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STRONG STABILIZATION OF (ALMOST) IMPEDANCE PASSIVE SYSTEMS BY STATIC OUTPUT FEEDBACK

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Abstract. The plant to be stabilized is a system node Σ with generating triple (A, B, C) and transfer function $\mathbf{G}$, where A generates a contraction semigroup on the Hilbert space X. The control and observation operators B and C may be unbounded and they are not assumed to be admissible. The crucial assumption is that there exists a bounded operator E such that, if we replace $\mathbf{G}(s)$ by $\mathbf{G}(s) + E$, the new system Σ_E becomes impedance passive. An easier case is when $\mathbf{G}$ is already impedance passive and a special case is when Σ has colocated sensors and actuators. Such systems include many wave, beam and heat equations with sensors and actuators on the boundary. It has been shown for many particular cases that the feedback $u = -\kappa y + v$, where u is the input of the plant and $\kappa > 0$, stabilizes Σ, strongly or even exponentially. Here, y is the output of Σ and v is the new input. Our main result is that if for some $E \in \mathcal{L}(U)$, Σ_E is impedance passive, and Σ is approximately observable or approximately controllable in infinite time, then for sufficiently small κ the closed-loop system is weakly stable. If, moreover, $\sigma(A) \cap i\mathbb{R}$ is countable, then the closed-loop semigroup and its dual are both strongly stable.

1. **The main results.** This paper is a continuation and extension of our paper [11], in the sense that we consider a more general class of linear infinite-dimensional systems (system nodes) under more general assumptions, as described below, and we are interested in stabilizing these system nodes by static output feedback. The observability assumption imposed here is weaker than in [11], and correspondingly, the stability properties of the closed-loop system will also be weaker. We shall refer often to background and results from [11], but the concepts and techniques needed to derive the results in this paper are quite different from those in [11].

To specify our terminology and notation, we recall that a *system node* Σ with input space U, state space X and output space Y (all Hilbert spaces) is determined by its generating triple (A, B, C) and its transfer function $\mathbf{G}$, where the operator

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$A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup of operators $\mathbb{T}$ on X and the possibly unbounded operators B and C are such that $C : \mathcal{D}(A) \to Y$ and $B^* : \mathcal{D}(A^*) \to U$. There are no well-posedness assumptions for a system node. Nevertheless, for input functions $u \in C^2([0,\infty), U)$ and for initial states $z_0 \in X$ that satisfy $Az_0 + Bu(0) \in X$, there exists a unique state trajectory $z \in C^1([0,\infty), X)$ satisfying $\dot{z} = Az + Bu$. The corresponding Y-valued output function y is defined by $y = C[z - (\beta I - A)^{-1}u] + \mathbf{G}(\beta)u$ (where $\beta \in \rho(A)$) and it is continuous. The space $\mathcal{D}_0$ of all the pairs (z_0, u) which are as described above is dense in $X \times L^2([0,\infty), U)$. For $(z_0, u) \in \mathcal{D}_0$, the functions z, y can also be expressed in terms of their Laplace transforms:

$$\hat{z}(s) = (sI - A)^{-1}[z_0 + B\hat{u}(s)],$$
$$\hat{y}(s) = C(sI - A)^{-1}z_0 + \mathbf{G}(s)\hat{u}(s),$$

for all $s \in \mathbb{C}$ with sufficiently large real part. For details on system nodes we refer to Malinen *et al* [26], Opmeer [29], Staffans [42], and a short introduction to system nodes will be given in Section 2.

The system node Σ is called *impedance passive* if $Y = U$ and for all (z_0, u) in the space $\mathcal{D}_0$ (defined a little earlier) and for all $\tau > 0$,

$$\|z(\tau)\|^2 - \|z_0\|^2 \leq 2\int_0^\tau \mathrm{Re}\langle u(t), y(t)\rangle \mathrm{d}t. \tag{1}$$

Necessary and sufficient conditions for passivity in terms of $A, B, C, \mathbf{G}$ will be recalled in Section 3. The concept of impedance passivity has been introduced in Willems [58] in the finite-dimensional context and has been generalized to the infinite-dimensional context (with modified terminology) by several researchers, see Staffans [39, 40] and the references therein.

In this paper we deal with systems (plants to be stabilized) that satisfy a relaxed version of (1): there exists an $E \in \mathcal{L}(U)$ such that

$$\|z(\tau)\|^2 - \|z_0\|^2 \leq 2\int_0^\tau \mathrm{Re}\langle u(t), y(t)\rangle \mathrm{d}t + 2\int_0^\tau \langle Eu(t), u(t)\rangle \mathrm{d}t, \tag{2}$$

for all $(z_0, u) \in \mathcal{D}_0$ and all $\tau \geq 0$. Equivalently, if we replace $\mathbf{G}$ by $\mathbf{G} + E$ (and keep A, B, C unchanged), then we obtain a modified system node Σ_E which is impedance passive. We call such system nodes *almost impedance passive*.

We remark that the existence of E such that (2) holds is obviously equivalent to the existence of $c \geq 0$ such that

$$\|z(\tau)\|^2 - \|z_0\|^2 \leq 2\int_0^\tau \mathrm{Re}\langle u(t), y(t)\rangle \mathrm{d}t + 2c\int_0^\tau \|u(t)\|^2 \mathrm{d}t.$$

This condition is simpler, but for certain arguments it is better to refer to (2).

In [11] we considered the class of well-posed linear systems with colocated actuators and sensors (meaning that $C = B^*$) and A was essentially skew-adjoint. In Section 3 we revisit this class but without the well-posedness assumption. We show that these systems satisfy (2) and moreover, for these systems we determine the minimal E so that (2) holds.

Although the A operator of an almost impedance passive system node always generates a contraction semigroup, it is not necessary that the actuators and sensors are colocated. Moreover, a system node with colocated actuators and sensors and a contraction semigroup need not be impedance passive. We give concrete examples that illustrate these facts in Sections 6, 7 and 8, where we investigate in detail the conditions for almost impedance passivity for some classes of second order systems.

In Section 4 we consider an impedance passive system node and we examine the effect of the static output feedback $u = -\kappa y + v$, where $\kappa > 0$ and v is the new input function, as shown in a block diagram in Figure 1. We give conditions under which this feedback results in a closed-loop system Σ^κ that is well-posed and has nice stability properties. To state our main results, we recall some stability concepts.

Definition 1.1. Let Σ be a well-posed linear system with input space U, state space X, output space Y, generating triple (A, B, C) and transfer function $\mathbf{G}$.

- Σ is *input stable* if for any $v \in L^2([0,\infty), U)$, the state trajectory of Σ_0 corresponding to the initial state zero and the input function v is bounded. This property is also known as *infinite-time admissibility* of B.
- Σ is *output stable* if for any $z_0 \in X$, the output function of Σ corresponding to the initial state z_0 and the input function zero is in $L^2([0,\infty), Y)$. This property is also known as *infinite-time admissibility* of C.
- Σ is *input-output stable* if for any $v \in L^2([0,\infty), U)$, the output function of Σ corresponding to the initial state zero and the input function v is in $L^2([0,\infty), U)$. Equivalently, $\mathbf{G} \in H^\infty(\mathcal{L}(U))$, the space of bounded analytic $\mathcal{L}(U)$-valued functions on the open right half-plane.
- Σ is *system stable* if it is input stable, output stable and input-output stable.

We shall also use the following standard stability concepts for a strongly continuous semigroup $\mathbb{T}$ on a Hilbert space X:

- *weak stability* means that $\langle \mathbb{T}_t z_0, z_1 \rangle \to 0$ as $t \to \infty$, for all $z_0, z_1 \in X$,
- *strong stability* means that $\mathbb{T}_t z_0 \to 0$ as $t \to \infty$, for all $z_0 \in X$,
- *exponential stability* means that there exist $M \geq 1$ and $\alpha > 0$ such that $\|\mathbb{T}_t\| \leq M e^{-\alpha t}$ for all $t \geq 0$.

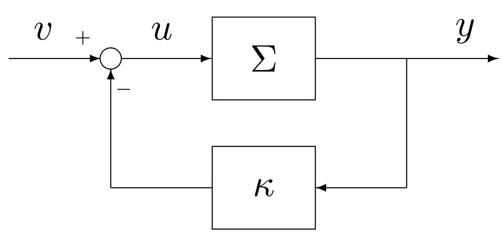

FIGURE 1. The open-loop system node Σ with static output feedback. If $\kappa > 0$ is sufficiently small, then this feedback results in a closed-loop system Σ^κ that is well-posed and system stable. Under suitable additional assumptions, the operator semigroup of Σ^κ and its dual are strongly stable.

A consequence of our results in Section 4 is the following.

Theorem 1.2. *Let Σ be a system node for which there exists $E \in \mathcal{L}(U)$ satisfying (2). Then there exists $\kappa_0 > 0$ (possibly $\kappa_0 = \infty$) such that for all $\kappa \in (0, \kappa_0)$, the feedback law $u = -\kappa y + v$ (where u and y are the input and the output of Σ) leads to a well-posed closed-loop system Σ^κ that is system stable.*

Moreover, the semigroup of Σ^κ is a contraction semigroup.

With the assumptions of the above theorem, denote $\operatorname{Re} E = \frac{1}{2}(E + E^*)$. If c is the smallest number in $[0, \infty)$ such that $\operatorname{Re} E \leq cI$, then we can take $\kappa_0 = 1/c$ (if $c = 0$ then $\kappa_0 = \infty$). In particular, if Σ is impedance passive then $\kappa_0 = \infty$.

In Section 5 we prove some results concerning weak and strong stability under mild observability and controllability assumptions. We say that (A, C) (or the system node Σ) is *approximately observable in infinite time* if for every $z_0 \in \mathcal{D}(A)$, $C\mathbb{T}_t z_0 = 0\ \forall t \geq 0$ implies $z_0 = 0$. We say that (A, B) (or the system node Σ) is *approximately controllable in infinite time* if (A^*, B^*) is approximately observable in infinite time. The following theorem is a consequence of our results in Section 5.

Theorem 1.3. *Under the notation and assumptions of Theorem 1.2, if (A, B, C) is either approximately controllable in infinite time or approximately observable in infinite time, then for every $\kappa \in (0, \kappa_0)$, the closed-loop semigroup $\mathbb{T}^\kappa$ is weakly stable. If, in addition, the intersection of the spectrum $\sigma(A)$ with the imaginary axis $i\mathbb{R}$ is at most countable, then $\mathbb{T}^\kappa$ and its dual $\mathbb{T}^{\kappa *}$ are strongly stable.*

The proof of the strong stability part of this theorem uses a famous result from Arendt and Batty [2] and Lyubich and Phong [24] (see also Arendt *et al* [3]).

In the literature we can find various particular examples or classes of passive systems stabilized by static output feedback. The result most closely related to the last theorem is the main result of Batty and Phong [7]. In that paper, it is assumed that A generates a contraction semigroup, B is bounded (i.e., it is in $\mathcal{L}(U, X)$), $C = B^*$ and the open-loop transfer function is $\mathbf{G}(s) = B^*(sI - A)^{-1}B$. Similar assumptions have been made in a series of earlier papers: The first PDE examples fitting into this framework were Bailey and Hubbard [4], Balakrishnan [5, 6], Russell [32], Slemrod [36, 37]. In this case, $\mathbf{G}$ is positive and (by a simple argument) this implies that the feedback $u = -\kappa y + v$ stabilizes in an input-output sense. Of course, the most desirable type of stability is exponential stability and for this, in the special case $A^* + A = 0$, we need the system to be exactly controllable (or equivalently, exactly observable). This is the setup studied in Haraux [19], Liu [22], Lasiecka and Triggiani [20], [21] and others.

However, early on it was realized that exponential stability is not achievable with a bounded B in the case that U is finite-dimensional and A has infinitely many eigenvalues on the imaginary axis. This is illustrated in Russell [32] with a PDE model of an undamped string. Similar results for a beam can be found in Slemrod [37]. More generally, it is known that if U is finite-dimensional, B is bounded and A has infinitely many unstable eigenvalues, we can never achieve exponential stability, see Gibson [16], Triggiani [45] or Curtain and Zwart [12, Theorem 5.2.6]. So for this class the best we can hope for is strong stability. Early results giving sufficient conditions under which $A - BB^*$ generates a weakly or strongly stable semigroup using LaSalle's principle can be found in Slemrod [35]. These were sharpened by Benchimol in [8] who used the canonical decomposition of contraction semigroups due to Szökefalvi-Nagy and Foias [27]. He showed that if A generates a contraction semigroup and $B \in \mathcal{L}(U, X)$, a sufficient condition for $A - BB^*$ to generate a weakly stable semigroup is that

$$\{x \in X \mid B^*\mathbb{T}_t^* x = 0, \quad \|\mathbb{T}_t x\| = \|x\| = \|\mathbb{T}_t^* x\| \qquad \forall\, t > 0\} = \{0\}. \tag{3}$$

In (3), $\mathbb{T}_t^*$ may be interchanged with $\mathbb{T}_t$. If, in addition, A has compact resolvents, then (3) implies strong stability. The above result for weak stability was also obtained by Batty and Phong [7]. They improved the above sufficient condition for strong stability, obtaining: if the spectrum of A has at most countably many points of intersection with the imaginary axis, then $A - BB^*$ generates a strongly stable semigroup if and only if (3) holds. It is worthwhile noting that, while the assumption that B be bounded is restrictive, it does not exclude PDEs with boundary control, see Slemrod [37], You [59] and Chapter 9 of Oostveen [28].

As already mentioned, there are systems that are impedance passive, but $C^* \neq B$. In [25] Luo, Guo and Morgul gave conditions for a class of PDEs to be *impedance energy preserving* (i.e., (1) holds with equality). The PDEs were in one spatial dimension with control and observation on the boundary and A was assumed to be skew-adjoint and to have compact resolvents. Using LaSalle's principle they showed that under an observability assumption, the feedback $u = -\kappa y + v$ with $\kappa > 0$ produces a strongly stable closed-loop system. Using a different approach Le Gorrec *et al* [17] defined a large class of hyperbolic-like systems in one spatial variable with boundary control and observation that were impedance energy preserving. In Zwart *et al* [48] it was shown, using a Lyapunov approach, that in the constant coefficients case these systems could be stabilized by static output feedback.

Most of the existing results on static output stabilization of PDE systems assume a skew-adjoint semigroup generator and colocated actuators and sensors. Theorems 1.2 and 1.3 include and generalize these results; neither a skew-adjoint generator nor colocated actuators and sensors are needed.

Historical note. This paper has been written over many years, up to August 2007. The authors wanted to improve certain things, and put the manuscript aside. Various events and projects distracted their attention and the manuscript remained unsubmitted for over a decade. In March 2018, Ruth died of lung cancer. In 2019, at the urging of guest editor Marius Tucsnak, the remaining author has made minimal adjustments and has submitted the manuscript to MCRF. The references remained as they were in 2007, except that papers with the status of "submitted" or "accepted" have been updated.

2. **Some background on system nodes and well-posed linear systems.**

In this section, we recall a rather general class of infinite-dimensional linear systems, called system nodes. System nodes do not satisfy any well-posedness assumption, but nevertheless they have well defined state trajectories and output functions corresponding to smooth input functions and compatible initial states, see Proposition 2.3. We also recall a few facts about well-posed linear systems.

In the semigroup approach to infinite-dimensional systems, we often encounter an operator semigroup $\mathbb{T}$ on a Hilbert space X which determines two additional Hilbert spaces, denoted by X_1 and X_{-1}. This construction is now part of standard operator semigroup theory, see for example Engel and Nagel [15], Staffans [42, Section 3.6] or Weiss [49, Section 3], so that we recall the main facts without proof:

Proposition 2.1. *Let X be a Hilbert space and let $A\colon \mathcal{D}(A)\to X$ be the infinitesimal generator of a strongly continuous semigroup $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ on X. Take $\alpha \in \rho(A)$.*
(1) *For each $x \in \mathcal{D}(A)$, define $\|x\|_1 = \|(\alpha I - A)x\|$. Then $\|\cdot\|_1$ is a norm on X_1 which makes X_1 into a Hilbert space, and $A \in \mathcal{L}(X_1, X)$. The operator $(\alpha I - A)^{-1}$ maps X isometrically onto X_1.*
(2) *Let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(\alpha I - A)^{-1}x\|$. Then X is a Hilbert space and A has a unique extension to an operator $A \in \mathcal{L}(X, X_{-1})$. $(\alpha I - A)^{-1}$ maps X_{-1} isometrically onto X.*
(3) *The restrictions of $\mathbb{T}_t$ to X_1 form a strongly continuous semigroup on X_1. The generator of $\mathbb{T}$ as a semigroup on X_1 is the restriction of A to $\mathcal{D}(A^2)$.*

(4) *The operators $\mathbb{T}_t$ have a unique extension to X_{-1} which form a strongly continuous semigroup on X_{-1}. The generator of this extended semigroup is A extended to X as in point* (2).
(5) *The choice of $\alpha \in \rho(A)$ does not change the spaces X_1 or X_{-1}, since different values of $\alpha \in \rho(A)$ lead to equivalent norms on X_1 and X_{-1}.*

The notations A and $\mathbb{T}_t$ will be used also for the various restrictions and extensions of these operators (as in the above proposition).

Below we give what we think to be the simplest formulation of the definition of a system node. Equivalent and related definitions can be found, for example, in Opmeer [29], Malinen *et al* [26], and Staffans [39, 42]. We refer to the same sources for the proofs of the results stated in this section.

Definition 2.2. Let U, X and Y be Hilbert spaces. Let A be the infinitesimal generator of a strongly continuous semigroup $\mathbb{T}$ on X, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$. Let $\mathbf{G} : \rho(A) \to \mathcal{L}(U, Y)$ be such that

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C[(sI-A)^{-1} - (\beta I - A)^{-1}]B\,, \tag{4}$$

for all $s, \beta \in \rho(A)$. Then $\Sigma = (A, B, C, \mathbf{G})$ is a *system node* on (U, X, Y).

U is the *input space* of Σ, X is its *state space*, Y is its *output space*, A is its *semigroup generator*, B is its *control operator*, C is its *observation operator* and $\mathbf{G}$ is its *transfer function.* (A, B, C) is the *generating triple* of Σ.

For a system node Σ it is useful to introduce the space

$$V = \left\{ \begin{bmatrix} x \\ v \end{bmatrix} \in X \times U \mid Ax + Bv \in X \right\}, \tag{5}$$

which is a Hilbert space with the norm

$$\left\| \begin{bmatrix} x \\ v \end{bmatrix} \right\|_V^2 = \|x\|^2 + \|v\|^2 + \|Ax + Bv\|^2 .$$

We recall the space $\mathcal{D}_0$ introduced in Section 1:

$$\mathcal{D}_0 = \left\{ \begin{bmatrix} x \\ v \end{bmatrix} \in X \times C^2([0,\infty), U) \mid \begin{bmatrix} x \\ v(0) \end{bmatrix} \in V \right\}. \tag{6}$$

The operator $C\&D \in \mathcal{L}(V, Y)$, called the *combined observation/feedthrough operator* of Σ, is defined by

$$C\&D \begin{bmatrix} x \\ v \end{bmatrix} = C\left[x - (\beta I - A)^{-1} Bv\right] + \mathbf{G}(\beta) v\,.$$

This makes sense, because for $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in V$ we have $x - (\beta I - A)^{-1}Bv \in \mathcal{D}(A)$. It is easy to verify (using (4)) that the above definition of $C\&D$ is independent of the choice of $\beta \in \rho(A)$. We have $\mathbf{G}(s) = C\&D \left[\begin{smallmatrix} (sI-A)^{-1}B \\ I \end{smallmatrix}\right]$. To make the connection with more familiar formulas from finite-dimensional linear systems theory, note the following particular case: if $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$, then the limit $D = \lim_{\lambda\to\infty} \mathbf{G}(\lambda)$ exists and we have $C\&D = [C \ D]$, $\mathbf{G}(s) = C(sI-A)^{-1}B + D$.

Suppose that Σ is a system node on (U, X, Y) with generating triple (A, B, C) and transfer function $\mathbf{G}$. Choose $\beta \in \rho(A)$ and introduce the Hilbert space $Z = \mathcal{D}(A) + (\beta I - A)^{-1}BU$, with the norm

$$\|z\|_Z^2 = \inf\{\|x\|_1^2 + \|v\|_U^2 \mid x \in \mathcal{D}(A),\ v \in U,\ z = x + (\beta I - A)^{-1}Bv\}.$$

(The space Z is independent of the choice of β.) The system node Σ is called *compatible* if C has an extension $\overline{C} \in \mathcal{L}(Z, Y)$.

Such an extension $\overline{C}$ is usually not unique. If Σ is compatible, then for every extension $\overline{C}$ we can find a unique $D \in \mathcal{L}(U, Y)$ such that

$$C\&D \begin{bmatrix} x \\ v \end{bmatrix} = \overline{C}x + Dv \qquad \forall \begin{bmatrix} x \\ v \end{bmatrix} \in V.$$

Hence, in this case, $\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B + D$ for every $s \in \rho(A)$.

The following proposition from [26, Sect. 2] shows that at least for suitably smooth input fuctions and compatible initial states, Σ has well defined state trajectories and output functions.

Proposition 2.3. *Let Σ be a system node on (U, X, Y), let A, B and V be as in (5) and let $C\&D$ be the combined observation/feedthrough operator of Σ. For all $\left[\begin{smallmatrix} z_0 \\ u \end{smallmatrix}\right] \in \mathcal{D}_0$ the equation*

$$\dot{z}(t) = Az(t) + Bu(t), \qquad\qquad z(0) = z_0, \tag{7}$$

has a unique (classical) solution satisfying $z \in C^1([0,\infty), X) \cap C^2([0,\infty), X_{-1})$, $\left[\begin{smallmatrix} z \\ u \end{smallmatrix}\right] \in C([0,\infty), V)$. The corresponding output function y, defined by

$$y(t) = C\&D \begin{bmatrix} z(t) \\ u(t) \end{bmatrix},$$

is in $C([0,\infty), Y)$.

It is easy to show that under the assumptions of Proposition 2.3, if $\ddot{u}(t) = O(e^{\omega t})$ as $t \to \infty$ for some $\omega \in \mathbb{R}$, then the Laplace transforms $\hat{u}$, $\hat{z}$, and $\hat{y}$ satisfy

$$\hat{z}(s) = (sI - A)^{-1}[z_0 + B\hat{u}(s)], \tag{8}$$

$$\hat{y}(s) = C(sI - A)^{-1}z_0 + \mathbf{G}(s)\hat{u}(s), \tag{9}$$

for all $s \in \mathbb{C}$ for which $\operatorname{Re} s$ is larger than ω and also larger than the growth bound of $\mathbb{T}$ (see, for example, [42, Lemma 4.7.11]). If $z_0 \in X$ and the input signal u is a distribution with a Laplace transform defined on some right half-plane, then (8) and (9) define the signals z and y as distributions that have polynomially bounded Laplace transforms on some right half-plane.

The *dual system node* Σ^d has the generating triple (A^*, C^*, B^*) and transfer function $\mathbf{G}^d$ defined by $\mathbf{G}^d(s) = \mathbf{G}(\bar{s})^*$.

We obtain the well-known class of *well-posed linear systems* by adding one more assumption to those in Definition 2.2.

Definition 2.4. Let Σ be a system node on (U, X, Y). We call Σ *well-posed* if for some (hence for every) $t > 0$ there exists an $M_t \geq 0$ such that

$$\|z(t)\|^2 + \|y\|^2_{L^2([0,t],Y)} \leq M_t\big(\|z_0\|^2 + \|u\|^2_{L^2([0,t],U)}\big), \tag{10}$$

for all z, y, z_0, and u satisfying the conditions of Proposition 2.3.

A well-posed system node is usually called a *well-posed linear system.* Necessary and sufficient conditions for well-posedness were given in Curtain and Weiss [10]. For alternative definitions, background and examples we refer to Salamon [33], Staffans [38], [42], [43], Weiss [50], [51], Weiss and Rebarber [54] and Weiss, Staffans and Tucsnak [55]. All well-posed systems are compatible, see Staffans and Weiss [43].

If the system node Σ is well-posed, then it defines a family of bounded operators parameterized by $t \geq 0$,

$$\Sigma_t = \begin{bmatrix} \mathbb{T}_t & \Psi_t \\ \Phi_t & \mathbb{F}_t \end{bmatrix} \tag{11}$$

from $\begin{bmatrix} X \\ L^2([0,t];U) \end{bmatrix}$ to $\begin{bmatrix} X \\ L^2([0,t];Y) \end{bmatrix}$, such that

$$\Sigma_t \begin{bmatrix} z_0 \\ \mathbf{P}_t u \end{bmatrix} = \begin{bmatrix} z(t) \\ \mathbf{P}_t y \end{bmatrix}$$

for all $z_0, z(t), u$ and y as in Proposition 2.3. Here, $\mathbf{P}_t u$ and $\mathbf{P}_t y$ are the restrictions of u and y to $[0,t]$. In fact, well-posed linear systems are usually defined via the operator family $(\Sigma_t)_{t\geq 0}$ by imposing certain algebraic conditions.

The state trajectories of the well-posed Σ are given by

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u, \tag{12}$$

where $u \in L^2_{loc}([0,\infty), U)$ is the input function. The operators Φ_t are called the *input maps* of Σ and they are given by

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma)\, d\sigma. \tag{13}$$

The output function of the well-posed Σ is given by

$$y = \Psi z_0 + \mathbb{F} u, \tag{14}$$

where $\Psi : X \to L^2_{loc}([0,\infty), Y)$ is called the (extended) *output map* of Σ and $\mathbb{F}$ is a continuous linear operator from $L^2_{loc}([0,\infty), U)$ to $L^2_{loc}([0,\infty), Y)$ called the (extended) *input-output map* of Σ. Ψ is given by

$$(\Psi z_0)(t) = C\mathbb{T}_t z_0 \qquad \forall\, z_0 \in \mathcal{D}(A) \tag{15}$$

and for every $t \geq 0$ we have $\Psi_t = \mathbf{P}_t \Psi$. $\mathbb{F}$ is given by

$$(\mathbb{F}u)(t) = C\&D \begin{bmatrix} \Phi_t u \\ u(t) \end{bmatrix},$$

for all $u \in C^2([0,\infty), U)$ such that $u(0) = 0$. For every $t \geq 0$ we have $\mathbb{F}_t = \mathbf{P}_t \mathbb{F} \mathbf{P}_t$. The transfer function of every well-posed system is *proper*, meaning that it is uniformly bounded on some right half-plane.

Let us now return to the more general case of a system node Σ on (U, X, Y), with input signal u and output signal y. We are interested in the feedback control law $u = Ky + v$, where $K \in \mathcal{L}(Y, U)$ and v is the new input signal. This corresponds to Figure 1, but with $-K$ in place of κ. Elementary manipulations lead to

$$[I - \mathbf{G}(s)K]\hat{y}(s) = C(sI - A)^{-1} z_0 + \mathbf{G}(s)\hat{v}(s) \tag{16}$$

and if the operators $I - K\mathbf{G}(s)$ have bounded inverses for all s in some right half-plane, then we can compute

$$\begin{aligned} \hat{z}(s) \;=\; & (sI - A)^{-1}[z_0 + BK(I - \mathbf{G}(s)K)^{-1} C(sI - A)^{-1} z_0] \\ & +(sI - A)^{-1} BK(I - \mathbf{G}(s)K)^{-1} \mathbf{G}(s)\hat{v}(s). \end{aligned} \tag{17}$$

In general, such a feedback need not result in a system node (even if the inverse of $I - \mathbf{G}(s)K$ is uniformly bounded on some right half-plane). In fact, a closed-loop semigroup generator may not exist. In this paper we are interested in the special case where the feedback results in a system node Σ^K, called the *closed-loop system*.

In this case we call K an *admissible feedback operator* for the system node Σ. A necessary condition for this is the bounded invertibility of the operators $I - K\mathbf{G}(s)$ for all s in some right half-plane. From (16) we see that the transfer function of Σ^K is

$$\mathbf{G}^K = \mathbf{G}(I - K\mathbf{G})^{-1} = (I - \mathbf{G}K)^{-1}\mathbf{G}$$

and its observation operator C^K is determined by

$$C^K(sI - A^K)^{-1} = (I - \mathbf{G}(s)K)^{-1}C(sI - A)^{-1}.$$

From (17) we see that the semigroup generator A^K is determined by

$$(sI - A^K)^{-1} = (sI - A)^{-1} + (sI - A)^{-1}BK(I - \mathbf{G}(s)K)^{-1}C(sI - A)^{-1}, \quad (18)$$

and the control operator B^K is determined by

$$(sI - A^K)^{-1}B^K = (sI - A)^{-1}B(I - K\mathbf{G}(s))^{-1}.$$

The last three formulas hold for all $s \in \rho(A) \cap \rho(A^K)$. The latter is not empty since both A and A^K are semigroup generators.

In particular, if the closed-loop system Σ^K is well-posed, then the $\mathcal{L}(U)$-valued function $(I - K\mathbf{G})^{-1}$ is proper and we call K a *well-posed feedback operator* for Σ. Well-posed feedback operators for well-posed systems were studied in [51], [42].

Proposition 2.5. *Let Σ be a system node on (U, X, Y) and let $K \in \mathcal{L}(Y, U)$. Then K is an admissible feedback operator for Σ if and only if K^* is an admissible feedback operator for the dual system node Σ^d. Similarly, K is a well-posed feedback operator for Σ if and only if K^* is a well-posed feedback operator for Σ^d.*

If K is an admissible feedback operator for Σ, resulting in the closed-loop system Σ^K, then the feedback operator K^* for Σ^d leads to the closed-loop system Σ^{dK^*}, which is the dual of Σ^K.

3. Impedance passive and scattering passive systems.

In this section, we give rigorous definitions for the concepts of scattering passive and impedance passive system nodes following the terminology in [39, 40, 41]. We start with the much simpler case of discrete-time systems.

Definition 3.1. Let U, X, Y be Hilbert spaces. A *discrete-time system* Σ_d on (U, X, Y) consists of four operators $A_d \in \mathcal{L}(X)$, $B_d \in \mathcal{L}(U, X)$, $C_d \in \mathcal{L}(X, Y)$ and $D_d \in \mathcal{L}(U, Y)$, also called the *generating operators* of Σ_d. If u is an *input signal* for the system (an arbitrary U-valued sequence defined for $k \in \{0, 1, 2, \ldots\}$), then a *state trajectory* of the system corresponding to this input is a solution z of the difference equation $z_{k+1} = A_d z_k + Bu_k$. This space trajectory is uniquely determined if we specify the *initial state* $z_0 \in X$. The corresponding *output signal* y is the Y-valued sequence defined by $y_k = Cz_k + Du_k$.

While the Laplace-transform is a common tool for analysing continuous-time systems, the analogous tool for discrete-time systems is the $\mathcal{Z}$-transform: if u is a sequence then its *$\mathcal{Z}$-transform* is $\hat{u}(z) = \sum_{k=0}^{\infty} u_k z^{-k}$. If the series converges for some $\zeta \in \mathbb{C}$, then it converges for every $z \in \mathbb{C}$ with $|z| > |\zeta|$. In this case, we call u *$\mathcal{Z}$-transformable*. With the notation of the last definition, if u is $\mathcal{Z}$-transformable and $z_0 = 0$, then also y is $\mathcal{Z}$-transformable and we have $\hat{y}(z) = \mathbf{G}_d(z)\hat{u}(z)$, for all $z \in \mathbb{C}$ with $|z|$ sufficiently large, where

$$\mathbf{G}_d(z) = C_d(zI - A_d)^{-1}B_d + D_d \qquad \forall\, z \in \rho(A).$$

The $\mathcal{L}(U, Y)$-valued function $\mathbf{G}_d$ is called the *transfer function* of Σ_d.

Definition 3.2. With the notation of the last definition, Σ_d is *scattering passive* if for all $z_0 \in X$, all input signals u and all $m \in \mathbb{N}$,

$$\|z_m\|^2 - \|z_0\|^2 \leq \sum_{k=0}^{m-1} \|u_k\|^2 - \sum_{k=0}^{m} \|y_k\|^2 . \tag{19}$$

Σ^d is *impedance passive* if for all $z_0 \in X$, all input signals u and all $m \in \mathbb{N}$,

$$\|z_m\|^2 - \|z_0\|^2 \leq 2\mathrm{Re} \sum_{k=0}^{m-1} \langle u_k, y_k \rangle . \tag{20}$$

In fact, it enough to verify (19) or (20) for $m = 1$, as it is easy to see. Hence, (19) holds if and only if $\begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix}$ is a contraction. In this case, A_d is a contraction and $\mathbf{G}_d$ satisfies $\|\mathbf{G}_d(z)\| \leq 1$ for all $z \in \mathbb{C}$ with $|z| > 1$.

It is readily verified that (20) is satisfied if and only if the following holds

$$\begin{bmatrix} A_d^* A_d & A_d^* B_d \\ B_d^* A_d & B_d^* B_d \end{bmatrix} \leq \begin{bmatrix} I & C_d^* \\ C_d & D_d + D_d^* \end{bmatrix} . \tag{21}$$

In this case, A_d is (again) a contraction and $\mathbf{G}_d$ satisfies $\mathbf{G}_d(z) + \mathbf{G}_d(z)^* \geq 0$ for all $z \in \mathbb{C}$ with $|z| > 1$.

Definition 3.3. With the notation of Definition 2.2, the *(internal) Cayley transform* of Σ for the parameter $\alpha \in \rho(A)$ is the following operator in $\mathcal{L}(X \times U, X \times Y)$:

$$\left[\begin{array}{cc} \mathfrak{A}(\alpha) & \mathfrak{B}(\alpha) \\ \mathfrak{C}(\alpha) & \mathbf{G}(\alpha) \end{array} \right] = \left[\begin{array}{cc} (\overline{\alpha} I + A)(\alpha I - A)^{-1} & \sqrt{2\mathrm{Re}\,\alpha}\,(\alpha I - A)^{-1} B \\ \sqrt{2\mathrm{Re}\,\alpha}\, C(\alpha I - A)^{-1} & \mathbf{G}(\alpha) \end{array} \right] . \tag{22}$$

Note that the Cayley transform of Σ defines a discrete-time linear system Σ_d with generating operators $A_d = \mathfrak{A}(\alpha)$, $B_d = \mathfrak{B}(\alpha)$, $C_d = \mathfrak{C}(\alpha)$, $D_d = \mathbf{G}(\alpha)$. Some algebraic computation (using (4)) shows that the transfer function of Σ_d is

$$\mathbf{G}_d(z) = \mathbf{G}\left(\frac{\alpha z - \overline{\alpha}}{z + 1} \right) \qquad \forall\, z \in \rho(A_d) = \left\{ \frac{\overline{\alpha} + \lambda}{\alpha - \lambda} \;\middle|\; \lambda \in \rho(A) \right\} .$$

The above formula for $\rho(A_d)$ refers to the usual case when A is unbounded. If A is bounded, then $\rho(A_d)$ contains in addition the point -1.

We denote by $\mathbb{C}_0$ the open right half-plane:

$$\mathbb{C}_0 = \{ s \in \mathbb{C} \mid \mathrm{Re}\ s > 0 \} ,$$

and we denote by $\mathcal{H}^2(\mathbb{C}_0)$ the usual Hardy space of analytic functions on $\mathbb{C}_0$, see Rudin [31]. If U is a Hilbert space, then the Hardy space $\mathcal{H}^2(\mathbb{C}_0; U)$ (containing analytic U-valued functions) is defined similarly, see for example Nagy and Foias [27] or Rosenblum and Rovniak [30]. The inner product on $\mathcal{H}^2(\mathbb{C}_0; U)$ is

$$\langle v, w \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle v(i\omega), w(i\omega) \rangle \mathrm{d}\, \omega ,$$

where the values $v(i\omega)$, $w(i\omega)$ are obtained as nontangential limits, for almost every $\omega \in \mathbb{R}$. The *Paley-Wiener theorem* states that the Laplace transformation is a unitary operator from $L^2([0, \infty), U)$ to $\mathcal{H}^2(\mathbb{C}_0; U)$.

Take $\alpha \in \mathbb{C}_0$. The functions $\varphi_k : \mathbb{C}_0 \to \mathbb{C}$ defined by

$$\varphi_k(s) = \frac{\sqrt{2\mathrm{Re}\,\alpha}}{\overline{\alpha} + s} \left(\frac{\alpha - s}{\overline{\alpha} + s} \right)^k ,$$

where $k \in \{0,1,2,\dots\}$, form an orthonormal basis in $\mathcal{H}^2(\mathbb{C}_0)$. (Indeed, for $\alpha = 1$ this is well-known, and for other α it follows by scaling and vertical shifting.) This implies that every $\hat{u} \in \mathcal{H}^2(\mathbb{C}_0; U)$ can be written in a unique way as

$$\hat{u}(s) = \sum_{k=0}^{\infty} u_k \varphi_k(s), \tag{23}$$

where $u_k \in U$ and

$$\|u\|^2 = \sum_{k=0}^{\infty} \|u_k\|^2.$$

Remark 3.4. One fascinating fact in infinite-dimensional systems theory is that the internal Cayley transformed system Σ^d inherits many important properties of Σ, and it also works in the opposite direction: many properties of Σ follow from properties of Σ^d. The reason for this is as follows:

Let Σ be a system node. Assume that its initial state is zero and

$$u \in C^2([0,\infty),U) \cap L^2([0,\infty),U)$$

is such that the corresponding output signal y (defined as in Proposition 2.3) satisfies $y \in L^2([0,\infty),Y)$. Then (according to (9)) we have $\hat{y}(s) = \mathbf{G}(s)\hat{u}(s)$, for all s in some right half-plane. Using the representation (23) for $\hat{u}$, and a similar one for $\hat{y}$, we obtain that for $\operatorname{Re} s$ sufficiently large,

$$\frac{\sqrt{2\operatorname{Re}\alpha}}{\overline{\alpha}+s} \sum_{k=0}^{\infty} y_k \left(\frac{\alpha - s}{\overline{\alpha}+s}\right)^k = \mathbf{G}(s) \frac{\sqrt{2\operatorname{Re}\alpha}}{\overline{\alpha}+s} \sum_{k=0}^{\infty} u_k \left(\frac{\alpha - s}{\overline{\alpha}+s}\right)^k.$$

Denoting $z = \frac{\overline{\alpha}+s}{\alpha - s}$ (so that $s = \frac{\alpha z - \overline{\alpha}}{z+1}$), we obtain

$$\sum_{k=0}^{\infty} y_k z^{-k} = \mathbf{G}\left(\frac{\alpha z - \overline{\alpha}}{z+1}\right) \sum_{k=0}^{\infty} u_k z^{-k} = \mathbf{G}_d(z) \sum_{k=0}^{\infty} u_k z^{-k}.$$

Thus, regarding (u_k) as the input signal to the Cayley transformed system Σ_d, with initial state zero, the sequence (y_k) will be the corresponding output signal. In other words, the input-output map of Σ_d corresponds to the input-output map of Σ through the unitary transformation that maps u into the sequence (u_k). Similar statements hold for the input-to-state map and the initial state-to-output map.

In the following definition we use the notation $\mathcal{D}_0$ from (6).

Definition 3.5. Let Σ be a system node with input space U, state space X and output space Y. Σ is called *scattering passive* if for all $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}_0$ and every $\tau > 0$,

$$\|z(\tau)\|^2 - \|z_0\|^2 \le \int_0^\tau \|u(t)\|^2 \,\mathrm{d}t - \int_0^\tau \|y(t)\|^2 \,\mathrm{d}t.$$

Here the state trajectory z and the output function $y(t)$ are defined as in Proposition 2.3. Σ is called *impedance passive* if $Y = U$ and for all $\tau > 0$, (1) holds.

From Definition 2.4 we see that scattering passive system nodes are automatically well-posed and system stable. The semigroup of any scattering or impedance passive system is a semigroup of contractions. We refer to Staffans [39], [41], [43, Section 7] and Weiss and Tucsnak [56, Section 4] for more details on scattering passive systems. The following theorem is contained in [43, Theorem 7.4] and [39, Theorems 3.3, 4.2]. If X, A and X_1 are as in Section 2, we denote by Z_{-1} the dual of X_1 with respect to the pivot space X. The spaces U and Y are identified with their duals.

Theorem 3.6. *Let Σ, U, X and Y be as in the last definition. We denote the generating triple of the system node Σ by (A, B, C) and its transfer function by $\mathbf{G}$.*

(a) *Σ is scattering passive if and only if for some (hence, for every) $s \in \rho(A)$*

$$\begin{bmatrix} A + A^* & (sI + A^*)(sI - A)^{-1}B \\ B^*(\bar{s}I - A^*)^{-1}(\bar{s}I + A) & B^*(\bar{s}I - A^*)^{-1}2(\operatorname{Re} s)(sI - A)^{-1}B \end{bmatrix}$$
$$+ \begin{bmatrix} C^*C & C^*\mathbf{G}(s) \\ \mathbf{G}(s)^*C & \mathbf{G}(s)^*\mathbf{G}(s) \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

(b) *Σ is impedance passive if and only if $Y = U$ and for some (hence, for every) $s \in \rho(A)$*

$$\begin{bmatrix} A + A^* & (sI + A^*)(sI - A)^{-1}B \\ B^*(\bar{s}I - A^*)^{-1}(\bar{s}I + A) & B^*(\bar{s}I - A^*)^{-1}2(\operatorname{Re} s)(sI - A)^{-1}B \end{bmatrix}$$
$$\leq \begin{bmatrix} 0 & C^* \\ C & \mathbf{G}(s) + \mathbf{G}(s)^* \end{bmatrix}.$$

(c) *Σ is impedance passive if and only $Y = U$ and for some (hence, for every) $\alpha \in \mathbb{C}_0$ the Cayley transform defined in (22) satisfies*

$$\begin{bmatrix} \mathfrak{A}(\alpha)^*\mathfrak{A}(\alpha) & \mathfrak{A}(\alpha)^*\mathfrak{B}(\alpha) \\ \mathfrak{B}(\alpha)^*\mathfrak{A}(\alpha) & \mathfrak{B}(\alpha)^*\mathfrak{B}(\alpha) \end{bmatrix} \leq \begin{bmatrix} I & \mathfrak{C}(\alpha)^* \\ \mathfrak{C}(\alpha) & \mathbf{G}(\alpha) + \mathbf{G}(\alpha)^* \end{bmatrix}$$

Note that all the above matrices in (a) *and* (b) *are self-adjoint operators in $\mathcal{L}(X_1 \times U, Z_{-1} \times U)$, so that they determine symmetric quadratic forms on $X_1 \times U$. The inequalities in* (a), (b) *are understood in the sense of these quadratic forms.*

Remark 3.7. By point (a) of the above theorem, the transfer function $\mathbf{G}$ of any scattering passive system is defined on $\mathbb{C}_0$ and it satisfies $\|\mathbf{G}(s)\| \leq 1$ for all $s \in \mathbb{C}_0$.

By point (b) of the above theorem the transfer function of any impedance passive system is necessarily *positive*, i.e.,

$$\mathbf{G}(s) + \mathbf{G}(s)^* \geq 0 \quad \text{for all} \quad s \in \mathbb{C}_0 .$$

Both converse statements are false, even in finite dimensions. This is because impedance or scattering passivity depends on the realisation of $\mathbf{G}$.

Remark 3.8. If equality holds in (1), then the system is called *impedance energy preserving* and the condition in part (b) of Theorem 3.6 becomes an equality. In this case A generates an isometric semigroup.

We recall that if $E \in \mathcal{L}(U)$ is self-adjoint, then U has a unique orthogonal decomposition into E-invariant subspaces $U = U^- \oplus U^+$ such that $\langle Ev, v\rangle \leq 0$ for all $v \in U^-$ and $\langle Ev, v\rangle > 0$ for all non-zero $v \in U^+$. The operator E^+ obtained by redefining E to be zero on U^- is called *the positive part* of E. Noting that $\|E\|I \geq E^+ \geq E$, we deduce the following corollary.

Corollary 3.9. *Let Σ be a system node with transfer function $\mathbf{G}$. Let Σ_E be the system node with the same generating operators and the transfer function $\mathbf{G} + E$, where $E = E^* \in \mathcal{L}(U)$. If Σ_E is impedance passive, then for all $c \geq ||E^+||$ the system node Σ_{cI} is impedance passive.*

Remark 3.10. For the special case that A is skew-adjoint and $C = B^*$, we see that the necessary and sufficient condition for impedance passivity reduces to

$$\mathbf{G}(s) + \mathbf{G}(s)^* - B^*(\bar{s}I - A)^{-1}(2\operatorname{Re} s)(sI - A)^{-1}B \geq 0\,, \tag{24}$$

for some (hence, for every) $s \in \rho(A)$. In particular, if there exists an $\omega \in \mathbb{R}$ such that $i\omega \in \rho(A)$, then (24) reduces to $\mathbf{G}(i\omega) + \mathbf{G}(i\omega)^* \geq 0$.

Remark 3.11. As in Staffans [39, p. 294], it is easily verified that the necessary and sufficient condition in (b) for a well-posed system with a bounded generating triple A, B, C to be impedance passive reduces to

$$\begin{bmatrix} -A - A^* & C^* - B \\ C - B^* & D + D^* \end{bmatrix} \geq 0\,,$$

where $D = \lim_{\lambda \to +\infty} \mathbf{G}(\lambda)$. In the case that $D + D^*$ is invertible, this is equivalent to $D + D^* > 0$ and $A + A^* + (C^* - B)(D + D^*)^{-1}(C - B^*) \leq 0$.

We have the following simple alternative condition for impedance passivity.

Proposition 3.12. *Let Σ be a system node with generating triple (A, B, C) and transfer function $\mathbf{G}$. Let Σ_E be the system node with the same generating operators and the transfer function $\mathbf{G} + E$, where $E = E^* \in \mathcal{L}(U)$. If $i\omega \in \rho(A)$, then Σ_E is impedance passive if and only if*

$$\begin{bmatrix} -A_\omega^{-1} - A_\omega^{-*} & A_\omega^{-1} B + A_\omega^{-*} C^* \\ B^* A_\omega^{-*} + C A_\omega^{-1} & 2E + \mathbf{G}(i\omega) + \mathbf{G}(i\omega)^* \end{bmatrix} \geq 0\,, \tag{25}$$

where $A_\omega = A - i\omega I$.

Proof. According to Theorem 3.6(b) with $s = i\omega$, Σ_E is impedance passive if and only if

$$\begin{bmatrix} A_\omega + A_\omega^* & -A_\omega^* A_\omega^{-1} B - C^* \\ -B^* A_\omega^{-*} A_\omega - C & -2E - \mathbf{G}(i\omega) - \mathbf{G}(i\omega)^* \end{bmatrix} \leq 0\,.$$

Premultiplying this with the block operator $\operatorname{diag}(A_\omega^{-*}, I)$ and postmultiplying with $\operatorname{diag}(A_\omega^{-1}, I)$ yields (25). □

Remark 3.13. Under the assumptions of Proposition 3.12 define $B_\omega = A_\omega^{-1} B$ and $C_\omega = -C A_\omega^{-1}$. Then (25) is the necessary and sufficient condition for the *reciprocal system* with bounded generating triple $(A_\omega^{-1}, B_\omega, C_\omega)$ and transfer function

$$\mathbf{G}^r(s) = \mathbf{G}(i\omega) + C_\omega (sI - A_\omega^{-1})^{-1} B_\omega$$

to be impedance passive. If $2E + \mathbf{G}(i\omega) + \mathbf{G}(i\omega)^*$ is invertible, then the necessary and sufficient condition (25) for the impedance passivity of Σ_E becomes

$$A_\omega^{-1} + A_\omega^{-*} + (A_\omega^{-1} B + A_\omega^{-*} C^*)(2E + \mathbf{G}(i\omega) + \mathbf{G}(i\omega)^*)^{-1}(B^* A_\omega^{-*} + C A_\omega^{-1}) \leq 0.$$

In particular, if A is skew-adjoint, then $A_\omega^* = -A_\omega$ and the above necessary and sufficient condition reduces to $B = C^*$ and

$$2E + \mathbf{G}(i\omega) + \mathbf{G}(i\omega)^* \geq 0. \tag{26}$$

An easy consequence of Proposition 3.12 is the following.

Corollary 3.14. *Suppose that Σ is a system node with generating triple (A, B, C) and transfer function $\mathbf{G}$, where A generates a contraction semigroup on X and there exists an $\omega \in \mathbb{R}$ such that $i\omega \in \rho(A)$ and*

$$B^*(i\omega I + A^*)^{-1} = C(i\omega I - A)^{-1}\,. \tag{27}$$

Then Σ is impedance passive if and only if

$$\mathbf{G}(i\omega) + \mathbf{G}(i\omega)^* \geq 0\,.$$

Hence, the minimal self-adjoint E for which Σ_E is impedance passive is

$$E = -\frac{1}{2}[\mathbf{G}(i\omega) + \mathbf{G}(i\omega)^*].$$

An interesting class of systems with colocated actuators and sensors was investigated in Oostveen [28]: A generates a contraction semigroup, B is bounded and $C = B^*$. It is well-known that under these assumptions $\mathbf{G}(s) = B^*(sI - A)^{-1}B$ is positive and below we show that the system is also impedance passive.

Proposition 3.15. *Let Σ be well-posed linear system on (U, X, U) with generating operators (A, B, B^*), where $B \in \mathcal{L}(U, X)$ and A generates a contraction semigroup on X. Then Σ is impedance passive.*

Proof. Rearranging the terms in part (b) of Theorem 3.6 we see that Σ is impedance passive if and only if for some $s \in \rho(A)$,

$$\begin{bmatrix} A + A^* & (A + A^*)(sI - A)^{-1}B \\ B^*(\bar{s}I - A^*)^{-1}(A + A^*) & B^*(\bar{s}I - A^*)^{-1}(A + A^*)(sI - A)^{-1}B \end{bmatrix} \leq 0, \quad (28)$$

and this can be factored as

$$\begin{bmatrix} I & 0 \\ 0 & B^*(\bar{s}I - A^*)^{-1} \end{bmatrix} \begin{bmatrix} A + A^* & A + A^* \\ A + A^* & A + A^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (sI - A)^{-1}B \end{bmatrix} \leq 0. \quad (29)$$

Since B is bounded, the left-hand side defines a symmetric quadratic form on $X_1 \times U$ and it is clearly non-negative. □

In Curtain and Weiss [11] it was shown by means of counter-examples that for B unbounded the above proposition is false. So it is interesting to ask if it is possible to find an explicit expression for the minimal E to ensure that Σ_E is impedance passive. In [11] this was done for the special class of well-posed systems satisfying the following two assumptions.

Assumption ESAD. *The operator A is essentially skew-adjoint and dissipative, which means that $\mathcal{D}(A) = \mathcal{D}(A^*)$ and there exists a $Q \in \mathcal{L}(X)$ with $Q \geq 0$ such that*

$$Ax + A^*x = -Qx \qquad \forall\, x \in \mathcal{D}(A). \quad (30)$$

Assumption COL. $Y = U$ *and* $C = B^*$.

Remark 3.16. For a well-posed system satisfying the above two assumptions, in [11] (in Theorem 5.2 and its proof) it was shown that the expression

$$E = -\frac{1}{2}[\mathbf{G}(s) + \mathbf{G}(s)^*] + \frac{1}{2}B^*(\bar{s}I - A^*)^{-1}\,[2(\mathrm{Re}s)I + Q]\,(sI - A)^{-1}B, \quad (31)$$

is independent of $s \in \mathbb{C}_0$ and this operator can be written also in the form

$$E = -\frac{1}{2}\lim_{\lambda \to \infty}[\mathbf{G}(\lambda)^* + \mathbf{G}(-\lambda)].$$

Under an additional smoothness assumption on B, it was shown in Proposition 5.5 of [11] that the above E is the minimal self-adjoint operator ensuring the positivity of the transfer function $\mathbf{G} + E$.

Now we show that the expression (31) is independent of s also for a system node with colocated actuators and sensors and, with this operator E, the system node Σ_E is impedance passive. Here we allow $s \in \rho(A)$.

Theorem 3.17. *Let Σ be a system node satisfying the assumptions* **ESAD** *and* **COL** *and let E be defined by (31) where $s \in \rho(A)$. Then E is independent of $s \in \rho(A)$ and the system node Σ_E with generating triple (A, B, B^*) and transfer function $\mathbf{G}_E = \mathbf{G} + E$ is impedance passive. Moreover, E is the smallest self-adjoint operator in $\mathcal{L}(U)$ which has this passivity property.*

Proof. First we note that the proof of Theorem 5.2 in [11] that the right-hand side of (31) is a constant also holds for a system node and for all $s \in \rho(A)$. For every $F = F^* \in \mathcal{L}(U)$, let us denote by Σ_F the system node with generating triple (A, B, B^*) and transfer function $\mathbf{G}_F = \mathbf{G} + F$. For simplicity we prove the result for a real $s = \lambda$. According to Theorem 3.6, Σ_F is impedance passive if and only if for some real $\lambda > 0$,

$$\begin{bmatrix} 2\lambda(\lambda I - A^*)^{-1}Q(\lambda I - A)^{-1} & \sqrt{2\lambda}(\lambda I - A^*)^{-1}Q(\lambda I - A)^{-1}B \\ \sqrt{2\lambda}B^*(\lambda I - A^*)^{-1}Q(\lambda I - A)^{-1} & \mathbf{P}_F(\lambda) \end{bmatrix} \geq 0, \tag{32}$$

where $\mathbf{P}_F(\lambda) = \mathbf{G}_F(\lambda)^* + \mathbf{G}_F(\lambda) - B^*(\lambda I - A^*)^{-1}2\lambda(\lambda I - A)^{-1}B$. If (32) holds for some $\lambda > 0$, then it holds for all $\lambda > 0$. Using (31) we find that

$$\mathbf{P}_F(\lambda) = 2(F - E) + B^*(\lambda I - A^*)^{-1}Q(\lambda I - A)^{-1}B. \tag{33}$$

We see from (32) that Σ_F is impedance passive if and only if for some $\lambda > 0$,

$$\begin{bmatrix} \sqrt{2\lambda}(\lambda I - A^*)^{-1} & 0 \\ 0 & B^*(\lambda I - A^*)^{-1} \end{bmatrix} \begin{bmatrix} Q & Q \\ Q & Q \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2\lambda}(\lambda I - A)^{-1} & 0 \\ 0 & (\lambda I - A)^{-1}B \end{bmatrix} \geq \begin{bmatrix} 0 & 0 \\ 0 & 2(E - F) \end{bmatrix}. \tag{34}$$

Again, if this holds for some $\lambda > 0$, then it holds for all $\lambda > 0$. It is clear that the left-hand expresssion is nonnegative and so we see that (34) holds for $F = E$, and that this is the smallest possible value for which Σ_F is impedance passive. $\square$

Remark 3.18. If $Q = 0$ (A is skew-adjoint) and $F = E$, then clearly we have equality in (32). According to [39, Theorem 4.6], it follows that if $Q = 0$, then Σ_E is impedance energy preserving (see Remark 3.8).

Remark 3.19. In the case that B is bounded and $\mathbf{G}(s) = D + B^*(sI - A)^{-1}B$, then taking limits as $\operatorname{Re} s \to \infty$ shows that $-2E = D + D^*$.

Another interesting class of systems is those with self-adjoint A.

Lemma 3.20. *Suppose that Σ is a system node and that $A \leq 0$ (i.e., A is self-adjoint and $\langle Ax, x\rangle \leq 0$ for all $x \in \mathcal{D}(A)$).*

1. *If $0 \in \rho(A)$ and $C = B^*$, then the system node Σ_E will be impedance passive if and only if*
$$2E + \mathbf{G}(0) + \mathbf{G}(0)^* \geq 0.$$
2. *Σ_E is impedance passive if for some $\alpha > 0$*
$$C^* = (\alpha I + A)(\alpha I - A)^{-1}B,$$
$$2E + \mathbf{G}(\alpha) + \mathbf{G}(\alpha)^* - 2\alpha B^*(\alpha I - A)^{-2}B \geq 0.$$
3. *If $C^* = B$ and $(\alpha I - A)^{-\frac{1}{2}}B$ is a bounded operator for some positive α, then there exists a minimal self-adjoint E for which Σ_E is passive. It is given by*
$$E = -\frac{1}{2}[\mathbf{G}(s) + \mathbf{G}(s)^*] + B^*(\bar{s}I - A)^{-1}[(\operatorname{Re} s)I - A](sI - A)^{-1}B, \tag{35}$$

which holds for all $s \in \mathbb{C}_0$.

Proof. We leave this to the reader as it is similar to previous proofs. □

4. **Impedance passive systems with feedback.** The next result describes a feedback transformation from impedance passive nodes to scattering passive well-posed systems. It is closely related to [39, Theorem 5.2] and to [52, Section 2].

Proposition 4.1. *Let* Σ^p *be an impedance passive system node on* (U, X, U) *with generating triple* (A, B, C) *and transfer function* $\mathbf{G}^p$ *and let* $k > 0$. *Suppose that* u^p *is a* C^2 *input function and* z_0 *an initial state satisfying* $Az_0 + Bu^p(0) \in X$, *and* z, y^p *are the resulting state trajectory and output function, as in Proposition 2.3. Then there is a unique scattering passive system node* Σ^s *with the following property: if we use the same initial state* z_0 *and the new input function* $u^s = \sqrt{\frac{k}{2}}(\frac{u^p}{k} + y^p)$ *for the system node* Σ^s, *then the state trajectory of* Σ^s *is again* z *and the output function of* Σ^s *is given by* $y^s = \sqrt{\frac{k}{2}}(\frac{u^p}{k} - y^p)$. *We have* $(I + k\mathbf{G}^p)^{-1} \in H^\infty$ *and the transfer function of* Σ^s *is*

$$\mathbf{G}^s = (I - k\mathbf{G}^p)(I + k\mathbf{G}^p)^{-1}.$$

Σ^s can be thought of as a closed-loop system node obtained from Σ^p as shown in Figure 2. In [39], k is chosen to be 1, while in [52], $\frac{1}{r}$ appears in place of k. Staffans in [39] calls the transformation from Σ^p to Σ^s the *diagonal transformation* (or the external Cayley transformation) and he traces it back to the work of M. Livsic [23].

Proof. Noting that $\mathbf{G}^p$ is positive, according to Proposition 2.1 in [11] (with $c = 0$), it follows that $(I + k\mathbf{G}^p)^{-1} \in H^\infty$. Thus the closed-loop transfer function

$$\mathbf{G}^s = (I - k\mathbf{G}^p)(I + k\mathbf{G}^p)^{-1} = 2(I + k\mathbf{G}^p)^{-1} - I$$

is in H^∞. Hence, inputs $u^s \in L^2([0, \infty), U)$ produce outputs $y^s \in L^2([0, \infty), U)$. Moreover, it is easy to check that the scattering passivity property holds, since

$$\|u^s(t)\|^2 - \|y^s(t)\|^2 = 2\mathrm{Re}\,\langle y^p(t), u^p(t)\rangle. \tag{36}$$

It remains to show that Σ^s is a system node. To do this we use the internal Cayley transform introduced in Section 3, in particular in (22). First we note that from Theorem 3.6 the Cayley-transformed system node produces the discrete-time linear system with generating operators A_d, B_d, C_d, D_d that satisfy (21). So the discrete-time system is impedance passive. As noted above, $I + k\mathbf{G}(\alpha)^p = I + kD_d$ is invertible for all $k > 0$ and $\alpha \in \mathbb{C}^+$. Hence the closed-loop discrete-time system obtained via the transformation is well-defined and its generating operators are given by $A_d^s, B_d^s, C_d^s, D_d^s$ are given by

$$A_d^s = A_d - B_d(I + kD_d)^{-1}kC_d, \quad B_d^s = \sqrt{2k}B_d(I + kD_d)^{-1},$$
$$C_d^s = -\sqrt{2k}(I + kD_d)^{-1}C_d, \quad D_d^s = (I + kD_d)^{-1}(I - kD_d)^{-1}.$$

Using (36) it is easy to see that it is scattering passive. Hence $\begin{bmatrix} A_d^s & B_d^s \\ C_d^s & D_d^s \end{bmatrix}$ is a contraction. In particular, this implies that A_d^s is a contraction. So its inverse Cayley transform will define the generator of a contraction semigroup if and only if -1 is not an eigenvalue of A_d^s. Using a contradiction argument we show that -1 cannot be an eigenvalue of A_d^s. Suppose, on the contrary, that $A_d^s x = -x$ for some non-zero $x \in X$. Then from the contraction property we obtain

$$\|x\|^2 + \|C_d^s x\|^2 = \|A_d^s x\|^2 + \|C_d^s x\|^2 \le \|x\|^2.$$

Hence $0 = C_d^s x = -\sqrt{2k}(I + kD_d)^{-1}C_d x$ and $C_d x = 0$. But

$$-x = A_d^s x = A_d x - B_d k(I + kD_d)^{-1}C_d x = A_d x$$

implies that -1 is an eigenvalue of A_d. Since we have assumed that A generates a contraction semigroup, we have arrived at a contradiction. The inverse Cayley transform of the closed-loop discrete-time system is Σ^s and it has a (contraction) semigroup generator. Thus Σ^s is a scattering passive system node. □

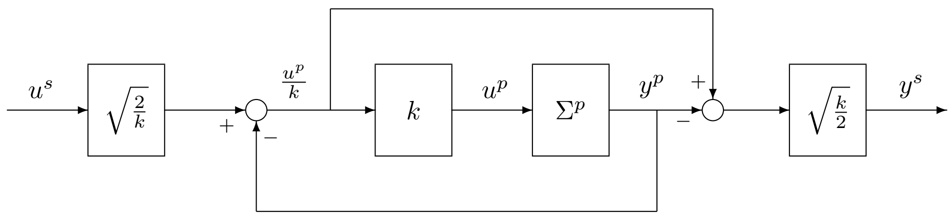

FIGURE 2. The scattering passive system Σ^s with input u^s and output y^s, obtained from the impedance passive system node Σ^p via the diagonal transformation, as in Proposition 4.1.

The relation between scattering passive and impedance passive system nodes in Proposition 4.1 allows us to prove the following stronger version of Theorem 1.2.

Theorem 4.2. *Let Σ be a system node on (U, X, U) with generating triple (A, B, C) and transfer function $\mathbf{G}$. For any $E \in \mathcal{L}(U)$ we denote by Σ_E the system node with the same generating triple and with the transfer function $\mathbf{G} + E$. Assume that $E^* = E$ is such that Σ_E is impedance passive. Denote*

$$c = \|E^+\|, \qquad \kappa_0 = \frac{1}{c},$$

where E^+ is the positive part of E (see Section 3).

Then for every $\kappa \in (0, \kappa_0)$, $K = -\kappa I$ is a well-posed feedback operator for Σ and the corresponding closed-loop system Σ^κ is system stable. Moreover, the semigroup of Σ^κ is a contraction semigroup on X.

Proof. The generating triple of Σ^κ is complicated to express directly in terms of $A, B, C, \mathbf{G}$ and κ (see for example [42]). Instead, we use an indirect approach involving several transformations from one system to another. Recall that the system node Σ has the transfer function $\mathbf{G}$. Let Σ^p be the impedance passive system node with transfer function $\mathbf{G}^p = \mathbf{G} + cI$ from Corollary 3.9. We apply Proposition 4.1 to Σ^p, which implies that for every $k > 0$, the closed-loop system node Σ^s with the transfer function $\mathbf{G}^s = (I - k\mathbf{G}^p)(I + k\mathbf{G}^p)^{-1}$ is scattering passive and hence well-posed. We have

$$I + k\mathbf{G}^p = (1 + kc)I + k\mathbf{G} = (1 + kc)\left(I + \frac{k}{1 + kc}\mathbf{G}\right). \tag{37}$$

Denote $\kappa = \frac{k}{1+kc}$, so that $\kappa \in (0, \kappa_0)$. We know from Proposition 4.1 that $(I + k\mathbf{G}^p)^{-1} \in H^\infty$, and this together with (37) shows that $(I + \kappa\mathbf{G})^{-1} \in H^\infty$. So $\mathbf{G}^\kappa = \mathbf{G}(I + \kappa\mathbf{G})^{-1} \in H^\infty$ and the input-output connection in Figure 1 is well-posed. It remains to show that $-\kappa I$ is a well-posed feedback operator for Σ with the closed-loop system node Σ^κ. We do this by clarifying its relationship with the three system nodes Σ, Σ^p and Σ^s. They are all obtained from the same basic system node Σ by defining new input and output signals via linear transformations applied to the original input signal u and the original output signal y of Σ (this does not affect the state trajectories). We start with the connection with Σ^κ: we denote its input by v (as in Figure 1). The output of Σ^κ is y, the same as for Σ. Then $u = v - \kappa y$ (see Figure 1), or in matrix form:

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I & -\kappa I \\ 0 & I \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix}.$$

The input and output signals of Σ^p are $u^p = u$ and $y^p = y + cu$. The input and output signals of Σ^s, denoted u^s and y^s, have been defined in terms of u^p and y^p in Proposition 4.1 (see Figure 2). Writing these formulas in matrix form, we have

$$\begin{bmatrix} u^p \\ y^p \end{bmatrix} = \begin{bmatrix} I & 0 \\ cI & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}, \qquad \begin{bmatrix} u^s \\ y^s \end{bmatrix} = \sqrt{\frac{k}{2}} \begin{bmatrix} \frac{1}{k}I & I \\ \frac{1}{k}I & -I \end{bmatrix} \begin{bmatrix} u^p \\ y^p \end{bmatrix}.$$

To obtain the relation between the signals of Σ^κ and Σ^s, we have to multiply the three 2×2 matrices appearing above, which yields

$$\begin{bmatrix} u^s \\ y^s \end{bmatrix} = \sqrt{\frac{k}{2}} \begin{bmatrix} (\frac{1}{k} + c)I & [1 - (\frac{1}{k} + c)\kappa]I \\ (\frac{1}{k} - c)I & -[1 + (\frac{1}{k} - c)\kappa]I \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha}I & 0 \\ \beta I & -\alpha I \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix},$$

where

$$\alpha = \sqrt{2\kappa(1 - \kappa c)}, \qquad \beta = \frac{1 - 2\kappa c}{\alpha},$$

so that $\alpha > 0$. Inverting the last 2×2 matrix, we have

$$\begin{bmatrix} v \\ y \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ \beta I & -\frac{1}{\alpha}I \end{bmatrix} \begin{bmatrix} u^s \\ y^s \end{bmatrix}. \tag{38}$$

This relation is illustrated in Figure 3.

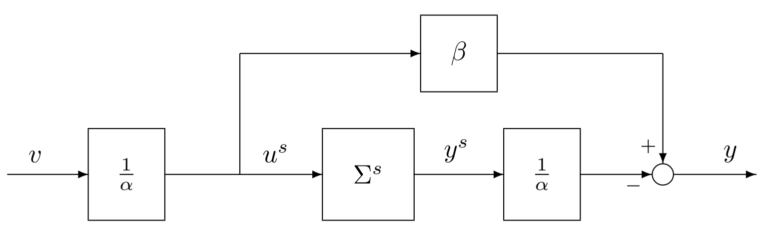

FIGURE 3. The feedback system Σ^κ from Figure 1, with input v and output y, as obtained from the scattering passive system Σ^s with input u^s and output y^s. Note that there is no feedback loop involved in this transformation.

We denote the generating triple of Σ^s by (A^s, B^s, C^s) and its transfer function by $\mathbf{G}^s$. Note from (38) and Figure 3 that there is no feedback loop involved in the transformation between the inputs and outputs of the scattering passive system Σ^s and of Σ^κ. So the latter is in fact a well-posed system and from (38) we deduce that its generating triple $(A^\kappa, B^\kappa, C^\kappa)$ and $\mathbf{G}^\kappa$ are given by

$$A^\kappa = A^s, \quad B^\kappa = \frac{1}{\alpha}B^s, \quad C^\kappa = -\frac{1}{\alpha}C^s, \quad \mathbf{G}^\kappa = \frac{\beta}{\alpha}I - \frac{1}{\alpha^2}\mathbf{G}^s. \tag{39}$$

Since the scattering passive linear system Σ^s is system stable, the above relations imply that Σ^κ is also system stable. Similarly, since A^s generates a contraction semigroup, so does A^κ. □

5. Weak and strong stabilization.

In this section, we continue our analysis of the stabilization of impedance passive system nodes. We need the following result about the spectrum of the closed-loop generator A^K under admissible feedback from Curtain and Jacob [13, Theorem 6.2], a generalization of Lemma 4.4 in Salamon [33], see also Weiss and Xu [57] for related results.

Theorem 5.1. *Let Σ be a system node with generating triple (A, B, C) and transfer function $\mathbf{G}$. Suppose that $K \in \mathcal{L}(Y, U)$ is an admissible feedback operator for Σ producing the closed-loop system node Σ^K. Denote its generating triple by (A^K, B^K, C^K) and its transfer function $\mathbf{G}(I - K\mathbf{G})^{-1}$ by $\mathbf{G}^K$.*

Then for $\lambda \in \rho(A)$ we have that $\lambda \in \rho(A^K)$ if and only if $I - K\mathbf{G}(\lambda)$ is invertible.

Actually, in [13] this is proved for operator nodes, a slightly more general concept than system nodes. The idea of the proof is to apply the Cayley transform to Σ and Σ^K and then to verify the corresponding discrete-time result.

For a system node Σ on (U, X, Y) with generating triple (A, B, C) we define the *unobservable space*

$$\mathcal{N} = \{x \in X \mid C(sI - A)^{-1}x = 0 \text{ for } \operatorname{Re} s > \omega\},$$

where ω is some real number larger than the growth bound of the semigroup generated by A. Clearly, the choice of ω is unimportant. Note that the approximate observability in infinite time of Σ is equivalent to $\mathcal{N} = \{0\}$. The corresponding space for the dual system node is

$$\mathcal{N}^d = \{x \in X \mid B^*(sI - A^*)^{-1}x = 0 \text{ for } \operatorname{Re} s > \omega\}.$$

Clearly the approximate controllability in infinite time of Σ is equivalent to $\mathcal{N}^d = \{0\}$. The main result of this section is a strengthened version of Theorem 1.3:

Theorem 5.2. *Suppose that Σ is a system node on (U, X, U) with generating triple (A, B, C) and transfer function $\mathbf{G}$. For any $E \in \mathcal{L}(U)$ we denote by Σ_E the system node with the same generating triple as Σ, but with the transfer function $\mathbf{G} + E$. Suppose that there exists $E = E^* \in \mathcal{L}(U)$ such that Σ_E is impedance passive, let $c = \|E^+\|$, where E^+ is the positive part of E and denote $\kappa_0 = \frac{1}{c}$.*

Then for every $\kappa \in (0, \kappa_0)$ the operator $K = -\kappa I$ is a well-posed feedback operator for Σ. If either of the following conditions holds,

$$\{x \in \mathcal{N} \mid \|\mathbb{T}_t x\| = \|x\| = \|\mathbb{T}_t^* x\| \quad \forall\, t > 0\} = \{0\}, \tag{40}$$

$$\left\{x \in \mathcal{N}^d \mid \|\mathbb{T}_t x\| = \|x\| = \|\mathbb{T}_t^* x\| \quad \forall\, t > 0\right\} = \{0\}, \tag{41}$$

then the semigroup $\mathbb{T}^\kappa$ of the closed-loop system Σ^κ is weakly stable.

Moreover, if (40) or (41) hold and $\sigma(A) \cap i\mathbb{R}$ is countable, then both semigroups $\mathbb{T}^\kappa$ and $\mathbb{T}^{\kappa *}$ are strongly stable.

Proof. (a) To prove weak stability, recall from Theorem 4.2 that Σ^κ is a well-posed linear system and $\mathbb{T}^\kappa$ is a semigroup of contractions. In Szökefalvi-Nagy and Foias [27] or Davies [14, Corollary 6.22] (see also Benchimol [8]) it is shown that $\mathbb{T}^\kappa$ is weakly stable if $X^u = \{0\}$, where

$$X^u = \{x \in X \mid \|\mathbb{T}_t^\kappa x\| = \|x\| = \|\mathbb{T}_t^{\kappa *} x\| \quad \forall\, t > 0\},$$

i.e., $\mathbb{T}^\kappa$ is completely non-unitary. We show that (40) implies $X^u = \{0\}$. Theorem 3.6 applies to the scattering passive system node Σ^s introduced in the proof of Theorem 4.2 (shown in Figure 2). Noting the close relationships betweeen the generating operators of Σ^κ and Σ^s from (39), we obtain

$$\langle A^\kappa z, z\rangle + \langle A^{\kappa *} z, z\rangle + \alpha^2 \langle C^{\kappa *} C^\kappa z, z\rangle \le 0 \qquad \forall\, z \in \mathcal{D}(A^\kappa),$$

where $\alpha^2 = 2\kappa(1-\kappa c) > 0$ and $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathcal{D}(A^\kappa)$ and its dual with respect to the pivot space X. We take $z = \mathbb{T}_t^\kappa x$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbb{T}_t^\kappa x\|^2 + \alpha^2 \|C^\kappa \mathbb{T}_t^\kappa x\|^2 \le 0 \qquad \forall\, x \in \mathcal{D}(A^\kappa).$$

Integrating from 0 to t gives $\|\mathbb{T}_t^\kappa x\|^2 + \alpha^2 \int_0^t \|C^\kappa \mathbb{T}_\sigma^\kappa x\|^2 \,\mathrm{d}\sigma \le \|x\|^2$, and this can be extended by continuity to all of X:

$$\|\mathbb{T}_t^\kappa x\|^2 + \alpha^2 \int_0^t \|(\Psi^\kappa x)(\sigma)\|^2 \,\mathrm{d}\sigma \le \|x\|^2 \qquad \forall\, x \in X,\; t \ge 0. \tag{42}$$

Suppose now that $x \in X^u$. Substituting this x into (42) implies that $\Psi^\kappa x = 0$. Consider Σ with the initial state $x \in X^u$ and input $u = 0$. Denote the corresponding output signal of Σ by y, which is defined by its Laplace transform

$$\hat{y}(s) = C(sI - A)^{-1} x \qquad \forall\, s \in \mathbb{C}_0$$

(see Section 2). The initial state of Σ^κ is the same x and its input signal is $v = \kappa y$ (see Figure 1). Its output signal satisfies $y = \Psi^\kappa x + \kappa \mathbb{F}^\kappa y = \kappa \mathbb{F}^\kappa y$. Taking Laplace transforms we obtain $(I - \kappa \mathbf{G}^\kappa)\hat{y} = 0$. Since $I - \kappa \mathbf{G}^\kappa = (I + \kappa \mathbf{G})^{-1}$, it follows that $\hat{y} = 0$, and so $v = 0$. Hence the state trajectory $z(\cdot)$ of Σ^κ is given by $z(t) = \mathbb{T}_t^\kappa x$. Since the systems Σ and Σ^κ have the same state trajectory $z(\cdot)$, we obtain $z(t) = \mathbb{T}_t^\kappa x = \mathbb{T}_t x$ for all $t \ge 0$. In particular, it follows that $\|\mathbb{T}_t x\| = \|x\|$ for all $t \ge 0$. We now redo this whole argument with the dual of Σ using Proposition 2.5. This shows that $\|\mathbb{T}_t^* x\| = \|x\| = \|\mathbb{T}_t x\|$ for all $t \ge 0$.

The output signal y of Σ with initial state $x \in X^u$ and with zero input equals the output signal of Σ^κ. We have seen earlier that $\hat{y} = 0$, i.e., $x \in \mathcal{N}$. According to (40) we obtain $x = 0$. Hence $X^u = \{0\}$, so that $\mathbb{T}^\kappa$ is weakly stable. Dual arguments show that (41) implies the weak stability of $\mathbb{T}^{\kappa *}$, which in turn is equivalent to the weak stability of $\mathbb{T}^\kappa$.

(b) The second step is to derive a useful estimate concerning the open-loop transfer function $\mathbf{G}$. Since $cI + \mathbf{G}$ is a positive transfer function (see Remark 3.7 and Corollary 3.9), for all $s \in \mathbb{C}_0$ and for all $u_0 \in U$ with $\|u_0\| = 1$ there holds

$$\begin{aligned} \left\| \left(\frac{1}{\kappa} I + \mathbf{G}(s) \right) u_0 \right\| &\ge \mathrm{Re} \left\langle \left(\frac{1}{\kappa} I + \mathbf{G}(s) \right) u_0, u_0 \right\rangle \\ &= \frac{1}{\kappa} - c + \mathrm{Re}\, \langle (cI + \mathbf{G}(s))\, u_0, u_0 \rangle \ge \frac{1}{\kappa} - c > 0. \end{aligned} \tag{43}$$

Suppose that $\omega \in \mathbb{R}$ is such that $i\omega_0 \in \rho(A)$. Then $(\frac{1}{\kappa} I + \mathbf{G}(s))u_0$ is analytic on a small neighbourhood $N_\varepsilon = \{z \mid \|z - i\omega_0\| < \varepsilon\}$ (and of course also on $\mathbb{C}_0$). By continuous extension we obtain from (43) that for $i\omega_0 \in \rho(A)$ and $\|u_0\| = 1$,

$$\left\| \left(\frac{1}{\kappa} I + \mathbf{G}(i\omega_0) \right) u_0 \right\| \ge \frac{1}{\kappa} - c > 0. \tag{44}$$

(c) To prove strong stability, first we show that

$$\sigma(A^\kappa) \cap i\mathbb{R} \subset \sigma(A) \cap i\mathbb{R}. \tag{45}$$

Since $-\kappa I$ is an admissible feedback operator for Σ that produces the well-posed closed-loop system Σ^κ, we can apply Theorem 5.1 to obtain that if $\lambda \in \sigma(A^\kappa) \cap \rho(A)$, then $-\frac{1}{\kappa} \in \sigma(\mathbf{G}(\lambda))$. In particular, suppose that for some $\omega \in \mathbb{R}$ we have $i\omega_0 \in \sigma(A^\kappa) \cap \rho(A)$, then $-\frac{1}{\kappa} \in \sigma(\mathbf{G}(i\omega_0))$. To prove (45), we have to show that such an ω_0 cannot exist. We consider the classical three cases:

If $-\frac{1}{\kappa}$ is in the point spectrum of $\mathbf{G}(i\omega_0)$, then there exists $u_0 \in U$ with $\|u_0\| = 1$ such that

$$\frac{1}{\kappa} u_0 + \mathbf{G}(i\omega_0) u_0 = 0 .$$

This is in direct contradiction with (44).

If $-\frac{1}{\kappa}$ is in the residual spectrum of $\mathbf{G}(i\omega_0)$, then it is in the point spectrum of $\mathbf{G}(i\omega_0)^*$. Then a similar argument applied to the dual system node with the dual transfer function $\mathbf{G}^d(s) = \mathbf{G}(\bar{s})^*$ also leads to a contradiction.

The remaining possibility is that $-\frac{1}{\kappa}$ is in the continuous spectrum of $\mathbf{G}(i\omega_0)$. In this case there exists a sequence $u_n \in U$ with $\|u_n\| = 1$ and

$$\left\| \frac{1}{\kappa} u_n + \mathbf{G}(i\omega_0) u_n \right\| \to 0 \text{ as } n \to \infty . \tag{46}$$

This is again in contradiction with (44). We have established (45).

This implies that (with the assumption in the last part of the theorem) $\sigma(A^\kappa) \cap i\mathbb{R}$ is countable. According to Theorem 4.2 $\mathbb{T}^\kappa$ is a contraction semigroup. Its weak stability (shown earlier in this proof) implies that A^κ and $A^{\kappa *}$ have no eigenvalues on $i\mathbb{R}$. So we can apply the main result from Arendt and Batty [2] to establish the strong stability of $\mathbb{T}^\kappa$. Since $\sigma(A^{\kappa *}) \cap i\mathbb{R} = \overline{\sigma(A^\kappa) \cap i\mathbb{R}}$, a similar argument shows that $\mathbb{T}^*$ is also strongly stable. $\square$

Remark 5.3. Note that (40) means that the unitary part of $\mathbb{T}$ (i.e., its restriction to the space X^u defined in the last proof) is approximately observable in infinite time. In particular, (40) holds if (A, C) is approximately observable in infinite time. Similar remarks hold for the dual condition (41). In particular, (41) is implied by the approximate controllability in infinite time of (A, B).

Remark 5.4. Theorem 5.2 generalizes Theorem 14 in Batty and Phong [7] to unbounded operators B and C, while eliminating the assumption that $C = B^*$.

Remark 5.5. Let Σ_i, $i = 1, 2$, be two impedance passive system nodes with the inputs u_i, outputs y_i and transfer functions $\mathbf{G}_i$, $i = 1, 2$. Hence we have

$$\|z_i(\tau)\|^2 - \|z_i(0)\|^2 \le 2 \int_0^\tau \mathrm{Re}\langle u_i(t), y_i(t)\rangle \mathrm{d}t .$$

Subsituting $u_1 = -y_2 + v_{cl}$, $\;u_2 = y_1$ we obtain

$$\|z_1(\tau)\|^2 - \|z_1(0)\|^2 \le 2 \int_0^\tau \mathrm{Re}\langle -y_2(t) + v_c(t), y_1(t)\rangle \mathrm{d}t ,$$

$$\|z_2(\tau)\|^2 - \|z_2(0)\|^2 \le 2 \int_0^\tau \mathrm{Re}\langle y_1(t), y_2(t)\rangle \mathrm{d}t ,$$

and adding these two equations yields

$$\|z_1(\tau)\|^2 + \|z_2(\tau)\|^2 - \|z_1(0)\|^2 - \|z_2(0)\|^2 \le 2 \int_0^\tau \mathrm{Re}\langle v_c(t), y_1(t)\rangle \mathrm{d}t .$$

Hence the above feedback connection produces another impedance passive system Σ_{cl} with the state $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, the input v_{cl}, the output $y_{cl} = y_1$ and the transfer function $\mathbf{G}_{cl} = \mathbf{G}_1(I - \mathbf{G}_2\mathbf{G}_1)^{-1}$ (see van der Schaft [34, Proposition 3.2.5]).

Hence, according to Theorem 4.2, for all $\varepsilon > 0$ the feedback control law $v_{cl} = -\varepsilon y_{cl} + v$ produces a well-posed closed-loop system Σ_ε that is system stable. Its transfer function is given by $\mathbf{G}_\varepsilon = \mathbf{G}_1(I + \mathbf{G}_2\mathbf{G}_1 + \varepsilon\mathbf{G}_1)^{-1}$. Note that Σ_ε can equally well be obtained by applying the transformation $u_1 = -(\varepsilon I + \mathbf{G}_2)y_1$ to Σ_1. In the finite-dimensional literature this is known as dynamic feedback stabilization.

6. **A class of damped second order systems with colocated actuators and sensors.** In this section we introduce a class of damped second order systems that are impedance passive system nodes (in the sense that (1) holds), where the actuators and sensors are colocated (in the sense that assumption **COL** holds).

Let U_0 and H be Hilbert spaces and let $A_0 : \mathcal{D}(A_0) \to H$ be positive and boundedly invertible on H. For every $\mu > 0$, we define $H_\mu = \mathcal{D}(A_0^\mu)$, with the norm $\|\varphi\|_\mu = \|A_0^\mu \varphi\|_H$, and we define $H_{-\mu} = H_\mu^*$ (duality with respect to the pivot space H). We denote $H_0 = H$ and $\|\varphi\|_0 = \|\varphi\|_H$. We assume that

$$C_0 \in \mathcal{L}(H_{\frac{1}{2}}, U_0), \quad M \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}}), \quad M \geq 0.$$

By $M \geq 0$ we mean that it defines a positive quadratic form on $H_{\frac{1}{2}}$. We identify U_0 with its dual, so that $C_0^* \in \mathcal{L}(U_0, H_{-\frac{1}{2}})$.

We wish to study the system described by the abstract second order equation

$$\ddot{q} + M\dot{q} + A_0 q = C_0^* u, \qquad y = C_0 \dot{q}. \tag{47}$$

This is a slight generalization of the class of systems studied in Tucsnak and Weiss [47, 56]. We introduce the state space $X = H_{\frac{1}{2}} \times H$ and we define

$$A : \mathcal{D}(A) \to X, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & -M \end{bmatrix}, \tag{48}$$

$$X_1 = \mathcal{D}(A) = \left\{ \left[\begin{smallmatrix} q \\ w \end{smallmatrix}\right] \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 q + Mw \in H \right\},$$

$$C = \begin{bmatrix} 0 & C_0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ C_0^* \end{bmatrix},$$

then $B \in \mathcal{L}(U_0, X_{-1})$, $B^*, C \in \mathcal{L}(X_1, U_0)$. It is also easy to verify that **COL** holds and

$$A^* : \mathcal{D}(A^*) \to X, \quad A^* = \begin{bmatrix} 0 & -I \\ A_0 & -M \end{bmatrix},$$

$$\mathcal{D}(A^*) = \left\{ \left[\begin{smallmatrix} q \\ w \end{smallmatrix}\right] \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid -A_0 q + Mw \in H \right\},$$

It can be shown that the operators A, B, C and D define a compatible system node Σ on $(U_{,0} X, U_0)$, whose state trajectories z and output functions y corresponding to C^2 input functions u and compatible initial conditions satisfy

$$\begin{cases} \dot{z} &= Az + Bu, \\ y &= \overline{C}z. \end{cases} \tag{49}$$

As in [56], [11], it is not difficult to prove that (47) is equivalent to (49), for any C^2 input signal u and compatible initial conditions $q(0)$ and $\dot{q}(0)$, i.e., $A_0 q(0) +$

$M\dot{q}(0) - C_0^* u(0) \in H$. Indeed, the connection is provided by

$$z = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}.$$

Note that

$$A + A^* = \begin{bmatrix} 0 & -0 \\ 0 & -2M \end{bmatrix},$$

and so **ESAD** is not satisfied if M is unbounded. However, since $0 \in \rho(A)$, it is relatively easy to show that this system is impedance passive by using Corollary 3.14. We have

$$A^{-1} = \begin{bmatrix} -A_0^{-1}M & -A_0^{-1} \\ I & 0 \end{bmatrix}, \; A^{-*} = \begin{bmatrix} -A_0^{-1}M & A_0^{-1} \\ -I & 0 \end{bmatrix},$$

and so

$$CA^{-1} = \begin{bmatrix} C_0 & 0 \end{bmatrix} = -B^* A^{-*}.$$

Moreover, $\mathbf{G}(0) = 0$ and the conditions of Corollary 3.14 are satisfied.

Example. In the literature there are many examples of systems of the type discussed above (see Luo *et al* [25]). We consider an example in Bontsema [9]. This is an idealized model of a large flexible satellite with a central hub. The partial differential equation model for a beam of length 2 is

$$\rho a \frac{\partial^2 w}{\partial t^2}(t,x) + \bar{E}I \frac{\partial^5 w}{\partial t \partial x^4}(t,x) + EI \frac{\partial^4 w}{\partial x^4}(t,x) = 0,$$

$$\frac{\partial^3 w}{\partial x^3}(-1,t) = 0 = \frac{\partial^3 w}{\partial x^3}(1,t), \quad \frac{\partial^2 w}{\partial x^2}(-1,t) = 0 = \frac{\partial^2 w}{\partial x^2}(1,t),$$

$$u_0(t) = EI[\frac{\partial^3 w}{\partial x^3}(0^+,t) - \frac{\partial^3 w}{\partial x^3}(0^-,t)] + \bar{E}I[\frac{\partial^4 w}{\partial t \partial x^3}(0^+,t) - \frac{\partial^4 w}{\partial t \partial x^3}(0^-,t)],$$

$$u_1(t) = -EI[\frac{\partial^2 w}{\partial x^2}(0^+,t) - \frac{\partial^2 w}{\partial x^2}(0^-,t)] + \bar{E}I[\frac{\partial^3 w}{\partial t \partial x^2}(0^+,t) - \frac{\partial^3 w}{\partial t \partial x^2}(0^-,t)],$$

$$y_0(t) = \frac{\partial w}{\partial t}(0,t), \quad y_1(t) = \frac{\partial^2 w}{\partial t \partial x}(0,t).$$

Here, $w(x,t)$ represents the vertical displacement of the beam at co-ordinate x along the beam at time t, a is the cross-sectional area of the beam, ρ its mass density, E is Young's modulus, I the moment of inertia of the beam per cross-section and $\bar{E}$ is a constant reflecting the stress-strain relation in the beam. $u_0(t)$ is the force and $u_1(t)$ is the moment acting on the centre of the beam at time t. $y_0(t)$ and $y_1(t)$ are the measurements of the velocity, respectively the angular velocity, in the middle of the beam at time t. Although the observation and control operators are admissible, the transfer function grows as $\sqrt{s}$ as $S \to \infty$ along the real axis and so it is not well-posed. Various stabilization schemes for this unstable system were developed in [9], but stabilization using colocated actuators and sensors was not treated. Here show that it can be strongly stabilized using static output feedback. Denote by $H^4(-1,1)$ the Sobolev space

$$H^4(-1,1) = \{f \in L_2(-1,1) \mid \frac{df}{dx}, \frac{df^2}{dx^2}\frac{d^3 f}{dx^3} \in L_2(-1,1)\},$$

where the derivatives are defined in terms of distributions. In [9] it was shown that the operator $A_0 = \frac{d^4}{dx^4}$ with domain

$$\mathcal{D}(A_0) = \{w \in H^4(-1,1) \mid \frac{d^2 w}{dx^2}(-1) = 0, \; \frac{d^3 w}{dx^3}(-1) = 0, \; \frac{d^2 w}{dx^2}(1) = 0, \; \frac{d^3 w}{dx^3}(1) = 0\}$$

is a densely defined, nonnegative, self-adjoint operator on $L_2(-1,1)$. Since it has a double eigenvalue at zero, to fit this example into the framework of (48) with $H := L_2(-1,1)$, $M = A_0$, we need to introduce the modified inner product on the state-space $X = \mathcal{D}(A_0) \oplus L_2(-1,1)$

$$\langle [\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}], [\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}] \rangle_X = \langle z_1, w_1 \rangle + \langle A_0^{\frac{1}{2}} z_1, A_0^{\frac{1}{2}} w_1 \rangle + \langle z_2, w_2 \rangle,$$

where the inner products are in $L_2(-1,1)$. Let $U_0 = \mathbb{C}^2$, $y = \begin{bmatrix} y_0 & y_1 \end{bmatrix}$, $u = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$ and define

$$C_0 w = \begin{bmatrix} \frac{\partial w}{\partial t}(0) & \frac{\partial^2 w}{\partial t \partial x}(0) \end{bmatrix}.$$

Although this example now fits into the framework (48)–(47), due to the eigenvalues at zero, Corollary 27 is not applicable. Instead we use Theorem 3.6 to show that it is impedance passive with the colocated pair u, y. First we compute

$$(sI - A)^{-1} = \; = \begin{bmatrix} V(s)(sI + M) & V(s) \\ -V(s)A_0 & sV(s) \end{bmatrix},$$

where $V(s) = (s^2 I + Ms + A_0)^{-1} \in \mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}})$.

$$(sI - A)^{-1} B = \begin{bmatrix} V(s) C_0^* \\ sV(s) C_0^* \end{bmatrix},$$

$$(sI + A^*)(sI - A)^{-1} B - B^* = (A + A^*)(sI - A)^{-1} B,$$
$$\mathbf{G}(s) = C(sI - A)^{-1} B = B^* sV(s) B.$$

From here,

$$\begin{aligned} &\mathbf{G}(s) + \mathbf{G}(s)^* - B^*(\overline{s}I - A)^{-1} 2\mathrm{Re}s(\overline{s}I - A)^{-1} B \\ = &\; - B^* V(\overline{s}) 2|s|^2 M V(s) B \\ = &\; B^*(\overline{s}I - A^*)^{-1}(A + A^*)(sI - A)^{-1} B. \end{aligned}$$

As in the proof of Proposition 3.15 the inequality in part (a) of Theorem 3.6 can be factored as (29) and the system Σ is impedance passive. This example is unstable due to a double eigenvalue at 0. In [9] it is shown that the spectrum of its generator is countable, it is approximately controllable in infinite time and approximately observable in infinite time. We have seen that it is impedance passive, hence for any $\kappa > 0$ the feedback $u = -\kappa y + v$ results in a strongly stable closed-loop system.

7. The effect of damping and feedthrough on a class of second order systems.

In this section we examine the conditions under which the following second order system will be almost impedance passive.

$$\ddot{q} + M\dot{q} + A_0 q = B_0 u,$$
$$y = C_0 \dot{q},$$

where A_0, M, C_0 are as in Section 6 and $B_0 \in \mathcal{L}(U_0, H_{-\frac{1}{2}})$. As in Section 6 it can be shown that this is equivalent to a compatible system node with everything defined as in Section 6, except for B which is given by

$$B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \text{ and } B^* = \begin{bmatrix} 0 & B_0^* \end{bmatrix}.$$

As before, $\mathbf{G}(0) = 0$. We apply Proposition 3.12 with $\omega = 0$ to obtain conditions for Σ_E to be impedance passive. We compute

$$A^{-1} + A^{-*} = \begin{bmatrix} -2A_0^{-1} M & 0 \\ 0 & 0 \end{bmatrix},$$

$$CA^{-1} + B^*A^{-*} = \begin{bmatrix} C_0 - B_0^* & 0 \end{bmatrix}, \text{ and}$$

$$A^{-*}C^* + A^{-1}B = \begin{bmatrix} A_0^{-1}(C_0^* - B_0) \\ 0 \end{bmatrix}.$$

So Σ_E is impedance passive if and only if

$$2\mathrm{Re}\langle u, (C_0 - B_0^*)x\rangle \le 2\langle Mx, x\rangle - \langle u, Eu\rangle$$

for all $x \in H_{\frac{1}{2}}$, $u \in U_0$. For simplicity suppose that M is invertible. Then by completing the square you can show that with $E = -\frac{1}{4}(C_0 - B_0^*)M^{-1}(C_0^* - B_0)$, Σ_E is impedance passive. So if M is invertible, Σ is almost impedance passive for an arbitrary choice of $C_0 - B_0^*$. The actuators and sensors need not be colocated. Note that colocated actuators and sensors are necessary for Σ to be impedance passive.

8. **A class of damped second order systems with colocated actuators and sensors, but not impedance passive.** In this section we show that colocated actuators and sensors need not imply impedance passivity. We then design a non-colocated input and output pair so that the system is almost impedance passive and we derive an explicit expression for the minimal E for which (2) holds. Following Section 6, we formulate our systems as compatible system nodes. Let U_0, U_1, H, be Hilbert spaces and $H_\mu = \mathcal{D}(A_0^\mu)$, $H_{-\mu} = H_\mu^*$ and M be as in Section 6. We assume that

$$C_0 \in \mathcal{L}(H_{\frac{1}{2}}, U_0), \qquad C_1 \in \mathcal{L}(H_1, U_1).$$

As before, we identify U_0 and U_1 with their duals, so that $C_0^* \in \mathcal{L}(U_0, H_{-\frac{1}{2}})$, $C_1^* \in \mathcal{L}(U_1, H_{-1})$.

We assume that C_0 and C_1 have extensions $\overline{C_0}$ and $\overline{C_1}$ such that the operators

$$\begin{aligned} D_0 &= \overline{C_0}A_0^{-1}C_1^* \in \mathcal{L}(U_1, U_0), \\ D_1 &= \overline{C_1}A_0^{-1}C_0^* \in \mathcal{L}(U_0, U_1) \\ C_2 &= \overline{C_1}A_0^{-1}M \in \mathcal{L}(H_{\frac{1}{2}}, U_1) \end{aligned}$$

exist. Moreover, we assume that C_2 has an extension $\overline{C_2}$ such that the following operator exists

$$D_2 = \overline{C_2}A_0^{-1}C_1^* \in \mathcal{L}(U_1, U_1).$$

We define again $X = H_{\frac{1}{2}} \times H$ and A is defined by (48),

$$C = \begin{bmatrix} 0 & C_0, \\ \overline{C_1} & 2C_2 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 0 & \overline{C_0}, \\ \overline{C_1} & 2\overline{C_2} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & A_0^{-1}C_1^* \\ C_0^* & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & D_0 \\ 0 & D_2 \end{bmatrix},$$

$$D_{col} = \begin{bmatrix} 0 & D_0 \\ 0 & 0 \end{bmatrix},$$

where $B \in \mathcal{L}(U, X_{-1})$, $B^*, C \in \mathcal{L}(X_1, U)$. It is easy to verify that

$$B^* = \begin{bmatrix} 0 & C_0 \\ \overline{C_1} & 0 \end{bmatrix}$$

and it can be shown that the operators $A, B, \overline{C}$ and D and the operators A, B, B^* and D_{col} define compatible system nodes Σ and Σ_{col}, respectively, on (U, X, U). We denote their transfer functions by $\mathbf{G}$, respectively, $\mathbf{G}_{col}$. The state trajectories z

and output functions y of Σ corresponding to C^2 input functions u and compatible initial conditions satisfy

$$\begin{cases} \dot{z} &= Az + Bu\,, \\ y &= \overline{C}z + Du\,. \end{cases} \tag{50}$$

The state trajectories z and output functions y of Σ_{col} corresponding to C^2 input functions u and compatible initial conditions satisfy

$$\begin{cases} \dot{z} &= Az + Bu\,, \\ y &= \overline{B^*}z + D_{col}u\,. \end{cases} \tag{51}$$

As in [11] it is straightforward to prove that (50) is equivalent to the following second order differential equation and the two output equations:

$$\ddot{q} + M\dot{q} + A_0 q = C_0^* u_0 + A_0^{-1} C_1^* \dot{u}_1\,, \tag{52}$$

$$y_0 = \overline{C_0}\,\dot{q}, \qquad y_1 = \overline{C_1}\,q + \overline{C_2}\,\dot{q} \tag{53}$$

for C^2 input signals u_0, u_1 and compatible initial conditions $q(0)$ and $\dot{q}(0)$. Indeed, the connection is provided by

$$z = \begin{bmatrix} q \\ w \end{bmatrix}, \quad u = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}, \quad w = \dot{q} - A_0^{-1} C_1^* u_1\,.$$

Similarly, (51) is equivalent to the same second order differential equation (52) and two output equations:

$$y_0 = \overline{C_0}\,\dot{q}, \qquad y_1 = \overline{C_1}\,q \tag{54}$$

for C^2 input signals u_0, u_1 and compatible initial conditions $q(0)$ and $\dot{q}(0)$. Applying Proposition 3.12 we obtain the following necessary and sufficient conditions for Σ_{col} to be impedance passive:

$$\langle \overline{u}, (\mathbf{G}_{col}(0) + \mathbf{G}(0)^*_{col})\overline{u}\rangle \geq 2\langle x, Mx\rangle + 2\mathrm{Re}\langle C_2 x, u_1\rangle$$

for all $x \in H_{\frac{1}{2}}, u_0 \in U_0, u_1 \in U_1$, where $\overline{u} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$. But

$$\mathbf{G}_{col}(0) + \mathbf{G}_{col}(0)^* = \begin{bmatrix} 0 & D_1^* \\ D_1 & D_2 + D_2^* \end{bmatrix}$$

and so Σ_{col} is not impedance passive. It will be almost impedance passive with

$$2E = \begin{bmatrix} 0 & -D_1^* \\ -D_1 & 2E_{22} - D_2 - D_2^* \end{bmatrix},$$

provided that

$$2\langle u_1, E_{22}u_1\rangle \geq 2\langle x, Mx\rangle + 2\mathrm{Re}\langle C_2 x, u_1\rangle.$$

for all $x \in H_{\frac{1}{2}}, u_1 \in U_1$. If M is invertible, then this is satisfied with $E_{22} \geq -\frac{1}{4}C_2 M^{-1} C_2^*$. Alternatively, if there exists a $Q \in \mathcal{L}(X, U_1)$ such that $QM^{\frac{1}{2}}z = \frac{1}{2}C_2 z$ for $x \in H_{\frac{1}{2}}$, then Σ_E is impedance passive with $E_{22} \geq -Q^*Q$. Otherwise, it is hard to see how to choose E_{11}. Instead we show that by choosing the output as in (53), it is clear how to choose an E so that Σ_E is impedance passive. By applying Corollary 3.14, we obtain sufficient conditions for Σ_E to be impedance passive.

Lemma 8.1. *Let Σ be the compatible system node from (50) and denote its transfer function by $\mathbf{G}$. Then for $E = E^* \in \mathcal{L}(U)$, Σ_E will be impedance passive if and only if*

$$\mathbf{G}(0) + \mathbf{G}(0)^* + 2E \geq 0 .$$

Moreover, the smallest E for which this holds is

$$E = -\frac{1}{2}\left[\begin{array}{cc} 0 & D_1^* \\ D_1 & 0 \end{array}\right]. \tag{55}$$

Proof . First we show that condition (27) holds. We have $0 \in \rho(A)$ and

$$CA^{-1} = \begin{bmatrix} C_0 & 0 \\ C_2 & -C_1 A_0^{-1} \end{bmatrix}$$

and

$$B^* A^{-*} = \begin{bmatrix} -C_0 & 0 \\ -C_2 & C_1 A_0^{-1} \end{bmatrix}.$$

So we have $CA^{-1} + B^* A^{-*} = 0$, and applying Corollary 3.14 to Σ_E proves that it is impedance passive.

The transfer function of Σ is given by $\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B + D$, which is easy to compute in terms of A_0, C_0 and C_1. We have

$$\overline{C}A^{-1}B = \left[\begin{array}{cc} 0 & \overline{C_0}A_0^{-1}C_1^* \\ -\overline{C_1}A_0^{-1}C_0^* & \overline{C_2}A_0^{-1}C_1^* \end{array}\right], \tag{56}$$

and

$$\mathbf{G}(0) + \mathbf{G}(0)^* = \left[\begin{array}{cc} 0 & D_1^* \\ D_1 & 0 \end{array}\right].$$

So the smallest E such that Σ_E is impedance passive is given by (55). □

An example of this type of system with no damping was given in Weiss and Curtain [53]. It was a model of a hinged elastic beam with 2 sensors, one measuring the curvature and one measuring the angular velocity at a point on the beam.

REFERENCES

[1] K. Ammari, Z. Liu and M. Tucsnak, Decay rates for a beam with pointwise force and moment feedback, *Mathematics of Control, Signals and Systems*, **15** (2002), 229–255.

[2] W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. of the American Mathematical Society*, **306** (1988), 837–852.

[3] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser Verlag, Basel, 2001.

[4] T. Bailey and J. E. Hubbard jr., Distributed piezoelectric polymer active vibration control of a cantilever beam, *AIAA Journal on Guidance, Control and Dynamics*, **8** (1985), 605–611.

[5] A. V. Balakrishnan, Compensator design for stability enhancement with collocated controllers, *IEEE Trans. Autom. Control*, **36** (1991), 994–1007.

[6] A. V. Balakrishnan, Shape control of plates with piezo actuators and collocated position / rate sensors, *Applied Math. and Comput.*, **63** (1994), 213–234.

[7] C. J. K. Batty and V. Q. Phong, Stability of individual elements under one-parameter semigroups, *Trans. Amer. Math. Soc.*, **322** (1990), 805–818.

[8] C. D. Benchimol, A note on weak stabilizability of contraction semigroups, *SIAM Journal on Control and Optim.*, **16** (1978), 373–379.

[9] J. Bontsema, *Dynamic Stabilization of Large Flexible Space Structures*, Ph.D.Thesis, Rijksuniversiteit Groningen, The Netherlands, 1989.

[10] R. F. Curtain and G. Weiss, Well-posedness of triples of operators (in the sense of linear systems theory), *Control and Estimation of Distributed Parameter Systems* (F. Kappel, K. Kunisch, W. Schappacher, eds.), 41–59, Birkhäuser-Verlag, Basel, 1989.

[11] R. F. Curtain and G. Weiss, Exponential stabilization of well-posed systems by colocated feedback, *SIAM J. Control and Optim.*, **45** (2006), 273–297.
[12] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.
[13] R. F. Curtain and B. Jacob, Spectral properties of pseudo-resolvents under structured perturbations, *Mathematics of Control, Signals and Systems*, **21** (2008), 21–50.
[14] E. B. Davies, *One-Parameter Semigroups*, Academic Press, London, 1980.
[15] K. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Math. vol. 194, Springer-Verlag, New York, 2000.
[16] J. S. Gibson, A note on stabilization of infinite dimensional linear oscillators by compact feedback, *SIAM J. Control and Optim.*, **18** (1980), 311–316.
[17] Y. Le Gorrec, H. J. Zwart and B. Maschke, Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM J. of Control and Optim.*, **44** (2005), 1864–1892.
[18] B.-Z. Guo and Z.-H. Luo, Controllability and stability of a second-order hyperbolic system with colocated sensor/actuator, *Systems* & *Control Letters*, **46** (2002), 45–65.
[19] A. Haraux, Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps, *Portugaliae Mathematica*, **46** (1989), 245–258.
[20] I. Lasiecka and R. Triggiani, $L_2(\Sigma)$-regular ity of the boundary to boundary operator B^*L for hyperbolic and Petrowski PDE's, *Abstract and Applied Analysis*, **2003** (2003), 1061–1139.
[21] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. II, Abstract Hyperbolic-type Systems over a Finite Time Horizon*, Encyclopedia of Mathematics and its Applications, **75**, Cambridge University Press, Cambridge, 2000.
[22] K. Liu, Local distributed control and damping for the conservative systems, *SIAM J. Control and Optim.*, **35** (1997), 1574–1590.
[23] M. S. Livšic, *Operators, Oscillations, Waves (Open Systems)*, volume 34 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1973.
[24] Y. I. Lyubich and V. Q. Phong, Asymptotic stability of linear differential equations in Banach spaces, *Studia Math.*, **88** (1988), 37–42.
[25] Z.-H. Luo, B.-Z. Guo and O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer-Verlag, London, 1999.
[26] J. Malinen, O. J. Staffans and G. Weiss, When is a linear system conservative?, *Quarterly of Applied Math.*, **64** (2006), 61–91.
[27] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970 (transl. of the French edition of 1967).
[28] J. C. Oostveen, *Strongly Stabilizable Infinite-Dimensional Systems*, Frontiers in Applied Mathematics, SIAM, Philadelphia, 2000.
[29] M. R. Opmeer, Infinite-dimensional linear systems: A distributional approach, *Proc. London Math. Society*, **91** (2005), 738–760.
[30] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford University Press, Oxford, 1985.
[31] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
[32] D. L. Russell, Linear stabilization of the linear oscillator in Hilbert space, *J. Math. and Applications*, **25** (1969), 663–675.
[33] D. Salamon, Infinite dimensional systems with unbounded control and observation: A functional analytic approach, *Trans. Amer. Math. Soc.,* **300** (1987), 383–431.
[34] A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, second (enlarged) edition, Springer-Verlag, London, 2000.
[35] M. Slemrod, A note on complete controllability and stabilizability for linear control systems in Hilbert space, *SIAM J. Control and Optim.*, **12** (1974), 500–508.
[36] M. Slemrod, Stabilization of boundary control systems, *J. of Differential Equations*, **22** (1976), 402–415.
[37] M. Slemrod, Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control, *Math. of Control, Signals and Systems*, **2** (1989), 265–285.
[38] O. J. Staffans, Quadratic optimal control of stable well-posed linear systems, *Trans. American Math. Society*, **349** (1997), 3679–3715.
[39] O. J. Staffans, Passive and conservative continuous-time impedance and scattering systems. Part I: Well-posed systems, *Math. Control, Signals and Systems*, **15** (2002), 291–315.

[40] O. J. Staffans, Stabilization by collocated feedback, *Directions in Mathematical Systems Theory and Optimization*, A. Rantzer and C.I. Byrnes, eds, LNCIS, vol. 286, Springer-Verlag, Berlin, 2003, 275–292.

[41] O. J. Staffans, Passive and conservative infinite-dimensional impedance and scattering systems (from a personal point of view), In *Mathematical Systems Theory in Biology, Communication, Computation, and Finance*, volume 134 of *IMA Volumes in Mathematics and its Applications*, pages 375–413. Springer-Verlag, New York, 2002.

[42] O. J. Staffans, *Well-Posed Linear Systems*, Cambridge University Press, Cambridge, UK, 2005.

[43] O. J. Staffans and G. Weiss, Transfer functions of regular linear systems, Part II: The system operator and the Lax-Phillips semigroup, *Trans. American Math. Society*, **354** (2002), 3229–3262.

[44] V. L. Syrmos, C. T. Abdallah, P. Dorato and K. Grigoriadis, Static output feedback - A survey, *Automatica*, **33** (1997), 125–137.

[45] R. Triggiani, Lack of uniform stabilization for noncontractive semigroups under compact perturbations, *Proc. Amer. Math. Soc.*, **105** (1989), 375–383.

[46] R. Triggiani, Wave equation on a bounded domain with boundary dissipation: an operator approach, *J. Math. Anal. Appl.*, **137** (1989), 438–461.

[47] M. Tucsnak and G. Weiss, How to get a conservative well-posed linear system out of thin air. Part II: Controllability and stability, *SIAM J. Control and Optim.*, **42** (2003), 907–935.

[48] J. A. Villegas, H. Zwart, Y. Le Gorrec, B. Maschke and A. J. van der Schaft, Stability and stabilization of a class of boundary control systems, *Proc. 44th IEEE Conference on Decision and Control and the European Control Conference*, Seville, Spain, (2005), 3850–3855.

[49] G. Weiss, Admissibility of unbounded control operators, *SIAM J. Control and Optim.*, **27** (1989), 527–545.

[50] G. Weiss, Transfer functions of regular linear systems, Part I: Characterizations of regularity, *Trans. Amer. Math. Society*, **342** (1994), 827–854.

[51] G. Weiss, Regular linear systems with feedback, *Mathematics of Control, Signals and Systems*, **7** (1994), 23–57.

[52] G. Weiss, Optimal control of systems with a unitary semigroup and with colocated control and observation, *Systems & Control Letters*, **48** (2003), 329–340.

[53] G. Weiss and R. F. Curtain, Exponential stabilization of a Rayleigh beam using colocated control, *IEEE Trans. on Automatic Control*, **53** (2008), 643–654.

[54] G. Weiss and R. Rebarber, Optimizability and estimatability for infinite-dimensional linear systems, *SIAM J. Control and Optim.*, **39** (2001), 1204–1232.

[55] G. Weiss, O. J. Staffans and M. Tucsnak, Well-posed linear systems -a survey with emphasis on conservative systems, *Int. J. Appl. Math. Comput. Sci.*, **11** (2001), 7–33.

[56] G. Weiss and M. Tucsnak, How to get a conservative well-posed linear system out of thin air. Part I: Well-posedness and energy balance, *ESAIM-COCV*, **9** (2003), 247–274.

[57] G. Weiss and C.-Z. Xu, Spectral properties of infinite-dimensional closed-loop systems, *Mathematics of Control, Signals and Systems*, **17** (2005), 153–172.

[58] J. C. Willems, Dissipative dynamical systems. Part I: General theory. Part II: Linear systems with quadratic supply rates, *Arch. Ration. Mech. Anal.*, **45** (1972), 321–351.

[59] Y. You, Dynamical boundary control of two-dimensional Petrovsky system: Vibrating rectangular plate, In A. Bensoussan and J.-L. Lions, editors, *Analysis and Optimization of Systems*, volume 111 of LNCIS, 519–530. Springer-Verlag, Heidelberg, 1988.

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