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Optimal regulation of flow networks with transient constraints

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ABSTRACT

This paper investigates the control of flow networks, where the control objective is to regulate the measured output (*e.g.* storage levels) towards a desired value. We present a distributed controller that dynamically adjusts the inputs and flows, to achieve output regulation in the presence of unknown constant disturbances, while satisfying given input and flow constraints. Optimal coordination among the controllers minimizing a suitable cost function of the inputs at the nodes, is achieved by exchanging information over a communication network. Exploiting an incremental passivity property, the desired steady state is proven to be globally asymptotically attractive under the closed loop dynamics. Two case studies (a district heating system and a super-conducting DC network) show the effectiveness of the proposed solution.

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1. Introduction

Flow networks (also known as distribution or transportation networks) consist of edges that are used to model the exchange of material (flow) between the nodes. The design and regulation of these networks received significant attention due to its many applications, including supply chains (Alessandri, Gaggero, & Tonelli, 2011), heating, ventilation and air conditioning (HVAC) systems (Gupta, Kar, Mishra, & Wen, 2015), data networks (Moss & Segall, 1982), traffic networks (Coogan & Arcak, 2015; Iftar, 1999) and compartmental systems (Blanchini, Franco, Giordano, Mardanlou, & Montessoro, 2016; Como, 2017). If the considered objective is static, the study of flow networks has a long history within the field of network optimization (Rockafellar, 1984). Many practical networks must on the other hand react dynamically on changes in the external conditions such as a change in the demand. In these cases continuous feedback controllers are

required, that dynamically adjust inputs at the nodes and the flows along the edges, and the design of such controllers is the subject of this work.

Since flow networks are ubiquitous in engineering systems, many solutions have been proposed to coordinate them, exploiting methodologies from *e.g.* passivity (Arcak, 2007) and model predictive control (Koeln & Alleyne, 2017). We focus on flow networks where the nodes can store the considered material (Kotnyek, 2003). A common objective in such networks is that the stored material needs to be regulated towards desired setpoints, despite the presence of an unknown demand. This is commonly achieved by actively controlling the flows on the edges (Bürger & De Persis, 2015; Wei & van der Schaft, 2013; Xiang, Li, & Hill, 2017) using *dynamic* flow controllers. These controllers on the edges generally provide a form of integral action, that shows some benefits over networks lacking these dynamics. For example, the presence of an integral action permits the achievement of output regulation, in contrast to approximate regulation (Giordano, 2016). Furthermore, in most cases, the capacity of the edges is constrained, requiring careful design of the flow controllers. Naturally, the control of flows only permits to distribute the material within the network. In case there is no possibility to adjust the input to the network, a necessary requirement for stability is that all uncontrollable inflows and outflows sum to zero (Wei, 2016). Since this is generally not the case, additional controllable inputs are required that might have their own capacity constraints.

1.1. Literature review and main contributions

In this work we focus on flow networks, where at various nodes, an *unknown* amount of material (disturbance) is supplied

to, or extracted from, the network. Despite these disturbances, we require the various storage levels at the nodes (or an 'output function' thereof) to be regulated towards desired values. We aim at achieving this so-called *output regulation*, by *optimally* allocating the required inputs among the nodes that possess a controllable external input. Here, only a subset of the nodes is assumed to have a controllable input, where a cost function relates the provided input to associated costs. We particularly propose a *distributed* control solution to enhance robustness to failures and to improve the scalability. Furthermore, the proposed solution respects capacity constraints that the inputs and flows might have.

Although various of these aspects have been addressed before, the way how we incorporate them within a coherent approach is new. We elaborate on some specific contributions below.

(i) In flow networks it is desirable to meet certain optimality criteria, prescribing *e.g.* the optimal flows within the network and the optimal inputs to the network. Examples of the former include a 'maximum flow', 'quickest flow' or 'minimum cost flow', and achieving them received a considerable amount of attention in the past (see Kotnyek (2003), Skutella (2009) and references therein). On the other hand, when optimal inputs are considered, costs are often associated with the amount of generated input (materials), and optimization thereof has been studied thoroughly within the setting of smart (electricity) grids (Trip, Bürger, & De Persis, 2016). In this paper we apply this idea to general flow networks (Scholten, De Persis, & Tesi, 2016), where only a subset of the nodes can generate an input. A communication network then connects the various nodes, where relevant information on the costs is exchanged.

(ii) The distributed controllers are designed to enjoy certain passivity properties. That passivity plays an outstanding role in the coordination of systems is well recognized (Arcak, 2007). Particularly, incremental passivity (Pavlov & Marconi, 2008) has been exploited to analyse the stability of flow networks (see, *e.g.* Bürger and De Persis (2015) and (Bürger, De Persis, & Allgöwer, 2015)). To prove asymptotic convergence to the desired state, generally, some form of strict output passivity (*e.g.* as a result of damping) is required. The considered flow networks in this work do not enjoy this property, due to the preservation of the material, making the controller design more challenging. We propose a 'dynamic extension' of previously considered integral-type controllers, to ensure convergence to a point, preventing the network to converge to a limit cycle, exhibiting oscillations. Although the approach is tailored to the system at hand, the design offers new perspectives on similar systems lacking dissipation. In case physical considerations forbid this dynamic extension, global convergence to the desired output can be achieved by carefully selecting nodes that have a controllable input. This selection is related to the zero forcing set of the underlying graph of the network (Monshizadeh, Zhang, & Camlibel, 2014; Trefois & Delvenne, 2015), and this work provides an interesting link between zero forcing sets and LaSalle's invariance principle.

(iii) The optimal control of flow networks considered in this paper was first tackled in Bauso, Blanchini, Giarr, and Pesenti (2013). To be specific, Bauso et al. (2013) proposed a distributed *static* state feedback control to practically stabilize the flow network, possibly in the presence of time-varying disturbances. Namely, the control in Bauso et al. (2013) guarantees convergence of the state to an ε-neighbourhood of the origin by suitably tuning a gain γ that is computed solving linear programmes depending on the convex and compact set to which the disturbance vector belongs or on the constrained set of the control inputs (depending on whether or not the optimal input has components strictly in the interior of the feasibility set (Bauso et al., 2013, Subsection 3.1)). Moreover, the control input, is shown to converge to the

optimal solution of a constrained quadratic problem that minimizes the costs associated with both the flow at the edges and the supplied material at the nodes. The problem of uniform global stabilization of a more general class of nonlinear compartmental models has been studied in Blanchini et al. (2016). In that paper, the state-input equilibrium pair is assumed given and the problem of regulating the state to a prescribed steady state value is not considered.

In our contribution we are interested to *asymptotically regulate* the state to a prescribed set-point in spite of unknown constant disturbances, while fulfilling constraints on the magnitude of both the flows and control inputs at the nodes, and forcing the latter to converge to the minimum of a linear quadratic cost function. The latter feature is relevant to those networks where transportation costs are negligible compared to the costs associated with the control at the nodes. To achieve the desired regulation goal, we propose *dynamical* feedback controllers that adjust the flow at the edges and the controllable external inputs. This is contrast with the majority of studies on compartmental systems (see the next paragraph for some exceptions), where flows and external control inputs are typically static maps of the states, see *e.g.*, Como (2017).

Setpoint regulation for (linear) compartmental systems has been studied before in Ahn, Kim, Lim, Lee, and Oh (2017) and Lee and Ahn (2015), but our approach is different. In the aforementioned works, the flows are adjusted by properly altering the system parameters of the network via projected (hence, discontinuous) integral controllers, whereas we consider here the parameters constant and dynamically adjust the flows at the edges and the in/outflow at some nodes, which allows us to enforce constraints on the flows. On the other hand, the approach in Ahn et al. (2017) leads to a closed-loop system which is a linear time-varying compartmental system, whose property can be exploited to show positivity of the system's states. Differently from Ahn et al. (2017), our controllers at the node exchange information to guarantee convergence to an optimal steady state. Another difference with respect to Ahn et al. (2017) is that in our model no term modelling state-dependent outflows is present, making our control design useful for those networks that are not strictly output passive. Various other control problems for compartmental systems are collected in the monograph (Haddad, Chellaboina, & Hui, 2010).

1.2. Outline

The paper is structured as follows. In Section 2 we introduce the considered flow network model. Next, in Section 3, we state our control objective of *optimal output regulation* and discuss various constraints under which the control objective should be achieved. In Section 4 we propose a distributed controller and study the feasibility of the control problem in more detail. Exploiting incremental passivity properties of the network and the controllers, the stability analysis of the closed loop system is carried out in Section 5. In Section 6, we study two modifications to the controlled flow network, widening the scope of this work. Two case studies are presented in Section 7. Finally, the conclusions and future directions are given in Section 8.

1.3. Notation

Let $\mathbf{0}$ be the vector of all zeros of suitable dimension and let $\mathbb{1}_n$ be the vector containing all ones of length n. The ith element of vector x is denoted by x_i or, if it enhances the readability, by $[x]_i$. We define $\mathcal{R}(f)$ to be the range of function $f(x)$. A steady state solution to system $\dot{x} = f(x)$, is denoted by $\bar{x}$, *i.e.* $\mathbf{0} = f(\bar{x})$. In case the argument of a function is clear from the context, we

Table 1
Description of various symbols.

Symbol	Description
$\mathcal{G}$	Graph of the network
$\mathcal{V}$	Set of nodes
$\mathcal{V}_e$	Set of nodes with controllable external input
$\mathcal{E}$	Set of edges
B	Incidence matrix of the network
E	Indicator matrix of controllable external inputs
$T_\star$	Constant (gain) matrix
L^{com}	Laplacian matrix of the communication graph
Q	Quadratic cost matrix
r	Linear cost vector
x	Storage / inventory level
y	Output ($y = h(x)$)
$\bar{y}$	Desired output
d	Disturbance / demand
u	Controllable external input ($u = g(\theta)$)
$\bar{u}$	Optimal input
λ	Flows on the edges ($\lambda = f(\mu)$)
ξ	Auxiliary flow controller state
ϕ	Auxiliary input controller state

occasionally write $f(x)$ as $f(\cdot)$. Let $A \in \mathbb{R}^{n \times m}$ be a matrix, then im(A) is the image of A and ker(A) is the kernel of A. In case A is a positive definite (positive semi-definite) matrix, we write $A \in \mathbb{R}^{n \times n}_{>0}$ ($A \in \mathbb{R}^{n \times n}_{\geq 0}$). Lastly, we denote the cardinality of a set $\mathcal{V}$ as $|\mathcal{V}|$. For convenience we provide, in Table 1, an overview of some important symbols appearing in this work.

2. Flow networks

In this paper we consider a network of physically interconnected *undamped* dynamical systems. The topology of the system is described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes and $\mathcal{E} = \{1, \ldots, m\}$ is the set of edges connecting the nodes. We represent the topology by its corresponding incidence matrix $B \in \mathbb{R}^{n \times m}$, where the entries of B are defined by arbitrarily labelling the ends of the edges in $\mathcal{E}$ with a '+' and a '−', and letting

$$b_{ik} = \begin{cases} +1 & \text{if node } i \text{ is the positive end of edge } k \\ -1 & \text{if node } i \text{ is the negative end of edge } k \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{V}_e \subseteq \mathcal{V}$ be the set of actuated nodes that are controlled by an *external* input and let $|\mathcal{V}_e| = p$. We define

$$e_i = \begin{cases} 1 & i \in \mathcal{V}_e \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The dynamics of node $i \in \mathcal{V}$ are given by[1]

$$T_{x_i} \dot{x}_i(t) = -\sum_{k \in \mathcal{E}_i} B_{ik} \lambda_k(t) + e_i u_i(t) - d_i \tag{2a}$$

$$y_i(t) = h_i(x_i(t)), \tag{2b}$$

where $x_i(t)$ is the storage (inventory) level, $u_i(t)$ the control input, $T_{x_i} \in \mathbb{R}_{>0}$ a constant,[2] d_i is a constant *unknown* disturbance and $y_i = h_i(x_i)$ the measured output with $h_i(\cdot)$ a continuously differentiable and strictly increasing function. Moreover, $\mathcal{E}_i$ is the

set of edges connected to node i and $\lambda_k(t)$ is the flow on edge k. We can represent the complete network compactly as[3]

$$T_x \dot{x} = -B\lambda + Eu - d \tag{3a}$$

$$y = h(x), \tag{3b}$$

where $T_x \in \mathbb{R}^{n \times n}_{>0}$, $B \in \mathbb{R}^{n \times m}$, $\lambda \in \mathbb{R}^m$, $u \in \mathbb{R}^p$ and $d \in \mathbb{R}^n$. Without loss of generality we assume that only the first p nodes have a controllable external input, i.e. $\{1, \ldots, p\} = \mathcal{V}_e$, and consequently $E \in \mathbb{R}^{n \times p}$ is of the form

$$E = \begin{bmatrix} I_{p \times p} \\ \mathbf{0}_{(n-p) \times p} \end{bmatrix}. \tag{4}$$

Furthermore, $y \in \mathbb{R}^n$ and $h(x) \in \mathbb{R}^n$ of which the ith component is given by $h_i(x_i)$. Throughout this work we will study the control of the inputs to the nodes and the control of the flows on the edges. We make two basic assumptions on the network that allows us to formulate the control objectives explicitly in the next section. First, in order to guarantee that each node can be reached from anywhere in the graph we make the following assumption on the topology:

Assumption 1 (*Connectedness*)**.** The graph $\mathcal{G}$ is connected.

Second, to compensate for the disturbances to the network, we assume that at least one node has a controllable external input, i.e. $p \geq 1$. An immediate consequence of Assumption 1 is that $\begin{bmatrix} B & E \end{bmatrix}$ is full row rank. Particularly, we will use the fact that the pseudoinverse of $\begin{bmatrix} B & E \end{bmatrix}$ constitutes a right inverse.

3. Optimal regulation with input and flow constraints

In this section we discuss two control objectives and the various input and flow constraints under which the objectives should be reached. We start with discussing the two objectives. The first objective is concerned with the output $y = h(x)$ in (3), at steady state.

Objective 1 (*Output Regulation*)**.** *Let $\bar{y}$ be a desired constant setpoint, then the output $y = h(x)$ of (3) asymptotically converges to $\bar{y}$, i.e. $\lim_{t \to \infty} \|h(x(t)) - \bar{y}\| = 0$.*

Since the function $h_i(x_i)$ is invertible, a desired output $\bar{y}_i$ at node i, prescribes the associated steady state value $\bar{x}_i = h_i^{-1}(\bar{y}_i)$ for all $i \in \mathcal{V}$. However, note that we do not assume the function $h_i(x_i)$ to be known in the remainder of this work. To ensure feasibility of Objective 1, the following assumption is made:

Assumption 2 (*Feasible Setpoint*)**.** The desired setpoint $\bar{y}$ satisfies $\bar{y}_i \in \mathcal{R}(h_i)$ for all $i \in \mathcal{V}$.

At a state where $\bar{x}$ is constant and satisfies $h(\bar{x}) = \bar{y}$ system (3a) necessarily satisfies

$$\mathbf{0} = -B\lambda + Eu - d. \tag{5}$$

Premultiplying (5) with $\mathbb{1}_n^T$ results in

$$0 = \mathbb{1}_n^T Eu - \mathbb{1}_n^T d = \mathbb{1}_p^T u - \mathbb{1}_n^T d = 0, \tag{6}$$

such that at a steady state the total input to the network needs to be equal to the total disturbance. If there are two or more inputs to the network (i.e. $p \geq 2$), it is natural to wonder if the total input can be coordinated optimally among the nodes. To this end, we assign a strictly convex linear-quadratic cost function $C_i(u_i)$ to

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[1] Here and throughout this work we do *not* require that any of the appearing functions is identical to another, e.g. it is permitted that $h_i \neq h_j$ for $i \neq j$.

[2] Usually we have $T_{x_i} = 1$ in the classical flow networks, where a material is transported. See however Section 7.2 for an example where $T_{x_i} \neq 1$.

[3] For the sake of simplicity, the dependence of the variables on time t is omitted in most of the remainder this paper.

each input of the form $C_i(u_i) = \frac{1}{2}q_i u_i^2 + r_i u_i + s_i$, with $q_i \in \mathbb{R}_{>0}$ and $r_i, s_i \in \mathbb{R}$. The total cost can be expressed as

$$C(u) = \sum_{i \in \mathcal{V}_e} C_i(u_i) = \frac{1}{2} u^T Q u + r^T u + s, \qquad (7)$$

where $Q = \text{diag}(q_1, \ldots, q_p)$, $r = (r_1, \ldots, r_p)^T$ and $s = \sum_{i \in \mathcal{V}_e} s_i$. Minimizing (7), while satisfying the equilibrium condition (5), gives rise to the following optimization problem:

$$\begin{aligned} &\underset{u, \lambda}{\text{minimize}} \quad C(u) \\ &\text{subject to} \quad \mathbf{0} = -B\lambda + Eu - d. \end{aligned} \qquad (8)$$

It is possible to explicitly characterize the solution to (8) and we do so in the following lemma:

Lemma 1 (*Solution to Optimization Problem* (8)). *The solution to* (8) *is given by*

$$\bar{u} = Q^{-1}(\kappa - r), \qquad (9)$$

where $\kappa = E^T \frac{\mathbb{1}_n \mathbb{1}_n^T}{\mathbb{1}_p^T Q^{-1} \mathbb{1}_p}(d + EQ^{-1}r)$.

Proof. The proof follows standard arguments from convex optimization and from realizing (Trip et al., 2016, Lemma 4) that the constraint in (8) can be equivalently replaced by (6). ∎

An immediate consequence of Lemma 1 is that the so-called marginal costs $\frac{\partial C_i(u_i)}{\partial u_i} = q_i u_i + r_i$ are identical for all $i \in \mathcal{V}_e$ when computed at the solution to (8). This observation motivates the addition of a consensus term in the controllers at the nodes (Eq. (13)). A possible value of λ, associated with the optimal input $\bar{u}$, can be obtained following Footnote 5. Restricting the cost function $C(u)$ to depend on the input u only and not on λ is needed to obtain the expression $\bar{u}$ in Lemma 1 for which we can design dynamic controllers whose outputs asymptotically converge to the optimal values. For static controllers that practically stabilize the network while minimizing a cost function of both u and λ we refer the reader to Bauso et al. (2013). We are now ready to state the second control objective.

Objective 2 (*Optimal Feedforward Input*). *The input at the nodes asymptotically converges to the solution to* (8), *i.e.* $\lim_{t \to \infty} \|u(t) - \bar{u}\| = 0$, *with* $\bar{u}$ *as in* (9).

We now turn our attention to possible constraints on the control inputs u and λ under which the objectives should be reached. First, in physical systems the input u is generally constrained by a minimum value (often zero, preventing a negative input) and a maximum value, representing *e.g.* a production capacity.

Constraint 1 (*Input Limitations*). *The inputs at the nodes satisfy*

$$u_i^- < u_i(t) < u_i^+ \quad \text{for all } i \in \mathcal{V}_e \text{ and all } t \geq 0, \qquad (10)$$

with $u_i^-, u_i^+ \in \mathbb{R}$ *being suitable constants.*

Second, the flows on the edges are often constrained to be unidirectional and to be within the capacity of the edges.

Constraint 2 (*Flow Capacity*). *The flows on the edges satisfy*

$$\lambda_k^- < \lambda_k(t) < \lambda_k^+ \quad \text{for all } k \in \mathcal{E} \text{ and all } t \geq 0, \qquad (11)$$

with $\lambda_k^-, \lambda_k^+ \in \mathbb{R}$ *being suitable constants.*

Note that physical limitations and safety requirements demand that the constraints should be satisfied *for all time* and not only at steady state.

In many applications it is desirable to have a distributed control architecture where controllers rely only on local information
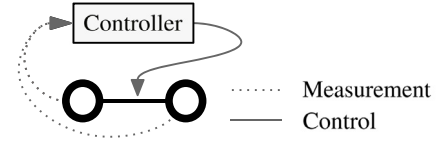

Fig. 1. The controller that is located at the edge k has access to the regulation error of its adjacent nodes, $h_i(x_i) - \bar{y}_i$ and $h_j(x_j) - \bar{y}_j$, where i, j are the two nodes connected by the edge k, as well as to the term $f_k(\mu_k) - \xi_k$ which only depends on local control variables. Using these measurements as inputs, the controller generates the flow rate λ_k on the edge.

to decrease communications, to increase robustness and to improve the scalability of the control scheme. We therefore require that the controllers to be designed, only depend on information available from adjacent nodes in the physical flow network or adjacent nodes in a digital communication network that is deployed to ensure optimality (see the next section). This leads to the following design problem:

Problem 1 (*Controller Design Problem*). Design distributed controllers that regulate the external inputs u at the nodes and the flows λ on the edges, such that Objectives 1 and 2 are achieved, while satisfying Constraints 1 and 2.

4. Controller design

4.1. Flow controller

We design a controller that regulates the flows on the edges, aiming at consensus in the error $y - \bar{y}$ (balancing), while obtaining a useful passivity property of the resulting closed loop system when interconnected with (3). Consider the following controller:

$$\begin{aligned} T_\mu \dot{\mu} &= B^T(h(x) - \bar{y}) - (f(\mu) - \xi) \\ T_\xi \dot{\xi} &= f(\mu) - \xi \\ \lambda &= f(\mu), \end{aligned} \qquad (12)$$

where $T_\mu, T_\xi \in \mathbb{R}_{>0}^{m \times m}$ are diagonal matrices with strictly positive entries, $\mu, \xi \in \mathbb{R}^m$ and the mapping $f(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$, with $f(\mu) = (f_1(\mu_1), \ldots, f_m(\mu_m))^T$, has suitable properties discussed in Assumptions 4 and 5. Moreover, B is the incidence matrix reflecting the topology of the physical network, which implies that the flow controller on edge k only requires information from its adjacent nodes (see also Fig. 1). Note that the term $[B^T(h(x) - \bar{y})]_k$ determines the difference in the output error of the two nodes adjacent to edge $k \in \mathcal{E}$. The controllers at the edges (12) are designed to induce a suitable passivity property when interconnected to the process (3), as shown in Lemma 3 in the next section. Furthermore, as will be discussed in Section 5, particularly in Remark 2, the state ξ is introduced to prove convergence to a constant flow, preventing oscillations and compensating for the lack of damping in the process dynamics (3).

4.2. Controller at the nodes

Next, we design an input controller u_i at each node i that adjusts the external input to the network. Inspired by the result in Trip and De Persis (2017), where a similar control problem is considered in the setting of power networks, we propose the controller

$$\begin{aligned} T_\theta \dot{\theta} &= -E^T(h(x) - \bar{y}) - (g(\theta) - \phi) \\ T_\phi \dot{\phi} &= g(\theta) - \phi - QL^{com}(Q\phi + r) \\ u &= g(\theta), \end{aligned} \qquad (13)$$
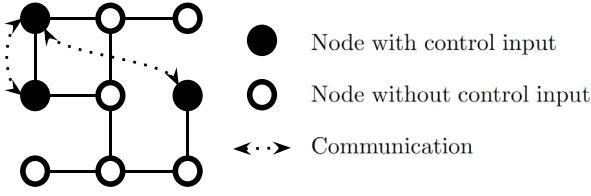
Fig. 2. Example of a flow network including a communication graph. The depicted communication graph is undirected, satisfying Assumption 3. At every node i the controller has access to the local regulation error $h_i(x_i) - \bar{y}_i$ and to the term $g_i(\theta_i) - \phi_i$, which depends on the node local control variables. Furthermore, node i receives information on the marginal costs $q_j\phi_j + r_j$ from the controllers located at the adjacent nodes and connected via the communication network.

where $T_\theta, T_\phi \in \mathbb{R}^{p\times p}_{>0}$ are diagonal matrices with strictly positive entries, $\theta, \phi \in \mathbb{R}^p$ and the mapping $g(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^p$, with $g(\theta) = (g_1(\theta_1)\dots g_p(\theta_p))^T$, has suitable properties discussed in Assumptions 4 and 5. Moreover, L^{com} is the Laplacian matrix reflecting the communication topology (see also Fig. 2). Similarly to the controller at the edges, controllers (13) are designed to enjoy a passivity property so that, when interconnected with the process (3) and the controllers (12) in a power preserving manner, passivity of the closed-loop system is preserved and stability can be inferred. Moreover, the consensus term enabled by the communication Laplacian L^{com} ensures that, at steady state, a consensus is obtained in the marginal costs, i.e. $Qg(\bar{\theta})+r \in \text{im}(\mathbb{1}_p)$. In order to guarantee that all marginal costs converge to the same value we make the following assumption on the communication network.

Assumption 3 (*Communication Network*). The graph reflecting the communication topology is balanced[4] and strongly connected.

An immediate consequence of Assumption 3 is that L^{com} is a positive semi-definite matrix and $\phi^T L^{com}\phi = 0$ if and only if $\phi \in \text{im}(\mathbb{1}_p)$. Again, we introduced an additional state ϕ, to ensure convergence to a constant point, whereas the term $[E^T(h(x) - \bar{y})]_i$ provides an integral action to reduce the output error at the node $i \in \mathcal{V}_e$.

Remark 1 (*Local and Exchanged Information*). According to (13), every controller at node $i \in \mathcal{V}_e$, measures $y_i = h_i(x_i)$ and compares it with the desired set point $\bar{y}_i$. Information on the marginal costs $(q_i\phi_i + r_i)$ is exchanged among neighbours over a communication network with a topology described by L^{com}. Controller (13) is therefore fully distributed. The output $g_i(\theta_i)$ is chosen to satisfy Constraint 1, and is discussed in more detail in the next subsection.

4.3. Feasibility of the control problem

To ensure feasibility of the controller design problem, we impose two assumptions on the controllers (12) and (13). The first assumption guarantees that the controllers are able to generate a (feedforward) control signal, that is required to attain a steady state of the system.

Assumption 4 (*Attainability of the Steady State*). Consider functions $f_k(\mu_k)$ and $g_i(\theta_i)$, in respectively (12) and (13). Let $\bar{u}$ be as in (9). There exists[5] an $\omega \in \mathbb{R}^m$, such that $[B^\dagger(E\bar{u}-d)+(I-B^\dagger B)\omega]_k \in \mathcal{R}(f_k)$ for all $k \in \mathcal{E}$. Furthermore, $\bar{u}_i \in \mathcal{R}(g_i)$ for all $i \in \mathcal{V}_e$.

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[4] A directed graph is balanced if the (weighted) in-degree is equal to the (weighted) out-degree of every node.

[5] If $B\bar{\lambda} = E\bar{u} - d$ has any solution $\bar{\lambda}$, then all solutions are given by $\bar{\lambda} = B^\dagger(E\bar{u} - d) + (I - B^\dagger B)\omega$, for an arbitrary vector $\omega \in \mathbb{R}^m$, where $B^\dagger$ denotes the Moore–Penrose pseudoinverse of B. The existence of a solution $\bar{\lambda}$ is shown in the proof of Lemma 2.

Moreover, the controllers (12) and (13) can be designed to satisfy constraints (10) and (11), by properly selecting $f(\mu)$ and $g(\theta)$. Since $\lambda = f(\mu)$ and $u = g(\theta)$, the following assumption is sufficient to ensure that the inputs and flows do not exceed their limitations.

Assumption 5 (*Controller Outputs*). Functions $f_k(\cdot)$ and $g_i(\cdot)$, in respectively (12) and (13), are continuously differentiable, strictly increasing and satisfy

$$\mathcal{R}(f_k) = (\lambda_k^-, \lambda_k^+)$$
$$\mathcal{R}(g_i) = (u_i^-, u_i^+), \tag{14}$$

for all $k \in \mathcal{E}$ and all $i \in \mathcal{V}_e$.

The property of $f_k(\mu_k)$ and $g_i(\theta_i)$ being continuously differentiable and strictly increasing functions, is exploited within the various proofs to establish the global convergence properties, and ensures e.g. the existence of an inverse function.

Before we analyse the stability of the system we investigate the properties of the steady state. To do so, we write system (3) in closed loop with controllers (12) and (13), obtaining

$$T_x\dot{x} = -Bf(\mu) + Eg(\theta) - d$$
$$T_\mu\dot{\mu} = B^T(h(x) - \bar{y}) - (f(\mu) - \xi)$$
$$T_\xi\dot{\xi} = f(\mu) - \xi \tag{15a}$$
$$T_\theta\dot{\theta} = -E^T(h(x) - \bar{y}) - (g(\theta) - \phi)$$
$$T_\phi\dot{\phi} = g(\theta) - \phi - QL^{com}(Q\phi + r). \tag{15b}$$

Any equilibrium of system (15) satisfies

$$0 = -Bf(\bar{\mu}) + Eg(\bar{\theta}) - d \tag{16a}$$
$$0 = B^T(h(\bar{x}) - \bar{y}) - (f(\bar{\mu}) - \bar{\xi}) \tag{16b}$$
$$0 = f(\bar{\mu}) - \bar{\xi} \tag{16c}$$
$$0 = -E^T(h(\bar{x}) - \bar{y}) - (g(\bar{\theta}) - \bar{\phi}) \tag{16d}$$
$$0 = (g(\bar{\theta}) - \bar{\phi}) - QL^{com}(Q\bar{\phi} + r). \tag{16e}$$

We will now show that under Assumptions 1–5 there exists at least one solution to (16) and all solutions (16) satisfy the control objectives.

Lemma 2 (*Equilibria*). Let Assumptions 1–4 hold. Then, there exists an equilibrium $(\bar{x}, \bar{\mu}, \bar{\xi}, \bar{\theta}, \bar{\phi})$ of system (15). Moreover, any equilibrium is such that $h(\bar{x}) = \bar{y}$ and $g(\bar{\theta}) = \bar{u}$, where $\bar{u}$ is the optimal control input given by (9).

Proof. To prove the statement, we first show that at least one equilibrium of system (15) exists. By Assumption 4, $\bar{u} \in \mathcal{R}(g)$, and we set $\bar{\theta} = g^{-1}(\bar{u})$. Also, we set $\bar{\phi} = \bar{u}$. Bearing in mind that $Q\bar{u} + r \in \text{im}(\mathbb{1}_p)$, we have that (16e) holds. Furthermore, by definition, $\bar{u}$ satisfies $\mathbb{1}_n^T(E\bar{u} - d) = 0$. Since the graph is connected (Assumption 1) and $\text{im}(B) = (\text{ker}(B^T))^\perp = (\text{im}(\mathbb{1}_n))^\perp$, we have that $E\bar{u} - d \in \text{im}(B)$. For this reason, there exists a $\bar{\lambda}$ satisfying $-B\bar{\lambda} + E\bar{u} - d = 0$, and any solution is given by $\bar{\lambda} = B^\dagger(E\bar{u} - d) + (I - B^\dagger B)\omega$, for an arbitrary vector $\omega \in \mathbb{R}^m$. By Assumption 4, there exists at least one ω such that $\bar{\lambda} \in \mathcal{R}(f_k)$. Taking such a $\bar{\lambda}$, setting $\bar{\xi} = \bar{\lambda}$ and $\bar{\mu} = f^{-1}(\bar{\lambda})$, shows that (16a), (16c) hold. Since $\bar{y} \in \mathcal{R}(h)$ (Assumption 2), setting $\bar{x} = h^{-1}(\bar{y})$ shows (16b) and (16d). Hence, there exists a state $(\bar{x}, \bar{\mu}, \bar{\xi}, \bar{\theta}, \bar{\phi})$ that satisfies Eqs. (16) and is therefore an equilibrium of (15).

Next, we show that any equilibrium $(\bar{x}, \bar{\mu}, \bar{\xi}, \bar{\theta}, \bar{\phi})$ necessarily satisfies $h(\bar{x}) = \bar{y}$ and $g(\bar{\theta}) = \bar{u}$, where $\bar{u}$ is the optimal control input given by (9). From (16c), $\bar{\xi} = f(\bar{\mu})$ holds and we will show that this implies that necessarily $h(\bar{x}) = \bar{y}$. By (16e), bearing in mind that L^{com} is the Laplacian of a balanced and strongly

connected graph (Assumption 3), we have that $\mathbb{1}_p^T Q^{-1}(g(\overline{\theta}) - \overline{\phi}) = 0$. This, together with (16d), implies that $\mathbb{1}_p^T Q^{-1} E^T(h(\overline{x}) - \overline{y}) = 0$. By (16b) and $\overline{\xi} = f(\overline{\mu})$, we also have $B^T(h(\overline{x}) - \overline{y}) = \mathbf{0}$.

Hence, $\begin{bmatrix} \mathbb{1}_p^T Q^{-1} E^T \\ B^T \end{bmatrix}(h(\overline{x}) - \overline{y}) = \mathbf{0}$. We now prove that necessarily $h(\overline{x}) - \overline{y} = \mathbf{0}$. Suppose, *ad absurdum*, that there exists $v \neq \mathbf{0}$ such that $\begin{bmatrix} \mathbb{1}_p^T Q^{-1} E^T \\ B^T \end{bmatrix} v = \mathbf{0}$. By Assumption 1, it follows that $v = \mathbb{1}_n v_*$ with v_* a scalar. Then $\mathbb{1}_p^T Q^{-1} E^T \mathbb{1}_n v_* = 0$, which is, by definition of E in (4) and the fact that $p \geq 1$, equivalent to $\mathbb{1}_p^T Q^{-1} \mathbb{1}_p v_* = 0$. This implies that $v_* = 0$, contradicting that $v = \mathbb{1}_n v^* \neq \mathbf{0}$. Hence, necessarily $h(\overline{x}) - \overline{y} = \mathbf{0}$ and by strict monotonicity of $h(\cdot)$, we must have that $\overline{x} = h^{-1}(\overline{y})$. Since $h(x) = \overline{y}$, it follows from (16d) that $g(\overline{\theta}) = \overline{\phi}$, and by strict monotonicity of $g(\theta)$, that $\overline{\theta} = g^{-1}(\overline{\phi})$. Moreover, from (16e) we obtain that $L^{com}(Q\overline{\phi} + r) = \mathbf{0}$, and since the communication graph is strongly connected due to Assumption 3, we have that $Q\overline{\phi} + r \in \text{im}(\mathbb{1}_p)$. Since $\mathbb{1}_n^T B = \mathbf{0}$, we obtain from (16a) that $\mathbb{1}_n^T(Eg(\overline{\theta}) - d) = 0$. Bearing in mind that $\overline{u}$ satisfies $Q\overline{u} + r \in \text{im}(\mathbb{1}_p)$ and $\mathbb{1}_n^T(E\overline{u} - d) = 0$, we have consequently that $g(\overline{\theta}) = \overline{\phi} = \overline{u}$, with $\overline{u}$ as in (9). ∎

As a consequence of Lemma 2 we have that if Assumptions 1–4 hold, system (15) is equivalent to

$$T_x \dot{x} = -B(f(\mu) - f(\overline{\mu})) + E(g(\theta) - g(\overline{\theta}))$$
$$T_\mu \dot{\mu} = B^T(h(x) - h(\overline{x}))$$
$$\quad - ((f(\mu) - f(\overline{\mu})) - (\xi - \overline{\xi})) \qquad (17a)$$
$$T_\xi \dot{\xi} = (f(\mu) - f(\overline{\mu})) - (\xi - \overline{\xi})$$

$$T_\theta \dot{\theta} = -E^T(h(x) - h(\overline{x}))$$
$$\quad - (g(\theta) - g(\overline{\theta})) + (\phi - \overline{\phi}) \qquad (17b)$$
$$T_\phi \dot{\phi} = (g(\theta) - g(\overline{\theta})) - (\phi - \overline{\phi})$$
$$\quad - QL^{com}Q(\phi - \overline{\phi}),$$

a form that will be exploited in the stability analysis.

5. Stability analysis

In this section we analyse the stability of the closed-loop system (15). The analysis is foremost based on LaSalle's invariance principle and exploits useful properties of interconnected incrementally passive systems. To facilitate the discussion, we first recall the following definition:

Definition 1 (*Incremental Passivity*). System $\dot{x} = f(x, u)$, $y = h(x)$, $x \in \mathcal{X}$, $\mathcal{X}$ the state space, $u, y \in \mathbb{R}^n$, is incrementally passive[6] with respect to a constant triplet $(\overline{x}, \overline{u}, \overline{y})$ satisfying $\mathbf{0} = f(\overline{x}, \overline{u})$, $\overline{y} = h(\overline{x})$, if there exists a continuously differentiable and radially unbounded function $V(x, \overline{x}) : \mathcal{X} \to \mathbb{R}$, such that for all $x \in \mathcal{X}$, $u \in \mathbb{R}^m$ and $y = h(x)$, $\overline{y} = h(\overline{x})$ it holds that $\dot{V}(\cdot) = \frac{\partial V}{\partial x} f(x, u) \leq (y - \overline{y})^T(u - \overline{u})$.

We now proceed with establishing the incremental passivity property of (15a), that is the proposed flow controller (12) renders the network dynamics (3) incrementally passive with respect to the input $Eg(\theta)$ and output $h(x)$.

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[6] With some abuse of terminology, we state the incremental passivity property with respect to a steady state solution. This is in contrast to the 'usual' definition where the incremental passivity property holds with respect to any solution (Pavlov & Marconi, 2008). Note that the incremental passivity property might hold with respect to multiple/all steady state solutions.

Lemma 3 (*Incremental Passivity of (15a)*). *Let Assumptions 1–4 hold. System (15a) with input $Eg(\theta)$ and output $h(x)$ is incrementally passive with respect to the constant $(\overline{x}, \overline{\mu}, \overline{\xi})$ satisfying (16a)–(16c). Namely, the radially unbounded storage function $V_1(x, \overline{x}, \mu, \overline{\mu}, \xi, \overline{\xi})$ satisfies*

$$\dot{V}_1(\cdot) = (h(x) - h(\overline{x}))^T E(g(\theta) - g(\overline{\theta}))$$
$$\quad - (f(\mu) - \xi)^T(f(\mu) - \xi), \qquad (18)$$

along the solutions to (15a).

Proof. Consider the storage function

$$V_1(x, \overline{x}, \mu, \overline{\mu}, \xi, \overline{\xi}) = \sum_{i \in \mathcal{V}} T_{x_i} \int_{\overline{x}_i}^{x_i} h_i(y) - h_i(\overline{x}_i) dy$$
$$+ \sum_{k \in \mathcal{E}} T_{\mu_k} \int_{\overline{\mu}_k}^{\mu_k} f_k(y) - f_k(\overline{\mu}_k) dy$$
$$+ \frac{1}{2}(\xi - \overline{\xi})^T T_\xi (\xi - \overline{\xi}).$$

Since $h_i(x_i)$ and $f_k(\mu_k)$ are strictly increasing functions, the incremental storage function $V_1(\cdot)$ is radially unbounded. Furthermore, bearing in mind that $f(\overline{\mu}) = \overline{\xi}$, $V_1(\cdot)$ indeed satisfies (18) along the solutions to (15a), or equivalently along the solutions to (17a). ∎

We now prove a similar result for (15b), that is the controller (13) is incrementally passive with respect to the input $-h(x)$ and output $Eg(\theta)$.

Lemma 4 (*Incremental Passivity of (15b)*). *Let Assumptions 1–4 hold. System (15b) with input $-h(x)$ and output $Eg(\theta)$ is incrementally passive with respect to $(\overline{\theta}, \overline{\phi})$ satisfying (16d)–(16e). Namely, the radially unbounded storage function $V_2(\theta, \overline{\theta}, \phi, \overline{\phi})$ satisfies*

$$\dot{V}_2(\cdot) = -(g(\theta) - \phi)^T(g(\theta) - \phi)$$
$$\quad - (\phi - \overline{\phi})^T QL^{com}Q(\phi - \overline{\phi}) \qquad (19)$$
$$\quad - (g(\theta) - g(\overline{\theta}))^T E^T(h(x) - h(\overline{x})),$$

along the solutions to (15b).

Proof. Consider the storage function

$$V_2(\theta, \overline{\theta}, \phi, \overline{\phi}) = \sum_{i \in \mathcal{V}} T_{\theta_i} \int_{\overline{\theta}_i}^{\theta_i} g_i(y) - g_i(\overline{\theta}_i) dy$$
$$+ \frac{1}{2}(\phi - \overline{\phi})^T T_\phi (\phi - \overline{\phi}). \qquad (20)$$

Note that since $g_i(\theta_i)$ is a strictly increasing function, the incremental storage function $V_2(\cdot)$ is radially unbounded. Furthermore, since $g(\overline{\theta}) = \overline{\phi}$, $V_2(\cdot)$ indeed satisfies (19) along the solutions to (15b), or equivalently along the solutions to (17b). ∎

Exploiting the previous lemmas, we are now ready to prove the main result of this paper.

Theorem 1 (*Solving Problem 1 for System (3)*). *Let Assumptions 1–5 hold. The solutions to system (3), in closed loop with (12) and (13), globally converge to a point in the set*

$$\Upsilon_1 = \left\{ x, \mu, \xi, \theta, \phi \; \middle| \; \begin{array}{l} B(f(\mu) - f(\overline{\mu})) = \mathbf{0}, \\ B(\xi - \overline{\xi}) = \mathbf{0}, \\ x = \overline{x}, \theta = \overline{\theta}, \phi = \overline{\phi} \end{array} \right\}, \qquad (21)$$

where $\lambda = f(\mu)$ is a constant, $h(x) = \overline{y}$ and where $u = g(\theta) = \overline{u}$, with $\overline{u}$ the optimal input given by (9). Moreover, $u = g(\theta)$ and $\lambda = f(\mu)$ satisfy constraints (10) and (11) for all $t \geq 0$. Therefore, controllers (12) and (13) solve Problem 1 for the flow network (3).

Proof. Satisfying constraints (10) and (11) for all $t \geq 0$ follows from the design of $g(\theta)$ and $f(\mu)$ and Assumption 5. Let

$$V(\cdot) = V_1(x, \bar{x}, \mu, \bar{\mu}, \xi, \bar{\xi}) + V_2(\theta, \bar{\theta}, \phi, \bar{\phi}), \tag{22}$$

with $V_1(\cdot)$ and $V_2(\cdot)$ given in Lemma 3 and Lemma 4, respectively. Consequently, $V(\cdot)$ satisfies

$$\begin{aligned}
\dot{V}(\cdot) = &- (\phi - \bar{\phi})^T Q L^{com} Q(\phi - \bar{\phi}) \\
&- (g(\theta) - \phi)^T (g(\theta) - \phi) \\
&- (f(\mu) - \xi)^T (f(\mu) - \xi),
\end{aligned} \tag{23}$$

along the solutions to (15). From (23) we have that $\dot{V}(\cdot) \leq 0$, and since $V(\cdot)$ is radially unbounded, the solutions to (15) approach the largest invariant set contained entirely in the set $\mathcal{S}_1$, where $\dot{V}(\cdot) = 0$. This set is characterized by

$$\mathcal{S}_1 = \left\{ x, \mu, \xi, \theta, \phi \;\middle|\; \begin{array}{l} \phi = g(\theta), \xi = f(\mu), \\ Q(\phi - \bar{\phi}) \in \operatorname{im}(\mathbb{1}) \end{array} \right\}, \tag{24}$$

where $Q(\phi + \bar{\phi}) \in \operatorname{im}(\mathbb{1}_p)$ follows from Assumption 3. On the set $\mathcal{S}_1$, system (15) therefore satisfies

$$T_x \dot{x} = - B(f(\mu) - f(\bar{\mu})) + E(\phi - \bar{\phi}) \tag{25a}$$
$$T_\mu \dot{\mu} = B^T(h(x) - h(\bar{x})) \tag{25b}$$
$$T_\xi \dot{\xi} = \mathbf{0} \tag{25c}$$
$$T_\theta \dot{\theta} = - E^T(h(x) - h(\bar{x})) \tag{25d}$$
$$T_\phi \dot{\phi} = \mathbf{0}. \tag{25e}$$

Due to (24), (25c) and (25e) we have that

$$\dot{\mu} = \left(\frac{\partial f(\mu)}{\partial \mu} \right)^{-1} \dot{\xi} = \mathbf{0} \tag{26}$$

$$\dot{\theta} = \left(\frac{\partial g(\theta)}{\partial \theta} \right)^{-1} \dot{\phi} = \mathbf{0}. \tag{27}$$

where we note that $\frac{\partial f(\mu)}{\partial \mu} \neq \mathbf{0}$ and $\frac{\partial g(\theta)}{\partial \theta} \neq \mathbf{0}$. It follows now from (25b), (25d), (26) and (27) that

$$\begin{bmatrix} B^T \\ -E^T \end{bmatrix} (h(x) - h(\bar{x})) = \mathbf{0}. \tag{28}$$

We recall that $\begin{bmatrix} B & -E \end{bmatrix}^T$ has full column rank and therefore has a left inverse. As a result, we have that necessarily $h(x) - h(\bar{x}) = \mathbf{0}$, i.e. $h(x) = \bar{y}$. By strict monotonicity of $h(x)$, it follows that on the invariant set $x = \bar{x}$ and that $\dot{x} = \mathbf{0}$.

Premultiplying both sides of (25a) by $\mathbb{1}_p^T$, yields $0 = \mathbb{1}_p^T(\phi - \bar{\phi})$ and since $Q(\phi - \bar{\phi}) \in \operatorname{im}(\mathbb{1}_p)$, where Q is a diagonal matrix with only strictly positive entries, it follows that on the set where $\dot{V} = 0$ necessarily $\phi = \bar{\phi}$. From (24) and (25a) it therefore follows that $B(f(\mu) - f(\bar{\mu})) = \mathbf{0}$ and $B(\xi - \bar{\xi}) = \mathbf{0}$. Moreover, since on the set $\mathcal{S}_1$, $\phi = g(\theta)$ and $\bar{\phi} = g(\bar{\theta})$, we also have that $g(\theta) = g(\bar{\theta}) = \bar{u}$ (see also Lemma 2). Consequently, system (15) indeed approaches the set Υ_1, where $h(x) = \bar{y}$ and where $u = g(\theta) = g(\bar{\theta}) = \bar{u}$, with $\bar{u}$ the optimal input given by (9). To prove convergence to a point in the set Υ_1, we note that Υ_1 consists of equilibria of (15). Since the incremental storage function $V(\cdot)$ can be defined with respect to any equilibrium in Υ_1, and since $V(\cdot) \leq 0$, every point in Υ_1 is a Lyapunov stable equilibrium of system (15). Consequently, every positive limit set associated with any solution to system (15) consists of Lyapunov stable equilibria. It then follows by Haddad and Chellaboina (2008, Theorem 4.20) that this positive limit set is a singleton, which proves convergence to a point. ∎

Remark 2 (*Avoiding Oscillations*)**.** In the proof of Theorem 1, we exploited the dynamics of the additional control variables ξ and ϕ to conclude that on the invariant set $\dot{\mu} = \dot{\theta} = \mathbf{0}$. It is natural to wonder if these additional controller states are essential to obtain the convergence result of Theorem 1. Therefore, we compare (12) and (13) with controllers of the form

$$\begin{aligned}
T_\mu \dot{\mu} &= B^T(h(x) - \bar{y}) \\
\lambda &= f(\mu)
\end{aligned} \tag{29}$$

$$\begin{aligned}
T_\theta \dot{\theta} &= - Q L^{com}(Qg(\theta) + r) - (h(x) - \bar{y}) \\
u &= g(\theta),
\end{aligned} \tag{30}$$

as both (12)–(13) and (29)-(30) admit a steady state where $h(\bar{x}) = \bar{y}$ and $g(\bar{\theta}) = \bar{u}$. However, in contrast to (15), for which we have proven global convergence to the desired state, system

$$\begin{aligned}
T_x \dot{x} &= - Bf(\mu) + Eg(\theta) - d \\
T_\mu \dot{\mu} &= B^T(h(x) - \bar{y}) \\
T_\theta \dot{\theta} &= - Q L^{com}(Qg(\theta) + r) - (h(x) - \bar{y}),
\end{aligned} \tag{31}$$

can converge (depending on E and Q) to a limit cycle exhibiting oscillatory behaviour as has been shown in Scholten et al. (2016). To illustrate this claim, consider the linear case, where $f(\mu) = \mu$, $g(\theta) = \theta$ and $h(x) = x$. Introducing $\tilde{x} = x - \bar{x}$, $\tilde{\mu} = \mu - \bar{\mu}$, $\tilde{\theta} = \theta - \bar{\theta}$, and assuming $E = I$, $Q = \bar{q}I$ with $\bar{q} \in \mathbb{R}$, system (31) is written as

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\mu}} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -B & I \\ B^T & \mathbf{0} & \mathbf{0} \\ -I & \mathbf{0} & -\bar{q}^2 L^{com} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\mu} \\ \tilde{\theta} \end{bmatrix}. \tag{32}$$

It can be readily confirmed that the solution to (32), with initial conditions $\tilde{x}(0) = \mathbf{0}$, $\tilde{\mu}(0) = \mathbf{0}$, and $\tilde{\theta}(0) = \mathbb{1}_n$, is given by $\tilde{x}(t) = \mathbb{1}_n \sin(t)$, $\tilde{\mu}(t) = \mathbf{0}$, $\tilde{\theta}(t) = \mathbb{1}_n \cos(t)$, which indeed clearly exhibits oscillatory behaviour.

6. Physical flow dynamics

In the previous discussion we focussed on the design of dynamical flow controllers. On the other hand, flows in networks might follow from underlying physical principles that are not accurately described by (12). An important example is the case where the flow λ_k directly depends on the states x_i of its adjacent nodes. This is common in *e.g.* compartmental systems (see *e.g.* Bauso et al. (2013), Blanchini et al. (2016) and Como (2017)). Another example is when a change of λ is induced by the dynamics of the system, instead of a controller that is up to design. We discuss in Section 6.2 an important example where the flow dynamics are induced by 'potential differences'. First we discuss how certain compartmental systems fit within the presented setting.

6.1. Compartmental systems

Since (3) shows similarities with those in compartmental systems, it is natural to wonder how these models are related. Compared to (3), compartmental systems have additional terms that model state dependent inflows, outflows and flows between nodes. In this section we incorporate such terms in our framework by augmenting (3), resulting in

$$T_x \dot{x} = \Psi(x) - B\lambda + Eu - d \tag{33a}$$
$$y = h(x), \tag{33b}$$

where $\Psi(x) = -B_c \gamma(B_c^T h(x)) - E_c \eta(E_c^T h(x))$. Here, B_c is the incidence matrix of a (not necessarily connected) graph $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}_c)$, representing the interconnection of the compartments (Blanchini

et al., 2016). Moreover, the set of nodes that have a state dependent inflow/outflow is given by $\mathcal{V}_c \subseteq \mathcal{V}$, with cardinality $p_c := |\mathcal{V}_c|$. Matrix $E_c \in \mathbb{R}^{n \times p_c}$ is used to indicate the locations of the p_c state dependent inflows/outflows and its entries are defined as

$$(e_c)_{ik} = \begin{cases} 1 & \text{if the } k\text{th flow is located at node } i \\ 0 & \text{otherwise.} \end{cases}$$

Let $l := |\mathcal{E}_c|$. The mapping $\gamma : \mathbb{R}^l \to \mathbb{R}^l$ is given by $\gamma(a) = (\gamma_1(a_1) \ldots \gamma_l(a_l))^T$, with $a = B_c^T h(x)$, $a_k = [B_c^T h(x)]_k$ and $\gamma_k(a_k)$ nondecreasing and continuously differentiable for all $k \in \mathcal{E}_c$. The term $B_c \gamma(B_c^T h(x))$ models the flow between nodes as a result of potential differences. The mapping $\eta : \mathbb{R}^{e_c} \to \mathbb{R}^{e_c}$ is given by $\eta(b) = (\eta_1(b_1) \ldots \eta_{e_c}(b_{e_c}))^T$, with $b = E_c^T h(x)$, $b_i = [E_c^T h(x)]_i$ and $\eta_i(b_i)$ is nondecreasing and continuously differentiable for all $i \in \mathcal{V}_c$. The term $E_c \eta(E_c^T h(x))$ models the inflow to or the outflow from the system, at a node, proportionally to the potential at the corresponding node (Riaza, 2017).

Remark 3 (*Interpretation of $\Psi(x)$*). System (33) models a compartmental system with additional actuated edges (*e.g.* flows controlled by a pump) and actuated inputs. The actuation allows us to achieve output regulation and an optimal coordination of the inputs among the nodes, in the presence of unknown disturbances. In absence of such actuation, most works on compartmental systems relied on some form of proportional control to achieve practical stabilization (Bauso et al., 2013) or uniform global stabilization (Blanchini et al., 2016; Como, 2017) of the origin. In this work, since we are interested in regulating the outputs to desired setpoints in spite of unmeasured constant disturbances and with optimal steady state control inputs at the nodes, we rely on a more complex control structure and, in contrast to Bauso et al. (2013), we do not minimize possible costs associated with the flows. In some cases, the flow on an edge is proportional to the potential of one of its adjacent nodes (e.g. the flow from a reservoir to another due to gravity). We do not consider this case here and leave the corresponding analysis to a future work.

The optimal control allocation problem (8) now becomes

$$\begin{aligned} & \underset{u, \lambda}{\text{minimize}} && C(u) \\ & \text{subject to} && \mathbf{0} = \Psi(\overline{x}) - B\lambda + Eu - d, \end{aligned} \tag{34}$$

where again $\overline{x}_i = h_i^{-1}(\overline{y}_i)$ for all $i \in \mathcal{V}$. Similar to Lemma 1, the following can be immediately shown:

Lemma 5 (*Solution to Optimization Problem* (34)). *The solution to* (34) *is given by*

$$\hat{u} = Q^{-1}(\hat{\kappa} - r), \tag{35}$$

where $\hat{\kappa} = E^T \frac{\mathbb{1}_n \mathbb{1}_n^T}{\mathbb{1}_p^T Q^{-1} \mathbb{1}_p}(\hat{d} + EQ^{-1}r)$, *and* $\hat{d} = d + E_c \eta(E_c^T h(\overline{x}))$.

Due to the new network dynamics (33) and optimal control input $\hat{u}$ in the network, Assumption 4 needs to be revisited.

Assumption 6 (*Attainability Revisited*). Consider functions $f_k(\cdot)$ and $g_i(\cdot)$, in respectively (12) and (13). Let $\hat{u}$ be as in (35). There exists an $\omega \in \mathbb{R}^m$, such that $[B^\dagger(\Psi(\overline{x}) + E\hat{u} - d) + (I - B^\dagger B)\omega]_k \in \mathcal{R}(f_k)$ for all $k \in \mathcal{E}$. Furthermore, $\hat{u}_i \in \mathcal{R}(g_i)$ for all $i \in \mathcal{V}_e$.

With the assumption above, we can prove, similarly as Lemma 2, the existence of a steady state for system (12), (13), (33). The argumentation is along the lines of the proof of Lemma 2 and we omit the details. We can now prove the following result:

Theorem 2 (*Solving Problem 1 for System* (33)). *Let Assumptions 1 – 3 and 5–6 hold. The solutions to system* (33), *in closed loop with* (12) *and* (13), *globally converge to point in the set*

$$\Upsilon_2 = \left\{ x, \mu, \xi, \theta, \phi \;\middle|\; \begin{array}{l} B(f(\mu) - f(\overline{\mu})) = \mathbf{0}, \\ B(\xi - \overline{\xi}) = \mathbf{0}, \\ x = \overline{x}, \theta = \overline{\theta}, \phi = \overline{\phi} \end{array} \right\}, \tag{36}$$

where $\lambda = f(\mu)$ *is a constant,* $h(x) = \overline{y}$ *and where* $u = g(\theta) = \hat{u}$, *with* $\hat{u}$ *given by* (35). *Moreover,* $u = g(\theta)$ *and* $\lambda = f(\mu)$ *satisfy constraints* (10) *and* (11) *for all* $t \geq 0$. *Therefore, controllers* (12) *and* (13) *solve Problem 1 for the flow network* (33).

Proof. First, the fulfilment of the constraints (10) and (11) for all $t \geq 0$ is guaranteed by the design of the controllers. Second, a straightforward adjustment of the arguments of Theorem 1 shows that the same incremental storage function (22), used in Theorem 1, now satisfies

$$\begin{aligned} \dot{V}(\cdot) = {} & (h(x) - h(\overline{x}))^T (\Psi(x) - \Psi(\overline{x})) \\ & - (\phi - \overline{\phi})^T Q L^{com} Q(\phi - \overline{\phi}) \\ & - (g(\theta) - \phi)^T (g(\theta) - \phi) \\ & - (f(\mu) - \xi)^T (f(\mu) - \xi), \end{aligned} \tag{37}$$

along the solutions to (33) in closed loop with (12) and (13). We continue by showing that the additional term in $\dot{V}(\cdot)$ (comparing with the expression of $\dot{V}(\cdot)$ in (23)) satisfies $(h(x) - h(\overline{x}))^T (\Psi(x) - \Psi(\overline{x})) \leq 0$. In fact, since $\gamma(\cdot)$ and $\eta(\cdot)$ are componentwise increasing mappings and by application of Hadamard's lemma we have that[7]

$$(h(x) - h(\overline{x}))^T (\Psi(x) - \Psi(\overline{x})) \tag{38}$$
$$\begin{aligned} = {} & -(h(x) - h(\overline{x}))^T B_c (\gamma(B_c^T h(x)) - \gamma(B_c^T h(\overline{x}))) \\ & -(h(x) - h(\overline{x}))^T E_c (\eta(E_c^T h(x)) - \eta(E_c^T h(\overline{x}))) \\ = {} & -(h(x) - h(\overline{x}))^T B_c \Gamma^b(x) B_c^T (h(x) - h(\overline{x})) \\ & -(h(x) - h(\overline{x}))^T E_c \Gamma^e(x) E_c^T (h(x) - h(\overline{x})) \leq 0, \end{aligned}$$

where $\Gamma^b(x)$ and $\Gamma^e(x)$ are diagonal matrices with entries

$$\Gamma_{kk}^b(x) = \int_0^1 \left.\frac{\partial \gamma_k(y_k)}{\partial y_k}\right|_{y_k = \tau(\chi_k^b(x) - \chi_k^b(\overline{x})) + \chi_k^b(\overline{x})} d\tau \tag{39}$$

$$\Gamma_{ii}^e(x) = \int_0^1 \left.\frac{\partial \eta_i(y_i)}{\partial y_i}\right|_{y_i = \tau(\chi_i^e(x) - \chi_i^e(\overline{x})) + \chi_i^e(\overline{x})} d\tau, \tag{40}$$

with $\chi_k^b(x) = [B_c^T h(x)]_k$ and $\chi_i^e(x) = [E_c^T h(x)]_i$. Note that $\Gamma_{kk}^b(x)$, $\Gamma_{ii}^e(x) \geq 0$ for any x, since $\gamma_k(\cdot)$ and $\eta_i(\cdot)$ are increasing functions for all $k \in \mathcal{E}_c$ and all $i \in \mathcal{V}_c$. Therefore, $V(\cdot)$ satisfies

$$\begin{aligned} \dot{V}(\cdot) \leq {} & -(\phi - \overline{\phi})^T Q L^{com} Q(\phi - \overline{\phi}) \\ & - (g(\theta) - \phi)^T (g(\theta) - \phi) \\ & - (f(\mu) - \xi)^T (f(\mu) - \xi), \end{aligned} \tag{41}$$

along the solutions to (33) in closed loop with (12) and (13). Note that expression (41) is identical to (23), that is used to prove Theorem 1. Similar to the proof of Theorem 1, we can argue that $x = \overline{x}$, by exploiting the relations (25b)–(25e). Therefore, on the invariant set where $\dot{V}(\cdot) = 0$,

$$T_x \dot{x} = \Psi(x) - \Psi(\overline{x}) - B(f(\mu) - f(\overline{\mu})) + E(\phi - \overline{\phi}), \tag{42}$$

reduces to (25a), such that system (33) in closed loop with (12) and (13), is on the invariant set identical to (25). From here, the proof follows the same steps as the proof of Theorem 1. ∎

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[7] According to Hadamard's lemma, one can write $\gamma(z) - \gamma(\overline{z}) = \Gamma(z)(z - \overline{z})$, where $\Gamma_i(z) \geq 0$ if $\gamma_i(z)$ is an increasing function. A similar argument applies to the mapping η.

6.2. Potential induced flow dynamics

In this subsection we study a network where the flow dynamics are given by the following expression:

$$T_\mu \dot{\mu} = B^T(h(x) - \overline{y})$$
$$\lambda = f(\mu), \tag{43}$$

that has been studied in the context of networked systems in *e.g.* Bürger et al. (2015), Bürger, Zelazo, and Allgöwer (2014) and van der Schaft and Wei (2012). Also, it describes the behaviour of inductive lines in an electric network (see also the case study on a super-conducting DC network in Section 7.2).

The dynamics (43) coincide with (12), if one neglects the terms depending on the now missing state ξ. In fact, (43) can generate the same steady state output as (12) and also shares an incremental passivity property. However, as we pointed out in Remark 2, the state ξ is essential to derive the convergence result in Theorem 1. On the other hand, by carefully selecting nodes that have a controllable external input, the controllers (13) and (43) still solve Problem 1 for the flow network (3). This choice is based on the notion of a zero forcing set (see *e.g.*, Hogben (2010), Monshizadeh et al. (2014), Trefois and Delvenne (2015)), which we review next.

Consider the graph $\mathcal{G}$ and let us initially colour each of its nodes either black or white. The colour of the nodes then changes according to the following colouring rule:

Graph colouring rule *If node i is coloured black and has exactly one neighbour j which is white, then the colour of node j is changed to black.*

Let $\mathcal{V}_0 \subseteq \mathcal{V}$ be the set of nodes which are initially coloured black, while the remaining ones are white, and let $\mathcal{C}(\mathcal{V}_0)$ be the set of black nodes obtained by applying the colour changing rule until no more changes are possible. A zero forcing set is then defined as:

Definition 2 (*Zero Forcing Set*). If $\mathcal{V}_0 \subseteq \mathcal{V}$ satisfies $\mathcal{C}(\mathcal{V}_0) = \mathcal{V}$ then $\mathcal{V}_0$ is a zero forcing set for $\mathcal{G}$.

We now make a connection between a zero forcing set and the set $\mathcal{V}_e$ of nodes that have actuation (*i.e.*, all nodes that correspond to the rows of E that contain a non-zero entry).

Assumption 7 ($\mathcal{V}_e$ *is a Zero Forcing Set*). The set $\mathcal{V}_e$ is a zero forcing set for $\mathcal{G}$.

An example of a zero forcing set is provided previously in Fig. 2, where the black nodes indeed form a zero forcing set for the physical network.

We are now ready to state the second result of this section.

Theorem 3 (*Solving Problem 1 with* (43)). *Let Assumptions 1–5 and 7 hold. The solutions to system (3), in closed loop with the controllers (13) and (43), globally converge to a point in the set*

$$\varUpsilon_3 = \left\{ x, \mu, \theta, \phi \,\middle|\, \begin{array}{l} B(f(\mu) - f(\overline{\mu})) = \mathbf{0}, \\ x = \overline{x}, \theta = \overline{\theta}, \phi = \overline{\phi} \end{array} \right\}, \tag{44}$$

where $\lambda = f(\mu)$ is a constant, $h(x) = \overline{y}$ and where $u = g(\theta) = \overline{u}$, with $\overline{u}$ given by (9). Moreover, $u = g(\theta)$ and $\lambda = f(\mu)$ satisfy constraints (10) and (11) for all $t \geq 0$. Therefore, controllers (13) and (43) solve Problem 1 for the flow network (3).

Proof. Following the argumentation of the proof of Theorem 1, using the same incremental storage function (22), allows us to conclude that the solutions to the system (3), (13), (43) approach

the largest invariant set contained in the set where $\dot{V}(\cdot) = 0$. This set, where $\dot{V}(\cdot) = 0$, is now characterized by

$$S_3 = \left\{ x, \mu, \theta, \phi \,\middle|\, \phi = g(\theta), Q(\phi - \overline{\phi}) \in \operatorname{im}(\mathbb{1}_p) \right\}. \tag{45}$$

System (3), (13), (43) satisfies on this set

$$T_x \dot{x} = -B(f(\mu) - f(\overline{\mu})) + E(\phi - \overline{\phi}) \tag{46a}$$

$$T_\mu \dot{\mu} = B^T(h(x) - h(\overline{x})) \tag{46b}$$

$$\mathbf{0} = -E^T(h(x) - h(\overline{x})) \tag{46c}$$

$$T_\phi \dot{\phi} = \mathbf{0}. \tag{46d}$$

We now prove by induction that $h_i(x_i) = h_i(\overline{x}_i)$ for all $i \in \mathcal{V}$. To this end, let us define the sequence of sets of nodes $\mathcal{V}_k \subseteq \mathcal{V}$, with $k \in \mathbb{N}_{\geq 0}$, having the properties:

(i) $\mathcal{V}_k$ is a zero forcing set;

(ii) on the largest invariant set for (3), (13), (43) contained in S_3, it holds that $h_i(x_i) = h_i(\overline{x}_i)$ for all $i \in \mathcal{V}_k$.

Let the cardinality of $\mathcal{V}_k$ be denoted by n_k. In order to show that $h_i(x_i) = h_i(\overline{x}_i)$ for all $i \in \mathcal{V}$ we will prove that there exists an index $\overline{k}$ such that $n_{\overline{k}} = n$, where $\mathcal{V}_{\overline{k}}$ satisfies properties (i) and (ii). Recall that $|\mathcal{V}| = n$.

First, we note that Assumption 7 and (46c) imply that $\mathcal{V}_e$ satisfies properties (i) and (ii). For this reason, we can set $\mathcal{V}_0 := \mathcal{V}_e$ and $n_0 := p > 0$ that satisfies properties (i) and (ii). If $n_0 = n$, then $\overline{k} = 0$, otherwise $n_0 < n$ and we proceed as follows.

For a $k \in \mathbb{N}_{\geq 0}$, we consider a set of nodes $\mathcal{V}_k$ of cardinality $0 < n_k < n$ satisfying properties (i) and (ii) above. We will show that this implies that there exists a set of nodes $\mathcal{V}_{k+1}$ that satisfies properties (i) and (ii) with $n_k < n_{k+1}$. Let us define $B^{(k)} = \begin{bmatrix} B^{\mathcal{B}(k)} \\ B^{\mathcal{W}(k)} \end{bmatrix}$, where the matrices $B^{\mathcal{B}(k)} \in \mathbb{R}^{n_k \times m}$ and $B^{\mathcal{W}(k)} \in \mathbb{R}^{(n-n_k) \times m}$ are obtained by collecting from B the rows indexed by $\mathcal{V}_k$ and $\mathcal{V} \setminus \mathcal{V}_k$, respectively. Note that $B^{(k)}$ is obtained from B by reordering of the rows, and that $B^{\mathcal{B}(k)}$ and $B^{\mathcal{W}(k)}$ are the rows of B corresponding to the black and white nodes, respectively. Similarly, for any vector $\chi \in \mathbb{R}^n$ let $\chi^{\mathcal{B}(k)} \in \mathbb{R}^{n_k}$ and $\chi^{\mathcal{W}(k)} \in \mathbb{R}^{n-n_k}$ be obtained by collecting from χ the elements indexed by $\mathcal{V}_k$ and $\mathcal{V} \setminus \mathcal{V}_k$ respectively. We note that, by property (ii), on the largest invariant set, the set $\mathcal{V}_k$ fulfils $(h(x) - h(\overline{x}))^{\mathcal{B}(k)} = \mathbf{0}$. More explicitly, $h_i(x_i) - h_i(\overline{x}_i) = 0$ for all $i \in \mathcal{V}_k$. By the strict monotonicity of $h_i(x_i)$, it follows that on the invariant set $x_i = \overline{x}_i$ for all $i \in \mathcal{V}_k$. Since $\frac{d}{dt}(\phi - \overline{\phi}) = \mathbf{0}$ due to (46d), on the invariant set we have, by (46a) and (46b), that

$$T_x \begin{bmatrix} \mathbf{0} \\ \ddot{x}^{\mathcal{W}(k)} \end{bmatrix} = -B^{(k)} \frac{\partial f(\mu)}{\partial \mu} B^{(k)T} \begin{bmatrix} \mathbf{0} \\ (h(x) - h(\overline{x}))^{\mathcal{W}(k)} \end{bmatrix},$$

from which it follows that

$$\mathbf{0} = -B^{\mathcal{B}(k)} \frac{\partial f(\mu)}{\partial \mu} B^{\mathcal{W}(k)T}(h(x) - h(\overline{x}))^{\mathcal{W}(k)}. \tag{47}$$

Note that $B^{\mathcal{B}(k)} \frac{\partial f(\mu)}{\partial \mu} B^{\mathcal{W}(k)T}$ is the right-upper block of the Laplacian matrix $B^{(k)} \frac{\partial f(\mu)}{\partial \mu} B^{(k)T}$ with strictly positive weight matrix, since $f_k(\mu_k)$ is strictly increasing, such that $\frac{\partial f_k(\mu_k)}{\partial \mu_k} > 0$ for all $k \in \mathcal{E}$. The non-zero entries in $B^{\mathcal{B}(k)} \frac{\partial f(\mu)}{\partial \mu} B^{\mathcal{W}(k)T}$ correspond to pairs of exactly one black and one white node that are connected via an edge. Therefore we have that each row i of $B^{\mathcal{B}(k)} \frac{\partial f(\mu)}{\partial \mu} B^{\mathcal{W}(k)T}$ (which corresponds to a black node) contains a strictly negative number at entry j if, and only if, node $n_k + j$ is a neighbour of the node i. By assumption we have that $\mathcal{V}_k$ is a zero forcing set and that $\mathcal{V}_k \subsetneq \mathcal{V}$, which implies that there exists at least one row of $B^{\mathcal{B}(k)} \frac{\partial f(\mu)}{\partial \mu} B^{\mathcal{W}(k)T}$ which contains exactly one non-zero entry. Let $\mathcal{U}_k$ be the set in which we collect the nodes that correspond to these rows and define $\mathcal{V}_{k+1} := \mathcal{V}_k \cup \mathcal{U}_k$. From (47), we have that $0 = h_i(x_i) - h_i(\overline{x}_i)$,

for all $i \in \mathcal{U}_k$ and therefore for all $i \in \mathcal{V}_{k+1}$. Moreover, since $\mathcal{V}_k \subset \mathcal{V}_{k+1}$, and since we assume that $\mathcal{V}_k$ is a zero forcing set for $\mathcal{G}$, also $\mathcal{V}_{k+1}$ is a zero forcing set for $\mathcal{G}$. This concludes the proof that there exists $\mathcal{V}_{k+1}$ that satisfies properties (i) and (ii), with $n_{k+1} > n_k$.

Since the number of nodes is finite, in a finite number of iterations $\bar{k}$ we arrive at a set $\mathcal{V}_{\bar{k}}$ where $n_{\bar{k}} = n$, i.e. $\mathcal{V}_{\bar{k}}$ coincides with $\mathcal{V}$ and has the property that on the largest invariant set for (3), (13), (43) contained in $\mathcal{S}_3$, $0 = h_i(x_i) - h_i(\bar{x}_i)$ for all $i \in \mathcal{V}$. From here, omitting the variable ξ, the proof follows, *mutatis mutandis*, the proof of Theorem 1, starting from the paragraph below (25). ∎

Remark 4 (*Relaxing Assumption 7*). In the case that $f(\mu) = \mu$ and $h(x) = x$, successive differentiations of (46c) yields

$$\mathbf{0} = \underbrace{\begin{bmatrix} -E^T \\ E^T Y \\ -E^T Y^2 \\ \vdots \\ (-1)^n E^T Y^{n-1} \end{bmatrix}}_{\mathcal{O}} (h(x) - h(\bar{x})), \qquad (48)$$

where $Y = T_x^{-1} B T_\mu^{-1} B^T$. To conclude that $h(x) = h(\bar{x})$, it is sufficient that the matrix $\mathcal{O}$ has full column rank, *i.e.* the pair (E^T, Y) is observable. Although, a similar argumentation can be performed with the nonlinear mappings $f(\mu)$ and $h(x)$, it does not immediately lead to a simple criterion that permits to conclude $h(x) = h(\bar{x})$.

After separately discussing the particular modifications to the flow network and controllers in Sections 6.1 and 6.2, we briefly discuss the combination of both in the corollary below:

Corollary 1 (*Combined Modifications*). *Let Assumptions 1– 3 and 5–7 hold. Consider the flow network (33) and let $\mathcal{V}_s \subseteq \mathcal{V}_c$ be defined as*[8]

$$\mathcal{V}_s = \{ i \in \mathcal{V}_c | \left. \frac{\eta_i(y_i)}{\partial y_i} \right|_{y_i = [E_c^T h(\bar{x})]_i} > 0 \}. \qquad (49)$$

If $\mathcal{V}_e \cup \mathcal{V}_s$ is a zero forcing set for $\mathcal{G}$, then the solutions to system (33), in closed loop with the controllers (13) and (43), globally converge to a point in the set

$$\Upsilon_4 = \left\{ x, \mu, \theta, \phi \middle| \begin{array}{l} B(f(\mu) - f(\bar{\mu})) = \mathbf{0}, \\ x = \bar{x}, \theta = \bar{\theta}, \phi = \bar{\phi} \end{array} \right\}, \qquad (50)$$

where $\lambda = f(\mu)$ is a constant, $h(x) = \bar{y}$ and where $u = g(\theta) = \bar{u}$, with $\bar{u}$ given by (9). Therefore, controllers (13) and (43) solve Problem 1 for the flow network (33).

Proof. Following a similar argumentation as in the proof of Theorem 3, $x_i = \bar{x}_i$ for all $i \in \mathcal{V}_e$. Moreover, the dynamics (33) give rise to an additional term in $\dot{V}(\cdot)$ in the same manner as in the proof of Theorem 2 (see (38)), namely: $-(h(x) - h(\bar{x}))^T E_c \Gamma^e(x) E_c^T (h(x) - h(\bar{x})) < 0$. Consequently, on the largest invariant set where $\dot{V}(\cdot) = 0$, also $x_i = \bar{x}_i$ for all $i \in \mathcal{V}_s$, since $\eta_i(y_i)$ is strictly increasing around $[E_c^T h(\bar{x})]_i$, for all $i \in \mathcal{V}_s$. From here the proof continues along the lines of the proof of Theorem 3. ∎
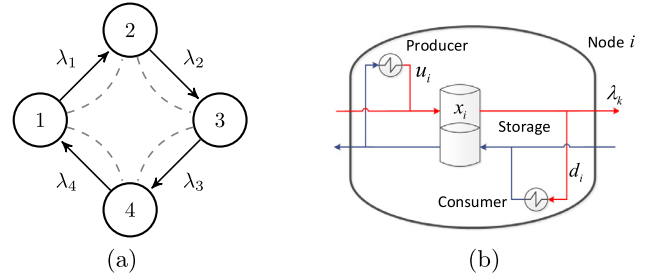
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[8] In Theorem 2, we only required $\eta_i(y_i)$ to be nondecreasing for all $i \in \mathcal{V}_c$, *i.e.* $\frac{\partial \eta_i(y_i)}{\partial y_i} \geq 0$.


Fig. 3. (a) Topology of the considered heat network. The arrows indicate the required flow directions in the heat network, while the dashed lines represent the communication network used by the controllers. (b) A node in the district heating network.

7. Case studies

To illustrate how physical systems can be regarded as a flow network and to show the performance of the proposed controllers we consider two case studies. The first case study considers a district heating system, whereas the second case study considers a super-conducting direct current (DC) network.

7.1. District heating system

Continuing our previous work in Scholten, De Persis, and Tesi (2015), we consider a district heating system with a topology as depicted in Fig. 3(a). Each node represents a producer, a consumer and a stratified storage tank (see Fig. 3(b)). The storage tank consists of a hot and a cold layer of water, both with variable volumes. We denote the volume of the hot layer of water at node i as x_i (m^3), which is also the measured output of the system, *i.e.* $h_i(x_i) = x_i$. The various nodes are interconnected via a pipe network $\mathcal{G}$. Following Scholten et al. (2015), the dynamics for the hot layer can be derived by applying mass conservation laws resulting in the following representation of the district heating system: $\dot{x} = -B\lambda + u - d$, where λ_k (m^3/h) denotes the flow through pipe k. Moreover, u_i (m^3/h) and d_i (m^3/h) are respectively the flow through the heat exchanger of the producer and the consumer at node i. It is immediate to see that the district heating system has identical dynamics as (3) if we set $T_x = I$. The controllers (12) and (13) are therefore applicable and we study the obtained closed-loop system.

We perform a simulation over a 40 h time interval in which we evaluate the response to a change in demand at $t = 12$ and change in setpoint at $t = 24$. The cost functions of the four producers are purely quadratic, *i.e.* $s = r = \mathbf{0}$. We take $Q = \text{diag}(\begin{array}{cccc} 10 & 9 & 7 & 6 \end{array})$. Initially the volume is $x(0) = \begin{bmatrix} 200 & 200 & 200 & 200 \end{bmatrix}^T$, which is also to the setpoint $\bar{x}(t)$ for all $t < 24$. The initial demand is given by $d(t) = \begin{bmatrix} 30 & 30 & 30 & 30 \end{bmatrix}^T$, for all $t < 12$, which is increased to $d(t) = \begin{bmatrix} 35 & 35 & 35 & 35 \end{bmatrix}^T$, for all $t \geq 12$. The setpoint for the volume $\bar{x}(t)$ is increased at $t = 24$ to $\bar{x}(t) = \begin{bmatrix} 210 & 210 & 210 & 210 \end{bmatrix}^T$, for all $t \geq 24$. To guarantee uni-directional flows and positive production we require $\lambda_k > 0$ and $u_i > 0$, for all $k, i \in \{1, 2, 3, 4\}$. Due to capacity constraints, we additional require them to be upper bounded by 14 m^3/h and 52 m^3/h, respectively. To enforce these constraints, the output of the controllers is designed as

$$\lambda_k = f_k(\mu_k) = 7(\tanh(\mu_k) + 1)$$
$$u_i = g_i(\theta_i) = 26(\tanh(\theta_i) + 1), \qquad (51)$$

where $\tanh(\cdot)$ is the hyperbolic tangent function. Finally, we let $T_\mu = I, T_\theta = I, T_\phi = 0.005 \cdot I$ and we set all the weights of L^{com} to 10 and we let it be undirected which implies that L^{com} is balanced.
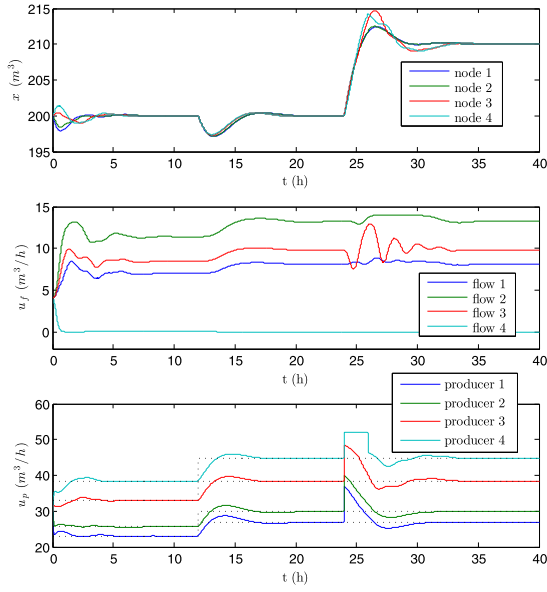
Fig. 4. Volumes, flows and productions of the district heating system during a 40 h period. The optimal production $\overline{u}_p$ as in (9) is indicated by dotted lines in the lower plot.

The resulting response of the system can be found in Fig. 4, where we can clearly see the effects of the increased demand at $t = 12$ and change in setpoint at $t = 24$. More specifically, in the upper plot we can see that the controllers indeed let the volumes in the four storage tanks to converge towards the desired setpoints of $200m^3$ ($t < 24$) and $210m^3$ ($t \geq 24$). In the middle plot we see that the flows in the pipes remain within the constraint $0 < \lambda_k < 14$ for all $k \in \{1, 2, 3, 4\}$ throughout the entire simulation. Finally, in the bottom plot, the production at the four nodes is given, where the optimal production is denoted by the dotted lines. We observe that the production converges towards the optimal value $\overline{u}$ and satisfies $0 < u_i < 52$ for all $i \in \{1, 2, 3, 4\}$, during the entire simulation period.

7.2. Super-conducting DC networks

As a second case study we consider super-conducting direct current (DC) networks that have been studied in *e.g.* Davies, Norman, Jones, Galloway, and Husband (2014), Doukas, Blatsi, Milioudis, Labridis, Harnefors, and Velotto (2015), Johnson, Lasseter, Alvarado, Divan, Singh, Chandorkar, and Adapa (1994) and Xiao, Dai, Lin, Zhang, and Zhang (2013). Particularly, the so-called 'high temperature super-conducting' (HTS) DC networks have received significant attention, as they provide means to transmit power over long distances with negligible losses at higher temperatures. For these networks it is noted that due to the absence of line resistances, undesired oscillations might occur, requiring tailored control schemes (Johnson & Hess, 1999). To illustrate the application of our results, we consider a network of four terminals (nodes) of which only terminals 2, 3 and 4 have a controllable current injection. The corresponding circuit is provided in Fig. 5, where $\mathcal{C}_i$ is the capacitance at terminal i, and $\mathcal{L}_k$ is the inductance of line k. The overall network dynamics are given by

$$\mathcal{C}\dot{V} = -B\mu + u - d$$
$$\mathcal{L}\dot{\mu} = B^T V,$$

(52)

where V are the voltages at the terminals, μ are the currents through the lines, d are *uncontrollable* current loads and u are the
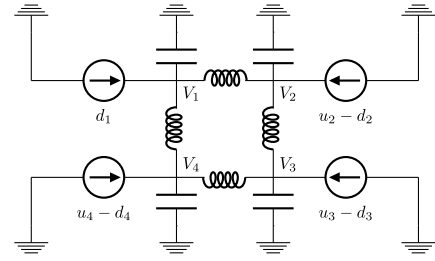

Fig. 5. Topology of a super-conducting DC network with four terminals. We take $\mathcal{C}_i = 80\mu F$ and $\mathcal{L}_k = 20mH$ for $i, k \in \{1, 2, 3, 4\}$.

controllable current injections. The first objective is to stabilize the voltage at terminal i around its desired setpoint $\overline{V}_i$, which is identical for each terminal. Therefore, $B^T V = B^T (V - \overline{V})$. The second objective is to share the controllable current injections equally among the terminals. Note that (52), is an example of the model studied in Section 6.2, and that the set of nodes with a controllable current injection is a zero forcing set for the considered network. Therefore, Assumption 7 is satisfied and it follows from Theorem 3 that asymptotic stability of the desired state is guaranteed, if the controllers (13) are applied to control the current injections. In this case study, the controllers (13) are applied, with $q_i = 1, s_i = 0, r_i = 0, T_{\theta i} = 100, T_{\phi i} = 0.02$, for all $i \in \{1, 2, 3, 4\}$. The underlying communication network is undirected and connects nodes 2–3 and 3–4, where each node has a weight of 10^4. The desired voltage is $\overline{V}_i = 245kV$ at all terminals throughout the simulation. Initially, all d_i have a value of $1kA$. At $t = 0.02s$, the value of d_2 is increased to $1.4kA$, whereas d_3 is decreased to $0.8kA$. To prevent low and high current injections during the transient we require at all terminals that $1.2kA \leq u_i(t) \leq 1.5kA$ is satisfied. To ensure this we let for all $i \in \{2, 3, 4\}$, $u_i = g_i(\theta_i) = 1350 + 150(\tanh(\theta_i) + 1)$. The response to the change in demand is given in Fig. 6, from where we conclude that the voltages converge towards their set point of $245kV$, while u satisfies its constraints at all time.

8. Conclusions and future directions

We presented a distributed controller that dynamically adjusts the inputs and flows in a flow network to regulate the measured output at the nodes towards the desired value. This is achieved in presence of unknown constant disturbances to the network. The use of nonlinear functions, bounding the controller outputs, guarantees that the inputs and the flows stay within their capacity limits. We only require that a subset of nodes have a controllable input to obtain output regulation throughout the complete network. Additionally, optimal coordination among the inputs, minimizing a suitable cost function, is achieved by exchanging information over a communication network. Based on Lyapunov arguments and an invariance principle, we have proven that the desired steady state is globally asymptotically attractive. We emphasized the connection to compartmental systems and we provided two case studies (a district heating system and a super-conducting direct current network) that show the effectiveness of the proposed solution.

There are multiple interesting directions to extend the presented results. We briefly discuss a few of them. It is currently assumed that the material can be instantaneously moved from one node to another, without costs. Incorporating the possibility to include a delay in this flow is desirable (Skutella, 2009), as well as extending the considered optimization problem to include
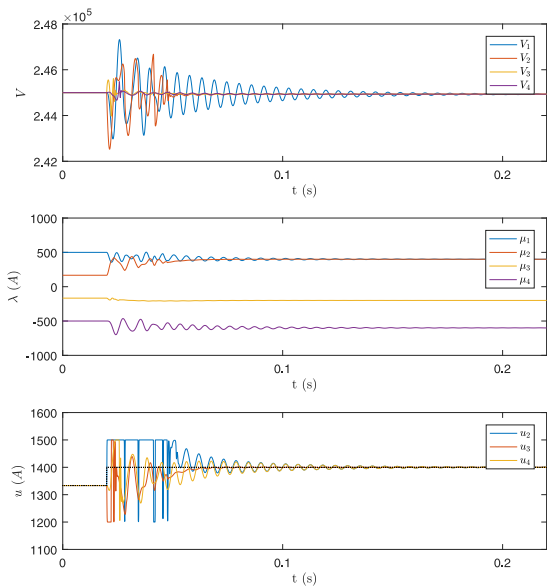
Fig. 6. Voltages, current flows and current injections for a super-conducting direct current network. The optimal production $\bar{u}$ as in (9) is indicated by dotted lines in the lower plot.

flow costs. To cover an even larger class of physical systems, it is worthwhile to include nodes that do not have storage capabilities, which can be modelled by algebraic relations, leading to an overall algebraic–differential system. Additionally, it is interesting to incorporate nodes dynamics of higher dimension as is *e.g.* discussed in Blanchini et al. (2016), as well as the possibility of output regulation in the presence of time-varying disturbances. Since the results are obtained without the common requirement of strict output passivity of the nodes, it is worth exploring if the proposed control structure can be applied to a wider class of systems than the considered flow networks.

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