Dissipative Systems Theory

van der Schaft, Arjan

Published in:
Communications and Control Engineering

DOI:
10.1007/978-3-319-49992-5_3

Publication date:
2017

Chapter 3
Dissipative Systems Theory

In this chapter the general theory of *dissipative systems* is treated, laying much of the foundation for subsequent chapters. The theory will be shown to provide a state space interpretation of the notions of finite L_2-gain and passivity for input–output maps as discussed in Chaps. 1 and 2, and to generalize the concept of *Lyapunov functions* for autonomous dynamical systems to systems with inputs and outputs.

3.1 Dissipative Systems

Throughout we consider state space systems with inputs and outputs of the general form

$$\Sigma \ : \ \begin{array}{ll} \dot{x} = f(x, u)\,, & u \in U \\ y = h(x, u)\,, & y \in Y \end{array} \tag{3.1}$$

where $x = (x_1, \ldots, x_n)$ are local coordinates for an n-dimensional state space manifold $\mathcal{X}$, and U and Y are linear spaces, of dimension m, respectively p. Throughout this chapter, as well as in the subsequent chapters, we will make the following assumption; see also the discussion in Sect. 1.3.

Assumption 3.1.1 There exists a unique solution trajectory $x(\cdot)$ on the infinite time interval $[0, \infty)$ of the differential equation $\dot{x} = f(x, u)$, for all initial conditions $x(0)$ and all input functions $u(\cdot) \in L_{2e}(U)$. Furthermore it will be assumed that the thus generated output functions $y(\cdot) = h(x(\cdot), u(\cdot))$ are in $L_{2e}(Y)$.

On the combined space $U \times Y$ of inputs and outputs consider a function

$$s \ : \ U \times Y \to \mathbb{R}\,, \tag{3.2}$$

called the *supply rate*. Denote as before $\mathbb{R}^+ = [0, \infty)$.

A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*,
Communications and Control Engineering, DOI 10.1007/978-3-319-49992-5_3

Definition 3.1.2 A state space system Σ is said to be *dissipative* with respect to the supply rate s if there exists a function $S : \mathcal{X} \to \mathbb{R}^+$, called the *storage function*, such that for all initial conditions $x(t_0) = x_0 \in \mathcal{X}$ at any time t_0, and for all allowed input functions $u(\cdot)$ and all $t_1 \geq t_0$ the following inequality holds[1]

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))dt \tag{3.3}$$

If (3.3) holds with *equality* for all x_0, $t_1 \geq t_0$, and all $u(\cdot)$, then Σ is *conservative* with respect to s. Finally, Σ is called *cyclo-dissipative* with respect to s if there exists a function $S : \mathcal{X} \to \mathbb{R}$ (not necessarily nonnegative) such that (3.3) holds.

The inequality (3.3) is called the *dissipation inequality*. It expresses the fact that the "stored energy" $S(x(t_1))$ of Σ at any *future* time t_1 is at most equal to the stored energy $S(x(t_0))$ at *present* time t_0, *plus* the total *externally supplied energy* $\int_{t_0}^{t_1} s(u(t), y(t))dt$ during the time interval $[t_0, t_1]$. Hence, there can be no internal "creation of energy"; only internal *dissipation* of energy is possible.

Remark 3.1.3 Note that cyclo-dissipativity implies

$$\int_{t_0}^{t_1} s(u(t), y(t))dt \geq 0 \tag{3.4}$$

for all trajectories $u(\cdot), x(\cdot), y(\cdot)$ of Σ on the time interval $[t_0, t_1]$ which are such that $x(t_1) = x(t_0)$ (whence the terminology *cyclo*-dissipative).

The two most important choices of supply rates will be seen to correspond to the notions of passivity, respectively finite L_2-gain, as treated for input–output maps in the preceding chapters.

For simplicity of exposition we will identify throughout this chapter the linear input and output spaces U and Y with $\mathbb{R}^m$, respectively $\mathbb{R}^p$, equipped with the standard Euclidean inner product and norm. Throughout the Euclidean inner product of two vectors $v, z \in \mathbb{R}^m$ will be denoted by $v^T z$, and the Euclidean norm of a vector $v \in \mathbb{R}^m$ by $\|v\|$.

Definition 3.1.4 A *state space system* Σ with $U = Y = \mathbb{R}^m$ is *passive* if it is dissipative with respect to the supply rate $s(u, y) = u^T y$. Σ is *input strictly passive* if there exists $\delta > 0$ such that Σ is dissipative with respect to $s(u, y) = u^T y - \delta||u||^2$. Σ is *output strictly passive* if there exists $\varepsilon > 0$ such that Σ is dissipative with respect to $s(u, y) = u^T y - \varepsilon||y||^2$. Finally, Σ is *lossless* if it is conservative with respect to $s(u, y) = u^T y$.

Definition 3.1.4 is directly seen to extend the definitions of (input/output) strict passivity for input–output maps G as given in the previous Chap. 2. Based on Assumption 3.1.1 we consider for every $x_0 \in \mathcal{X}$ the input–output map $G_{x_0} : L_{2e}(U) \to$

[1]Here it is additionally assumed that for allowed input functions $u(\cdot)$ and generated output functions $y(\cdot)$ the integral $\int_{t_0}^{t_1} s(u(t), y(t))dt$ is well defined.

$L_{2e}(Y)$, given as the map from allowed input functions $u(\cdot)$ on $[0, \infty)$ to output functions $y(\cdot)$ on $[0, \infty)$ specified as $y(t) = h(x(t), u(t))$, where $x(t)$ is the state at time $t \geq 0$ resulting from initial condition $x(0) = x_0$ and input function $u(\cdot)$. Note that all these input–output maps G_{x_0}, $x_0 \in \mathcal{X}$, are *causal* (Definition 1.1.3), and *time invariant* (Definition 1.1.5).

Proposition 3.1.5 *Consider the state space system Σ with $U = Y = \mathbb{R}^m$, and consider for any x_0 the input–output map G_{x_0}. If Σ is passive, input strictly passive, respectively output strictly passive, then so are the input–output maps G_{x_0} for every x_0.*

Proof Suppose Σ is dissipative with respect to the supply rate $s(u, y) = u^T y$. Then for some function $S \geq 0$

$$\int_0^T u^T(t)y(t)dt \;\geq\; S(x(T)) - S(x(0)) \;\geq\; -S(x(0)) \tag{3.5}$$

for all $x(0) = x_0$, and all $T \geq 0$ and all input functions $u(\cdot)$. This means precisely that the input–output maps G_{x_0} of Σ, for every $x_0 \in \mathcal{X}$, are passive in the sense of Definition 2.2.1 (with bias β given as $S(x_0)$). The (input or output) strict passivity case follows similarly.

□

A second important class of supply rates is

$$s(u, y) = \frac{1}{2}\gamma^2||u||^2 - \frac{1}{2}||y||^2 \;, \qquad \gamma \;\geq\; 0\,, \tag{3.6}$$

where $||u||$ and $||y||$ denote the Euclidian norms on $U = \mathbb{R}^m$, respectively $Y = \mathbb{R}^p$.

Definition 3.1.6 A *state space system* Σ with $U = \mathbb{R}^m, \; Y = \mathbb{R}^p$ has L_2-*gain* $\leq \gamma$ if it is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2}\gamma^2||u||^2 - \frac{1}{2}||y||^2$. The L_2-gain of Σ is defined as $\gamma(\Sigma) := \inf\{\gamma \mid \Sigma \text{ has } L_2\text{-gain} \leq \gamma\}$. Σ is said to have L_2-gain $< \gamma$ if there exists $\tilde{\gamma} < \gamma$ such that Σ has L_2-gain $\leq \tilde{\gamma}$. Finally Σ is called *inner* if it is conservative with respect to $s(u, y) = \frac{1}{2}||u||^2 - \frac{1}{2}||y||^2$.

Definition 3.1.6 is immediately seen to extend the definition of finite L_2-gain from Chaps. 1 and 2.

Proposition 3.1.7 *Suppose Σ is dissipative with respect to $s(u, y) = \frac{1}{2}\gamma^2||u||^2 - \frac{1}{2}||y||^2$ for some $\gamma > 0$. Then all input–output maps $G_{x_0} : L_{2e}(U) \to L_{2e}(Y)$ have L_2-gain $\leq \gamma$. Furthermore, the infimum of the L_2-gains of G_{x_0} over all x_0 is equal to the L_2-gain of Σ.*

Proof If Σ is dissipative with respect to $s(u, y) = \frac{1}{2}\gamma^2||u||^2 - \frac{1}{2}||y||^2$ then there exists $S \geq 0$ such that for all $T \geq 0,\; x(0)$, and $u(\cdot)$

$$\frac{1}{2}\int_0^T (\gamma^2||u(t)||^2 - ||y(t)||^2)dt \geq S(x(T)) - S(x(0))|! \geq S(x(0)) \tag{3.7}$$

and thus

$$\int_0^T ||y(t)||^2 dt \ \leq \ \gamma^2 \int_0^T ||u(t)||^2 dt + 2S(x(0)) \tag{3.8}$$

This implies by Proposition 1.2.7 that the input–output maps G_{x_0} for every initial condition $x(0) = x_0$ have L_2-gain $\leq \gamma$. The rest of the statements follows directly.

□

Remark 3.1.8 Note that by considering supply rates $s(u, y) = \tilde{\gamma}||u||^q - ||y||^q$ we may also treat L_q-gain for $q \neq 2$; this will not be further discussed.

In the subsequent chapters we will elaborate on the special cases $s(u, y) = u^T y$ and $s(u, y) = \frac{1}{2}\gamma^2||u||^2 - \frac{1}{2}||y||^2$ corresponding to, respectively, passivity (Chap. 4) and finite L_2-gain (Chap. 8), in much more detail. Instead, in the current chapter we will focus on the *general* theory of dissipative systems.

Before doing so we mention one immediate generalization of the definition of dissipativity. In Chaps. 1 and 2 we already noticed that the notions of finite L_2-gain and passivity can be extended from input–output *maps* to *relations*. In the same vein, the definition of dissipativity for input–state–output systems Σ as in (3.1) can be extended to state space systems described by a mixture of differential *and* algebraic equations, where we do not distinguish a priori between input and output variables. That is, we may consider systems of the general form

$$F(x, \dot{x}, w) = 0, \tag{3.9}$$

where $x = (x_1, \ldots, x_n)$ are local coordinates for an n-dimensional state space manifold $\mathcal{X}$, and $w \in W = \mathbb{R}^s$ denotes the total vector of *external variables*. Note that this entails two generalizations of (3.1): (i) we replace the combined vector $(u, y) \in Y \times U$ by a vector $w \in W$ (where we do not make an a priori splitting into input and output variables), and (ii) we replace the explicit differential and algebraic equations in (3.1) by a general mixture, called a set of *differential–algebraic equations* (DAEs). Note that systems (3.9) include *implicit* and *constrained* state space systems. In this more general context the supply rate s is now simply defined as a function

$$s : W \to \mathbb{R},$$

while the DAE system (3.9) is called dissipative with respect to s if there exists a function $S : \mathcal{X} \to \mathbb{R}^+$ such that

$$S(x(t_1)) \ \leq \ S(x(t_0)) + \int_{t_0}^{t_1} s(w(t))dt, \tag{3.10}$$

for all[2] solutions $x(\cdot)$, $w(\cdot)$ of (3.9).

[2]Here we naturally restrict to continuous solutions.

Let us now return to the definition of dissipativity given in Definition 3.1.2. First, we notice that in general the storage function of a dissipative system is far from unique. Nonuniqueness already arises from the fact that we may always add a nonnegative constant to a storage function, and so obtain another storage function. Indeed, the dissipation inequality (3.3) is invariant under addition of a constant to the storage function S. However, apart from this rather trivial nonuniqueness, more often than not dissipative systems will admit really *different* storage functions. Furthermore, if S_1 and S_2 are storage functions then any convex combination $\alpha S_1 + (1-\alpha)S_2$, $\alpha \in [0, 1]$, is also a storage function, as immediately follows from substitution into the dissipation inequality. Hence the set of storage functions is always a *convex* set.

The storage function *is* guaranteed to be unique (up to a constant) in case the system is *conservative* and a controllability condition is met, as formulated in the following proposition.

Proposition 3.1.9 *Consider a system Σ that is conservative with respect to some supply rate s. Assume that the system is "connected" in the sense that for every two states x_a, x_b, there exists a number of intermediate states $x_1, x_2, \ldots, x_m$ with $x_1 = x_a, x_m = x_b$, such that for every pair x_i, x_{i+1} either x_i can be steered (by the application of a suitable input function) to x_{i+1}, or, conversely, x_{i+1} can be steered to x_i, $i = 1, 2\ldots, m$. Then the storage function is unique up to a constant.*

Proof Let S^1, S^2 be two storage functions. By the dissipation equality, the difference $S^2 - S^1$ is constant along any state trajectory of the system. By the above property of "connectedness" this constant is the same for every state trajectory. Hence $S^2 = S^1$ up to this constant. □

Remark 3.1.10 A simple physical example of a dissipative system that is not conservative, but still has unique (up to a constant) storage function will be provided in Example 4.1.7 in Chap. 4.

A fundamental question is how we may *decide* if Σ is dissipative with respect to a given supply rate s. The following theorem gives an intrinsic *variational* characterization of dissipativity.

Theorem 3.1.11 *Consider the system Σ and supply rate $s(u, y)$. Then Σ is dissipative with respect to s if and only if*

$$S_a(x) := \sup_{\substack{u(\cdot)\\ T\geq 0}} \; -\int_0^T s(u(t), y(t))dt\,, \qquad x(0) = x, \tag{3.11}$$

is finite ($<\infty$) for all $x \in \mathcal{X}$. Furthermore, if S_a is finite for all $x \in \mathcal{X}$ then S_a is a storage function, called the available storage, and all other possible storage functions S satisfy

$$S_a(x) \;\leq\; S(x) - \inf_x S(x)\,, \qquad \forall\, x \in \mathcal{X} \tag{3.12}$$

Moreover,

$$\inf_x S_a(x) = 0 \tag{3.13}$$

Proof Suppose S_a is finite. Clearly $S_a \geq 0$ (take $T = 0$ in (3.11)). Compare now $S_a(x(t_0))$ with $S_a(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t))dt$, for a given $u : [t_0, t_1] \to \mathbb{R}^m$ and resulting state $x(t_1)$. Since S_a is given as the *supremum* over all $u(\cdot)$ in (3.11) it immediately follows that

$$S_a(x(t_0)) \ \geq \ S_a(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t))dt \,, \tag{3.14}$$

and thus S_a is a storage function, proving that Σ is dissipative with respect to the supply rate s.

Suppose conversely that Σ is dissipative. Then there exists $S \geq 0$ such that for all $u(\cdot)$

$$S(x(0)) + \int_0^T s(u(t), y(t))dt \ \geq \ S(x(T)) \ \geq \ 0 \,, \tag{3.15}$$

which shows that

$$S(x(0)) \ \geq \ \sup - \int_0^T s(u(t), y(t))dt \ = \ S_a(x(0)) \,, \tag{3.16}$$

proving finiteness of S_a. On the other hand, $S' := S - \inf_x S(x)$ is a storage function as well (since we have just subtracted the constant $\inf_x S(x)$ from S and thus the dissipation inequality remains to hold, while clearly $S' \geq 0$). Hence also $S'(x_0) \geq S_a(x_0)$ for all x_0, proving (3.12). Moreover, since $\inf_x S'(x) = 0$, also $\inf_x S_a(x) = 0$. □

The quantity $S_a(x_0)$ can be interpreted as the maximal "energy" which can be extracted from the system Σ starting at initial condition x_0. Theorem 3.1.11 thus states that Σ is dissipative if and only if this "extractable energy" is finite for every initial condition.

Under additional conditions the following equivalent characterizations of the available storage S_a can be obtained.

Proposition 3.1.12 (i) *Assume the system Σ and supply rate $s(u, y)$ are such that for any x there exists $u(x)$ such that*

$$s(u(x), h(x, u(x))) \leq 0, \qquad x \in \mathcal{X} \tag{3.17}$$

Then

$$S_a(x) = \sup_{u(\cdot)} \ - \int_0^\infty s(u(t), y(t))dt, \qquad x(0) = x \tag{3.18}$$

(ii) Assume Σ *and* $s(u, y)$ *are such that there exists a state feedback* $u(x)$ *such that* (3.17) *holds, while furthermore* x^* *is a globally asymptotically stable equilibrium of the closed-loop system*[3] $\dot{x} = f(x, u(x))$. *Then*

$$S_a(x) = \sup_{u(\cdot),\, x \to x^*} - \int_0^\infty s(u(t), y(t))dt, \qquad x(0) = x \tag{3.19}$$

Proof (i) By letting $T \to \infty$ in (3.11) we have $\sup_{u(\cdot)} - \int_0^\infty s(u(t), y(t))dt \leq S_a(x)$. Conversely, note that

$$\begin{aligned} - \int_0^\infty s(u(t), y(t))dt &= - \int_0^T s(u(t), y(t))dt - \int_T^\infty s(u(t), y(t))dt \\ &\geq - \int_0^T s(u(t), y(t))dt\,, \end{aligned} \tag{3.20}$$

whenever $u(\cdot)$ is such that $s(u(t), y(t)) \leq 0$ for all $t \in [T, \infty)$. Hence for any $\bar{u} : [0, T] \to U$ there exists $u : [0, \infty) \to U$ with $u_T = \bar{u}$ such that $- \int_0^\infty s(u(t), y(t))dt \geq - \int_0^T s(\bar{u}(t), y(t))dt$. Therefore, by taking the supremum at both sides of this inequality we obtain the inequality $\sup_{u(\cdot)} - \int_0^\infty s(u(t), y(t))dt \geq S_a(x)$.

(ii) As in the proof of part (i) we have $\sup_{u(\cdot),\, x \to x^*} - \int_0^\infty s(u(t), y(t))dt \leq S_a(x)$. For the reverse inequality we apply the same reasoning as in the proof of part (i), by considering extensions of $\bar{u} : [0, T] \to U$ to $u : [0, \infty) \to U$ which are such that $x(t) \to x^*$ for $t \to \infty$. □

Remark 3.1.13 Note that part (i) of Proposition 3.1.12 applies to the (input strict or output strict) passivity supply rate and to the L_2-gain supply rate by taking $u = 0$. Furthermore, part (ii) applies whenever Σ has a globally asymptotically stable equilibrium x^* for $u = 0$.

The next proposition shows that if the system is *reachable* from some state, then the finiteness of extractable energy needs only to be checked for this initial condition.

Proposition 3.1.14 *Assume that* Σ *is* reachable *from* $x^* \in \mathcal{X}$. *Then* Σ *is dissipative if and only if* $S_a(x^*) < \infty$.

Proof (*Only if*) Trivial. (If) Suppose there exists $x \in \mathcal{X}$ such that $S_a(x) = \infty$. Since by reachability we can steer x^* to x in finite time, this would imply (using time invariance) that also $S_a(x^*) = \infty$. □

Corollary 3.1.15 *Assume that* Σ *is reachable from* $x^* \in \mathcal{X}$. *Then* Σ *is passive if and only if the input–output map* G_{x^*} *is passive, and* Σ *has* L_2*-gain* $\leq \gamma$ *if and only if* G_{x^*} *has* L_2*-gain* $\leq \gamma$, *while* $\gamma(\Sigma) = \gamma(G_{x^*})$. *Furthermore, if* G_{x^*} *is passive with zero bias or has* L_2*-gain* $\leq \gamma$ *with zero bias, then* $S_a(x^*) = 0$.

[3] Here it is assumed that $\dot{x} = f(x, u(x))$ has unique solutions on $[0, \infty)$ for all initial conditions.

Proof Suppose the input–output map G_{x^*} is passive, then $\exists\ \beta < \infty$ such that (cf. Definition 2.2.1)

$$\int_0^T u^T(t)y(t)dt \ \geq\ -\beta \tag{3.21}$$

for all $u(\cdot),\ T \geq 0$. Therefore

$$S_a(x^*) = \sup_{u(\cdot),\,T\geq 0}\ -\int_0^T u^T(t)y(t)dt\ \leq\ \beta\ <\ \infty\,, \qquad x(0) = x^* \tag{3.22}$$

and by Proposition 3.1.14 Σ is passive. If $\beta = 0$ then $S_a(x^*) = 0$.

Similarly, let G_{x^*} have L_2-gain $\leq \gamma$, then (cf. Proposition 1.2.7) for all $\tilde{\gamma} > \gamma$ there exists a constant c such that

$$\int_0^T ||y(t)||^2 dt\ \leq\ \tilde{\gamma}^2 \int_0^T ||u(t)||^2 dt + c \tag{3.23}$$

yielding (with $x(0) = x^*$)

$$S_a(x^*)\ =\ \sup_{u(\cdot),\,T\geq 0}\ -\int_0^T \left(\frac{1}{2}\tilde{\gamma}^2||u(t)||^2 - \frac{1}{2}||y(t)||^2\right) dt\ \leq\ \frac{c}{2}\,, \tag{3.24}$$

implying that Σ has L_2-gain $\leq \tilde{\gamma}$ for all $\tilde{\gamma} > \gamma$. If $c = 0$, then clearly $S_a(x^*) = 0$. It also follows that $\gamma(\Sigma) = \gamma(G_{x^*})$. □

If Σ is reachable from a state x^* then, in addition to the available storage S_a, there exists another canonically defined storage function. Contrary to the available storage, which is the *minimal* storage function (see (3.12)), this storage function has a *maximality* property, in the following sense.

Theorem 3.1.16 *Assume that Σ is reachable from $x^* \in \mathcal{X}$. Define the required supply (from x^*) $S_r : \mathcal{X} \to \mathbb{R} \cup \{-\infty\}$ as*

$$S_r(x) := \inf_{u(\cdot),\,T\geq 0} \int_{-T}^0 s(u(t), y(t))dt\ , \quad x(-T) = x^*,\ x(0) = x \tag{3.25}$$

Then S_r satisfies the dissipation inequality (3.3). Furthermore, Σ is dissipative if and only if there exists $K > -\infty$ such that $S_r(x) \geq K$ for all $x \in \mathcal{X}$. Moreover, if S is a storage function for Σ, then

$$S(x)\ \leq\ S_r(x) + S(x^*)\,, \qquad \forall x \in \mathcal{X}, \tag{3.26}$$

and $S_r(x) + S(x^*)$ is itself a storage function. In particular, $S_r(x) + S_a(x^*)$ is a storage function.

Proof The fact that S_r satisfies the dissipation inequality (3.3) follows from the variational definition of S_r in (3.25). Indeed, in taking the system from x^* at $t = -T$ to $x(t_1)$ at time t_1 we can restrict to those input functions $u(\cdot) : [-T, t_1] \to U$ which first take x^* to $x(t_0)$ at time $t_0 \leq t_1$, and then are equal to a given input $u(\cdot) : [t_0, t_1] \to U$ transferring $x(t_0)$ to $x(t_1)$. This will a be suboptimal control policy, whence

$$S_r(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))dt \;\geq\; S_r(x(t_1)) \tag{3.27}$$

For the second claim, note that by definition of S_a and S_r

$$S_a(x^*) = \sup_x -S_r(x), \tag{3.28}$$

from which by Proposition 3.1.14 it follows that Σ is dissipative if and only if $\exists\, K > -\infty$ such that $S_r(x) \geq -K$ for all x.

Finally, let S satisfy the dissipation inequality (3.3). Then for any $u(\cdot) : [-T, 0] \to \mathbb{R}^m$ transferring $x(-T) = x^*$ to $x(0) = x$ we have by the dissipation inequality

$$S(x) - S(x^*) \;\leq\; \int_{-T}^{0} s(u(t), y(t))dt \tag{3.29}$$

Taking the infimum on the right-hand side over all those $u(\cdot)$ yields (3.26). Furthermore if $S \geq 0$, then by (3.26) $S_r + S(x^*) \geq 0$, and by adding $S(x^*)$ to both sides of (3.27) it follows that also $S_r + S(x^*)$ satisfies the dissipation inequality. □

Remark 3.1.17 Let Σ be reachable from x^*. Then under the additional assumption of existence of u^* such that $f(x^*, u^*) = 0,\ h(x^*, u^*) = 0$ it can be verified that the required supply is equivalently given as

$$S_r(x) = \lim_{t_1 \to -\infty} \inf_{u(\cdot),\, x(t_1)=x^*,\, x(0)=x} \int_{t_1}^{0} s(u(t), y(t))dt \tag{3.30}$$

Furthermore, we note that in case Σ is dissipative with a storage function S which attains its global *minimum* at some point $x^* \in \mathcal{X}$, then also $S - S(x^*)$ will be a storage function, which is *zero* at x^*. Hence in this case any motion starting from x^* at time 0 satisfies by the dissipation inequality

$$\int_0^T s(u(t), y(t))dt \;\geq\; 0\,, \qquad x(0) = x^*,\ \forall T \geq 0 \tag{3.31}$$

Thus if we start from the state of "minimal energy" x^* then the net supply flow is always directed *into* the system.[4] This leads to the following *alternative* definition of dissipativity.

Definition 3.1.18 Consider a system Σ and supply rate s. The system is called *dissipative from* x^* if (3.31) holds.

Proposition 3.1.19 *Let Σ be dissipative with storage function S satisfying $S(x^*) = 0$. Then the system is also dissipative from x^*. Conversely, if the system is dissipative from x^* then $S_a(x^*) = 0$. If additionally the system is reachable from x^* then the system is dissipative, while its required supply satisfies $S_r(x^*) = 0$.*

Proof The fact that dissipativity with storage function S satisfying $S(x^*) = 0$ implies (3.31) was already observed in (3.31). Conversely, assume that the system is dissipative from x^*. Then by definition of S_a in (3.11) it directly follows that $S_a(x^*) = 0$. Furthermore by Proposition 3.1.14 it follows that the system is dissipative, while clearly $S_r(x^*) = 0$. □

Hence, if Σ is dissipative as well dissipative from x^*, then both S_a and S_r attain their minimum 0 at x^*. Under an additional assumption it can be shown that all other storage functions attain their minimum at x^* as well, as formulated in the following proposition.

Proposition 3.1.20 *Let Σ be dissipative and dissipative from x^*. Suppose furthermore the supply rate s is such that there exists a feedback $u(x)$ satisfying (3.17) for which x^* is a globally asymptotically equilibrium for the closed-loop system $\dot{x} = f(x, u(x))$. Then any storage function S attains its minimum at x^*, implying that $S(x) - S(x^*)$ is a storage function that is zero at x^*. Furthermore*

$$S_a(x) \leq S(x) - S(x^*), \qquad x \in \mathcal{X} \tag{3.32}$$

Proof Consider the dissipation inequality for any storage function S, rewritten as

$$-\int_0^T s(u(t), y(t))dt \leq S(x) - S(x(T)) \tag{3.33}$$

with $x(0) = x$. Extend $u(\cdot) : [0, T] \rightarrow U$ to the infinite time interval $[0, \infty)$ by considering on (T, ∞) a feedback $u(x)$ as in (3.17) such that x^* is a globally asymptotically equilibrium $\dot{x} = f(x, u(x))$. It follows from (3.20) and convergence of $x(t)$ to x^* for $t \rightarrow \infty$ that

$$-\int_0^T s(u(t), y(t))dt \leq S(x) - S(x^*) \tag{3.34}$$

[4] Note however that there does not always exist such a state of minimal internal energy. In particular $\inf_x S_a(x) = 0$ but not necessarily the minimum is attained.

Hence by taking the supremum at the left-hand side over all $u(\cdot) : [0, T] \to U$ and $T \geq 0$ we obtain (3.32), also implying that S attains its minimum at x^*. □

The above developments can be summarized as follows.

Corollary 3.1.21 *Consider a system (3.1) that is dissipative from x^*, reachable from x^*, and for which there exists a feedback $u(x)$ satisfying (3.17) such that x^* is a globally asymptotically equilibrium of $\dot{x} = f(x, u(x))$. Then any storage function S attains its minimum at x^* and the storage function $S'(x) := S(x) - S(x^*)$ satisfies*

$$S_a(x) \leq S'(x) \leq S_r(x), \ \textit{for all } x \in \mathcal{X}, \tag{3.35}$$

where $S_a(x^*) = S_r(x^*) = 0$.

Remark 3.1.22 For a linear system $\dot{x} = Ax + Bu,\ y = Cx + Du$ with $x^* = 0$ satisfying the assumptions of Corollary 3.1.21 it can be proved by standard optimal control arguments [351] that S_a and S_r are given by quadratic functions $\frac{1}{2}x^T Q_a x$, respectively $\frac{1}{2}x^T Q_r x$, with Q_a, Q_r symmetric matrices satisfying $Q_a \leq Q_r$.

3.2 Stability of Dissipative Systems

In this section we will elaborate on the close connection between dissipative systems theory and the theory of *Lyapunov functions* for autonomous dynamical systems $\dot{x} = f(x)$.

Consider the dissipation inequality (3.3), where we assume throughout this section that the storage functions S are C^1 (continuously differentiable); see the discussion in the Notes for this chapter for generalizations. By dividing the dissipation inequality by $t_1 - t_0$, and letting $t_1 \to t_0$ we see that (3.3) is equivalent to

$$S_x(x) f(x, u) \ \leq \ s(u, h(x, u)) \ , \qquad \text{for all } x, u, \tag{3.36}$$

with $S_x(x)$ denoting the *row vector* of partial derivatives

$$S_x(x) = \left(\frac{\partial S}{\partial x_1}(x), \ldots, \frac{\partial S}{\partial x_n}(x) \right) \tag{3.37}$$

The inequality (3.36) is called the *differential dissipation inequality*, and is much easier to check than (3.3) since we do not have to compute the system trajectories (which for most nonlinear systems is even not possible).

In order to make the connection with the theory of Lyapunov functions we recall some basic notions and results from Lyapunov stability theory. Consider the set of differential equations

$$\dot{x} = f(x) \tag{3.38}$$

Here x are local coordinates for an n-dimensional manifold $\mathcal{X}$, and thus (3.38) is the local coordinate expression of a *vector field* on $\mathcal{X}$. Throughout we assume that f is locally Lipschitz continuous; implying existence and uniqueness of solutions of (3.38), at least for small time. The solution of (3.38) for initial condition $x(0) = x_0$ will be denoted as $x(t; x_0)$, with $t \in [0, T(x_0))$ and $T(x_0) > 0$ maximal.

Definition 3.2.1 Let x^* be an equilibrium of (3.38), that is $f(x^*) = 0$, and thus $x(t; x^*) = x^*$, for all t. The equilibrium x^* is

(a) *stable*, if for each $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that

$$\| x_0 - x^* \| < \delta(\varepsilon) \Rightarrow \| x(t; x_0) - x^* \| < \varepsilon, \quad \forall t \geq 0 \tag{3.39}$$

(b) *asymptotically stable*, if it is stable and additionally there exists $\bar{\delta} > 0$ such that

$$\| x_0 - x^* \| < \bar{\delta} \Rightarrow \lim_{t \to \infty} x(t, x_0) = x^* \tag{3.40}$$

(c) *globally asymptotically stable*, if it is stable and $\lim_{t \to \infty} x(t; x_0) = x^*$ for all $x_0 \in \mathcal{X}$.

(d) *unstable*, if it is not stable.

Remark 3.2.2 If x^* is a globally asymptotically stable equilibrium then necessarily $\mathcal{X}$ is diffeomorphic to $\mathbb{R}^n$.

An important tool in the stability analysis of equilibria are *Lyapunov functions*.

Definition 3.2.3 Let x^* be an equilibrium of (3.38). A C^1 function $V : \mathcal{X} \to \mathbb{R}^+$ satisfying

$$V(x^*) = 0, \quad V(x) > 0, \quad x \neq x^* \tag{3.41}$$

(that is, V is *positive definite* at x^*), as well as

$$\dot{V}(x) := V_x(x) f(x) \leq 0, \quad x \in \mathcal{X}\,, \tag{3.42}$$

is called a *Lyapunov function* for the equilibrium x^*.

Theorem 3.2.4 *Let x^* be an equilibrium of (3.38). If there exists a Lyapunov function for the equilibrium x^*, then x^* is a stable equilibrium. If moreover*

$$\dot{V}(x) < 0, \quad \forall x \in \mathcal{X}, \; x \neq x^*, \tag{3.43}$$

then x^* is an asymptotically stable equilibrium, which is globally asymptotically stable if V is proper (that is, the sets $\{x \in \mathcal{X} \mid 0 \leq V(x) \leq c\}$ are compact for every $c \in \mathbb{R}^+$).

Remark 3.2.5 Theorem 3.2.4 can be also applied to any neighborhood $\tilde{\mathcal{X}}$ of x^*. In particular, if (3.41) and (3.42), or (3.41) and (3.43) hold on a neighborhood of x^*, then x^* is still a stable, respectively, asymptotically stable, equilibrium.

Remark 3.2.6 For $\mathcal{X} = \mathbb{R}^n$ the requirement of properness amounts to V being *radially unbounded*; that is, $V(x) \to \infty$ whenever $\|x\| \to \infty$.

With the aid of Theorem 3.2.4 the following stability result for dissipative systems is readily established.

Proposition 3.2.7 *Let $s(u, y)$ be a supply rate, and $S : \mathcal{X} \to \mathbb{R}^+$ be a C^1 storage function for Σ. Assume that s satisfies*

$$s(0, y) \;\leq\; 0\,, \qquad \forall y \in Y \tag{3.44}$$

Assume furthermore that $x^* \in \mathcal{X}$ is a strict local minimum for S. Then x^* is a stable equilibrium of the unforced system $\dot{x} = f(x, 0)$ with Lyapunov function $V(x) := S(x) - S(x^*)$ for x around x^*, while $s(0, h(x^*, 0)) = 0$. If additionally, $\dot{S}(x) < 0$, for all $x \neq x^*$, then x^* is an asymptotically stable equilibrium.

Proof By (3.36) and (3.44) $S_x(x) f(x, 0) \leq s(0, h(x, 0)) \leq 0$, and thus S is nonincreasing along solutions of $\dot{x} = f(x, 0)$. Since S has a strict minimum at x^* this implies $f(x^*, 0) = 0$, and thus $s(0, h(x^*, 0)) = 0$. The rest follows directly from Theorem 3.2.4. □

An important weakness in the asymptotic stability statement of Proposition 3.2.7 concerns the condition $\dot{S}(x) < 0$ for all $x \neq x^*$. In general, this condition can*not* be inferred from the dissipation inequality (unless e.g., $y = x$). An important generalization of Theorem 3.2.4 to remedy this weakness is based on *LaSalle's Invariance principle*. Recall that a set $\mathcal{N} \subset \mathcal{X}$ is *invariant* for $\dot{x} = f(x)$ if $x(t; x_0) \in \mathcal{N}$ for all $x_0 \in \mathcal{N}$ and for all $t \in \mathbb{R}$, and is *positively invariant* if this holds for all $t \geq 0$, where $x(t; x_0),\ t \geq 0$, denotes the solution of $\dot{x} = f(x)$ for $x(0) = x_0$.

Theorem 3.2.8 *Let $V : \mathcal{X} \to \mathbb{R}$ be a C^1 function for which $\dot{V}(x) := V_x(x) f(x) \leq 0$, for all $x \in \mathcal{X}$. Suppose there exists a compact set $\mathcal{C}$ which is positively invariant for $\dot{x} = f(x)$. Then for any $x_0 \in \mathcal{C}$ the solution $x(t; x_0)$ converges for $t \to \infty$ to the largest subset of $\{x \in \mathcal{X} \mid \dot{V}(x) = 0\} \cap \mathcal{C}$ that is invariant for $\dot{x} = f(x)$.*

The usual way of applying Theorem 3.2.8 is as follows. Since $\dot{V}(x) \leq 0$, the connected component of $\{x \in \mathcal{X} \mid V(x) \leq V(x_0)\}$ containing x_0 is positively invariant. If additionally V is assumed to be *positive definite* at x^* then the connected component of $\{x \in \mathcal{X} \mid V(x) \leq V(x_0)\}$ containing x_0 will be compact for x_0 close enough to x^*, and hence may serve as the compact set $\mathcal{C}$ in the above theorem.

Using this reasoning Theorem 3.2.8 yields the following connection between dissipativity and asymptotic stability.

Proposition 3.2.9 *Let $S : \mathcal{X} \to \mathbb{R}^+$ be a C^1 storage function for Σ. Assume that the supply rate s satisfies*

$$s(0, y) \;\leq\; 0\,, \qquad \textit{for all } y \tag{3.45}$$

Assume that $x^* \in \mathcal{X}$ is a strict local minimum for S. Furthermore, assume that no solution of $\dot{x} = f(x, 0)$ other than $x(t) \equiv x^*$ remains in $\{x \in \mathcal{X} \mid s(0, h(x, 0)) = 0\}$ for all t. Then x^* is an asymptotically stable equilibrium of $\dot{x} = f(x, 0)$, which is globally asymptotically stable if $V \geq 0$ is proper.

Proof Note that $\dot{S}(x) = 0$ implies $s(0, h(x, 0)) = 0$, which by assumption implies $h(x, 0) = 0$. The statement now directly follows from LaSalle's Invariance principle. □

Remark 3.2.10 The requirement $s(0, y) \leq 0$ for all y is satisfied by the (output and input strict) passivity and L_2-gain supply rates.

A main condition on the storage function S in the previous statements is the requirement that S has a *strict* (local) minimum at the equilibrium x^*. This is *not* part of the standard definition of a storage function. On the other hand, in case $S(x^*) = 0$ (see also Proposition 3.1.20), then the property of a strict (local) minimum may be sometimes *derived* making use of an additional *observability* condition.

In the rest of this section we assume that Σ has no *feedthrough terms*, i.e., $y = h(x)$. Without loss of generality take $x^* = 0$. Moreover, assume $h(0) = 0$.

Definition 3.2.11 Σ with $y = h(x)$ is *zero-state observable* if $u(t) = 0$, $y(t) = 0, \forall t \geq 0$, implies $x(t) = 0, \forall t \geq 0$.

Proposition 3.2.12 *Let $S \geq 0$ be a C^1 storage function with $S(0) = 0$ for a supply rate s satisfying $s(0, y) \leq 0$ for all y, and such that $s(0, y) = 0$ implies $y = 0$. Suppose Σ_a is zero-state observable, then $S(x) > 0$ for all $x \neq 0$.*

Proof By substituting $u = 0$ in (3.3) we obtain

$$S(x(T)) - S(x(0)) \leq \int_0^T s(0, y(t))dt$$

implying, since $S(x(T)) \geq 0$,

$$S(x(0)) \geq \int_0^T s(0, y(t))dt$$

which is > 0 for $x(0) \neq 0$. □

Remark 3.2.13 The same result follows for any supply rate $s(u, y)$ for which there exists an output feedback $u = \alpha(y)$ such that $s(\alpha(y), y) \leq 0$ for all y, and $s(\alpha(y), y) = 0$ implies $y = 0$. Just consider in the above proof $u = \alpha(y)$ instead of $u = 0$.

Remark 3.2.14 Note that the L_2-gain and output strict passivity supply rate satisfy the conditions in the above Proposition 3.2.12, while the passivity supply rate satisfies the conditions of Remark 3.2.13 (take $u = -y$).

A weaker property of observability, called *zero-state detectability*, is instrumental for proving asymptotic stability based on LaSalle's Invariance principle.

Definition 3.2.15 Σ_a is *zero-state detectable* if $u(t) = 0,\ y(t) = 0,\ \forall t \geq 0$, implies $\lim_{t\to\infty} x(t) = 0$.

Proposition 3.2.16 *Let S be a C^1 storage function with $S(0) = 0$ and $S(x) > 0$, $x \neq 0$, for a supply rate s satisfying $s(0, y) \leq 0$ and such that $s(0, y) = 0$ implies $y = 0$, where $h(0) = 0$. Suppose that Σ_a is zero-state detectable. Then $x = 0$ is an asymptotically stable equilibrium of $\dot{x} = f(x, 0)$. If additionally S is proper then* 0 *is globally asymptotically stable.*

Proof By Proposition 3.2.9 $x = 0$ is a stable equilibrium of $\dot{x} = f(x, 0)$. Furthermore

$$\dot{S}(x) = S_x(x) f(x) \leq s(0, h(x, 0)),$$

and asymptotic stability follows by LaSalle's Invariance principle, since $\dot{S}(x) = 0$ implies $h(x, 0) = 0$. □

Finally, let us investigate the case that the storage function S has a local minimum at x^*, which is however *not* a strict minimum. In this case, $S(x) - S(x^*)$ is *not* a standard Lyapunov function, and thus stability, let alone asymptotic stability, of x^* is not guaranteed. Nevertheless, even in this case one can still obtain (asymptotic) stability, provided additional conditions are satisfied. The tool for doing this is formulated in the following theorem; see the references in the Notes for this chapter.

Theorem 3.2.17 *Let x^* be an equilibrium of $\dot{x} = f(x)$, and let $V : \mathcal{X} \to \mathbb{R}^1$ be a C^1 function which is positive semi-definite at x^*, that is,*

$$V(x^*) = 0, \quad V(x) \geq 0 \tag{3.46}$$

Furthermore, suppose that $\dot{V}(x) := V_x(x) f(x) \leq 0$, for all $x \in \mathcal{X}$.

(i) Define $\mathcal{V}_0 := \{x \in \mathcal{X} \mid V(x) = 0\}$. If x^* is asymptotically stable conditionally to $\mathcal{V}_0$, that is (3.39) and (3.40) hold for $x_0 \in \mathcal{V}_0$, then x^* is a stable equilibrium of $\dot{x} = f(x)$.

(ii) Define $\mathcal{V} := \{x \in \mathcal{X} \mid \dot{V}(x) = 0\}$, and let $\mathcal{V}^*$ be the largest positively invariant (with respect to $\dot{x} = f(x)$) set contained in $\mathcal{V}$. Then x^* is an asymptotically stable equilibrium of $\dot{x} = f(x)$ if and only if x^* is an asymptotically stable equilibrium conditionally to $\mathcal{V}^*$, that is, (3.39) and (3.40) hold for $x_0 \in \mathcal{V}^*$.

Remark 3.2.18 By replacing the condition $\dot{V}(x) = V_x(x) f(x) \leq 0$ by the condition that the function V is nonincreasing along solution trajectories, the above theorem also holds for functions V which are not C^1.

With the aid of Theorem 3.2.17 we obtain the following stability result extending Proposition 3.2.9.

Proposition 3.2.19 *Let* $S \geq 0$ *with* $S(x^*) = 0$ *be a solution to the dissipation inequality, where the supply rate* $s(u, y)$ *is such that*

$$s(0, y) \leq 0 \text{ for all } y, \;\; s(0, y) = 0 \text{ if and only if } y = 0$$

Let $\mathcal{H}^*$ *be the largest positively invariant set contained in the set* $\mathcal{H} := \{x \mid h(x, 0) = 0\}$*. If* x^* *is asymptotically stable conditionally to* $\mathcal{H}^*$*, then* x^* *is an asymptotically stable equilibrium.*

Proof In view of the dissipation inequality $\dot{S}(x) \leq s(0, h(x, 0)) \leq 0$. Since $s(0, y) = 0$ if and only if $y = 0$, it follows that the largest positively invariant set where $\dot{S}(x) = 0$ is contained in $\mathcal{H}^*$. Application of Theorem 3.2.17 yields the claim. □

Remark 3.2.20 Note that the L_2-gain and output strict passivity supply rates satisfy the conditions of Proposition 3.2.19.

Remark 3.2.21 The property of $x^* = 0$ being asymptotically stable conditionally to the largest positively invariant set contained in the set $\{x \mid h(x, 0) = 0\}$ is very close to zero-state detectability. In fact, this latter property implies that $\lim_{t \to \infty} x(t) = 0$ whenever $y(t) = 0, \; t \geq 0$, for all initial conditions x_0 close to 0.

For later use we state the following closely related result.

Proposition 3.2.22 *Consider the* C^1 *system*

$$\dot{x} = f(x) + g(x)k(x), \quad f(x^*) = 0, \quad k(x^*) = 0, \tag{3.47}$$

and assume that x^* *is an asymptotically stable equilibrium of* $\dot{x} = f(x)$*, and that there exists a* C^1 *function* $S \geq 0$ *which is positive semi-definite at* x^* *and satisfies*

$$S_x(x) \left[f(x) + g(x)k(x) \right] \leq -\varepsilon ||k(x)||^2 , \tag{3.48}$$

for some $\varepsilon > 0$*. Then* x^* *is an asymptotically stable equilibrium of (3.47).*

Proof Similarly to the proof of Proposition 3.2.19, let $\mathcal{K}^*$ be the largest positively invariant set contained in $\mathcal{K} := \{x \mid k(x) = 0\}$. Since x^* is an asymptotically stable equilibrium of $\dot{x} = f(x)$ it follows that x^* is asymptotically stable conditionally to $\mathcal{K}^*$. Since $S_x(x)[f(x) + g(x)k(x)] = 0$ implies $k(x) = 0$, the rest of the proof is the same as that of Proposition 3.2.19. □

Remark 3.2.23 Note that the condition of x^* being an asymptotically stable equilibrium of $\dot{x} = f(x)$ can be regarded as a *zero detectability* assumption on $\dot{x} = f(x) + g(x)k(x), \;\; y = k(x)$.

3.3 Interconnections of Dissipative Systems

Dissipative systems theory can be viewed as an extension of Lyapunov function theory to systems with external variables (inputs and outputs). Furthermore, it provides a systematic way to construct Lyapunov functions for large-scale interconnected systems by starting from the storage functions of the component systems, and requiring a compatibility between the interconnection equations and the supply rates of the component systems. In fact, this will be a leading theme in the state space versions of the passivity theorems in Chap. 4 and the small-gain theorems in Chap. 8. Also this is a continuing thread in the theory of port-Hamiltonian systems in Chap. 6. While in these subsequent chapters the attention will be confined to the passivity supply rate (in the case of passive and port-Hamiltonian systems) and the L_2-gain supply rate (in the case of the small-gain theorems) the current section will be devoted to a *general* theory of interconnections of dissipative systems.

Consider k systems Σ_i of the form (3.1) with input, state, and output spaces $U_i, \mathcal{X}_i, Y_i, i = 1, \dots, k$. Suppose Σ_i are dissipative with respect to the supply rates

$$s_i(u_i, y_i), \quad u_i \in U_i, \ \ y_i \in Y_i, \quad i = 1, \dots, k, \tag{3.49}$$

and storage functions $S_i(x_i), \ i = 1, \dots, k$.

Now consider an interconnection of $\Sigma_i, i = 1, \dots, k$, defined through an *interconnection subset*

$$I \subset U_1 \times Y_1 \times \cdots \times U_k \times Y_k \times U^e \times Y^e \tag{3.50}$$

where U^e, Y^e are spaces of external input and output variables u^e, y^e. This defines an interconnected system Σ_I with state space $\mathcal{X}_1 \times \cdots \times \mathcal{X}_k$ and inputs and outputs u^e, y^e, by imposing the interconnection equations

$$\big((u_1, y_1), \dots, (u_k, y_k), (u^e, y^e)\big) \in I \tag{3.51}$$

Note that in general the interconnected system Σ_I is of the DAE form (3.9). The following result is immediate.

Proposition 3.3.1 *Suppose the supply rates $s_1, \dots, s_k$ and the interconnection subset I are such that there exists a supply rate $s^e : U^e \times Y^e \to \mathbb{R}$ for which*

$$\begin{aligned} &s_1(u_1, y_1) + \cdots + s_k(u_k, y_k) \leq s^e(u^e, y^e), \\ &\quad \textit{for all} \ \ ((u_1, y_1), \dots, (u_k, y_k), (u^e, y^e)) \in I \end{aligned} \tag{3.52}$$

Then the interconnected system Σ_I is dissipative with respect to the supply rate s^e, with storage function

$$S(x_1, \dots, x_k) := S_1(x_1) + \cdots + S_k(x_k) \tag{3.53}$$

Proof Just add the k dissipation inequalities

$$S_i(x_i(t_1)) \ \leq \ S_i(x_i(t_0)) + \int_{t_0}^{t_1} s_i(u_i(t), y_i(t))dt, \quad i = 1, \ldots, k$$

and invoke the inequality (3.52). □

Note that for the purpose of stability analysis of the interconnected system Σ_I the external inputs and outputs u^e, y^e and the supply rate s^e can be left out, in which case (3.52) reduces to

$$\begin{array}{c} s_1(u_1, y_1) + \cdots + s_k(u_k, y_k) \leq 0\,, \\ \text{for all } \ ((u_1, y_1), \ldots, (u_k, y_k)) \in I \end{array} \tag{3.54}$$

Example 3.3.2 Consider a system having inputs and outputs (u_c, y_c) accessible to control interaction, and another set of inputs and outputs (u_e, y_e) via which the system interacts with its environment. Suppose the system is passive, with respect to the combined set of variables (u_c, y_c) and (u_e, y_e); that is, there exists a storage function S such that

$$\frac{dS}{dt} \leq u_c^T y_c + (u^e)^T y^e$$

An example is a robotic mechanism interacting with its environment via generalized forces u_e and generalized velocities y_e, and controlled by collocated sensors (generalized velocities y_c) and actuators (generalized forces u_c). Closing the loop with a passive controller with storage function S_c, that is,

$$\frac{dS_c}{dt} \leq -y_c^T u_c,$$

results in a system which is passive with respect to (u^e, y^e), since

$$\frac{d}{dt}(S + S_c) \leq (u^e)^T y^e$$

Note that the storage function of the interconnected system Σ_I in Proposition 3.3.1 is simply the *sum* of the storage functions of the component systems Σ_i. A useful extension of Proposition 3.3.1 is obtained by allowing instead for *weighted* combinations of the storage functions of the component systems. For simplicity we will only consider the case without external inputs and outputs u^e, y^e.

Proposition 3.3.3 *Suppose the supply rates $s_1, \ldots, s_k$ and the interconnection subset I are such that there exist positive constants $\alpha_1, \ldots, \alpha_k$ for which*

$$\begin{array}{c} \alpha_1 s_1(u_1, y_1) + \cdots + \alpha_k s_k(u_k, y_k) \leq 0, \\ \textit{for all } \ ((u_1, y_1), \ldots, (u_k, y_k)) \in I \end{array} \tag{3.55}$$

Then the nonnegative function

$$S_\alpha(x_1, \dots, x_k) := \alpha_1 S_1(x_1) + \cdots + \alpha_k S_k(x_k) \tag{3.56}$$

satisfies $\frac{d}{dt} S_\alpha \leq 0$ *along all solutions of the interconnected system* Σ_I.

Proof Multiply each i-th dissipation inequality

$$S_i(x_i(t_1)) \leq S_i(x_i(t_0)) + \int_{t_0}^{t_1} s_i(u_i(t), y_i(t))dt$$

by α_i, $i = 1, \dots, k$, add them, and use the inequality (3.55). □

In Sect. 8.2 we will see how this proposition underlies the small-gain theorem and extensions of it. Furthermore, it will appear naturally in the network interconnection of passive systems in Sect. 4.4.

3.4 Scattering of State Space Systems

The generalization (3.10) of the definition of dissipativity to DAE systems without a priori splitting of the external variables into inputs u and y is also useful in discussing the extension of the notion of *scattering*, as treated in Sect. 2.4 for input–output maps, to the state space system context.

Consider a state space system Σ given in standard input–state-output form (3.1), which is assumed to be passive, i.e.,

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} u^T(t)y(t)dt \tag{3.57}$$

for some storage function $S \geq 0$. Consider the scattering representation (v, z) of (u, y) defined as, see (2.41),

$$v = \frac{1}{\sqrt{2}}(u + y) \, , \qquad z = \frac{1}{\sqrt{2}}(-u + y) \tag{3.58}$$

The inverse transformation of (3.58) is $u = \frac{1}{\sqrt{2}}(v - z),\ y = \frac{1}{\sqrt{2}}(v + z)$, and substitution of these expressions in (3.57) yields

$$S(x(t_1)) - S(x(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} \left(||v(t)||^2 - ||z(t)||^2 \right) dt \tag{3.59}$$

This shows that Σ is passive with respect to u and y if and only if Σ with transformed external variables v and z has L_2-gain ≤ 1 from v to z, while the storage function remains the same. Similarly, it follows that Σ is lossless with respect to u and y if and only if it is inner with respect to v and z.

Note that in general the transformed system Σ with external variables $w = (v, z)$ is in the format (3.9). In case the original Σ is an affine system without feedthrough term, i.e., of the form

$$\Sigma_a \; : \; \begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3.60}$$

the substitution $u = \frac{1}{\sqrt{2}}(v - z)$, $y = \frac{1}{\sqrt{2}}(v + z)$, leads to an input–state–output representation in the wave vectors v, z, namely

$$\Sigma_s \; : \; \begin{aligned} \dot{x} &= f(x) - g(x)h(x) + \sqrt{2}g(x)v \\ z &= \sqrt{2}h(x) - v \end{aligned} \tag{3.61}$$

Summarizing

Proposition 3.4.1 *Σ_a is passive (lossless) with storage function S if and only if Σ_s has L_2-gain ≤ 1 (is inner) with storage function S.*

3.5 Dissipativity and the Return Difference Inequality

Dissipative systems theory turns out to provide an insightful framework for the study of the *Inverse problem of optimal control*, as originally introduced in [155] for the linear quadratic optimal control problem.

Consider the nonlinear optimal control problem (see Sects. 9.4 and 11.2 for further information)

$$\min_u \int_0^\infty (||u(t)||^2 + \ell(x(t)))dt \; , \tag{3.62}$$

for the system

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0, \tag{3.63}$$

where $\ell \geq 0$ is a cost function with $\ell(0) = 0$.

Denote the minimal cost (*value*) defined by (3.62) for initial condition $x(0) = x_0$ by $V(x_0)$. The function $V : \mathcal{X} \to \mathbb{R}^+$ is called the *value function*. Suppose that the value function V is well defined for all initial conditions and is C^1. Then it is known from optimal control theory that V is a nonnegative solution to the Hamilton–Jacobi–Bellman equation

$$V_x(x)f(x) - \frac{1}{2}V_x(x)g(x)g^T(x)V_x^T(x) + \frac{1}{2}\ell(x) = 0, \; V(0) = 0 \tag{3.64}$$

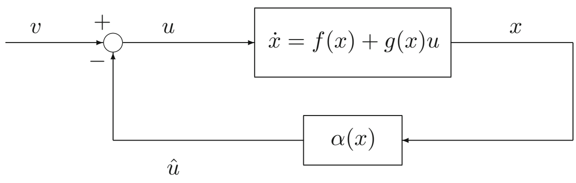

Fig. 3.1 Feedback system

Furthermore the optimal control is given in feedback form as

$$u = -\alpha(x) := -g^T(x)V_x^T(x) \tag{3.65}$$

If additionally $V(x) > 0$ for $x \neq 0$ and the system $\dot{x} = f(x)$, $y = \ell(x)$ is *zero-state detectable* (cf. Definition 3.2.15), it follows from LaSalle's Invariance principle that this optimal feedback is actually stabilizing, since (3.64) can be rewritten as

$$V_x(x)[f(x) - g(x)\alpha(x)] = -\frac{1}{2}\alpha^T(x)\alpha(x) - \ell(x)\,, \tag{3.66}$$

and thus asymptotic stability of $x = 0$ follows as in Proposition 3.2.16.

As we will now show, the optimal control feedback $u = -\alpha(x) := -g^T(x)V_x^T(x)$ has a direct *dissipativity interpretation*. Indeed, (3.64) and (3.65) can be rewritten as

$$\begin{aligned} V_x(x)f(x) - \tfrac{1}{2}\alpha^T(x)\alpha(x) &= -\ell(x) \;\leq\; 0 \quad V(0) = 0 \\ V_x(x)g(x) &= \alpha^T(x) \end{aligned} \tag{3.67}$$

implying that the following system (the "loop transfer" from u to minus the optimal feedback $-\alpha(x)$)

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \hat{u} &= \alpha(x) \end{aligned} \tag{3.68}$$

is dissipative with respect to the supply rate

$$s(u, \hat{u}) := \frac{1}{2}||\hat{u}||^2 + \hat{u}^T u \tag{3.69}$$

This leads to the following interesting consequence. By further rewriting the supply rate $s(u, \hat{u}) = \frac{1}{2}||\hat{u}||^2 + \hat{u}^T u$ as

$$\frac{1}{2}||\hat{u}||^2 + \hat{u}^T u = \frac{1}{2}||u + \hat{u}||^2 - \frac{1}{2}||u||^2 = \frac{1}{2}||v||^2 - \frac{1}{2}||u||^2\,, \tag{3.70}$$

it means that the feedback system in Fig. 3.1 with external inputs v satisfies the property

$$\frac{1}{2}\int_0^T ||u(t)||^2 dt \leq \frac{1}{2}\int_0^T ||v(t)||^2 dt + V(x(0)) - V(x(T))$$
$$\leq \frac{1}{2}\int_0^T ||v(t)||^2 dt + V(x(0)) \,, \tag{3.71}$$

for all initial conditions $x(0)$, and for all external input signals v. Thus the *L_2-gain from the external inputs v to the internal inputs u is less than or equal to one*.

In the linear case the frequency domain version of the inequality (3.71) is called the *return difference inequality*. The inequality expresses the favorable property that in the closed-loop system of Fig. 3.1 the L_2-norm of the optimal feedback $u = -\alpha(x)$ is *attenuated* with regard to the L_2-norm of any *external* control signal $v(\cdot)$.

Conversely, it can be shown that any stabilizing feedback $u = -\alpha(x)$ for which (3.71) holds is actually *optimal* with respect to *some* cost function $\ell \geq 0$ with $\ell(0) = 0$. Indeed, consider $u = -\alpha(x)$ such that (3.71) is satisfied for some function $V \geq 0$, with $V(0) = 0$. Equivalently, the system (3.68) is dissipative with respect to the supply rate $s(u, \hat{u}) = \frac{1}{2}||\hat{u}||^2 + \hat{u}^T u$, with storage function $V \geq 0$, $V(0) = 0$. Then it follows that (assuming V is C^1)

$$\begin{aligned} V_x(x)f(x) - \tfrac{1}{2}\alpha^T(x)\alpha(x) &\leq 0 \\ V_x(x)g(x) &= \alpha^T(x) \end{aligned} \tag{3.72}$$

Hence we may *define* the cost function $\ell \geq 0$ as

$$\ell(x) := -V_x(x)f(x) + \frac{1}{2}\alpha^T(x)\alpha(x), \tag{3.73}$$

satisfying $\ell(0) = 0$. It follows that V is actually a nonnegative solution of the Hamilton–Jacobi–Bellman equation (3.64) of the optimal control problem (3.62) for this cost function ℓ. As will be shown in Sect. 11.2, it follows that V is the value function of this optimal control problem and that $u = -\alpha(x)$ is the optimal feedback control.

Summarizing, we have obtained the following theorem.

Theorem 3.5.1 *Consider the system (3.63). Let $\ell \geq 0$ be a cost function with $\ell(0) = 0$ such that $\dot{x} = f(x)$, $y = \ell(x)$, is zero-state detectable, and that the value function V of the optimal control problem is well defined, C^1, and satisfies $V(x) > 0, x \neq 0$. Then the feedback $u = -\alpha(x) := -g^T(x)V_x^T(x)$ is stabilizing, and the resulting feedback system in Fig. 3.1 satisfies property (3.71).*

Conversely, $u = -\alpha(x)$ is a stabilizing feedback such that the feedback system in Fig. 3.1 satisfies property (3.71) for a C^1 function $V \geq 0$ with $V(0) = 0$, then $u = -\alpha(x)$ is the optimal control for (3.62) with the cost function $\ell(x) := -V_x(x)f(x) + \frac{1}{2}\alpha^T(x)\alpha(x)$.

Thus, loosely speaking, a feedback $u = -\alpha(x)$ is optimal with regard to *some* optimal control problem of the form (3.62) *if and only if* the return difference inequality (3.71) holds.

Remark 3.5.2 In view of Proposition 3.1.14, we additionally note that if the system $\dot{x} = f(x) + g(x)u$ is *reachable* from $x = 0$, then property (3.74) holds for a feedback $u = -\alpha(x)$ and a $V \geq 0$ with $V(0) = 0$, if and only if

$$\int_0^T ||u(t)||^2 dt \;\leq\; \int_0^T ||v(t)||^2 dt\,, \qquad x(0) = 0, \tag{3.74}$$

for all external inputs v, and all $T \geq 0$.

Remark 3.5.3 Notice that the optimal regulator in Fig. 3.1, that is the system

$$\begin{aligned} \dot{x} &= [f(x) - g(x)\alpha(x)] + g(x)v \\ \hat{u} &= \alpha(x) = g^T(x)V_x^T(x) \end{aligned} \tag{3.75}$$

is *output strictly passive*, as follows directly from (3.66).

3.6 Notes for Chapter 3

1. The main part of the theory exposed in Sect. 3.1 is based on Willems' seminal and groundbreaking paper [350]. The developments around Propositions 3.1.19, 3.1.20 and Corollary 3.1.21 are relatively new, although inspired by similar arguments in Willems [349, 351].

2. Other expositions of (parts of) the theory of dissipative systems can be found in Hill & Moylan [126], Moylan [225], Brogliato, Lozano, Maschke & Egeland [52], as well as Isidori [139], Arcak, Meissen & Packard [11].

3. We refer to e.g., Weiland & Willems [346], and Trentelman & Willems [338, 339], Willems & Trentelman [353] for developments on dissipative systems theory within a (linear) behavioral systems theory framework; generalizing the concept of quadratic supply rates to quadratic differential forms that may also involve derivatives of the inputs and outputs. Note that in these papers "dissipativity" is often used in the meaning of "cyclo-dissipativity".

4. The first part of Sect. 3.2 is also mainly based on Willems' paper [350], together with important contributions due to Hill & Moylan [123–126]. The exposition on stability in Sect. 3.2 was influenced by Sepulchre, Jankovich & Kokotovic [312].

5. The theory of dissipative systems is closely related to the theory of *Integral Quadratic Constraints* (IQCs); see e.g., Megretski & Rantzer [215], Jönsson [152] and the references quoted therein.
   Basically, in the theory of Integral Quadratic Constraints (IQCs) the system Σ_1 denotes the given *linear* "nominal" part of the system to be studied, specified by a transfer matrix G, while $\Delta := \Sigma_2$ denotes the "troublemaking" (nonlinear,

time-delay, time-varying, uncertain) components. In order to assess stability of the overall system one searches for IQCs for Δ. These are given by a Hermitian matrix valued function $\Pi(j\omega)$, $\omega \in \mathbb{R}$, such that

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}_2(j\omega) \\ \widehat{y}_2(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}_2(j\omega) \\ \widehat{y}_2(j\omega) \end{bmatrix} d\omega \geq 0$$

for all L_2 signals u_2, y_2 compatible with Δ. Here $\widehat{}$ denotes Fourier transform, and * is complex conjugate and transpose. For rational Π that are bounded on the imaginary axis, the time domain version of the IQC is

$$\int_0^{\infty} \sigma(x_\pi(t), y_2(t), u_2(t))dt \geq 0$$

for a certain quadratic form σ, where x_π is solution of an auxiliary system

$$\dot{x}_\pi = A_\pi x_\pi + B_{y_2} y_2 + B_{u_2} u_2, \quad x_\pi(0) = 0.$$

The main theorem (Theorem 1 in Megretski & Rantzer [215]) states that if we can find a Π such that for every $\tau \in [0, 1]$ the interconnection of G and $\tau\Delta$ is well posed and Π is an IQC for $\tau\Delta$, while there exists a $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I, \quad \forall \omega \in \mathbb{R}$$

then the closed-loop system is stable. Compared with the setup of dissipative systems theory there are two major differences. One is that Π is not necessarily a constant matrix, and that therefore in the time domain formulation the function σ, which replaces the supply rate s of dissipative systems theory, also depends on an auxiliary dynamical system (acting as an additional filter for the signals u_2, y_2). Secondly, the IQC should only hold for all L_2 signals u_2, y_2, and therefore in the time domain version the integral is from 0 to ∞, instead of from 0 to any $T \geq 0$, as in dissipative systems theory formulation. The first aspect constitutes a major extension with respect to dissipative systems theory. The second difference is more of a technical nature, closely related to an extension of dissipativity to cyclo-dissipativity. The L_2-stability problems caused by the second difference are taken care of by an ingenious homotopy argument based on the variation of τ from 0 to 1 (nominal value). In Veenman & Scherer [341] it has been shown how in most situations IOC stability analysis can be proved by dissipative systems theory.

On a methodological level, the philosophy of the theory of IQCs is somewhat different from dissipative systems theory in the sense that in IQC theory the emphasis is on stability analysis by splitting between nominal linear dynamics and "troublemaking" nonlinearities or time delays, whose disturbing properties are sought to be bounded by a suitable IQC. Dissipative systems theory, on the other hand, is primarily a compositional theory of complex systems (rooted in network dynamics), where nonlinear dynamical components are not necessarily considered to be detrimental. The theory of IQCs is especially useful for stability analysis of systems with "small-scale" nonlinearities or time delays. It yields sharp results on the classical cases of "noncausal multipliers" and the Popov criterion.

6. In Sect. 3.2 we have assumed throughout that there exist storage functions which are continuously differentiable (C^1), in order to make the link with Lyapunov stability theory, and, very importantly, in order to be able to rewrite the dissipation inequality (3.3) as the differential dissipation inequality (3.36). Now, for Lyapunov stability theory the Lyapunov functions do not necessarily have to be C^1, see e.g., Sontag [317]. Moreover, often storage functions for nonlinear systems are *not* everywhere differentiable (in particular this may happen for the available storage S_a and the required supply S_r, being solutions to an optimal control problem). Since it is much easier to work with differential dissipation inequalities than with dissipation inequalities in integral form, it would thus be desirable to have a *generalized solution* concept for differential dissipation inequalities (3.36), admitting solutions S that are *not* everywhere differentiable. In fact, this is possible using the concept of a *viscosity solution* (see e.g., Fleming & Soner [99], for a clear exposé), as shown in James [144] (see also James & Baras [145], Ball & Helton [22]). We also like to refer to Clarke, Ledyaev, Stern & Wolenski [67, 68] for a broader discussion of generalized solution concepts for Hamilton–Jacobi inequalities or equalities, showing equivalence between apparently different solution concepts.
7. See e.g., Khalil [159] for a coverage of Lyapunov stability theory and LaSalle's Invariance principle.
8. Theorem 3.2.17 is due to Iggidr, Kalitine & Outbib [134]. I thank Laurent Praly for pointing out an error in the presentation of the consequences of this theorem in the second edition of this book.
9. Proposition 3.2.22 is due to Imura, Sugie & Yoshikawa [132], Imura, Maeda, Sugie & Yoshikawa [131], where an alternative proof is given.
10. Section 3.3 is largely based on Willems [350], Moylan & Hill [227], see also Moylan [225].
11. The stability analysis of an interconnected system the approach taken in Sect. 3.3 can be turned around as well. Given the component systems $\Sigma_1, \ldots, \Sigma_k$ and the interconnection subset I, one may *search* for (suitably defined) supply rates $s_1, \ldots, s_k$, for which the systems are dissipative and (3.54) holds. This point of view is already (implicitly) present in classical papers on dissipative systems such as Moylan & Hill [227], see also Moylan [225], and was recently emphasized and explored in Meissen, Lessard, Arcak & Packard [217], Arcak, Meissen, Packard [11].

Furthermore, in Jokic & Nakic [151] the *converse* result is obtained stating that if an interconnected linear system has an additive quadratic Lyapunov function (a sum of terms only depending on the state variables of the subsystems) then there exist interconnection neutral supply rates with respect to which the subsystems are dissipative.

12. Section 3.5 is an extended and simplified exposition of basic ideas developed in Moylan & Anderson [226]. For related work on the inverse optimal control problem and its applications to robust control design we refer to Sepulchre, Jankovic & Kokotovic [312], and Freeman & Kokotovic [106].
13. The differential dissipation inequality (3.36) admits the following *factorization* perspective. For concreteness, assume that $f(0, 0) = 0$, $h(0, 0) = 0$ as well as $s(0, 0) = 0$. Then, under technical assumptions (see Chap. 9), satisfaction of (3.36) will imply that there exists a map $\bar{h} : \mathcal{X} \times \mathbb{R}^m \to \mathbb{R}^{\bar{p}}$ such that

$$S_x(x)f(x, u) - s(u, h(x, u)) = -\|\bar{h}(x, u)\|^2 \tag{3.76}$$

Equivalently, the system is *conservative* with respect to the new supply rate

$$\bar{s}(u, y, \bar{y}) := s(u, y) - \|\bar{y}\|^2, \tag{3.77}$$

involving, next to y, the new output $\bar{y} = \bar{h}(x, u)$. This factorization perspective will be further discussed in the context of the passivity supply rate in Sect. 4.1, and will be key in the developments on the L_2-gain supply rate in Sect. 9.4.