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Brief paper

Distributed averaging integral Nash equilibrium seeking on networks ☆

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ABSTRACT

Continuous-time gradient-based Nash equilibrium seeking algorithms enjoy a passivity property under a suitable monotonicity assumption, which has been exploited to design distributed Nash equilibrium seeking algorithms. We further exploit the passivity property to interconnect the algorithms with distributed nonlinear averaging integral controllers that tune on-line the weights of the communication graph. The main advantage is to guarantee convergence to a Nash equilibrium without requiring a strong coupling condition on the algebraic connectivity of the communication graph over which the players exchange information, nor a global fixed high-gain.

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1. Introduction

In the literature on optimization and control, continuous-time algorithms that solve optimization problems are gaining increasing attention for several reasons (Cherukuri, Mallada, & Cortés, 2016; Goebel, 2017). First, the control-theoretic properties are more easily unveiled in a continuous-time setting and this permits to naturally establish connections with control methodologies, eventually leading to technical advances in the analysis and design of the optimization algorithms themselves. Second, a way to achieve optimal performance in control systems is to interconnect the physical process with optimization algorithms and study the stability and optimality of the resulting closed-loop system (Stegink, Cherukuri, De Persis, van der Schaft, & Cortés, 2019). In the case of complex network systems, such as power and social networks, multiple agents or players make decisions to optimize their own objective functions. This leads to a game-theoretic setup, where algorithms are commonly designed with the purpose of converging to Nash equilibria, possibly using limited or local information. The local nature of the information is defined by the topology of the network over which the game takes place. With similar motivations as before, more attention is currently being paid to Nash equilibrium seeking algorithms in continuous-time.

Related literature: Nash-equilibrium seeking algorithms have a long history and solutions in continuous-time were proposed in classical early work on game-theoretic problems (Rosen, 1965). Game-theoretic problems have also attracted the interest of the control community already decades ago (Li & Basar, 1987). Extensions to the case of games with coupling constraints where generalized Nash equilibria are of interest, has been the subject of investigation since Rosen (1965), with a wide variety of results available (Arslan, Demirkol, & Yüksel, 2015; Belgioioso & Grammatico, 2017; Kulkarni & Shanbhag, 2012; Pavel, 2007; Yin, Shanbhag, & Mehta, 2011). For general N-player games, the implementation of algorithms for Nash computation requires each player to access information regarding their own objective functions as well as the decisions taken by the other players in the game, information which might not be available. A way to remedy this lack of information is provided by model-free methods inspired by extremum seeking algorithms, see e.g. Frihauf, Krstic, and Basar (2012), Stankovic, Johansson, and Stipanovic (2012) and references therein. Other solutions have been proposed to solve game equilibria in a distributed fashion (Gharesifard & Cortés, 2013; Koshal, Nedić, & Shanbhag, 2016; Li & Marden, 2013; Salehisadaghiani & Pavel, 2016; Ye & Hu, 2017). The approach to Nash equilibrium distributed computation which is of major interest for this paper is the one suggested by Gadjov and Pavel (2019), which reconstructs the non-local information concerning the other players in the game based on a communication graph where each player communicates only with its neighbors. From a methodological point of view, the paper (Gadjov & Pavel, 2019) has pointed out the passivity property of Nash equilibrium seeking algorithms and used this property to design and analyze a consensus-based algorithm that uses local information only.

Paper contribution: In this paper, we further exploit the passivity property revealed in Gadjov and Pavel (2019) to enrich the

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features of Nash equilibrium seeking algorithms. We propose a new passivity-based algorithm that allows us to relax the requirements on the knowledge of the algebraic connectivity of the graph or the use of a controller with a fixed high-gain, which were the solutions proposed in Gadjov and Pavel (2019). Even though the high-gain controller of Gadjov and Pavel (2019) is devised with the purpose of avoiding an *a priori* knowledge of the algebraic graph connectivity, it still uses a global parameter (its high-enough fixed gain), which might be difficult to estimate or implement in a network system. Motivated by Kim and De Persis (2017), the algorithm we propose tunes on-line the weights of the graph over which the players exchange the coordinating information via a distributed integral control action, thus enabling the convergence to Nash equilibria without assuming any information on the graph nor resorting to global parameters. Distributed averaging integral algorithms for Nash equilibria computation of the kind studied in this paper are new to the best of our knowledge. In De Persis and Grammatico (2018), different integral control laws are proposed to solve aggregative games in continuous-time. The results are discussed in a crescendo of complexity, from unconstrained games (Section 3), to games with local constraints (Section 4).

Basic notation: I denotes the identity matrix. $\mathbf{0}$ ($\mathbf{1}$) denotes a matrix/vector with all elements equal to 0 (1); to improve clarity, we may add the dimension of these matrices/vectors as subscript. $A \otimes B$ denotes the Kronecker product between matrices A and B. $\|A\|$ denotes the maximum singular value of matrix A. Given N vectors $x_1, \dots, x_N$, we define $x := \operatorname{col}(x_1, \dots, x_N) = \left[x_1^\top \cdots x_N^\top\right]^\top$, and, for each i, $x_{-i} := \operatorname{col}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$. Similarly, given N sets $\Omega_1, \dots, \Omega_N$, we define $\boldsymbol{\Omega} := \Omega_1 \times \cdots \times \Omega_N$ and, for each i, $\boldsymbol{\Omega}_{-i} := \Omega_1 \times \cdots \times \Omega_{i-1} \times \Omega_{i+1} \times \cdots \times \Omega_N$. For a differentiable function $v : \mathbb{R}^n \to \mathbb{R}$, we denote the vector of partial derivatives as $\nabla v(x) := \operatorname{col}\left(\frac{\partial v(x)}{\partial x_1}, \dots, \frac{\partial v(x)}{\partial x_N}\right) \in \mathbb{R}^n$.

Operator-theoretic definitions: The mapping $\iota_{\mathcal{S}} : \mathbb{R}^n \to \{0, \infty\}$ denotes the indicator function for the set $\mathcal{S} \subseteq \mathbb{R}^n$, i.e., $\iota_{\mathcal{S}}(x) = 0$ if $x \in \mathcal{S}$, ∞ otherwise. The set-valued mapping $N_{\mathcal{S}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the normal cone operator for the set $\mathcal{S} \subseteq \mathbb{R}^n$, i.e., $N_{\mathcal{S}}(x) = \emptyset$ if $x \notin \mathcal{S}$, $\left\{v \in \mathbb{R}^n \mid \sup_{z \in \mathcal{S}} v^\top (z - x) \le 0\right\}$ otherwise. The set-valued mapping $T_{\mathcal{S}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the tangent cone operator for the set $\mathcal{S} \subseteq \mathbb{R}^n$. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is ℓ-Lipschitz continuous if, for all $x, y \in \operatorname{dom}(F)$, $\|F(x) - F(y)\| \le \ell \|x - y\|$. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is (strictly) monotone if, for all $x, y \in \operatorname{dom}(F)$, $(x - y)^\top (F(x) - F(y)) \, (>) \ge 0$; F is ϵ-strongly monotone, with $\epsilon > 0$, if, for all $x, y \in \operatorname{dom}(F)$, $(x - y)^\top (F(x) - F(y)) \ge \epsilon \|x - y\|^2$. Given a closed convex set $\mathcal{C} \subseteq \mathbb{R}^n$ and a single-valued mapping $F : \mathcal{C} \to \mathbb{R}^n$, the variational inequality problem $\mathrm{VI}(\mathcal{C}, F)$, is the problem to find $x^* \in \mathcal{C}$ such that $(y - x^*)^\top F(x^*) \ge 0$ for all $y \in \mathcal{C}$. The mapping $\operatorname{proj}_\Omega$ denotes the projection operator for the set $\Omega \subseteq \mathbb{R}^n$, i.e., $\operatorname{proj}_\Omega(x) := \operatorname{argmin}_{y \in \Omega} \|y - x\|$; $\Pi_\Omega(x, v)$ denotes the projection of the vector v onto the tangent cone of Ω at x, i.e., $\Pi_\Omega(x, v) := \operatorname{proj}_{T(x)}(v)$.

2. Mathematical background

In this section, we recall a few known results to set the ground for the analysis in the rest of the paper.

2.1. Nash equilibrium problem

A game, denoted by $G(\mathcal{I}, \{J_i\}_i, \{\Omega_i\}_i)$ consists of N players indexed by the set $\mathcal{I} := \{1, 2, \dots, N\}$, where each agent i can decide on a strategy vector $x_i \in \Omega_i \subseteq \mathbb{R}^{n_i}$, with the aim to minimize its the cost function $J_i(x_i, x_{-i})$. Let us define $M := \sum_{i=1}^N n_i \ge N$.

Definition 1 (*Nash Equilibrium*)**.** A vector of strategies $x^* \in \boldsymbol{\Omega} \subseteq \mathbb{R}^M$ is a Nash equilibrium if, for all $i \in \mathcal{I}$,

$$J_i(x_i^*, x_{-i}^*) \le \inf_{x_i \in \Omega_i} J_i(x_i, x_{-i}^*). \quad \square$$

Assumption 1 (*Convexity Gadjov & Pavel, 2019, Assumption 2*)**.** Either condition (i) or (ii) below holds.

(i) For each $i \in \mathcal{I}$, $\Omega_i = \mathbb{R}^{n_i}$, and $J_i(x_i, x_{-i})$ is $\mathcal{C}^2$, strictly convex and radially unbounded in x_i for every $x_{-i} \in \boldsymbol{\Omega}_{-i}$.

(ii) For each $i \in \mathcal{I}$, $\Omega_i \subset \mathbb{R}^{n_i}$ is a nonempty, convex and compact set, $J_i(x_i, x_{-i})$ is $\mathcal{C}^1$ and convex in x_i for every $x_{-i} \in \boldsymbol{\Omega}_{-i}$. $\square$

It follows from Başar and Olsder (1995, Cor. 4.2) that under Assumption 1(i), a Nash equilibrium x^* exists, and it satisfies $\frac{\partial J_i}{\partial x_i}(x_i^*, x_{-i}^*) = 0$, for all $i \in \mathcal{I}$. In fact, for the existence of a Nash equilibrium, continuity of J_i is sufficient (Rosen, 1965, Th. 1).

Let us introduce the so-called pseudo-gradient dynamics:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{bmatrix} = - \begin{bmatrix} \frac{\partial J_1}{\partial x_1}(x_1, x_{-1}) \\ \vdots \\ \frac{\partial J_N}{\partial x_N}(x_N, x_{-N}) \end{bmatrix} =: -F(x). \tag{1}$$

Assumption 2 (*Strictly Monotone Pseudo-gradient*)**.** The pseudo-gradient mapping F in (1) is strictly monotone. $\square$

Due to Assumption 2, it follows from Scutari, Facchinei, Pang, and Palomar (2014, Th. 3) that there exists a unique Nash equilibrium for the game $G(\mathcal{I}, \{J_i\}_i, \{\Omega_i\}_i)$. Moreover, due to Assumptions 1 and 2, the equilibrium of the dynamics in (1) is the unique Nash equilibrium and is globally asymptotically stable.

2.2. Nash equilibrium problem with partial information: Distributed seeking via static consensus

Whenever an agent i has no access to the full information x_{-i}, the authors in Gadjov and Pavel (2019) propose to augment the pseudo-gradient dynamics. Specifically, each agent shall estimate the states of all the other agents, i.e., implement the following dynamics:

$$\begin{aligned} \dot{x}_j^i &= u_j^i \quad \forall j \neq i \\ \dot{x}_i &= -\frac{\partial J_i}{\partial x_i}\left(x_i, \boldsymbol{x}_{-i}^i\right) + u_i^i \end{aligned} \tag{2}$$

where $\boldsymbol{x}^i := \operatorname{col}(x_1^i, \dots, x_{i-1}^i, x_i, x_{i+1}^i, \dots, x_N^i)$ is the state vector of agent i, and $u_j^i \in \mathbb{R}^{n_j}$, for all $j \in \mathcal{I}$, are the control inputs for agent i to be designed. Note that in (2), each agent i relies on the estimated strategies $\boldsymbol{x}_{-i}^i$, which is available locally, not on the true strategies $\boldsymbol{x}_{-i}$. This approach comes at the expenses that each agent has a local copy of the state of each other agent in the game. Whenever confusion does not arise, let us use both x_i^i and x_i for the same variable.

In compact form, the dynamics for agent i read as

$$\dot{\boldsymbol{x}}^i = -\mathcal{R}_i^\top \frac{\partial J_i}{\partial x_i}\left(\boldsymbol{x}^i\right) + \boldsymbol{u}^i,$$

where $\mathcal{R}_i \in \mathbb{R}^{n_i \times M}$ is the matrix

$$\mathcal{R}_i := \left[\mathbf{0}_{n_i \times n_1} \cdots \mathbf{0}_{n_i \times n_{i-1}} \, I_{n_i \times n_i} \, \mathbf{0}_{n_i \times n_{i+1}} \cdots \mathbf{0}_{n_i \times n_N}\right].$$

To induce the local variables x_j^i, $j \neq i$, to converge towards the true values x_j, a consensus protocol is used, i.e.,

$$\boldsymbol{u}^i = \sum_{j=1}^N a_{i,j}\left(\boldsymbol{x}^j - \boldsymbol{x}^i\right), \tag{3}$$

where $a_{i,j}$ are the entries of the adjacency matrix of a graph over which the agents exchange information.

Assumption 3. The adjacency matrix $A = [a_{i,j}]$ defines an undirected and connected graph. □

Now, if we define $\boldsymbol{x} = \mathrm{col}\left(\boldsymbol{x}^1, \dots, \boldsymbol{x}^N\right)$,

$$\boldsymbol{F}(\boldsymbol{x}) = \mathrm{col}\left(\frac{\partial J_1}{\partial x_1}(\boldsymbol{x}^1), \dots, \frac{\partial J_N}{\partial x_N}(\boldsymbol{x}^N)\right) \tag{4}$$

and $\mathcal{R}^\top = \mathrm{diag}\left(\mathcal{R}_1^\top, \dots, \mathcal{R}_N^\top\right)$, then the closed-loop system reads in compact form as

$$\dot{\boldsymbol{x}} = -\mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}) - (L \otimes I_M)\boldsymbol{x}, \tag{5}$$

where L is the Laplacian matrix associated with the information-exchange graph. It follows from Gadjov and Pavel (2019, Lemma 2) that an equilibrium $\bar{\boldsymbol{x}}$ of (5) satisfies $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$, with x^* being a Nash equilibrium.

2.3. Stability analysis

For the stability analysis of the dynamics in (5), we consider the following assumptions.

Assumption 4 (*Strongly Monotone Pseudo-gradient*)**.** The pseudo-gradient mapping F in (1) is μ-strongly monotone, with $\mu > 0$. □

Assumption 5 (*Lipschitz Pseudo-gradient*)**.** The pseudo-gradient mapping F in (1) is ℓ_F-Lipschitz continuous, with $\ell_F > 0$; the extended pseudo-gradient mapping $\boldsymbol{F}$ in (4) is $\ell_{\boldsymbol{F}}$-Lipschitz continuous, with $\ell_{\boldsymbol{F}} > 0$. Define $\ell := \max\{\ell_F, \ell_{\boldsymbol{F}}\}$. □

The following result due to Gadjov and Pavel (2019) establishes global asymptotic convergence of the dynamics in (5) to the unique Nash equilibrium, under a strong coupling condition.

Lemma 1 (*Gadjov & Pavel, 2019, Th. 2*)**.** *Let Assumptions 1(i), 3, 4, 5 hold. If*

$$\lambda_2(L) > \ell + \ell^2/\mu, \tag{6}$$

then, for any initial condition, the solution to (5) is bounded and converges exponentially to the unique Nash equilibrium, i.e., $\lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{1}_N \otimes x^*$. □

3. Distributed averaging integral Nash equilibrium seeking

In this section, we propose a novel Nash equilibrium seeking algorithm on networks. Instead of the static consensus coupling $\boldsymbol{u} = -(L \otimes I_M)\boldsymbol{x}$ in (3), we propose an integral loop that tunes the control gains to compensate for the possible lack of strong coupling, i.e., without requiring condition (6) on the algebraic connectivity of the graph:

$$\begin{aligned} \dot{k}_i &= \gamma_i \left\|\boldsymbol{\rho}^i\right\|^2 \\ \boldsymbol{\rho}^i &= \sum_{j=1}^N a_{i,j}\left(\boldsymbol{x}^j - \boldsymbol{x}^i\right) \\ \boldsymbol{u}^i &= -\sum_{j=1}^N a_{i,j}(k_j\boldsymbol{\rho}^j - k_i\boldsymbol{\rho}^i) \end{aligned} \tag{7}$$

where, for all $i \in \mathcal{I}$, $k_i \in \mathbb{R}$ is the state of the ith integrator, $\boldsymbol{u}^i, \boldsymbol{\rho}^i \in \mathbb{R}^M$, and $\gamma_i > 0$ is a constant parameter. Note that the control algorithm in (7) requires the agents to exchange the variables $\boldsymbol{x}^i$, k_i, and $\boldsymbol{\rho}^i$. In vector form, we have

$$\begin{aligned} \dot{\boldsymbol{k}} &= D(\boldsymbol{\rho})^\top(\Gamma \otimes I_M)\boldsymbol{\rho} \\ \boldsymbol{u} &= -(LKL \otimes I_M)\boldsymbol{x} \end{aligned} \tag{8}$$

where

$$\begin{aligned} &\boldsymbol{k} := \mathrm{col}(k_1, \dots, k_N), && \boldsymbol{\rho} := \mathrm{col}(\boldsymbol{\rho}^1, \dots, \boldsymbol{\rho}^N), \\ &\Gamma := \mathrm{diag}(\gamma_1, \dots, \gamma_N) && K := \mathrm{diag}(k_1, \dots, k_N), \\ &D(\boldsymbol{\rho}) := \mathrm{block.diag}(\boldsymbol{\rho}^1, \dots, \boldsymbol{\rho}^N), \end{aligned}$$

and we used that $\boldsymbol{\rho} = -(L \otimes I_M)\boldsymbol{x}$ and $\boldsymbol{u} = (L \otimes I_M)(K \otimes I_M)\boldsymbol{\rho}$. Then, the resulting closed-loop system has the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= -\mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}) - (LKL \otimes I_M)\boldsymbol{x} \\ \dot{\boldsymbol{k}} &= D(\boldsymbol{\rho})^\top(\Gamma \otimes I_M)\boldsymbol{\rho}, \quad \boldsymbol{\rho} = -(L \otimes I_M)\boldsymbol{x}. \end{aligned} \tag{9}$$

Next, inspired by Kim and De Persis (2017), we show convergence of $\boldsymbol{x}(t)$ in (9) to the Nash equilibrium without the assumption of strong coupling of the information exchange graph.

Theorem 1 (*Convergence to Nash Equilibrium*)**.** *Let Assumptions 1(i), 3, 4, 5, hold. Then, for any initial condition, the solution to system (9) is bounded and its* $\boldsymbol{x}$*-component globally asymptotically converges to the Nash equilibrium, i.e.,* $\lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{1}_N \otimes x^*$. □

Proof. See Appendix A. ■

4. Projected distributed averaging integral Nash equilibrium seeking

In this section, we postulate Assumption 1(ii), that is, we consider compact local constraints. Moreover, by Facchinei and Pang (2007, Prop. 1.4.2), a vector is the Nash equilibrium if and only if it satisfies the variational inequality $\mathrm{VI}(\Omega, F)$. Due to the presence of the constraint sets, in Gadjov and Pavel (2019), the authors consider the projected pseudo-gradient dynamics

$$\begin{aligned} \dot{x}_j^i &= u_j^i \quad \forall j \neq i \\ \dot{x}_i &= \Pi_{\Omega_i}\left(x_i\,,\, -\tfrac{\partial J_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}^i) + u_i^i\right) \end{aligned} \tag{10}$$

that in compact form read as

$$\dot{\boldsymbol{x}}^i = \mathcal{R}_i^\top \Pi_{\Omega_i}\left(x_i, -\tfrac{\partial J_i}{\partial x_i}(\boldsymbol{x}^i) + \mathcal{R}_i\boldsymbol{u}^i\right) + (I_M - \mathcal{R}_i^\top\mathcal{R}_i)\boldsymbol{u}^i \tag{11}$$

where

$$(I_M - \mathcal{R}_i^\top\mathcal{R}_i)\boldsymbol{u}^i = \begin{bmatrix} \mathrm{col}\left(u_1^i, \dots, u_{i-1}^i\right) \\ \mathbf{0}_{n_i} \\ \mathrm{col}\left(u_{i+1}^i, \dots, u_N^i\right) \end{bmatrix}.$$

Similarly to (5), the overall extended pseudo-gradient dynamics can be written as

$$\dot{\boldsymbol{x}} = \mathcal{R}^\top \Pi_{\boldsymbol{\Omega}}\left(\mathcal{R}\boldsymbol{x}\,,\, -\boldsymbol{F}(\boldsymbol{x}) + \mathcal{R}\boldsymbol{u}\right) + (I_M - \mathcal{R}^\top\mathcal{R})\boldsymbol{u}. \tag{12}$$

We recall the following equivalence between the equilibrium of (12) and the Nash equilibrium:

Lemma 2. *$x^* \in \Omega$ is the Nash equilibrium if and only if the pair $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) = (\mathbf{1}_N \otimes x^*, \mathbf{0}_M)$ is the equilibrium of (12), i.e.,*

$$\mathbf{0} = \mathcal{R}^\top \Pi_{\boldsymbol{\Omega}}\left(\mathcal{R}\bar{\boldsymbol{x}}\,,\, -\boldsymbol{F}(\bar{\boldsymbol{x}})\right). \quad \square \tag{13}$$

Proof. Eq. (13) holds if and only if $\boldsymbol{x}^*$ is a solution to the $\mathrm{VI}(\Omega, F)$. The proof follows from Facchinei and Pang (2007, Prop. 1.4.2). ■

We now rephrase a key result from Gadjov and Pavel (2019).

Lemma 3 (*Gadjov & Pavel, 2019, Lemma 8*)**.** *The storage function $V(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2$ satisfies, for all $\boldsymbol{x}, \boldsymbol{y}$ such that $\mathcal{R}\boldsymbol{x}, \mathcal{R}\boldsymbol{y} \in \boldsymbol{\Omega}$ and all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^M$,*

$$\begin{aligned} &\nabla V(\boldsymbol{x}, \boldsymbol{y})^\top \left[\begin{smallmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{y}} \end{smallmatrix}\right] \\ &\leq -(\boldsymbol{x} - \boldsymbol{y})\mathcal{R}^\top(\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{y})) + (\boldsymbol{x} - \boldsymbol{y})^\top(\boldsymbol{u} - \boldsymbol{v}), \end{aligned}$$

where $\dot{\boldsymbol{x}}$ is the right-hand side of (12) and $\dot{\boldsymbol{y}}$ equals

$$\mathcal{R}^\top \Pi_\Omega\left(\mathcal{R}\boldsymbol{y}\,,\ -\boldsymbol{F}(\boldsymbol{y})+\mathcal{R}\boldsymbol{v}\right)+\left(I_M-\mathcal{R}^\top\mathcal{R}\right)\boldsymbol{v}. \quad \square \tag{14}$$

We remark that in Lemma 3 the dissipation inequality is intended to hold point-wise, which dispenses us to specify the notion of solution at this stage. The dissipation inequality highlighted in Lemma 3 is important for our purposes because it allows us to derive a Lyapunov inequality for the feedback interconnection of the project dynamical system in (12) with the distributed averaging integral control in (8), namely for the closed-loop system:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \mathcal{R}^\top\Pi_\Omega(\mathcal{R}\boldsymbol{x}\,,\ -\boldsymbol{F}(\boldsymbol{x})-\mathcal{R}(LKL\otimes I_M)\boldsymbol{x})\\ &\quad -(I_M-\mathcal{R}^\top\mathcal{R})(LKL\otimes I_M)\boldsymbol{x},\\ \dot{\boldsymbol{k}} &= D(\boldsymbol{\rho})^\top(\Gamma\otimes I_M)\boldsymbol{\rho},\quad \boldsymbol{\rho}=-(L\otimes I_M)\boldsymbol{x}.\end{aligned} \tag{15}$$

We can now show the following inequality for the closed-loop system in (15).

Lemma 4. *Let Assumption 1(ii) hold. Then, the Lyapunov function*

$$W(\boldsymbol{x},\boldsymbol{k})=\tfrac{1}{2}\|\boldsymbol{x}-\bar{\boldsymbol{x}}\|^2+\tfrac{1}{2}\left\|\boldsymbol{k}-\bar{\boldsymbol{k}}\right\|^2_{\Gamma^{-1}},$$

where $\bar{\boldsymbol{x}}=\mathbf{1}_N\otimes x^*$, $\bar{\boldsymbol{k}}=k^*\mathbf{1}_N$ and $k^*\in\mathbb{R}$ to determine, satisfies, for all $\boldsymbol{x}$ such that $\mathcal{R}\boldsymbol{x}\in\Omega$ and all $\boldsymbol{k}\in\mathbb{R}^N$,

$$\begin{aligned}\nabla W(\boldsymbol{x},\boldsymbol{k})\left[\begin{smallmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{k}}\end{smallmatrix}\right] \le\ & -(\boldsymbol{x}-\bar{\boldsymbol{x}})\mathcal{R}^\top(\boldsymbol{F}(\boldsymbol{x})-\boldsymbol{F}(\bar{\boldsymbol{x}}))\\ & -(\boldsymbol{x}-\bar{\boldsymbol{x}})^\top(LK^*L\otimes I_M)(\boldsymbol{x}-\bar{\boldsymbol{x}}),\end{aligned} \tag{16}$$

where $K^*=k^*I_N$ and $\left[\begin{smallmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{k}}\end{smallmatrix}\right]$ denotes the right-hand side of (15). $\square$

Proof. Assumption 1(ii) implies the existence of a Nash equilibrium $x^*\in\Omega$, which is equivalent to $\mathcal{R}\bar{\boldsymbol{x}}\in\Omega$. Thus, we can apply Lemma 3 with $\bar{\boldsymbol{x}}$ and $\mathbf{0}$ in place of $\boldsymbol{y}$ and $\boldsymbol{v}$, respectively. In view of (13), we obtain that

$$\begin{aligned}\frac{\partial V(\boldsymbol{x},\bar{\boldsymbol{x}})}{\partial\boldsymbol{x}}^\top\dot{\boldsymbol{x}} \le\ & -(\boldsymbol{x}-\bar{\boldsymbol{x}})\mathcal{R}^\top(\boldsymbol{F}(\boldsymbol{x})-\boldsymbol{F}(\bar{\boldsymbol{x}}))\\ & +(\boldsymbol{x}-\bar{\boldsymbol{x}})^\top\boldsymbol{u}.\end{aligned}$$

Now, for $\boldsymbol{u}=-(LKL\otimes I_M)\boldsymbol{x}$, since $\frac{\partial V}{\partial\boldsymbol{x}}=\frac{\partial W}{\partial\boldsymbol{x}}$, the inequality above becomes

$$\begin{aligned}\frac{\partial W}{\partial\boldsymbol{x}}^\top\dot{\boldsymbol{x}} &\le -(\boldsymbol{x}-\bar{\boldsymbol{x}})\mathcal{R}^\top(\boldsymbol{F}(\boldsymbol{x})-\boldsymbol{F}(\bar{\boldsymbol{x}}))\\ &\quad -(\boldsymbol{x}-\bar{\boldsymbol{x}})^\top(LKL\otimes I_M)\boldsymbol{x}.\end{aligned} \tag{17}$$

Furthermore, as in the proof of Theorem 1,

$$\frac{\partial W}{\partial\boldsymbol{k}}^\top\dot{\boldsymbol{k}}=\boldsymbol{x}^\top(L(K-K^*)L\otimes I_{nN})\boldsymbol{x}. \tag{18}$$

Since the sum of the second addend on the right-hand side of (17) and the term on the right-hand side of (18) is $-(\boldsymbol{x}-\bar{\boldsymbol{x}})^\top(LK^*L\otimes I_M)(\boldsymbol{x}-\bar{\boldsymbol{x}})$, the thesis follows. ■

Similarly to Lemma 3, we remark that the Lyapunov inequality in Lemma 4 holds point-wise. We now use Lemma 4 and an invariance principle for projected dynamical systems to infer convergence for the closed-loop system.

We first note that the closed-loop system (15) can be written as a projected dynamical system:

$$\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{k}}\end{bmatrix}=\Pi_\Xi\left(\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{k}\end{bmatrix},\ \boldsymbol{g}(\boldsymbol{x},\boldsymbol{k})\right). \tag{19}$$

Specifically, let us define $\Omega^1:=\Omega^1\times\mathbb{R}^n\times\cdots\times\mathbb{R}^n$, $\Omega^2:=\mathbb{R}^n\times\Omega^2\times\mathbb{R}^n\times\cdots\times\mathbb{R}^n,\ldots,\Omega^N:=\mathbb{R}^n\times\cdots\times\mathbb{R}^n\times\Omega^N$, the closed convex set $\Xi:=\Omega^1\times\cdots\times\Omega^N\times\mathbb{R}^N$, for all $i\in\mathcal{I}$, the mapping

$$f^i(\boldsymbol{x}^i,\boldsymbol{u}^i):=\mathcal{R}_i^\top\frac{\partial J_i}{\partial x_i}(x_i,\boldsymbol{x}^i_{-i})+\boldsymbol{u}^i, \tag{20}$$

and finally the mapping

$$\boldsymbol{g}(\boldsymbol{x},\boldsymbol{k}):=\begin{bmatrix} f^1\left(\boldsymbol{x}^1\,,\ -\sum_{j=1}^N a_{1,j}(k_j\boldsymbol{\rho}^j-k_1\boldsymbol{\rho}^1)\right)\\ \vdots\\ f^N\left(\boldsymbol{x}^N\,,\ -\sum_{j=1}^N a_{N,j}(k_j\boldsymbol{\rho}^j-k_N\boldsymbol{\rho}^N)\right)\\ \gamma_1\left\|\boldsymbol{\rho}^1\right\|^2\\ \vdots\\ \gamma_N\left\|\boldsymbol{\rho}^N\right\|^2\end{bmatrix},$$

where we recall that, for all $i\in\mathcal{I}$, $\boldsymbol{\rho}^i=\sum_{j=1}^N a_{i,j}\left(\boldsymbol{x}^j-\boldsymbol{x}^i\right)$ from Eq. (7).

The projected dynamical system in (19) has a discontinuous right-hand side, and its solutions must be intended in a Carathèodory sense. It is known (Nagurney & Zhang, 1996, Th. 2.5), (Cherukuri et al., 2016, Prop. 2.2), that if the vector field $\boldsymbol{g}$ is Lipschitz continuous on the closed convex set Ξ, then for any initial condition $(\boldsymbol{x}_0,\boldsymbol{k}_0)\in\Xi$, there exists a unique Carathèodory solution to (19) from $(\boldsymbol{x}_0,\boldsymbol{k}_0)$ that is defined on the entire interval $[0,\infty)$, satisfies $(\boldsymbol{x}(t),\boldsymbol{k}(t))\in\Xi$ for all $t\in[0,\infty)$, and is uniformly continuous with respect to the initial condition. The latter property is essential to prove the invariance of the positive limit set associated to any initial condition, as stated next.

Lemma 5 (*Brogliato & Goeleven, 2005, Proof Th. 4, Part 1*)**.** *If the mapping $\boldsymbol{g}$ is Lipschitz continuous on the closed convex set Ξ, then, for any $\boldsymbol{y}_0\in\Xi$, the positive limit set $\Lambda(\boldsymbol{y}_0)$ of the solution to (19) starting from $\boldsymbol{y}_0$ is forward invariant.* $\square$

Proof. Take any point $\boldsymbol{z}\in\Lambda(\boldsymbol{y}_0)$. By definition, there exists a diverging sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\boldsymbol{y}(t_k,\boldsymbol{y}_0)=\boldsymbol{z}$. By the Lipschitz continuity, uniform continuity of the solution with respect to the initial condition holds, and therefore $\boldsymbol{y}(t,\boldsymbol{z})=\lim_{k\to\infty}\boldsymbol{y}(t,\boldsymbol{y}(t_k,\boldsymbol{y}_0))$. Then, one can proceed as in Brogliato and Goeleven (2005, Proof of Th. 4, Part 1). ■

Once we have guaranteed invariance of the limit set, we establish the following invariance principle for systems with Carathèodory solutions. The proof is a variation of the arguments in Brogliato and Goeleven (2005), Cherukuri et al. (2016) and Bacciotti and Ceragioli (2006) adapted to our case study and is included for the sake of completeness.

Theorem 2. *Consider a projected dynamical system $\dot{x}=\Pi_K(f(x))$, where the set $K\subset\mathbb{R}^n$ is closed and convex, and the mapping $f:K\to\mathbb{R}^n$ is continuously differentiable. Let $\Psi\subset\mathbb{R}^n$ be a compact set such that the intersection $K\cap\Psi$ is an invariant set for $\dot{x}=\Pi_K(f(x))$. Suppose that there exists a continuously differentiable function $V:\mathbb{R}^n\to\mathbb{R}$ such that*

$$\sup_{x\in K\cap\Psi}\nabla V(x)^\top\Pi_K(f(x))\le 0. \tag{21}$$

Then, for any initial condition in $K\cap\Psi$, there exists a unique Carathèodory solution to $\dot{x}=\Pi_K(f(x))$, which remains in $K\cap\Psi$ and converges to the largest invariant set contained in $\{x\in K\cap\Psi\mid\nabla V(x)^\top\Pi_K(f(x))=0\}$. $\square$

Proof. See Appendix B. ■

We are ready to show the main result of this section, the global asymptotic convergence to a Nash equilibrium of the projected dynamical system in (19), technically, the same convergence

result as in Gadjov and Pavel (2019, Th. 5), but without assuming a lower bound on the algebraic connectivity of the graph. Our proof relies on an invariance principle for projected dynamical systems, not on Barbalat's lemma as in Gadjov and Pavel (2019).

Theorem 3. *Under Assumptions 1(ii), 3–5, for any initial condition in Ξ, there exists a unique Carathèodory solution to system (15), which belongs to Ξ for all $t \geq 0$, such that its $\boldsymbol{x}$-component converges asymptotically to the Nash equilibrium, i.e.,* $\lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{1}_N \otimes x^*$. □

Proof. See Appendix C. ■

5. Discussion

5.1. On the proposed distributed averaging algorithm

The proposed algorithm in (7) comprises a distributed integral action averaging the local decisions of the agents whose state variable k_i, $i \in \mathcal{I}$, is used as a tuning weight of the input u coupling the agents' dynamics. The control gains k_i are updated with positive rates that vanish as the states $\boldsymbol{x}^i$ approach the Nash equilibrium. The update compensates for the lack of knowledge of the algebraic connectivity, as inferred from (A.7)–(A.8) in Appendix A. Consequently, the proposed algorithm requires the exchange of the vector $k_i\boldsymbol{\rho}^i$ in addition to the vector $\boldsymbol{x}^i$. By following Kim and De Persis (2017), an alternative control algorithm may be given by

$$\dot{k}_i = \gamma_i \left\|\boldsymbol{\rho}^i\right\|^2, \quad \boldsymbol{\rho}^i = \sum_{j=1}^{N} a_{i,j}(\boldsymbol{x}^j - \boldsymbol{x}^i), \quad \boldsymbol{u}^i = k_i\boldsymbol{\rho}^i \tag{22}$$

where the local control $\boldsymbol{u}^i$ uses the local average $\boldsymbol{\rho}^i$ only, as opposed to (7). This alternative approach requires the use of a different Lyapunov function and to establish additional technical results for the boundedness of the solutions. Similarly, the results in Kim and De Persis (2017) suggest that, if the control input $\boldsymbol{u}$ is affected by an exosystem-generated additive disturbance $\boldsymbol{d}$, a suitably modified internal-model-based version of the controller (8) could still guarantee the convergence to the Nash equilibrium in spite of the disturbance. We leave these lines of investigation for future research.

5.2. On the algebraic connectivity

We note that for the unconstrained case, Gadjov and Pavel (2019, Th. 1) establishes asymptotic stability without a condition on the graph algebraic connectivity but under a monotonicity assumption on the extended pseudo-gradient $\boldsymbol{F}$, an assumption which is stronger than the Lipschitz continuity considered in Assumption 5 (cf., Gadjov & Pavel, 2019, Remark 1). On the other hand, under the same assumption on the Lipschitz continuity of the extended pseudo-gradient $\boldsymbol{F}$ considered in our Theorem 1, Gadjov and Pavel (2019, Th. 3) relaxes the condition on the algebraic connectivity using a singular perturbation approach. The result, which guarantees exponential stability, requires however the use of a gain $1/\epsilon$ which is global to all the agents and must be larger than a bound $1/\epsilon^*$. See Gadjov and Pavel (2019, Remark 4) for additional discussion on the singular perturbation approach to the problem. The approach we propose in Theorem 1 removes the need of a global parameter. Similarly, for the constrained case, compared with Gadjov and Pavel (2019, Th. 5), our Theorem 3 relaxes the need of a condition on the algebraic connectivity.

6. Conclusion

Computing a Nash equilibrium in a distributed fashion, with unknown algebraic connectivity of the information exchange graph, nor imposing a fixed high-gain, is possible via dynamic (rather than static) consensus and pseudo-gradient dynamics, mainly under strong monotonicity and Lipschitz continuity assumptions. Our analysis and design are based on passivity and an invariance principle for projected dynamical systems. Solving the generalized Nash equilibrium problem is left as future work.

Appendix A. Proof of Theorem 1

For the stability analysis, we consider the Lyapunov function

$$W(\boldsymbol{x}, \boldsymbol{k}) = \tfrac{1}{2}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|^2 + \tfrac{1}{2}\left\|\boldsymbol{k} - \bar{\boldsymbol{k}}\right\|^2_{\Gamma^{-1}}$$

where $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$ and $\bar{\boldsymbol{k}} = k^*\mathbf{1}_N$, with $k^* \in \mathbb{R}_{\geq 0}$ to determine. The time derivative of W is then written as

$$\begin{aligned}\dot{W}(\boldsymbol{x}, \boldsymbol{k}) = &-(\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}) \\ &-(\boldsymbol{x} - \bar{\boldsymbol{x}})^\top (LKL \otimes I_M)\boldsymbol{x} \\ &+(\boldsymbol{k} - \bar{\boldsymbol{k}})^\top \Gamma^{-1} D(\boldsymbol{\rho})^\top (\Gamma \otimes I_M)\boldsymbol{\rho}.\end{aligned} \tag{A.1}$$

Now, we show that the first addend is bounded as follows:

$$\begin{aligned}&-(\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}) \leq \\ &-\begin{bmatrix}\|(P_N \otimes I_M)\,\boldsymbol{x}\| \\ \|\mathrm{avg}(\boldsymbol{x}) - x^*\|\end{bmatrix}^\top \begin{bmatrix}-\ell_F & \ell \\ \ell & \mu\end{bmatrix} \begin{bmatrix}\|(P_N \otimes I_M)\,\boldsymbol{x}\| \\ \|\mathrm{avg}(\boldsymbol{x}) - x^*\|\end{bmatrix},\end{aligned} \tag{A.2}$$

where $P_N := I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$ and $\mathrm{avg}(\boldsymbol{x}) := \frac{1}{N}\sum_{i=1}^N \boldsymbol{x}^i = (\frac{1}{N}\mathbf{1}_N^\top \otimes I_M)\boldsymbol{x}$. In fact, following Gadjov and Pavel (2019) with minor modifications, let us define $\boldsymbol{x}^\perp := (P_N \otimes I_M)\boldsymbol{x}$ and $\boldsymbol{x}^\parallel := \mathbf{1}_N \otimes \mathrm{avg}(\boldsymbol{x})$. Then, the first addend in (A.1) reads as

$$\begin{aligned}-(\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}) = &-\boldsymbol{x}^{\perp\top} \mathcal{R}^\top (\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}^\parallel)) \\ &-\boldsymbol{x}^{\perp\top} \mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}^\parallel) \\ &-(\boldsymbol{x}^\parallel - \bar{\boldsymbol{x}})^\top \mathcal{R}^\top (\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}^\parallel)) \\ &-(\boldsymbol{x}^\parallel - \bar{\boldsymbol{x}})^\top \mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}^\parallel).\end{aligned}$$

Since $\|\mathcal{R}^\top\| = 1$, by Assumption 5, we have

$$\left\|\boldsymbol{x}^{\perp\top} \mathcal{R}^\top (\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}^\parallel))\right\| \leq \ell_{\boldsymbol{F}} \|(P_N \otimes I_M)\boldsymbol{x}\|^2. \tag{A.3}$$

Then, we note that $\boldsymbol{F}(\boldsymbol{x}^\parallel) = F(\mathrm{avg}(\boldsymbol{x}))$. Similarly, since $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$, it holds that $\boldsymbol{F}(\bar{\boldsymbol{x}}) = F(x^*) = \mathbf{0}$. Thus, by the ℓ_F-Lipschitz continuity of F, it holds that

$$\begin{aligned}-\boldsymbol{x}^{\perp\top} \mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}^\parallel) = &-\boldsymbol{x}^{\perp\top} \mathcal{R}^\top \left(F(\mathrm{avg}(\boldsymbol{x})) - F(x^*)\right) \\ \leq &\ \ell_F \|\boldsymbol{x}^\perp\| \|\mathrm{avg}(\boldsymbol{x}) - x^*\|.\end{aligned} \tag{A.4}$$

Furthermore, we note that $\mathcal{R}\boldsymbol{x}^\parallel = \mathrm{avg}(\boldsymbol{x})$, $\mathcal{R}\bar{\boldsymbol{x}} = x^*$, and therefore, by Assumption 5, the following inequality is true

$$\begin{aligned}&-(\boldsymbol{x}^\parallel - \bar{\boldsymbol{x}})^\top \mathcal{R}^\top (\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}^\parallel)) \\ &= -(\mathrm{avg}(\boldsymbol{x}) - x^*)^\top (\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}^\parallel)) \\ &\leq \ell_{\boldsymbol{F}} \|\mathrm{avg}(\boldsymbol{x}) - x^*\| \|\boldsymbol{x}^\perp\|.\end{aligned} \tag{A.5}$$

By the strong monotonicity stated in Assumption 4, we obtain that

$$\begin{aligned}&-(\boldsymbol{x}^\parallel - \bar{\boldsymbol{x}})^\top \mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}^\parallel) \\ &= -(\boldsymbol{x}^\parallel - \bar{\boldsymbol{x}})^\top \mathcal{R}^\top (\boldsymbol{F}(\boldsymbol{x}^\parallel) - \boldsymbol{F}(\bar{\boldsymbol{x}})) \\ &= -(\mathrm{avg}(\boldsymbol{x}) - x^*)^\top \left(F(\mathrm{avg}(\boldsymbol{x})) - F(x^*)\right) \\ &\leq -\mu \|\mathrm{avg}(\boldsymbol{x}) - x^*\|^2.\end{aligned} \tag{A.6}$$

The second addend on the right-hand side in (A.1) can be rewritten as $-\boldsymbol{x}^\top (LKL \otimes I_M)\boldsymbol{x}$. Finally, we rewrite the third addend in (A.1) as

$$\begin{aligned}&(\boldsymbol{k} - \bar{\boldsymbol{k}})^\top \Gamma^{-1} D(\boldsymbol{\rho})^\top (\Gamma \otimes I_M)\boldsymbol{\rho} = \sum_{i=1}^N (k_i - k^*)\boldsymbol{\rho}^{i\top}\boldsymbol{\rho}^i \\ &= \boldsymbol{\rho}^\top ((K - K^*) \otimes I_M)\boldsymbol{\rho} = \boldsymbol{x}^\top (L(K - K^*)L \otimes I_M)\boldsymbol{x},\end{aligned}$$

where $K^* := k^* I_N$. Thus, the sum of the second and the third addend is equal to $-\boldsymbol{x}^\top (LK^*L \otimes I_M)\boldsymbol{x}$, from which (Kim & De Persis, 2017), we conclude that

$$\boldsymbol{x}^\top (LK^*L \otimes I_M)\boldsymbol{x} \geq k^* \lambda_2(L)^2 \|(P_N \otimes I_M)\boldsymbol{x}\|^2. \tag{A.7}$$

By replacing the bounds (A.3)–(A.7) in (A.1), we obtain the following inequality:

$$\dot{W}(\boldsymbol{x}, \boldsymbol{k}) \leq -\left\| \begin{bmatrix} \|(P_N \otimes I_M)\boldsymbol{x}\| \\ \|\mathrm{avg}(\boldsymbol{x}) - x^*\| \end{bmatrix} \right\|^2_{\mathcal{M}} \tag{A.8}$$

where

$$\mathcal{M} = \begin{bmatrix} -\ell_{\boldsymbol{F}} + k^* \lambda_2(L)^2 & \ell \\ \ell & \mu \end{bmatrix} \tag{A.9}$$

and the free design parameter k^* is chosen such that $\mathcal{M} \succ 0$.

Since the Lyapunov function W is radially unbounded, the inequality in (A.8) shows boundedness of the solutions and convergence to the largest invariant set where $(P_N \otimes I_M)\boldsymbol{x} = \boldsymbol{0}$ and $\mathrm{avg}(\boldsymbol{x}) = x^*$. On this invariant set, we have $\boldsymbol{x} = \boldsymbol{1}_N \otimes x^*$. Thus, $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{1}_N \otimes x^*$. ■

Remark 6 (*Passivity Interpretation*)**.** The design of (8) and the proof above are inspired by passivity-based arguments. The first part of the proof establishes the shifted passivity property of the dynamics $\dot{\boldsymbol{x}} = -\mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{u}$, namely

$$\dot{V}(\boldsymbol{x}) \leq (\boldsymbol{x} - \overline{\boldsymbol{x}})^\top \boldsymbol{u} - \begin{bmatrix} \|(P_N \otimes I_M)\,\boldsymbol{x}\| \\ \|\mathrm{avg}(\boldsymbol{x}) - x^*\| \end{bmatrix}^\top \begin{bmatrix} -\ell_{\boldsymbol{F}} & \ell \\ \ell & \mu \end{bmatrix} \begin{bmatrix} \|(P_N \otimes I_M)\,\boldsymbol{x}\| \\ \|\mathrm{avg}(\boldsymbol{x}) - x^*\| \end{bmatrix},$$

where $V(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x} - \overline{\boldsymbol{x}}\|^2$. On the other hand, writing the control gain dynamics as $\dot{\boldsymbol{k}} = \boldsymbol{v}$, with $\boldsymbol{v}$ to design, the storage function $U(\boldsymbol{k}) = \frac{1}{2}\left\|\boldsymbol{k} - \overline{\boldsymbol{k}}\right\|^2_{\Gamma^{-1}}$ satisfies the shifted dissipation inequality $\dot{U}(\boldsymbol{k}) \leq (\boldsymbol{k} - \overline{\boldsymbol{k}})^\top \boldsymbol{v}$. Taking as storage function for the overall system the function $W(\boldsymbol{x}, \boldsymbol{k}) = V(\boldsymbol{x}) + U(\boldsymbol{k})$, the design of $\boldsymbol{u}$ and $\boldsymbol{v}$ is guided by the idea that the cross-terms $(\boldsymbol{x} - \overline{\boldsymbol{x}})^\top \boldsymbol{u}$ and $(\boldsymbol{k} - \overline{\boldsymbol{k}})^\top \boldsymbol{v}$ should provide the extra damping term $-(\boldsymbol{x} - \overline{\boldsymbol{x}})^\top (LK^*L \otimes I_M)(\boldsymbol{x} - \overline{\boldsymbol{x}})$, leading to the desired convergence result. Similar comments apply to the projected version of the algorithm (Section 4). ■

Appendix B. Proof of Theorem 2

By the compactness of $K \cap \Psi$, the convexity of the set K, and the continuous differentiability of f, f is Lipschitz continuous on $K \cap \Psi$. Thus, by the invariance of $K \cap \Psi$ under $\dot{x} = \Pi_K(f(x))$, for any initial condition x_0 in $K \cap \Psi$, there exists a unique Carathèodory solution defined on $[0, \infty)$ that remains in $K \cap \Psi$. The derivative $\dot{V}(x(t))$ exists for almost all t because $x(t)$ is absolutely continuous, and by (21), it satisfies $\dot{V}(x(t)) \leq 0$ for almost all t. Since $V(x(t))$ is absolutely continuous and $x(t)$ belongs to the compact set $K \cap \Psi$, then $V(x(t))$ is bounded from below, and the last property, along with $\dot{V}(x(t)) \leq 0$ for almost all t, implies that $\lim_{t\to\infty} V(x(t; x_0)) = V_*$, for some $V_* \in \mathbb{R}$. Consider now a point p in the limit set $\Lambda(x_0)$. Note that $\Lambda(x_0)$ is non-empty because $x(t; x_0)$ is bounded, as a consequence of the Bolzano–Weierstrass theorem. Then, by definition of limit set, $V(p) = V_*$, and since p is a generic point in $\Lambda(x_0)$, $V(p) = V_*$ for all $p \in \Lambda(x_0)$. It is also known that $\lim_{t\to\infty} \mathrm{dist}(x(t; x_0), \Lambda(x_0)) = 0$.

By Lemma 5, any solution $x(t, p)$ with $p \in \Lambda(x_0)$ remains in $\Lambda(x_0)$ by the Lipschitz continuity of f on $K \cap \Psi$. Thus, since $V(x)$ is constant on $\Lambda(x_0)$, we have $0 = \dot{V}(x(t, p)) = \nabla V(x(t, p))^\top \Pi_K(f(x(t, p)))$ for almost all $t \in \mathbb{R}_{\geq 0}$. Now, since the function $\nabla V(x)^\top \Pi_K(f(x))$ is continuous, we have

$$\begin{aligned} 0 &= \lim_{t\to 0^+} \nabla V(x(t, p))^\top \Pi_K(f(x(t, p))) \\ &= \nabla V(p)^\top \Pi_K(f(p)). \end{aligned} \tag{B.1}$$

Since p is a generic point in $\Lambda(x_0)$, the equality above shows that $\nabla V(p)^\top \Pi_K(f(p)) = 0$ for all $p \in \Lambda(x_0)$. We conclude that $\Lambda(x_0)$ is contained in the largest invariant set contained in $\{y \in K \cap \Psi \mid \nabla V(y)^\top \Pi_K(f(y)) = 0\}$. The proof then follows since $\lim_{t\to 0} \mathrm{dist}(x(t; x_0), \Lambda(x_0)) = 0$. ■

Appendix C. Proof of Theorem 3

In view of Lemmas 4 and 5, similarly to the proof of Theorem 1, we obtain that

$$\begin{aligned} &\nabla W(\boldsymbol{x}, \boldsymbol{k})^\top \Pi_{\Xi}\left(\mathrm{col}(\boldsymbol{x}, \boldsymbol{k}),\, \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{k})\right) \\ &\quad \leq -\left\| \begin{bmatrix} \|(P_N \otimes I_{nN})\boldsymbol{x}\| \\ \|\mathrm{avg}(\boldsymbol{x}) - x^*\| \end{bmatrix} \right\|^2_{\mathcal{M}} \end{aligned} \tag{C.1}$$

where $\mathcal{M}$ is as in (A.9) and $M \succ 0$ for large enough k^*.

Having fixed k^*, let Ψ be any compact sublevel set of the function W which contains the initial condition in Ξ. The intersection $\Xi \cap \Psi$ is a compact convex set and therefore $\boldsymbol{g}$ is Lipschitz continuous on it. The last property and the inequality in (C.1) imply that there exists a unique Carathèodory solution to (15), which belongs to $\Xi \cap \Psi$ for all time and, by Theorem 2, converges to the largest invariant set contained in $\{(\boldsymbol{x}, \boldsymbol{k}) \in \Xi \cap \Psi : \dot{W}(\boldsymbol{x}, \boldsymbol{k}) = 0\}$, where, by an abuse of notation, $\dot{W}(\boldsymbol{x}, \boldsymbol{k})$ denotes the right-hand side of (C.1). As in Theorem 1, on this invariant set, we have $\boldsymbol{x} = \boldsymbol{1}_N \otimes x^*$, which yields the thesis. ■

References

Arslan, G., Demirkol, M. F., & Yüksel, S. (2015). On games with coupled constraints. *IEEE Transactions on Automatic Control*, *60*(2), 358–372.

Bacciotti, A., & Ceragioli, F. (2006). Nonpathological Lyapunov functions and discontinuous Carathèodory systems. *Automatica*, *42*(3), 453–458.

Başar, T., & Olsder, G. J. (1995). *Dynamic noncooperative game theory*. SIAM.

Belgioioso, G., & Grammatico, S. (2017). Semi-decentralized Nash equilibrium seeking in aggregative games with coupling constraints and non-differentiable cost functions. *IEEE Control Systems Letters*, *1*(2), 400–405.

Brogliato, B., & Goeleven, D. (2005). The Krakovskii-LaSalle invariance principle for a class of unilateral dynamical systems. *Mathematics of Control, Signals, and Systems*, *17*(1), 57–76.

Cherukuri, A., Mallada, E., & Cortés, J. (2016). Asymptotic convergence of constrained primal–dual dynamics. *Systems & Control Letters*, *87*, 10–15.

De Persis, C., & Grammatico, S. (2018). Continuous-time integral dynamics for aggregative game equilibrium seeking. In *2018 European Control Conference (ECC)* (pp. 1042–1047).

Facchinei, F., & Pang, J.-S. (2007). *Finite-dimensional variational inequalities and complementarity problems*. Springer Science & Business Media.

Frihauf, P., Krstic, M., & Basar, T. (2012). Nash equilibrium seeking in noncooperative games. *IEEE Transactions on Automatic Control*, *57*(5), 1192–1207.

Gadjov, D., & Pavel, L. (2019). A passivity-based approach to Nash equilibrium seeking over networks. *IEEE Transactions on Automatic Control*, *64*(3), 1077–1092.

Gharesifard, B., & Cortés, J. (2013). Distributed convergence to Nash equilibria in two-network zero-sum games. *Automatica*, *49*(6), 1683–1692.

Goebel, R. (2017). Stability and robustness for saddle-point dynamics through monotone mappings. *Systems & Control Letters*, *108*, 16–22.

Kim, H., & De Persis, C. (2017). Adaptation and disturbance rejection for output synchronization of incrementally output–feedback passive systems. *International Journal of Robust and Nonlinear Control*, *27*(17), 4071–4088.

Koshal, J., Nedić, A., & Shanbhag, U. (2016). Distributed algorithms for aggregative games on graphs. *Operations Research*, *64*(3), 680–704.

Kulkarni, A. A., & Shanbhag, U. (2012). On the variational equilibrium as a refinement of the generalized Nash equilibrium. *Automatica*, *48*, 45–55.

Li, S., & Basar, T. (1987). Distributed algorithms for the computation of noncooperative equilibria. *Automatica*, *23*(4), 523–533.

Li, N., & Marden, J. (2013). Designing games for distributed optimization. *IEEE Journal of Selected Topics in Signal Processing*, *7*(2), 230–242.

Nagurney, A., & Zhang, D. (1996). *Projected dynamical systems and variational inequalities with applications*. Kluwer Academic Publishers.

Pavel, Lacra (2007). An extension of duality to a game-theoretic framework. *Automatica*, *43*(2), 226–237.

Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave N-person games. *Econometrica*, *33*(3), 520–534.

Salehisadaghiani, F., & Pavel, L. (2016). Distributed Nash equilibrium seeking: A gossip-based algorithm. *Automatica*, *72*, 209–216.

Scutari, G., Facchinei, F., Pang, J.-S., & Palomar, D. (2014). Real and complex monotone communication games. *IEEE Transaction on Information Theory*, *60*(7), 4197–4231.

Stankovic, M. S., Johansson, K. H., & Stipanovic, D. M. (2012). Distributed seeking of Nash equilibria with applications to mobile sensor networks. *IEEE Transactions on Automatic Control*.

Stegink, T., Cherukuri, A., De Persis, C., van der Schaft, A., & Cortés, J. (2019). Hybrid interconnection of iterative bidding and power network dynamics for frequency regulation and optimal dispatch. *IEEE Transactions on Control of Network Systems*, *6*(2), 572–585.

Ye, M., & Hu, G. (2017). Distributed Nash equilibrium seeking by a consensus based approach. *62*(9), 4811–4818.

Yin, H., Shanbhag, U. V., & Mehta, P. G. (2011). Nash equilibrium problems with scaled congestion costs and shared constraints. *IEEE Transactions on Automatic Control*, *56*(7), 1702–1708.

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