Student Development in Logical Reasoning

Bronkhorst, Hugo; Roorda, Gerrit; Suhre, Cor; Goedhart, Martin

Published in:
Canadian Journal of Science, Mathematics and Technology Education

DOI:
10.1007/s42330-021-00148-4

Can. J. Sci. Math. Techn. Educ. (2021) 21:378–399
https://doi.org/10.1007/s42330-021-00148-4

Student Development in Logical Reasoning: Results of an Intervention Guiding Students Through Different Modes of Visual and Formal Representation

Hugo Bronkhorst[1] · **Gerrit Roorda**[2] · **Cor Suhre**[2] · **Martin Goedhart**[1]

Accepted: 7 April 2021 / Published online: 23 April 2021

Abstract Due to growing interest in twenty-first-century skills, and critical thinking as a key element, logical reasoning is gaining increasing attention in mathematics curricula in secondary education. In this study, we report on an analysis of video recordings of student discussions in one class of seven students who were taught with a specially designed course in logical reasoning for non-science students (12th graders). During the course of 10 lessons, students worked on a diversity of logical reasoning tasks: both closed tasks where all premises were provided and everyday reasoning tasks with implicit premises. The structure of the course focused on linking different modes of representation (enactive, iconic, and symbolic), based on the model of concreteness fading (Fyfe et al., 2014). Results show that students easily link concrete situations to certain iconic referents, such as formal (letter) symbols, but need more practice for others, such as Venn and Euler diagrams. We also show that the link with the symbolic mode, i.e. an interpretation with more general and abstract models, is not that strong. This might be due to the limited time spent on further practice. However, in the transition from concrete to symbolic via the iconic mode, students may take a step back to a visual representation, which shows that working on such links is useful for all students. Overall, we conclude that the model of concreteness fading can support education in logical reasoning. One recommendation is to devote sufficient time to establishing links between different types of referents and representations.

Résumé En raison de l'intérêt grandissant porté aux compétences arrimées au 21e siècle ainsi que du rôle central qu'y joue la pensée critique, le raisonnement logique gagne sans cesse en importance dans le programme d'enseignement des mathématiques au secondaire. Dans cette étude, nous présentons l'analyse d'enregistrements vidéo de discussions qui ont eu lieu dans une classe de sept élèves à qui l'on a donné une formation spécialement élaborée en raisonnement logique et destinée aux élèves de 12e an-

✉ Hugo Bronkhorst
h.bronkhorst@rug.nl

1 Faculty of Science and Engineering, Institute for Science Education and Communication, University of Groningen, Groningen, The Netherlands

2 Teacher Education, Faculty of Behavioural and Social Sciences, University of Groningen, Groningen, The Netherlands

née qui ne sont pas en sciences. Pendant une période s'étendant sur 10 leçons, les élèves ont exécuté diverses tâches de raisonnement logique, incluant des tâches dites «fermées» où toute l'information nécessaire est donnée explicitement et des tâches de tous les jours faisant appel au raisonnement par inférence implicitement. La structure du cours a porté essentiellement sur l'établissement de liens entre différents modes de représentation (énactive, iconique, et symbolique), en se fondant sur le modèle de «*concreteness fading* (l'atténuation du concret)» (Fyfe et al., 2014). Les résultats indiquent que les élèves établissent facilement des liens entre des situations concrètes et certains référents iconiques tels que des symboles (de lettres) formels, mais qu'ils ont besoin de s'entraîner davantage avec d'autres comme les diagrammes de Venn et d'Euler. Nous démontrons aussi que le lien qui s'établit en mode symbolique c'est-à-dire par la représentation de modèles plus génériques et abstraits n'est pas très marqué. Ceci peut être dû au manque d'entraînement. Toutefois, dans la transition qui s'effectue du concret au symbolique en mode iconique, les élèves peuvent faire un pas en arrière et choisir une représentation visuelle, ce qui démontre que de travailler sur ces liens s'avère utile pour tous les élèves. Notre conclusion générale est que le modèle de «*concreteness fading* (l'atténuation du concret)» peut soutenir l'enseignement en ce qui concerne le raisonnement logique. Nous recommandons d'allouer suffisamment de temps à l'établissement de liens entre les différents types de référents et de représentations.

Keywords Twenty-first-century skills · Concreteness fading · Logical reasoning · Mathematics education · Secondary school

Introduction

It is generally accepted that the development of twenty-first-century skills is essential for success in work and life, and this should be an important objective in all stages of education. One key element of twenty-first-century skills is critical thinking (Brookhart, 2010; P21, 2015; Vincent-Lancrin et al., 2019). In an earlier article, we stressed the importance of logical reasoning for the development of critical thinking skills (Bronkhorst et al., 2020a). Liu et al. (2015) even claim that logical reasoning is the "core foundation" (p. 337) of critical thinking. However, one unresolved question is how students with relatively little experience in logical reasoning can be taught to reason logically in various situations and recognise logical fallacies. Based on research about effective instruction in mathematics education, we developed an intervention for non-science students with specific emphasis on the use of visual and formal representations in logical reasoning tasks, such as syllogism tasks, tasks with if–then statements, and argument analysis tasks. In previous work, we found that student development of logical reasoning was supported by the use of visual and formal representations. In this study, we explicitly focus on how students developed the ability to use these representations effectively over the course of the intervention.

Theoretical Background

First, we will elaborate on our definition of logical reasoning, which includes formal and informal reasoning. Formal reasoning is considered to occur within a system of predefined rules and symbols, based on unchanging premises. Valid conclusions are reached if the rules are followed, for example, rules of logic and mathematics (e.g. Schoenfeld, 1991; Teig & Scherer, 2016). Informal or everyday reasoning is often considered to be reasoning that is expressed in ordinary language and used to construct an argument where the reasoning and conclusions are context-dependent without strict validity (e.g. Bronkhorst et al., 2020a; Johnson & Blair, 2006; Kuhn, 1991; Voss et al., 1991). Because both formal and informal reasoning are important to accomplish critical thinking, logical reasoning should not be considered synonymous with formal reasoning alone, but needs a broader definition. In this vein, Nunes (2012) defines logical reasoning as "a form of thinking in which premises and relations between premises are used in a rigorous manner to *infer* [emphasis added] conclusions that are entailed (or implied) by the premises and the relations" (p. 2066). The book, *How People Learn II: Learners, Contexts, and Cultures* (National Academies of Sciences, 2018), refers to inferential reasoning as "making logical connections between pieces of information in order to organize knowledge for understanding and to drawing conclusions through deductive reasoning, inductive reasoning, and abductive reasoning" (p. 93; based on: Seel, 2012). As we intend to emphasise the importance of making connections between information and using formal and informal reasoning, we define logical reasoning as "selecting and interpreting information from a given context, making connections, and verifying and drawing conclusions based on provided and interpreted information and the associated rules and processes" (Bronkhorst et al., 2020a, p. 1676), which we will use in this study.

Perhaps because of the influence of the twenty-first-century skills movement, mathematics curricula from around the world stipulate that apart from developing students' formal logical reasoning applied within mathematics tasks, mathematics education should foster reasoning that can be applied beyond the classroom (e.g. cTWO, 2012; Liu et al., 2015; McChesney, 2017; National Council of Teachers of Mathematics, 2009). In the Netherlands, the domain of "logical reasoning" has recently been introduced into the mathematics curriculum for pre-university non-science students (College voor Toetsen en Examens, 2016).

Since we stress that logical reasoning is applied within a diversity of contexts and thus should be used in a variety of tasks in the classroom, we make a distinction between formal reasoning tasks and everyday reasoning tasks, following Galotti (1989, p. 335). The key elements of formal reasoning tasks are that "all premises are provided, problems are self-contained, there is typically one correct answer, [and] it is typically unambiguous when the problem is solved" (Galotti, 1989, p. 335). For everyday reasoning tasks, the key elements are that "some premises are implicit, and some are not supplied at all, problems are not self-contained, there are typically several possible answers that vary in quality, [and] it is often unclear whether the current 'best' solution is good enough" (Galotti, 1989, p. 335). For the readability of this article, we will refer to formal reasoning tasks as *closed tasks*. Consider the conclusions in the following two examples as reasoning in closed syllogism tasks:

I (1) All *A* are *B*. (2) All *B* are *C*. (So) All *A* are *C*.
II (1) All *humans* are *mammals*. (2) All *mammals* are *animals*. (So) All *humans* are *animals*.

Although the examples are presented differently, with formal letter symbols in an abstract model (I) versus concrete objects (II), both conclusions follow logically from the given premises, they are valid, and they are conclusive. An example of an everyday reasoning task is the analysis of the argument in a newspaper article. In such tasks, not all premises are provided; therefore, the reader must make some implicit assumptions and review the most likely outcome.

Concreteness Fading

The model of "concreteness fading" (CF) provides a useful framework (Fyfe et al., 2014) to describe the phases of our intervention with a course in logical reasoning. The model is inspired by Bruner's theory of instruction (Bruner, 1966), which distinguishes three stages, with students using different modes of representation that are applied in successive stages of skill learning. In the first stage of learning, the enactive mode, students rely on concrete knowledge and actions to achieve satisfactory outcomes. In the second stage, students start using iconic modes of representation, such as images or graphical representations. In the final stage, the symbolic mode, students use abstract representations, such as symbols and logical propositions used in reasoning with certain rules or laws. The strength of a representation may be different for each individual, depending on their understanding, but Bruner states that every problem situation can always be transformed in a recognisable way for the learner.

Bruner's model has been translated into the Concrete Representational Abstract (CRA) instruction framework, commonly used in the USA (Butler et al., 2003), or the Concrete Pictorial Abstract (CPA) framework as adopted in, for example, Singapore (Kim, 2020). Although the description of the different modes in the CRA and CPA frameworks is similar to the stages in CF, we will use the terminology of CF because it explicitly focuses on consecutively establishing the links between the three stages. All stages are equally important, and by fading from the concrete information through the use of various representations, students gradually move to the iconic and symbolic stages in their development.

Fyfe et al. (2014) emphasise that spending sufficient time on making the connections between the stages should be the strength of the model. Although much research concerning this model focuses on elementary students (e.g. Fyfe et al., 2015) and the fact that many textbooks in middle and higher secondary school do not address the sequence correctly (Witzel et al., 2008, p. 272), there are some studies that show positive effects of CF in comparison with other approaches within mathematics education at other levels (e.g. Kim, 2020; McNeil & Fyfe, 2012; Ottmar & Landy, 2017). The different stages of this model and the focus on successful transitions between the different stages guide the activities in our intervention.

In terms of the teaching of logical reasoning, the enactive mode refers to reasoning in concrete situations that stimulates learners to explore the situations in ordinary language. As soon as unrelated context is removed or formal symbols are introduced, a learner leaves the enactive mode and enters the iconic mode. Using schematic representations may help students to interiorise schemata that can be used in abstract reasoning (Chu et al., 2017). These representations are called graphic pictorial models in CF and should not be confused with concrete pictorial drawings (e.g. Hegarty & Kozhevnikov, 1999), which are concrete representations aimed at representing an authentic and complete image of a situation with unnecessary details (Chu et al., 2017). These are part of the enactive mode and are called "drawings" in this study. The symbolic mode is the most abstract level and refers to, for example, general rules of logic, such as *modus ponens* and *modus tollens*.

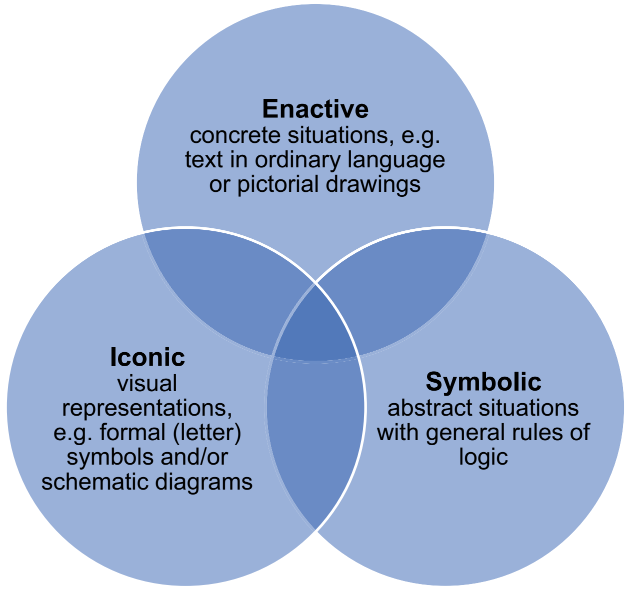
Following the development represented by the model, students move from the enactive mode to the iconic and, finally, to the symbolic mode. However, students should be capable of translating reasoning used in the symbolic mode back to the concrete world and vice versa. More specifically, while diagrams provide a means for students to apply the rules of logic to everyday situations, students should also learn to link these conclusions to the everyday situations and might even use representations from all three modes in their reasoning. This is visualised in Fig. 1. The overlapping areas show the possible links (see also Tondevold, 2019).

Formal and Visual Representations

Prior research among university students indicates that teaching formal reasoning can be beneficial for the development of reasoning skills in general (e.g. Lehman et al., 1988; Stenning, 1996); however, Stenning (2002) also acknowledges that not all teaching in formal reasoning and representations is beneficial and that informal methods might be sufficient. Representations taught to students should capture relevant aspects of contexts and leave out irrelevant details to support their thinking (McKendree et al., 2002). Hegarty and Kozhevnikov (1999) conclude that instruction in visual representations "should encourage students to construct spatial representations of the relations between objects in a problem and discourage them from representing irrelevant pictorial details" (p. 688).

Research in secondary education suggests that the use of formal representations improves student reasoning and can be taught (Adey & Shayer, 1993; Van Aalten & De Waard, 2001). Based on Halpern (2014) and Van Gelder (2005), as well as on our own findings (Bronkhorst et al., 2018, 2020a), we

Fig. 1 Different modes of representation with links in all directions



conjecture that diagrams (such as Venn and Euler diagrams), scheme-based methods, and knowledge of formal logical rules will be highly beneficial for all sorts of reasoning tasks for our target group. The use of such representations is illustrated in Fig. 2 for the syllogism: "(1) All *humans* are *mammals*. (2) All *mammals* are *animals*. (So) All *humans* are *animals*." On the left, the Euler diagram offers a visual representation of the context provided, while the diagram on the right is more general and even further formalised with the formulas on the right-hand side. The conclusion $A \Rightarrow C$ can be verified by using the *modus ponens (m.p.)* rule.

Intervention

The intervention consisted of 10 50-min lessons on logical reasoning. Here, we provide an overview of the intervention with some task examples first, before showing how these lessons are linked to modes of representation in CF. In the design, the first two lessons were devoted to an exploration of reasoning in concrete tasks, mainly short newspaper articles, as an introduction. In the following lessons, students practised creating and working with visual and formal representations in small, mainly closed and meaningful tasks, with specific attention paid to links between the different modes of representation: first from enactive to iconic and later from iconic to symbolic. This was done with all sorts of syllogisms and several if–then claims, with specific attention paid to the students' own

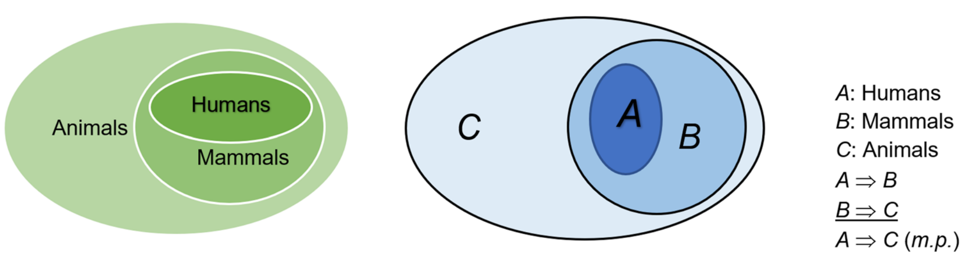


Fig. 2 Syllogism schematised on the left, more general on the right

solution methods. Recognising the importance of discourse in mathematics education, opportunities to discuss and justify their methods in pairs, in groups, and as a class were provided (Gravemeijer, 2020; Grouws & Cebulla, 2000; National Research Council, 1999). Figure 3 provides an example. On the left, the syllogism is stated in ordinary language (concrete version). In an earlier task, students were asked to find a general structure for this syllogism, after which letter symbols (first step iconic mode) were introduced. In this task, students were asked to use a visualisation for this syllogism (further exploration of representations in iconic mode) before Venn and Euler diagrams were introduced in the lesson materials. If–then statements, such as "If it rains, the street gets wet", were used to make the connection between iconic and symbolic modes of representation. Using Euler diagrams (iconic), formal notations with logical symbols ($\wedge$, $\vee$, $\Rightarrow$, and $\neg$) were explored to discover the rules of *modus ponens* ($A \Rightarrow B$. A, so B) and *modus tollens* ($A \Rightarrow B$. $\neg B$, so $\neg A$) (symbolic). During the final lessons, students were encouraged to apply and combine the representations of the different modes they had learned in a task such as that shown in Fig. 4.

Figure 5 shows the structure of the intervention based on the three modes of representation. After two lessons of explorations in concrete situations, two lessons were aimed at establishing the link between enactive and iconic modes of representation (arrow 1). Subsequently, two lessons aimed at linking iconic and symbolic modes of representation (arrow 2) and enactive and symbolic modes of representation (arrow 3). Afterwards, two lessons offered students opportunities to use and link all three modes (area 4). The last two lessons consisted of further practice.

The intervention was developed via two iterative cycles (Van den Akker et al., 2013) in collaboration with a group of teachers. After a pilot study and evaluation, adjustments were made, mainly to provide students with sufficient time to develop their own solutions, for discussion in small groups or with the whole class, and for additional practice. During the sessions with the teachers, materials and implementation guidelines were discussed extensively. These guidelines were also provided in a teacher manual. In particular, attention to the links between the different modes of representation of CF and the importance of classroom discussions about the various representations were emphasised during the meetings.

Research Question

In an earlier experimental study, we found a significant increase in the use of visual and formal representations among students in the experimental group but not in the control group (Bronkhorst et al., 2020b). In the experimental group, the use of Venn and Euler diagrams positively correlated with the scores on closed tasks. We also found that, in the post-test, students from the experimental group used Venn and Euler diagrams much more frequently than symbolic logical rules. In this article, the focus is on student development through the different modes of representation in the classroom. We focus on the way students developed effective use of visual and formal representations over the course of the intervention. In evaluating student development during the different parts of our intervention, our study was guided by the following research question: How do students use and apply visual and formal representations (iconic and symbolic) in logical reasoning tasks?

Method

In this article, we will mainly use the analysis of video recordings from one group of students (12th graders) who were taught with the specially designed course in logical reasoning described in the introduction of this article. From the 10 lessons, the third through the seventh were videotaped by

Exercise 18:

We will have a look again at the first syllogism, for which we made the structure below.

All humans are mortal.	All *A* are *B*.
Socrates is human.	*C* is an *A*.
So: Socrates is mortal.	So: *C* is *B*.

The question to you is:
How could we visualise the syllogism above? Give/draw your visualisation below.

Fig. 3 Visualising task for a syllogism

Exercise 40c:

Next part newspaper article "It starts with one glass a day"
By: ***Rinskje Koelewijn*** *– 5 October 2016 – NRC Handelsblad*

The Health Council has now amended the advice: "do not drink alcohol, and if you cannot resist it, no more than one glass a day." Kahn: "In most studies, the (moderate) drinkers are compared with abstainers. But the point is: the abstainers have never been asked why they do not drink. If you are going to figure that out, you will see that people have all kinds of reasons to abandon alcohol: they want to live a healthy life, they are religious, they do not like the taste. These groups of non-drinkers do not die before moderate drinkers. The increased mortality among the non-drinkers is caused by the non-drinkers who did not drink for health reasons, or who stopped drinking after an alcohol problem. That is the group that causes the crazy curl in the hockey stick, from which the wrong conclusion has been drawn that one glass is better than none. "

How would you best visualize Kahn's assertion? Then show your scheme/diagram/representation and compare your visualisation with three or four others. Discuss the differences.

Fig. 4 Analysis statement newspaper article (based on: Koelewijn, 2016)

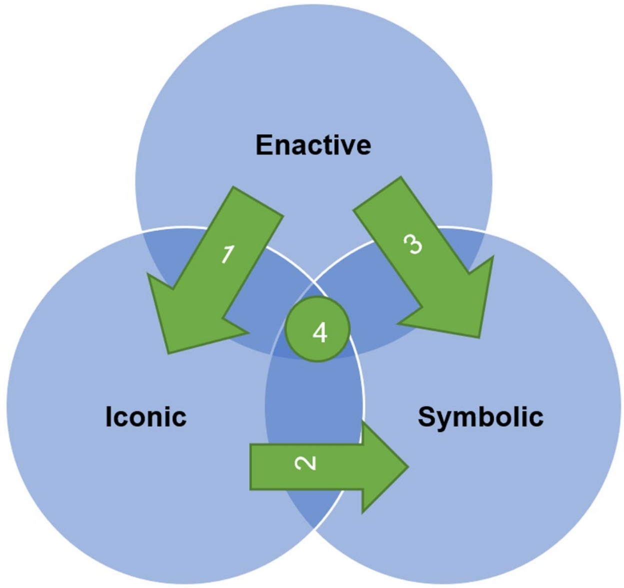


Fig. 5 Structure of intervention

the first author of this article. These five lessons were selected because of the central focus on linking different modes of representations as important in CF.

Participants

The recorded group consisted of seven students from a school in the northern part of the Netherlands. The students were in their last year of pre-university education (12th graders): there were four boys (Adam, Daniel, Liam, and Owen) and three girls (Julia, Nora, and Riley). The small class size is common for this mathematics course because the course with logical reasoning is an elective for non-science students (College voor Toetsen en Examens, 2016). Their mathematics teacher has long-standing experience and has taught at this school for 32 years. All students and the teacher agreed to the video and audio recordings. An informed consent release was sent to all participating students and their parents, which was approved by the ethics committee of the authors' university.

Data Collection

Five lessons of the experimental group were recorded. Classroom discussions were videotaped and interactions between students during work in pairs or groups of three were recorded with voice recorders. For some tasks, student worksheets were collected. After each lesson, the teacher filled out a logbook and rated statements concerning the implementation of the intervention (Likert scale 1–5).

Analysis

The teacher's logbook was used to verify whether the lessons were implemented according to plan. We used the video and audio recordings to analyse students' statements and discussions. The recordings and corresponding transcripts were analysed in Dutch by the first two authors of this article. For this article, selected excerpts of these conversations have been translated into English.

Discussions among students and classroom discourse were analysed qualitatively in an interpretive way (e.g. Cohen et al., 2007) and categorised based on the different modes of representation of CF and the links between them, as represented in Figs. 1 and 5. If students' answers were concrete, with text in ordinary language or concrete pictorial drawings, it was categorised as reasoning in the enactive mode. Iconic modes of representation were identified by (1) the use or introduction of formal symbols as abstract referents, such as letter symbols, logical symbols, and arrows, but without manipulating them or applying general rules, or (2) the use of schematic diagrams and visual representations such as Venn and Euler diagrams. Figure 2, for the example discussed above, shows iconic representations: (1) letter symbols *A*, *B*, and *C* to represent humans, mammals, and animals respectively and (2) Euler diagrams. If abstract referents were used in a model to discover structural patterns or to apply general formal rules, the students' reasoning was categorised as symbolic. Examples are the abstract rules *modus ponens* and *modus tollens*, which are shown in the bottom right corner of our example in Fig. 2.

Results

According to the teacher's reports in the logbooks, we concluded that the lessons were implemented according to our intentions. The video recordings confirmed that the teacher provided opportunities for the students to work in pairs or groups of three on the tasks (about 50% of the lesson time). The teacher reported high student participation and that different solutions were discussed in student groups and the classroom (about 25% of the lesson time). We observed much more discussion among students in the second half of the intervention. The different phases of the intervention will be described in detail below according to the sequential structure of the intervention, as shown in Fig. 5.

From Enactive to Iconic Modes of Representation

Below, we describe two activities that aimed at establishing the link between the enactive and iconic modes of representation (see arrow 1 in Fig. 5).

Letter Symbols

After some explorations of the meaning of logical reasoning, students were introduced to syllogisms and explored the truth and validity of these short arguments. A typical example of these syllogism tasks was the following:

Premise 1: All humans are mortal.
Premise 2: Socrates is human.
Conclusion: Socrates is mortal.

In an open task, students were asked to find "a structure" for this syllogism individually and to compare their "structure" with others. However, Julia and Riley immediately started discussing this and introduced the symbols *P* and *Q* at the beginning of their conversation to abbreviate the premises, later using *A* and *B* as well.

[1] Julia: oh, do we have to do something with *P*, *Q*, at least that is all I can think of now
[2] Riley: yes, then it is *P*, *Q*
[3] Julia: *Q*, *P*
[4] Riley: *P*, so *Q*

[5] Julia: huh? Wait, why *P*, *Q*?
[6] Riley: because, those are just the things they always use
[7] Julia: no, there are several forms, right?
[8] Riley: I can do that *P* and *Q* ...[inaudible]…, it doesn't matter what you use
[9] Julia: no, I mean the form
[10] Riley: yes, but you might say *A*, *P*. *A*, so *P*. It doesn't matter what you say, right? Or am I saying something stupid now?
[11] Julia: *P*, *Q*, *P* are humans then?
[12] Riley: yes, *P* is humans, and *Q* is mortal. He is human so he is mortal.
[13] Julia: ah, wow
[14] Riley: right? Or *A*, *B*, *A*, *B*, you know, you have to decide yourself.

From this transcript, we observe that Julia introduced the letter symbols *P* and *Q* (line [1]) and Riley agreed with this (line [2]), linking the concrete situation in the task to iconic representations. In the conversation, Riley made new reasoning steps (even numbered lines), while Julia asked questions or confirmed Riley's reasoning (odd numbered lines). Riley understood that the letter symbols chosen were arbitrary (lines [8] and [14]) and that concrete meaning (here: humans for *P* and mortal for *Q*) could be assigned to them (line [12]), which shows an initial understanding of the general form of a syllogism with letter symbols as an abstract model. Julia confirmed that she understood the link between the syllogism with letter symbols and the concrete example (line [13]).

Visual Representations

After further practice with letter symbols, students were asked to individually come up with their own visual representations of the syllogism about Socrates and then compare their ideas with their peers (see Fig. 3). The goal was that students would not only be able to generalise these syllogisms into a form with letter symbols, as in the previous task, but would also be able to use other forms of iconic representation such as Venn diagrams.[1]

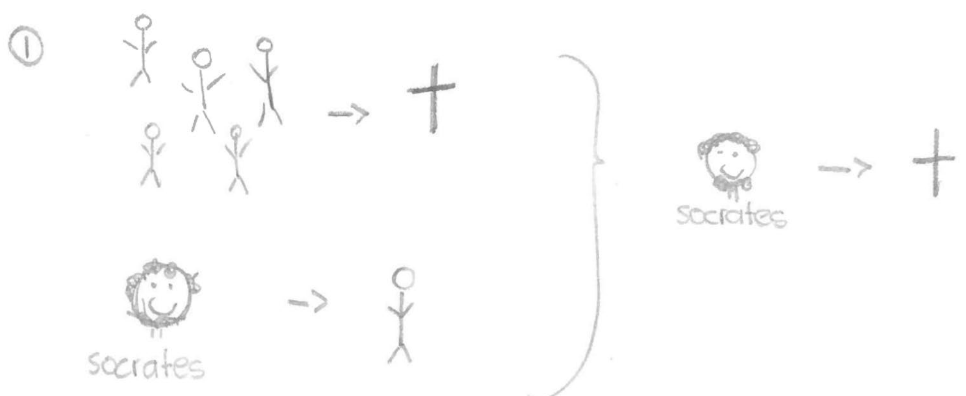


Fig. 6 Nora's visualisation for the Socrates syllogism

[1] We will use the word "Venn" for all Venn and Euler diagrams in the "Results" section, because the lesson materials and the students used the term Venn for both.

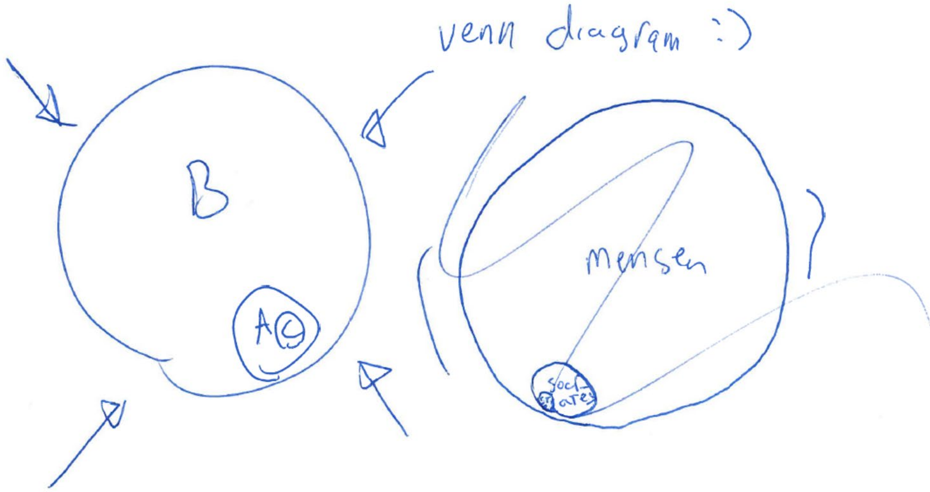


Fig. 7 Daniel's visualisation for the Socrates syllogism

Nora and Daniel completed the first part of the task individually. Figures 6 and 7 show their answers. In the transcript below, Nora's work is discussed (see Fig. 6). She literally tried to visualise the situation presented in the syllogism.

[1] Teacher: you have very different things, did you have a look at each other's work?
[2] Nora: yes, then it is, I drew some dummies
[3] Daniel: also nice
[4] Nora: also nice. I drew both premises separately, so that those are very clear and I have derived the conclusion from there. So I quite literally translated the premises with the symbols into pictures.
[5] Teacher: yes, okay, [...] but then you stay really close to the example, right?
[6] Nora: yes, true, shouldn't I have done that?
[7] Teacher: the aim was actually, the way of reasoning, so, this is in general, such a syllogism, in fact for *A* you can take humans, but also other things, how would you visualise that?
[8] Nora: oh
[9] Daniel: yes, you can just do the same thing, but leave out the dummies, you can put
[10] Nora: *A*'s there
[11] Daniel: just one *A*
[12] Nora: that's basically what I
[13] Daniel: you just do the same instead of drawings

Here, we observe that Nora made a drawing to represent the syllogism about Socrates. She visualised the meaning of the words literally in a pictorial drawing (see Fig. 6) and used arrows to schematise the implications, as she explained in line [4]. Apart from the arrows, the rest of her drawing was limited to the real situation described in the task and thus an enactive representation. The teacher tried to convince her to link her concrete model to more abstract referents (lines [5] and [7]). Nora thought that she could just replace the dummies by *A*'s (line [10]), but Daniel stated that one letter *A* for the whole set was enough (lines [9] and [11]). This suggests that Daniel tried to discover a more general structural pattern.

After this conversation, Nora made a second visualisation (see Fig. 8) and the teacher asked her to explain it, but Nora was not able to.

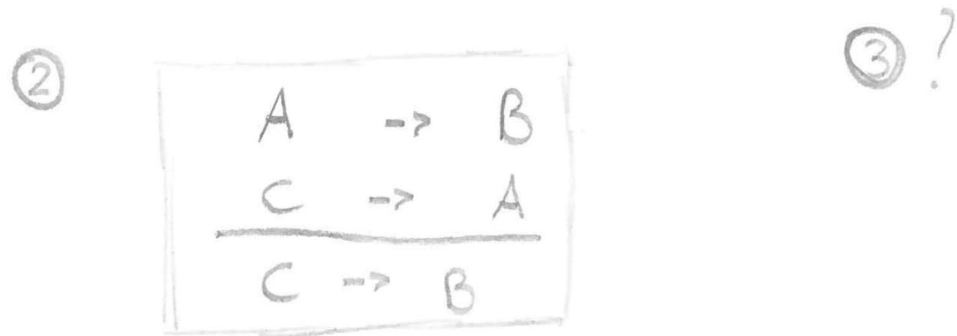


Fig. 8 Nora's second and third attempts to visualise the syllogism

[14] Teacher: please explain it to me, *A* arrow *B*
[15] Nora: yes, that is let's say, a fixed reasoning pattern with all *A* are *B* or *A* is *B*, *C* is *A*, so *C* is *B*
[16] Teacher: okay, but the information all or one, is that still important? Or can you just leave it out?
[17] Nora: I don't really get it right now anymore, so I try something new, these are just variables that you can add or not

The teacher wanted Nora to explain the meaning of the arrows (line [14]) and the difference between "all *A*" and a single *C* (line [16]), because in her structure, both premises look the same. Nora only translated the conjugations of the verb "to be" into an implication arrow (line [15]), and she seemed confused by the meaning of the letter symbols as variables (line [17]). Although she said she was giving it another try (line [17]), she only wrote a question mark behind the 3 (Fig. 8).

Nora's transformations from the concrete situation in this task to a visual representation started with an enactive representation (pictorial drawing), before she tried to link her drawing to an iconic representation. The following transcript shows how Daniel progressed from the use of letter symbols to the use of circles as a visualisation in another iconic representation (see Fig. 7).

[18] Daniel: well, I think mine is the most suitable, because it's just really cool.
[19] Nora: but is it clear? If I have a look at it
[20] Daniel: isn't it clear to you?
[21] Nora: no
[22] Daniel: why? you clearly see that all *A* are *B*, and all *C* are *A*, so all *C* are also *B*.
[23] Nora: mmm, okay, I understand where you are going to, but if I see all those circles, I wouldn't say that

We observe that Daniel tried to convince Nora (lines [20] and [22]), but she did not accept Daniel's visualisation (line [23]). Later, the teacher asked Daniel to further explain his diagram. Daniel: "Well, okay, you do have *B*, so you have, okay, all *A* are *B*, so everything from *A* is part of *B*, then you also have *C*, which is part of *A* and then, so *C* is always part of *B*." Notable is the use of the phrase "is part of" instead of using "are" as in his earlier explanation (line [22]). This shows that he understood his Venn diagram in a general way, which might help him to make the link to symbolic representations.

Summary: from Enactive to Iconic Modes of Representation

These transcripts show that the tasks stimulated students to link concrete situations with iconic representations. We found that the students linked a concrete representation of a logical reasoning problem to a situation with letter symbols, but when asked for a visualisation, they had different interpretations. It was apparent that Nora knew that formal letter symbols could be used to represent a concrete model, but that she could not yet establish the exact links between the concrete and abstract referents. Daniel's visualisation showed that students may come up with a Venn diagram as a representation for a concrete situation. Daniel easily changed his vocabulary to words that connected the Venn diagram to an abstract referent, while Nora only acknowledged his use of circles and, at that moment, clearly needed more guidance and practice to link concrete situations to an abstract pictorial model.

Towards Symbolic Modes of Representation

Below, we describe two tasks in which students were encouraged to take steps towards symbolic modes of representation. The first task concerned the relation between if–then statements and Venn diagrams, and aimed at linking iconic and symbolic modes of representation (arrow 2 in Fig. 5). The second task concerned similarities between if–then statements, and it intended to link enactive with symbolic representations (arrow 3 in Fig. 5).

Linking Iconic and Symbolic Modes of Representation

To explore the relation between if–then statements and a corresponding Venn diagram, students were provided with the following situation taken from a newspaper article about a court case.

A: The baby is poisoned.
B: The baby turns blue.
Statement: If the baby is poisoned, then the baby turns blue.

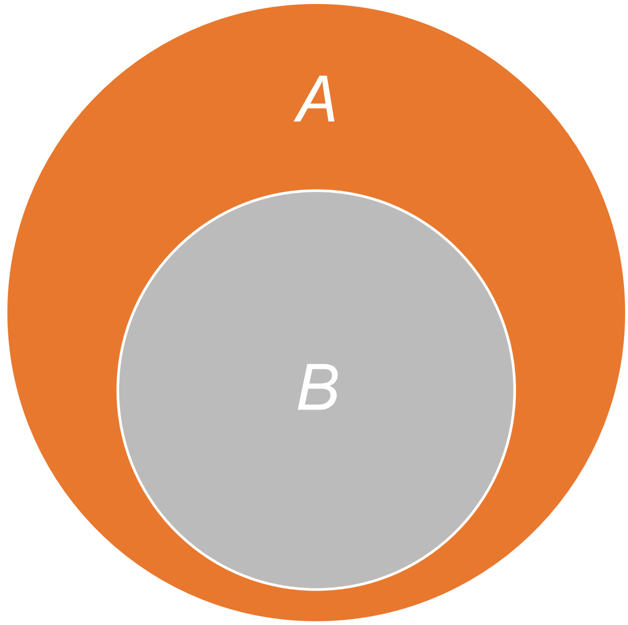
Students were asked to generalise the concretely formulated if–then statement and to provide a Venn diagram. Liam and Owen translated the statement into "If *A*, then *B*" and discussed this expression with the teacher.

[1] Liam: I have a question, how do you put if-then in a Venn diagram?
[2] Teacher: yes, that's a tricky one, isn't it? Owen has something, what did you do?
[3] Owen: I put the *B* in *A*
[4] Teacher: the *B* in *A*
[5] Owen: if *A* then certainly *B*
[6] Teacher: ok, we will have a look at it if we are all done, but are you convinced?
[7] Liam: no, not yet
[8] Owen: me neither, but this seems the most logical to me

Owen put *B* in *A* (line [3] and Fig. 9), which is incorrect, and expressed that "if *A*, then certainly *B*" (line [5]). The teacher did not agree or disagree but asked Liam if he was convinced by Owen's explanation (line [6]). Both students expressed their doubts (lines [7] and [8]).

They did not discuss this further in this setting; however, the teacher started a classroom discussion about the connection between if–then statements and Owen's Venn diagram (see Fig. 9), because he saw that other students had drawn similar diagrams.

Fig. 9 Owen's Venn diagram



[13] Teacher: yes, okay, but if *A* is true then *B*, so if you are in set *A* [points to A] then you are also in this set [points to B]
[14] Nora: we should have switched the order
[15] Adam: no, wait a minute, because if the baby is poisoned, then it will turn blue, or should I have done it the other way around, indeed?
[16] Teacher: if he is poisoned, then he will turn blue
[17] Nora: I think we should switch *B* and *A*
[18] Adam: no, we must change the order, indeed, ah, rubbish

Here, we observe that when the teacher pointed to the different areas of the Venn diagram (line [13]), this triggered Nora to exclaim that *A* and *B* should be switched (line [14]). This was not immediately clear to Adam, so he needed it translated back to the concrete example of the baby before he was convinced (lines [15] and [18]) about the correct positioning of the circles in the Venn diagram.

Linking Enactive and Symbolic Modes of Representation

The students were already introduced to valid and invalid conclusions in if–then statements before they were asked about the similarities in if–then statements. They were provided with the following two statements:

Statement 1 for a set of stones with pictures of animals on one side and astronomical objects on the other side: "If there is a moon on one side, then there is a fish on the other side."
Statement 2: "If I rob the Dutch national bank, I will be rich."

Both statements represent concrete scenarios. To make a judgement about their similarities, it would be useful to translate them into a symbolic expression, which Nora did quickly, clearly showing the structural pattern: "If you just translate this to regular symbols, then they both are if *A* then *B*."

Adam and Liam experienced more difficulties understanding why these two if–then statements were similar and mainly reasoned with the concrete information, although Liam shortened the first statement to "Moon = Fish" in his notebook, not visualising the direction of the statement. Adam and Liam had the following conversation, which demonstrates that Adam did not agree with the equals sign.

[1] Liam: is it true, that the moon cannot be combined with another animal?
[2] Adam: I think so, if you say that if there is a moon on one side, then you have fish on the other side, and you say there is a butterfly, then there may still be a moon
[3] Liam: it is still possible, then any astronomical object is possible
[4] Adam: because moon means fish, but fish does not automatically mean moon

Line [4] shows that Adam did not accept the reversibility of the given statement, and thus that Fish on one side does not necessarily imply Moon on the other side. Near the end of their discussion, Liam concluded: "If this statement is true [refers to moon-fish statement], then this [Statement 2] is just like this one." Later, during the classroom discussion, the teacher wrote the correct expression on the board using an implication arrow "Moon ⇒ Fish", and he only indicated that the two statements were similar because they both were "just if–then statements" without elaborating on this or verifying the students' understanding.

Summary: Towards Symbolic Modes of Representation

These transcripts show that the tasks stimulated the students to link referents from the iconic mode with abstract rules from the symbolic mode, but the link between if–then statements and a correct iconic visualisation was not made automatically. We saw that students put the consequent in the antecedent in their Venn diagrams. Only during the classroom discussion and after some guidance by the teacher did one of the students (Nora) recognise the invalid conditions in the diagram. Another student, Adam, needed a translation back to the concrete situation to verify the correctness of the diagram. Moreover, in the second task, Nora quickly used general rules to conclude that the concrete statements were similar, but not all students accepted this and did not use the general form $A \Rightarrow B$ to derive conclusions for the concrete situations.

Linking Enactive, Iconic, and Symbolic Modes of Representation

In the last phase of the intervention, students were challenged to use their acquired knowledge and establish links between enactive, iconic, and symbolic modes of representation to verify their reasoning both in closed tasks and everyday reasoning tasks about newspaper articles (see number 4 in Fig. 5). In this section, we describe the students' reasoning in a closed if–then task and a task presenting an argument from a newspaper article.

In the closed task, the following arguments were provided (based on: College voor Toetsen en Examens, 2017):

(I): "If you are strong, then you go to bed late. You are not strong, so you do not go to bed late."
We can represent the statement in the first sentence with symbols as follows: $S \Rightarrow L$.
(II): (1) "If you are strong, then you go to bed late."
(2) "If you are weak, then you do not go to bed late."

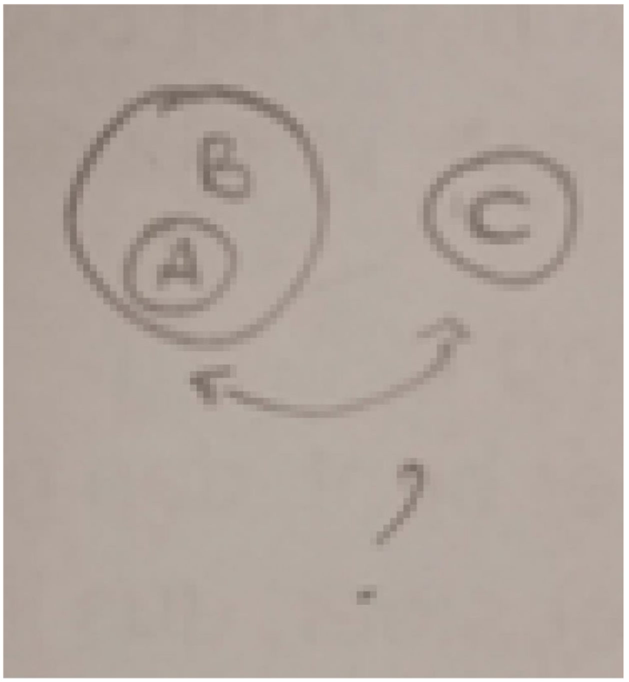


Fig. 10 Nora's Venn diagram

First, students were asked to show that the second statement in argument (I) does not follow from the first statement and is in fact an incorrect conclusion. Riley and Nora discussed this and quickly switched to terminology connected to the symbolic mode.

[1] Riley: well, that is not *S* is not *L*, isn't it?
[2] Nora: here it says if *A* then *B*, or if *S* then *L*
[…]
[3] Nora: but, look, you have, let's say, two of those, this is not modus ponens, but modus tollens, so it should be not *B,* so not *A*
[4] Riley: yes indeed
[5] Nora: so it is a fallacy
[6] Riley: yes
[7] Nora: that's like
[8] Riley: yes exactly
[9] Nora: even if you are not strong, you can still go to bed late

Here, we observe that Riley translated the second proposition and used the letter symbols *S* and *L* as provided in the task (line [1]). Nora showed that she could switch easily from Riley's letter symbols to a general form with *A* and *B* (line [2]). Subsequently, Nora applied general rules to support her argument (line [3]) with *modus tollens* and thus showed why the order was wrong. With that information Nora easily translated that part to the concrete context again (line [9]) and Riley confirmed all her steps.

For the second subtask (II), students were asked if the conclusion, "If you are weak, then you are not strong," is allowed on the basis of statements (1) and (2). Nora and Riley discussed this first in the concrete context and were not sure how to approach this argument, but then Nora heard another group using a Venn diagram and convinced Riley to use a Venn diagram as well.

[10] Nora: no wait, I want to draw a Venn diagram, I heard those guys doing that, I think that's quite a good idea!
[11] Riley: oh yes
[12] Nora: look, it would be right! Because then you have if you are strong [A], you go to bed late [B]. If you are weak (so C), you do not go to bed late. Is now actually separate. Conclusion: so if you are weak, you are not strong. If you see it like this, it is possible. [see Fig. 10]
[13] Riley: wait a moment, *B*, *A* and then
[14] Nora: just say *C* is separated from that
[15] Riley: *C* is not *B*. If *C* is separated from it, yes then *C* is also not *A*. Yes it is.
[16] Nora: yes

Here, we observe that Nora quickly came to the right conclusion (line [12]) by using a Venn diagram, which is an iconic representation, and translated the conclusion back to the concrete context. She considered the space outside B as $\neg B$, which is correct, and concluded that C lays in that area. Noteworthy is her use of general letter symbols A, B, and C, which could be used in an abstract model, although Nora and Riley did not reason with logical rules in an abstract model.[2]

We do not have discussion data about the reasoning on the newspaper article tasks. However, five worksheets for one of the subtasks were available, where students had been asked to visualise, schematise, or create a diagram for a paragraph of a newspaper article. The full task is shown in Fig. 4. Four of the five students used a Venn diagram, but none used letter symbols or logical symbols. One student tried to make a Venn diagram, but crossed it out and made an argument in ordinary language. From these answers, we conclude that they were able to use iconic representations for an everyday reasoning task but did not demonstrate any symbolic representations.

Summary: Linking Enactive, Iconic, and Symbolic Modes of Representation

Based on the closed subtasks, we found that Nora and Riley used abstract rules (symbolic mode) for statements with one step (argument (I) in the first task), but if they had to take two steps (argument (II) in the first task), they used alternatives, such as a Venn diagram (iconic representation), which was sufficient help in many of the tasks. During the classroom discussion, the teacher showed the solution using symbolic expressions, but did not verify whether the students had understood the steps. In the newspaper article task, the students did not use abstract models, but almost all of them used Venn diagrams, perhaps because of the implicit nature of the task, as we will discuss in the next section.

Conclusions and Discussion

In this article, we reported on student development of logical reasoning. The students were guided through the different stages of CF, working mainly in pairs or groups of three on logical reasoning problems. Our study addressed the research question: "How do students use and apply visual and formal representations (iconic and symbolic) in logical reasoning tasks?".

Our main conclusion is that students were able to establish the link between enactive and iconic modes of representation. They rapidly resorted to the use of letter symbols in concrete tasks. Most often they chose general letter symbols (A, B, C or P, Q, R), even before they were introduced in the teaching materials or by the teacher. In later phases, most students directly introduced letter symbols to start

[2] Using the letter symbols, they could have verified with logical rules in an abstract model that the conclusion is allowed in two steps: $C \Rightarrow \neg B$ (statement 2), $\neg B \Rightarrow \neg A$ (*modus tollens*, statement 1), so: $C \Rightarrow \neg A$.

their reasoning. Although this was stimulated during the lessons in logical reasoning, this might also be explained by the fact that students are used to doing this for other mathematics topics.

We also showed that before the students were introduced to the use of Venn and Euler diagrams for logical reasoning, some created a correct generalised diagram as a visual representation. However, for other students, more practice or guidance was needed before they saw the merits of Venn diagrams and started using them. Nevertheless, we conclude that after the intervention, the students had all added Venn and Euler diagrams to their reasoning toolboxes. Moreover, we saw that students sometimes preferred a Venn diagram over formal logical rules. This might be explained by the concreteness of the tasks, but may also indicate that the link with symbolic modes of representation is less well developed, as more time was spent on iconic representations than on formal logical rules in the intervention. Additionally, the way and speed in which students made the transition from enactive and iconic representations to symbolic representations differed between students. We saw that taking a step back to the iconic mode can help scaffold their reasoning. In summary, we conclude that visual representations play a major role in solving logical reasoning problems.

However, the use of symbolic modes of representation was not completely absent. We observed that some students used *modus tollens*, although only in one-step reasoning. The students experienced difficulties when they had to apply more than one rule in a task. This is consistent with the results of the post-test from our earlier study (Bronkhorst et al., 2020b), in which students rarely used general, abstract expressions and logical rules. One possible explanation for using iconic representations instead of symbolic ones in everyday reasoning tasks is that students consider the former as more useful or appropriate for the tasks. It is possible that the implicit nature of the everyday reasoning tasks caused confusion or doubt among the students (Galotti, 1989), which results in the choice of a visual rather than an abstract model to gain a better understanding or overview. More practice time and tasks might lead to a better understanding and acceptance of other representations and support the relevance of abstract tools for everyday reasoning problems (Witzel et al., 2008, p. 275).

Our results reveal the importance of the interplay between the different modes of representation, as shown in Figs. 1 and 5. Over time, the link between enactive and iconic modes of representation became stronger and, as mentioned above, this might also be the case for the link with symbolic modes of representation if students are given more time and practice.

Our study was limited to a specific target group of non-science students, for whom correct logical reasoning has societal relevance. In general, our target group was not strong in mathematics and did not like using abstract rules, and this would probably also apply to logical reasoning problems. Febriana et al. (2019) showed that for elementary school students, whether they reached the iconic or symbolic stage depended on their mathematical ability. We cannot verify this for our target group, but it is recommended that further research compares groups with different mathematical abilities and adjusts the teaching of logical reasoning for both groups.

Our analysis of the interactions between the students indicated that conversations often led them to another representation or better understanding. Although the role of the teacher was not a separate object of research in this study, the way he guided the pair and group work was important for students' conversations. As we showed, he did this by encouraging students to explain their solutions and to elaborate on them. In classroom discussions, he often tried to show the connection between the different representations, but did not always verify whether the students had understood the symbolic modes of representation. Perhaps more explicit attention to abstract reasoning would result in better understanding and use of the general rules.

Thus far, we have only discussed links and transfer between the different modes of representation. Another interesting question would be whether the students' use of representations transfers to other contexts. Although we used meaningful everyday reasoning tasks, such as newspaper articles, the students still fulfil tasks within the mathematics classroom and expect that there should be one correct solution, as is common in mathematics exercises (e.g. Jäder et al., 2017). While we showed CF's contribution to student development of logical reasoning in the context of mathematics education, we need more research on the transfer of their logical reasoning abilities to daily life contexts, as an important indicator of their development of twenty-first-century skills (Liu et al., 2015).

Recommendations

In the "Introduction" section, we stressed the importance of logical reasoning for twenty-first-century learning (P21, 2015). CF appears to be a useful framework (Fyfe et al., 2014) to develop teaching materials that work gradually from concrete reasoning to more abstraction. The combined approach of students working on their own solutions and discussing them in small groups and at the classroom level shows promise as a way to strengthen the links between the different modes of representation. However, as not all kinds of representations were internalised by students in this study, we not only recommend further research but also that teachers provide more practice time for tasks that improve the link with the symbolic mode. Finally, while the participants in our research were part of a specific group of non-science students, we recommend the teaching of logical reasoning skills to other secondary education students as well, as the fostering of such skills should be a major goal for all students.

Funding This work is part of the research programme Doctoral Grant for Teachers with project number 023.007.043, which is (partly) financed by the Netherlands Organisation for Scientific Research (NWO).

Declarations

Ethical Approval This work is approved by The Research Ethics Committee (CETO) of the University of Groningen with application number 58772634.

Conflict of Interest The authors declare no competing interests.

References

Adey, P., & Shayer, M. (1993). An exploration of long-term far-transfer effects following an extended intervention program in the high school science curriculum. *Cognition and Instruction*, *11*(1), 1–29.

Bronkhorst, H., Roorda, G., Suhre, C., & Goedhart, M. (2018). Secondary students' logical reasoning abilities. In E. Bergqvist, O. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education, 5,* 271. PME.

Bronkhorst, H., Roorda, G., Suhre, C., & Goedhart, M. (2020a). Logical reasoning in formal and everyday reasoning tasks. *International Journal of Science and Mathematics Education*, *18*(8), 1673–1694. https://doi.org/10.1007/s10763-019-10039-8

Bronkhorst, H., Roorda, G., Suhre, C., & Goedhart, M. (2020b). *Students' use of formalisations for improved logical reasoning* [Manuscript submitted for publication]. Institute for Science Education and Communication, University of Groningen.

Brookhart, S. M. (2010). *How to assess higher-order thinking skills in your classroom*. ASCD.

Bruner, J. S. (1966). *Toward a theory of instruction*. Belknap Press of Harvard University Press.
Butler, F. M., Miller, S. P., Crehan, K., Babbitt, B., & Pierce, T. (2003). Fraction instruction for students with mathematics disabilities: comparing two teaching sequences. *Learning Disabilities Research & Practice*, *18*(2), 99–111. https://doi.org/10.1111/1540-5826.00066
Chu, J., Rittle-Johnson, B., & Fyfe, E. R. (2017). Diagrams benefit symbolic problem-solving. *British Journal of Educational Psychology*, *87*(2), 273–287. https://doi.org/10.1111/bjep.12149
Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education* (6th ed.). Routledge/Taylor & Francis Group.
College voor Toetsen en Examens. (2016). *Wiskunde C vwo, syllabus centraal examen 2018 (bij het nieuwe examenprogramma) nader vastgesteld 2*. https://www.examenblad.nl/examenstof/syllabus-2018-wiskunde-c-vwo/2018/vwo/f=/syllabus_wiskunde_C_2_versie_vwo_2018_nader_vastgesteld2_def.pdf
College voor Toetsen en Examens. (2017). Wiskunde C (pilot) examen vwo 2017. https://static.examenblad.nl/9336117/d/ex2017/VW-1026-f-17-1-o.pdf
cTWO. (2012). *Denken & doen: wiskunde op de havo en vwo per 2015*. Utrecht, the Netherlands: Commissie Toekomst Wiskunde Onderwijs.
Febriana, D. F., Amin, S. M., & Wijayanti, P. (2019). Concreteness fading process of elementary school students based on mathematical ability. *Journal of Physics: Conference Series*, *1157*(4). https://doi.org/10.1088/1742-6596/1157/4/042049
Fyfe, E. R., McNeil, N. M., & Borjas, S. (2015). Benefits of "concreteness fading" for children's mathematics understanding. *Learning and Instruction*, *35*, 104–120. https://doi.org/10.1016/j.learninstruc.2014.10.004
Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: a systematic review. *Educational Psychology Review*, *26*(1), 9–25. https://doi.org/10.1007/s10648-014-9249-3
Galotti, K. M. (1989). Approaches to studying formal and everyday reasoning. *Psychological Bulletin*, *105*(3), 331–351.
Gravemeijer, K. (2020). A socio-constructivist elaboration of realistic mathematics education. In M. V. den Heuvel-Panhuizen (Ed.), *National reflections on the Netherlands didactics of mathematics: Teaching and learning in the context of realistic mathematics education* (pp. 217–233). Springer International Publishing. https://doi.org/10.1007/978-3-030-33824-4
Grouws, D. A., & Cebulla, K. J. (2000). *Improving student achievement in mathematics. Educational practices series 4* (ED445925). ERIC. https://eric.ed.gov/?id=ED445925
Halpern, D. F. (2014). *Thought and knowledge: An introduction to critical thinking* (5th ed.). Psychology Press.
Hegarty, M., & Kozhevnikov, M. (1999). Types of visual–spatial representations and mathematical problem solving. *Journal of Educational Psychology*, *91*(4), 684–689. https://doi.org/10.1037/0022-0663.91.4.684
Jäder, J., Sidenvall, J., & Sumpter, L. (2017). Students' mathematical reasoning and beliefs in non-routine task solving. *International Journal of Science and Mathematics Education*, *15*(4), 759–776. https://doi.org/10.1007/s10763-016-9712-3
Johnson, R. H., & Blair, J. A. (2006). *Logical self-defence*. Idebate press.
Kim, H. (2020). Concreteness fading strategy: a promising and sustainable instructional model in mathematics classrooms. *Sustainability*, *12*(6), 2211. https://doi.org/10.3390/su12062211
Koelewijn, R. (2016). *Het begint bij een glas per dag.* NRC Media. https://www.nrc.nl/nieuws/2016/10/04/het-begint-bij-een-glas-per-dag-4637333-a1524797
Kuhn, D. (1991). *The skills of argument*. Cambridge University Press.
Lehman, D. R., Lempert, R. O., & Nisbett, R. E. (1988). The effects of graduate training on reasoning: Formal discipline and thinking about everyday-life events. *American Psychologist*, *43*(6), 431–442. https://doi.org/10.1037/0003-066X.43.6.431
Liu, H., Ludu, M., & Holton, D. (2015). Can K-12 math teachers train students to make valid logical reasoning? In X. Ge, D. Ifenthaler, & J. M. Spector (Eds.), *Emerging technologies for STEAM education: full STEAM ahead* (pp. 331–353). Springer International Publishing. https://doi.org/10.1007/978-3-319-02573-5_18
McChesney, J. (2017). Searching the New Zealand curriculum landscape for clarity and coherence: some tensions in mathematics and statistics. *Curriculum Matters*, *13*, 115–131.
McKendree, J., Small, C., Stenning, K., & Conlon, T. (2002). The role of representation in teaching and learning critical thinking. *Educational Review*, *54*(1), 57–67. https://doi.org/10.1080/00131910120110884
McNeil, N. M., & Fyfe, E. R. (2012). "Concreteness fading" promotes transfer of mathematical knowledge. *Learning and Instruction*, *22*(6), 440–448. https://doi.org/10.1016/j.learninstruc.2012.05.001
National Academies of Sciences, Engineering, and Medicine. (2018). *How people learn II: learners, contexts, and cultures*. The National Academies Press. https://doi.org/10.17226/24783
National Research Council. (1999). *How people learn: Brain, mind, experience, and school* (J. Bransford, A. L. Brown, & R. R. Cocking, Eds.). The National Academies Press.
National Council of Teachers of Mathematics. (2009). *Focus in high school mathematics. Reasoning and sense making*. National Council of Teachers of Mathematics.

Nunes, T. (2012). Logical reasoning and learning. In N. M. Seel (Ed.), *Encyclopedia of the sciences of learning* (pp. 2066–2069). Springer. https://doi.org/10.1007/978-1-4419-1428-6_790
Ottmar, E., & Landy, D. (2017). Concreteness fading of algebraic instruction: effects on learning. *Journal of the Learning Sciences*, *26*(1), 51–78. https://doi.org/10.1080/10508406.2016.1250212
P21. (2015). *P21 framework definitions*. http://www.p21.org/storage/documents/docs/P21_Framework_Definitions_New_Logo_2015.pdf.
Schoenfeld, A. H. (1991). On mathematics as sense-making: an informal attack on the unfortunate divorce of formal and informal mathematics. In J. F. Voss, D. N. Perkins, & J. W. Segal (Eds.), *Informal reasoning and education* (pp. 311–343). L. Erlbaum Associates.
Seel, N. M. (2012). Inferential learning and reasoning. In N. M. Seel (Ed.), *Encyclopedia of the sciences of learning* (pp. 1550–1555). Springer. https://doi.org/10.1007/978-1-4419-1428-6_583
Stenning, K. (1996). Embedding logic in communication: lessons from the logic classroom. In J. van Benthem, F. H. van Eemeren, R. Grootendorst, & F. Veltman (Eds.), *Logic and argumentation* (pp. 227–240). Royal Netherlands Academy.
Stenning, K. (2002). *Seeing reason: image and language in learning to think*. Oxford University Press.
Teig, N., & Scherer, R. (2016). Bringing formal and informal reasoning together—a new era of assessment? *Frontiers in Psychology*, *7*, 1097. https://doi.org/10.3389/fpsyg.2016.01097
Tondevold, C. (2019). What is the concrete representational abstract approach. The Recovering Traditionalist. https://www.therecoveringtraditionalist.com/concrete-representational-abstract-approach/
Van Aalten, P., & De Waard, K. (2001). *Denklessen*. APS.
Van den Akker, J., Bannan, B., Kelly, A. E., Nieveen, N., & Plomp, T. (2013). *Educational design research. Part A: An introduction.* SLO.
Van Gelder, T. (2005). Teaching critical thinking: some lessons from cognitive science. *College Teaching*, *53*(1), 41–48. https://doi.org/10.3200/CTCH.53.1.41-48
Vincent-Lancrin, S., González-Sancho, C., Bouckaert, M., Luca, F. de, Fernández-Barrerra, M., Jacotin, G., Urgel, J., & Vidal, Q. (2019). *Fostering students' creativity and critical thinking: what it means in school.* Centre for Educational Research and Innovation. https://www.oecd-ilibrary.org/content/publication/62212c37-en.
Voss, J. F., Perkins, D. N., & Segal, J. W. (Eds.). (1991). *Informal reasoning and education*. L. Erlbaum Associates.
Witzel, B., Riccomini, P., & Schneider, E. (2008). Implementing CRA with secondary students with learning disabilities in mathematics. *Intervention in School and Clinic*, *43*(5), 270–276. https://doi.org/10.1177/1053451208314734

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.