mathbb{P}(X_0 > u_n) = \lim_{n \rightarrow \infty} n(1 - u_n) = \lambda > 0.$$

To that end, observe that we have the following inclusion:

$$(u_n, 1) \cap f^{-j}((u_n, 1)) \supset \left(1 - \frac{1 - u_n}{\beta^j}, 1\right).$$

This gives the inequality

$$\mathbb{P}(X_0 > u_n, X_j > u_n) = \text{Leb}((u_n, 1) \cap f^{-j}((u_n, 1))) \geq \text{Leb}\left(1 - \frac{1 - u_n}{\beta^j}, 1\right) = \frac{1 - u_n}{\beta^j},$$

which implies that

$$n \sum_{j=1}^{\lfloor n/k \rfloor} \mathbb{P}(X_0 > u_n, X_j > u_n) \geq n(1 - u_n) \sum_{j=1}^{\lfloor n/k \rfloor} \frac{1}{\beta^j} = n(1 - u_n) \cdot \frac{1 - \beta^{-\lfloor n/k \rfloor}}{\beta - 1}.$$

Finally, it follows that

$$\lim_{k \rightarrow \infty} \left(\limsup_{n \rightarrow \infty} n \sum_{j=1}^{\lfloor n/k \rfloor} \mathbb{P}(X_0 > u_n, X_j > u_n) \right) \geq \frac{\lambda}{\beta - 1} > 0,$$

which shows that the $D'(u_n)$ condition is not satisfied.

Appendix B. Clustering and the extremal index

Extremes in the Rényi map process can form clusters. Let $u = \beta^{-k}$ for some $k \in \mathbb{N}$. The probability of having a cluster of q consecutive variables X_i exceeding the threshold $1 - u$ is given by

$$\frac{\mathbb{P}(X_0, \dots, X_{q-1} > 1 - u, X_q \leq 1 - u)}{\mathbb{P}(X_0 > 1 - u)} = \frac{\text{Leb}(E_0 \cap \dots \cap E_{q-1} \cap E_q^c)}{\text{Leb}(E_0)}.$$

Observe that

$$E_0 \cap \dots \cap E_{q-1} = \left[\frac{\beta^{q-1} - u}{\beta^{q-1}}, 1 \right) \quad \text{and} \quad E_q^c = \bigcup_{j=1}^{\beta^q} \left[\frac{j-1}{\beta^q}, \frac{j-u}{\beta^q} \right),$$

which implies that

$$E_0 \cap \dots \cap E_{q-1} \cap E_q^c = \left[\frac{\beta^{q-1} - u}{\beta^{q-1}}, \frac{\beta^q - u}{\beta^q} \right).$$

Hence, the probability of the occurrence of a cluster of length q is given by

$$\frac{\mathbb{P}(X_0, \dots, X_{q-1} > 1 - u, X_q \leq 1 - u)}{\mathbb{P}(X_0 > 1 - u)} = \frac{\beta - 1}{\beta} \cdot \frac{1}{\beta^{q-1}}.$$

Using that $1 + 2x + 3x^2 + \dots = 1/(1 - x)^2$ for $|x| < 1$ implies that the mean cluster size is given by

$$\begin{aligned} \mathbb{E}(\text{cluster size}) &= \sum_{q=1}^{\infty} q\mathbb{P}(\text{cluster of size } q) = \frac{\beta - 1}{\beta} \sum_{q=1}^{\infty} \frac{q}{\beta^{q-1}} \\ &= \frac{\beta - 1}{\beta} \left(\frac{\beta}{\beta - 1} \right)^2 = \frac{\beta}{\beta - 1}. \end{aligned}$$

Finally, by taking the reciprocal of the mean cluster size we obtain the extremal index $\theta = (\beta - 1)/\beta$. It is precisely the clustering of extremes which violates the $D'(u_n)$ condition that was discussed in the previous section.

References

- [1] M. Abadi, Sharp error terms and necessary conditions for exponential hitting times in mixing processes, *Ann. Probab.* 32 (2004) 243–264.
- [2] T. Addabbo, M. Alioto, A. Fort, A. Pasini, S. Rocchi, V. Vignoli, A class of maximum-period nonlinear congruential generators derived from the Rényi chaotic map, *IEEE Trans. Circuits Syst.* 54 (2007) 816–828.
- [3] A. Boyarsky, P. Góra, *Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimension*, Birkhäuser, 1997.
- [4] H.W. Broer, F. Takens, *Dynamical Systems and Chaos*, in: *Applied Mathematical Sciences*, vol. 172, Springer, 2011.
- [5] G.H. Choe, *Computational Ergodic Theory*, Springer, 2005.
- [6] P. Collet, Statistics of closest return for some non-uniformly hyperbolic systems, *Ergodic Theory Dynam. Systems* 21 (2001) 401–420.
- [7] G.P.B. Dresden, Z. Du, A simplified Binet formula for the k -generalized Fibonacci numbers, *J. Integer Seq.* 17 (2014) 14.4.7.
- [8] A.C.M. Freitas, Statistics of the maximum of the tent map, *Chaos Solitons Fractals* 42 (2009) 604–608.
- [9] A.C.M. Freitas, J.M. Freitas, On the link between dependence and independence in extreme value theory for dynamical systems, *Statist. Probab. Lett.* 78 (2008) 1088–1093.
- [10] A.C.M. Freitas, J.M. Freitas, M. Todd, Hitting time statistics and extreme value theory, *Probab. Theory Related Fields* 146 (2010) 675–710.
- [11] A.C.M. Freitas, J.M. Freitas, M. Todd, The extremal index, hitting time statistics and periodicity, *Adv. Math.* 231 (2012) 2626–2665.
- [12] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, Wiley, 1978.
- [13] L. de Haan, A. Ferreira, *Extreme Value Theory. An Introduction*, Springer, 2006.
- [14] G. Haiman, Extreme values of the tent map process, *Statist. Probab. Lett.* 65 (2003) 451–456.
- [15] G. Haiman, Level hitting probabilities and extremal indexes for some particular dynamical systems, *Methodol. Comput. Appl. Probab.* 20 (2018) 553–562.
- [16] N. Haydn, Y. Lacroix, S. Vaienti, Hitting and return times in ergodic dynamical systems, *Ann. Probab.* 33 (2005) 2043–2050.
- [17] N. Haydn, S. Vaienti, The compound Poisson distribution and return times in dynamical systems, *Probab. Theory Related Fields* 144 (2009) 517–542.
- [18] M. Hirata, Poisson law for Axiom A diffeomorphisms, *Ergodic Theory Dynam. Systems* 13 (1993) 533–556.
- [19] M. Hirata, B. Saussol, S. Vaienti, Statistics of return times: a general framework and new applications, *Comm. Math. Phys.* 206 (1999) 33–55.
- [20] C. Kalle, D. Kong, N. Langeveld, W. Li, The β -transformation with a hole at 0, *Ergodic Theory Dynam. Systems* 40 (2020) 2482–2514.
- [21] M.R. Leadbetter, G. Lindgren, H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, 1980.
- [22] C. Levesque, On m th order linear recurrences, *Fibonacci Quart.* 23 (1985) 290–293.
- [23] V. Lucarini, D. Faranda, A.C.M. Freitas, J.M. Freitas, M. Holland, T. Kuna, M. Nicol, M. Todd, S. Vaienti, *Extremes and Recurrence in Dynamical Systems*, Wiley, 2016.
- [24] E.P. Miles Jr., Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly* 67 (1960) 745–752.
- [25] M.D. Miller, On generalized Fibonacci numbers, *Amer. Math. Monthly* 78 (1971) 1108–1109.
- [26] A. Rényi, Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hungar.* 8 (1957) 477–493.
- [27] S.I. Resnick, *Extreme Values, Regular Variation, and Point Processes*, in: *Applied Probability*, vol. 4, Springer-Verlag, New York, 1987.
- [28] J. Rousseau, Hitting time statistics for observations of dynamical systems, *Nonlinearity* 27 (2014) 2377–2392.
- [29] W.R. Spickerman, R.N. Joyner, Binet’s formula for the recursive sequence of order k , *Fibonacci Quart.* 22 (1984) 327–331.
- [30] D.A. Wolfram, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* 36 (1998) 129–145.