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Inventory control with seasonality of lead times ☆

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A B S T R A C T

The practical challenges posed by the seasonality of lead times have largely been ignored within the inventory control literature. The length of the seasons, as well as the length of the lead times during a season, may demonstrate cyclical patterns over time. This study examines whether inventory control policies that anticipate seasonal lead-time patterns can reduce costs. We design a framework for characterizing different seasonal lead-time inventory problems. Subsequently, we examine the effect of deterministic and stochastic seasonal lead times within periodic review inventory control systems. We conduct a base case analysis of a deterministic system, enabling two established and alternating lead-time lengths that remain valid through known intervals. We identify essential building blocks for developing solutions to seasonal lead-time problems. Lastly, we perform numerical experiments to evaluate the cost benefits of implementing an inventory control policy that incorporates seasonal lead-time lengths. The findings of the study indicate the potential for cost improvements. By incorporating seasonality in length of seasons and length of lead times within the season into the control models, inventory controllers can make more informed decisions when ordering their raw materials. They need smaller buffers against lead-time variations due to the cyclical nature of seasonality. Reductions in costs in our experiments range on average between 18.9 and 26.4% (depending on safety time and the probability of the occurrence of stock out). Therefore, inventory control methods that incorporate seasonality instead of applying large safety stock or safety time buffers can lead to substantial cost reductions.

1. Introduction

Lead-time seasonality is frequently encountered in the practical context of supply chain management. Accounting for seasonality entails benefits for customers while reducing costs incurred in the supply chain through the introduction of opportunities for anticipating future changes [1]. Seasonal lead times exhibit a cyclical pattern of variation over time. For example, seasonally changing weather conditions during an annual cycle may prompt lead times of differing lengths over the course of a year that are almost identical during the following year. Cycles can also be quarterly, monthly, weekly, or even daily (e.g., rush hours). According to the definition of seasonality of [2], the essential characteristic of a seasonal time series is a systematic but not necessary identical or regular repetition of a pattern over time. Thus, the question that arises is what induces systematic patterns.

To address this question, it is necessary to distinguish between problems related to transportation and suppliers, which are primary causes of seasonal lead times. Whereas, transportation, directly influences lead-time length [3], transportation time may vary during different seasons. For example, [4] reports that in Sweden some roads are closed during the spring (thawing) season, leading to longer lead times during these months, and [5] discusses the effect of the climate and weather on access to waterways for transportation over the course of the year. Rain, snow, frost, fog, droughts, and storms may also occur as seasonal weather phenomena that have a significant impact on transportation time. Therefore, the ability to predict the timing and impact of such conditions on transportation networks would enable seasonal lead-time patterns to be anticipated in advance.

Forces of nature are not the only factors that generate seasonal lead-time patterns relating to transportation over the course of a cycle. Holidays and other non-working days, such as weekends, may also result in a limited availability of required facilities or personnel, which temporarily leads to longer transportation lead times. While companies such as Amazon now offer same-day delivery in many regions globally, Saturday or Sunday deliveries were initially unavailable in many areas, resulting in changing lead times during the week. Moreover, contracts with logistical service providers may lead to different lead-time lengths over the course of the cycle. For example, Avebe, an internationally operating company that manufactures starch products, transports items by ship

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every Monday from the Netherlands to Scandinavia, incurring a transportation lead time of five days. On other working days of the week, truck-based transport is available, which entails a transportation time of just two days. These variations in lead times occur on a weekly basis. Hence, a seasonal lead-time pattern is generated through the combination of available transportation modes and different lead times.

A second causative factor contributing to seasonal lead times is supply. Studies on multiple supplier issues (e.g. [6]) have generally entailed an assumption that supplier lead times differ because of the geographical spread of the suppliers' locations. A major reason for maintaining this geographical spread of suppliers within fresh food supply chains may be the limited timeframe for obtaining supplies from each location, for example, because of the lengths of the harvesting seasons or governmental regulations, such as closed seasons for fishing or hunting. These limitations generate seasonal availability of fresh food supplies within each location. A strategy of sourcing fresh products from suppliers located in different areas ensures continued supplies of these products. However, the lead times differ in these locations, resulting in a discernable seasonal lead-time pattern during the year. An illustrative example is a case study of a tobacco company confronted with the problem of seasonal lead times that is conducted by Riezebos [7].

Constraints relating to suppliers' capacities are another supply-related factor contributing to seasonal lead times. For make-to-order products (i.e., products that cannot be stored in advance), suppliers may decide to increase the lead time on a temporary basis during the high season to enhance their capacities to serve all of their customers. Retailers that order items from make-to-order furniture-producing companies face such seasonal lead-time patterns. These companies therefore extend the lead time for some of their products during holiday or high-peak seasons when they face capacity constraints, and decrease it at the end of the season. The retailer is able to anticipate this pattern because it is repeated annually.

Our review of the literature on inventory control relating to variations in seasonal lead times reveals that seasonal lead times, surprisingly, are not discussed in classical textbooks on inventory control [8–11]. These textbooks focus narrowly on demand and/or supply seasonality, and lead time variability in general. In journal papers, various causes of lead-time variations have been discussed, for example, dual sourcing [12], outsourcing [13], or stochasticity [14]. However, similar patterns that are repeated at the onset of new cycles have generally received little attention. The only exceptions that we encountered in our review are studies by Song et al. [15,16]. Song and Zipkin [15] models variation in lead-time length as an exogenous stationary process. Although the authors do not explore seasonality in their study, their model could be applied in an examination of specific seasonal lead-time patterns. Silver and Zufferey [16] proposes a heuristic solution for a stochastic seasonal lead-time pattern of a saw mill. We did not find any other papers that examine non-stationary cyclic lead-time processes within the reviewed literature. This gap is striking, as we would expect these processes to be accounted for within inventory control policies in case a substantial proportion of the total lead-time variance can be attributed to seasonal patterns. Therefore, this study aims to explore the phenomenon of lead-time seasonality and how it can be used for improving inventory control.

We will explore the following ideas on how to benefit from seasonal lead times. First, we will examine what families of seasonal lead-time problems exist, as every family of problems might require specific solution approaches. Next, we will identify seasonal shifts in lead time lengths over time in order to anticipate these shifts. Anticipation may lead to either postpone ordering till the new season has started or submit a larger order if the lead-time length is expected to increase in the near future. The policy will be developed for both deterministic and stochastic seasonal variations. Another idea is to reduce the required safety buffers in case of stochastic variations of lead-time lengths. By developing new methods to anticipate seasonal shifts of lead-time lengths and simultaneously using smaller safety buffers, it might be possible to reduce operational costs significantly. As this is the first paper that explores lead-time seasonality in inventory control, it will focus on providing a foundation for future research directions in this area.

Section 2 presents a literature review. Section 3 proposes a framework to distinguish various families of seasonal lead-time problems. Section 4 identifies building blocks for developing methods to solve these issues on the smallest possible seasonal lead-time inventory control system, namely, a system with two alternating lead-time lengths within a setting entailing deterministic periodic reviews. The results of this model yield key insights on when to order in advance and when to postpone ordering, according to the particular seasonal lead-time pattern that is being observed. Extending the analysis to models entailing random lengths of seasons, we obtain other key insights related to the occurrence and treatment of order crossovers. Section 5 presents a numerical analysis of the dimension of stochastic magnitude using the previously identified solution-building blocks. The results of our numerical experiments indicate potential cost savings ranging on average between 18.9% and 26.4% when using inventory control methods that incorporate seasonality instead of applying traditional buffering methods using safety stock and safety time. The final section presents our conclusions and offers suggestions for future research.

2. Literature review on seasonal lead times

Seasonal lead times are a special type of lead-time variation. Two types of variations can be distinguished: changes that are known to occur (deterministic) and changes that are not known in advance (stochastic). We begin this section with a discussion of deterministic changes in lead times. If lead-time variations are known before they actually occur, their effects can be included in a model of inventory control. Examples of inventory control models in which lead-time variations are known in advance include negotiable lead-time models [17], multiple supplier models [18,19], emergency models [20], multiple sourcing models [21], and endogenous lead-time inventory models [22]. However, the inclusion of known lead-time changes may result in transient effects that constrain the development of optimal solutions. For example, [23] examines the effect of an anticipated lead-time change on the timing of changes in the optimal order-up-to levels. It appears that transient effects are observable in case of future lead-time reductions, impeding the determination of the optimal time when the order-up-to-level should be reduced to accommodate the future change in lead-time length. Axsäter [23] notes that despite their prevalence in practice, due to processual changes and disruptions in supplies, these transient effects have not been studied extensively. However, [23] does not examine seasonal lead times.

In addition to the above-mentioned deterministic models of lead times, stochastic models of variations in lead-time lengths have also been developed. The underlying assumption in these models is that every order faces an uncertain lead-time length that remains unknown up to the time of the order's arrival [24–27]. In many of the existing stochastic inventory models, lead times are considered to be independent and identically distributed. One of the insights emerging from stochastic models is that splitting an order into multiple smaller orders submitted to different suppliers may be advantageous, as this reduces the expected lead time up to the time of the first arrival of an order (deemed the effective lead time) [28–30]. However, as already pointed out by Hadley and Whitin [8], such an assumption may lead to the occurrence of order crossovers, in which the sequence of orders does not match

Table 1
Comparison of literature on lead-time variability.

Study	Deterministic	Stochastic	Dynamic	Seasonal
Wright [17]	x			
Zalkind [24]		x		
Nevison and Burstein [25]		x		
Zipkin [26]		x		
Moinzadeh and Nahmias [18]	x			
Harvey and Snyder [32]	x	x	x	
Ramasesh et al. [28]		x		
Song and Zipkin [15]	x			
Tagaras and Vlachos [20]	x			
Silver and Zufferey[16]		x		x
Riezebos [7]	x		x	
Hayya et al. [31]		x		
Huang and Kucukyavuz [27]		x		
Wang and Tomlin [37]	x			
Axsäter [23]	x		x	
Hayya et al. [38]		x		
Jansen et al. [22]	x			
Srinivasan et al. [39]	x			
Fang et al. [40]		x		
Riezebos and Zhu [19]	x		x	
Disney et al. [14]		x		
Liu et al. [1]	x			
Our paper	x	x	x	x

their arrivals. Models that assume independent and identically distributed lead times generally ignore order crossovers because of the associated complexities that hinder the discovery of optimal solutions relating to these models [15]. Riezebos et al. [7,31] and more recently [14] have examined stochastic lead times and order crossovers. Other scholars have applied heuristic solutions, considering the possibility of order crossovers [16].

Hybrid lead-time models entail stochastic as well as dynamic components. This type of models is useful for modeling seasonal patterns using time series demand data. Harvey and Snyder [32] provides an overview of such models. Subsequently, Proietti [33] draws a distinction between fixed (deterministic) seasonal patterns and stochastic patterns in seasonal demand time series models. The pattern's repetition over time is modelled using a deterministic component that changes over time according to the identified seasonal pattern, while the stochastic component models the remaining variation during the seasons. Song and Zipkin [15] applies a slightly different approach by modelling the supply system as a discrete time Markov process. To prevent order crossovers, Song and Zipkin [15] assumes that the moment of transition to a different state of the supply system (e.g., a different season) is known at the start of the previous season, but the length of the lead time is a random variable that depends on the state of the supply system (the season) and the length of the previous lead time. These assumptions make it possible to model various classes of inventory control problems, but they do not encompass the broad category of identifiable seasonal inventory control problems.

We do not find any specific studies focusing on seasonal lead times within the inventory control literature. The only study that explicitly covers seasonal lead times [16] focuses on the practical context of a saw mill that procured wet logs needed to be dried before being further processed. The drying time is modeled as a stochastic lead time. Over the course of the year the average length of the drying time changes because of weather conditions. Therefore, for each period another distribution of the lead-time length is used. Silver and Zufferey [16] allows for some order crossovers in the heuristic analysis, limiting the number of orders that cross over simultaneously to three. The tabu search heuristic that they developed is not designed to explore the seasonal pattern itself because of the complex situation of this saw mill.

Demand seasonality has been studied extensively within the forecasting literature. See [34] for an overview of the findings in the previous 25 years. This literature has developed seasonal estimates for groups of items exhibiting similar seasonal patterns. Insights derived from the field of forecasting have been applied within studies on inventory control to forecast lead-time demands. However, Nielsen et al. [35] argues that the forecasting of lead-time lengths has been largely ignored, notwithstanding their influence on the bull whip effect [36].

In light of our review of the literature, summarized in Table 1, we conclude that the seasonality of lead time variability has not received sufficient attention within the literature. The next two sections apply insights emerging from the literature on both deterministic and stochastic lead-time variations and the related field of forecasting to develop a framework of seasonal lead-time variations (Section 3) as well as guidelines on how to handle these variations (Section 4).

3. Characterization of seasonal lead-time problems

Lead-time seasonality implies variability but not necessarily stochasticity. We distinguish several classes of stochastic lead-time patterns by determining which parameters remain deterministic (following [33]), consequently identifying seasonality in lead times. We apply this notion of classes to develop a framework for characterizing families of seasonal lead-time problems. Strategies deployed for coping with lead-time seasonality may differ among the families of problems. In this section, we describe how we develop this framework using characteristics of lead-time data. We provide illustrative examples of companies that face seasonal lead-time problems relating to the identified classes.

The first step entails the identification of the various parameters of lead-time series data, as shown in Fig. 1. Parameter L denotes the length of a lead time. Changes in substantial different lead time lengths, observed over time, are represented in the following time series: L_j; $j = 1, \dots, N$, where N denotes the number of seasons during a cycle ($N \geq 2$). Parameter I denotes the observed periodicity of this time series indicating that after a time interval I_j a lead-time change to length L_{j+1} can be observed. The subsequent interval lengths between lead-time changes also form a time series:$I_j : j = 1, \dots, N$. The resulting time series $\Lambda_t : t = 1, \dots, n$,

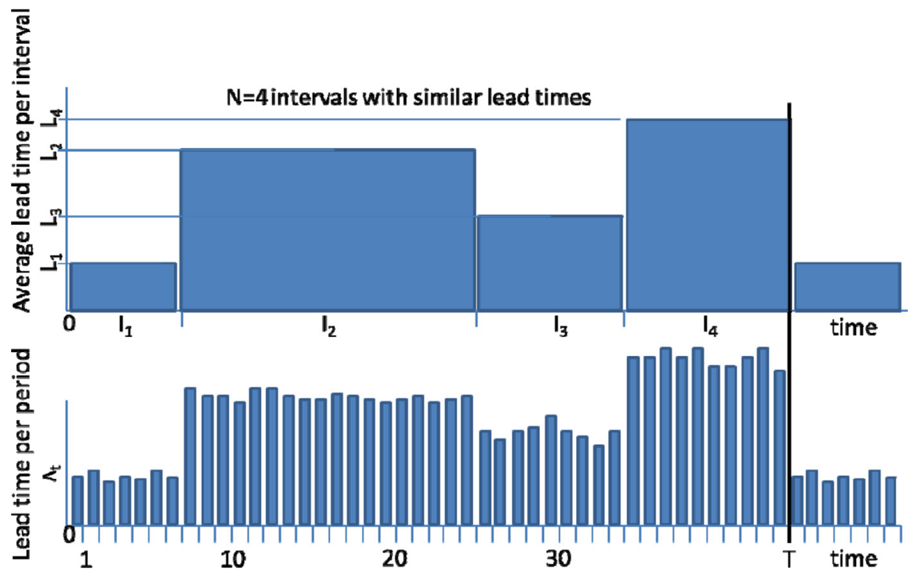

Fig. 1. Parameters of time series with seasonal lead times.

where n denotes the total number of periods in the planning horizon, needs to be examined in conjunction with t, that is, a time series with fixed time periods to determine seasonality.

We examine patterns that emerge for the dimensions of time as well as magnitude relating to the lead-time data. The magnitude dimension reflects the length of the lead time$L$. This length changes over time and a recurring pattern can be observed in the sequence and heights of different lead-time lengths. The time dimension reflects the length of the time interval I during which a specific lead time is available. It should be noted that each of these parameters may be either stochastic or deterministic.

The first step in our analysis entails identifying a cycle length T that is less then n, where n is the number of periods in the time series The cycle length denotes the time span, following which a seasonal lead-time pattern is repeated. The cycle length may be either stochastic or fixed. Next, we determine the number of seasons N and the length of seasons $I_j : j = 1..N$. The number of seasons during a cycle may be either stochastic or fixed. Thus, the parameters constituting a seasonal lead time pattern, which can be either stochastic or deterministic, are $L_j, I_j, N,$ and T. Parameter L_j is the only one of these parameters that relates to magnitude; all of the other parameters relate to time.

Prior to attempting to identify patterns in the data, we find it helpful to set up the scene and to characterize the type of seasonal lead-time problems encountered with this setting. We also distinguish between stochastic and deterministic parameter settings for the time and magnitude dimensions of the lead-time data. Accordingly, we develop a two-by-two framework for analyzing seasonal lead-time problems (Fig. 2).

If all of the parameters are deterministic, then the type of seasonal lead-time problem is categorized as ‘all fixed’. For these deterministic patterns, the number of seasons N within a cycle is fixed, and the cycle length T remains constant. While the length of the seasonal intervals is also deterministic, the values may change dynamically over time. However, these variations are known in advance and are not therefore stochastic in nature. Lead times are similarly fixed but not constant, demonstrating predictable fluctuations over time. This is the first class of seasonal patterns in which both the magnitude and time dimensions of the time series are deterministic, enabling accurate anticipation of the prevailing seasonal patterns.

Stochasticity may relate to either or both dimensions of time or magnitude. If it only applies to magnitude, then although the

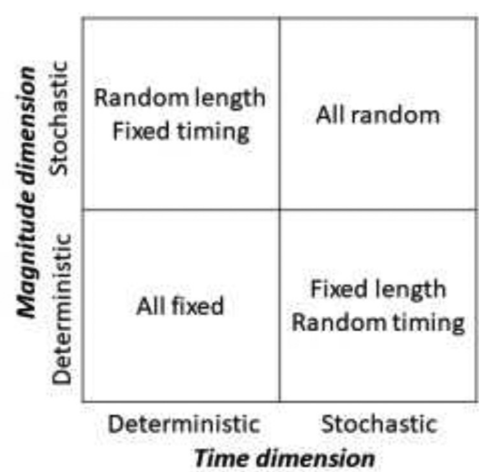

Fig. 2. Framework for characterizing seasonal lead-time problems.

lead-time lengths are stochastic, the precise times when the lead times will change substantially are known in advance, as the beginnings and endings of seasons of long or short lead times are already known. This class is termed ‘Random length, fixed timing’.

If the time dimension is stochastic, then one or more of the parameters $N, I,$ or T will be stochastic. A discussion on possible combinations of these three parameters that may occur within this class of problems is pertinent. Given that the cycle length T generally follows from the characteristics of N and $I,$ our focus here will be on these latter two parameters.

If N is fixed, the number of seasons that can be expected is known in advance, but if the length of the seasons I becomes stochastic, the beginnings and endings of the various seasons become less easily discernable. For example, within the fashion industry, some years entail short summer seasons and long winter seasons while the reverse is the case for other years. Thus, the issue of when the new season will be launched is not discernible in advance.

If, however, N is stochastic, but I is fixed and known, the number of seasons within a cycle is not constant, but the timing of lead-time changes is known in advance. In this case, the lack of certainty relates to whether or not the new lead time will belong

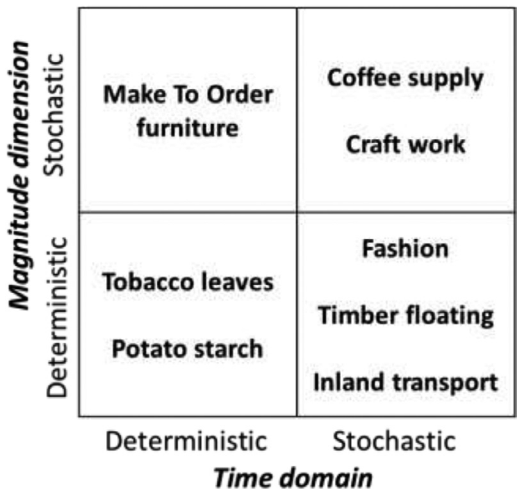

Fig. 3. Examples of seasonal lead-time problems.

to another season. This situation is practically encountered when, for example, traditional seasons are sometimes skipped. [41] notes that the fashion industry typically entails four to five seasons annually, whereas there may be only two seasons for seasonal items for which year-round demand is more stable.

Stochasticity of N and/or I is likely to be indicative of the stochasticity of T, that is, the total cycle length, unless the ending of the previous season has been predetermined in advance in light of a known external factor, that results in the fixing of T. For example, within the fashion industry, it is well known that the launch of the next haute couture cycle coincides with a time that marks a highlight in the year, such as immediately after the fashion week [42].

The 'fixed length, random timing' category entails stochasticity in relation to the time dimension that may be accompanied by deterministic lead-time lengths. In this case, problems are encountered in cases where seasonality relates to lead-time changes, the magnitudes of which are known in advance but the timings of which are uncertain. An example of this seasonal pattern is one in which a company switches to another supplier with a longer lead time at a point in time when its own supply is exhausted.

The final 'all random' category that we distinguish entails stochasticity in terms of both dimensions of time and magnitude. Although clear seasonal patterns may still be evident in this category, they are less obvious and are more difficult to identify in comparison of the other three categories.

Fig. 3 depicts the framework to characterize seasonal lead-time problems. For example, the ordering problem relating to tobacco leaves, analyzed by Riezebos (2006), is an example of an 'all fixed' seasonal problem that is known in advance on a monthly basis at the start and conclusion of harvest seasons in various regions globally. Accordingly, the time taken for the raw materials to be received in the production facility is predetermined. Similarly, the problem of transporting potato starch (see Section 2) belongs to the same category, as the day in the week associated with the lead time is known in advance. The case studies presented by Song and Zipkin (1996) also belong to this class of seasonal lead-time problems.

The 'fixed length and random timing' problem class (shown in lower right-hand corner of Fig. 3) is widely encountered within the fashion industry, where there is a high degree of uncertainty regarding the timing of lead-time changes [42,43]. Other examples of industries where this problem is encountered are timber floating in Scandinavia [44] and inland transport [5]. The latter industries are strongly affected by the availability of waterways for transportation, which is contingent on weather conditions. Hence, the uncertainty entailed in these examples mainly relates to the time dimension, while the magnitude remains relatively predictable.

Examples of uncertainty relating to the magnitude dimension can be found in make-to-order production activities. For example, a furniture manufacturer in the Netherlands, who builds furniture according to customers' requirements, knows in advance when long or short lead times can be offered to the customers, in light of repetitive patterns of peak demand before the summer and the non-availability of personnel during the summer vacation. However, the actual lead time that pertains to a customer depends on the sequence of colors applied in the factory relating to those that have been ordered. Hence, while the timing of lead-time changes is known in advance, the lead-time lengths depend on the sequencing and demand for other colors.

Shukla and Naim [45] presents an example of the all-random class lead-time seasonality in a coffee supply chain, although the seasonal patterns in that paper can mainly be attributed to endogenous effects. Another example relates to customers' orders submitted to a small-scale producer of handicrafts who faces problems relating to limited capacity and work-load estimation. The issue of stochasticity applies to these customers in terms of both dimensions of time and magnitude of the lead times, as predetermined future changes associated with the lead times offered cannot be anticipated in advance. At the same time, there may be significant disparities between the lead times that are offered and those that are realized. These disparities result from difficulties in estimating creative processes.

In light of our application of the above-discussed framework to characterize seasonal lead-time problems, we now turn to the as yet unanswered question of how to contend with the different causes of seasonal lead-time variations. Knowing the root causes enables the development of specific inventory-control models. In the case of deterministic variations, future changes, either in the timing or length of a lead-time change, can be anticipated. However, in the case of stochasticity, the main cause of seasonality may relate to the magnitude and/or time dimensions. In the following section, we propose inventory-control policies that explicitly entail the assumption of seasonal lead times.

4. Inventory models in deterministic magnitude dimension

Our aim in this section is to identify building blocks for developing solutions that address seasonal lead-time problems relating to the deterministic magnitude dimension, namely the 'all fixed' and 'fixed length and random timing' categories. The problems considered here are very elementary to enable the identification of the effects of seasonality on lead times. However, valuable theoretical and practical insights can be gained from this analysis.

4.1. Fixed lengths and the timing of seasonal lead times

We develop a model for the 'all fixed' class of seasonal lead times within a periodic-review setting with integer-valued interval and lead-time lengths and a review period length of 1. In this model, demand, lead times, and interval lengths are deterministic, but could vary dynamically over time, which is a characteristic feature of dynamic lot-size models. Our aim is to develop an optimal ordering policy for a repetitive cycle. Without incurring a loss of generality, we confine our attention to problems entailing lead times that do not exceed the cycle time.

Differences in lead times are essential aspects of seasonal lead-time problems. To explore their impact on cost performance, we introduce two types of lead-time-related costs per item per period: in-transit cost related to pipeline inventory and holding cost related to in-stock inventory. The in-transit cost is incurred from the time of ordering up to the time of arrival of stock. Hence, this cost is charged per period over the inventory position minus the inventory level of that period. We calculate the inventory position and inventory level at the end of the period (after accounting for arrivals and demand). The in-transit and holding cost is computed at the end of a period. Our key assumptions, aiming at gaining elementary insights into the cyclical behavior of the inventory policy, are that stationary in-transit and holding cost per period is known and stock outs are disallowed, as has been commonly reported within the inventory literature (see [11,23]). Given our focus on the effect of seasonal lead times, we do not include ordering and purchasing costs. In Section 6, we propose extensions of our model that include these costs.

The following notations are included in this cyclical problem:

- q_t order size in period t,t=1,...,T
- ip_t inventory position at the end of period t
- il_t inventory level at the end of period t
- D_t demand in period t
- I_j interval length of jth subcycle, j=1,...,N
- T total cycle length ($T=\sum_{j=1}^{N} I_j$)
- L_j lead time length that holds in jth season
- Λ_t lead time that holds when ordering q_t in period t
- c in-transit costs per period charged over the pipeline inventory, i.e., $(ip_t - il_t)$
- h holding costs per period charged over inventory level il_t (normally h>c)

We assume, without loss of generality, that $L_1 = \min_{j=1\ldots N}\{L_j\}$, that is, the cycle commences at the time when the shortest lead time becomes available for ordering stock. For the brevity of the notations, we assume that the input parameters ($D_t, L_j, I_j, N,$ etc.) have identical values for each cycle. This model can easily be extended more generally to cases entailing different values during each cycle.

Lead-time lengths are expressed for each period t and each cycle j. The relation between these two variables within a deterministic setting is expressed as $\Lambda_t = L_j$, where t satisfies $\sum_{l=1}^{j-1} I_l < t \le \sum_{l=1}^{j} I_l$, and $\Lambda_t = \sum_{j=1}^{N} (L_j \cdot 1\{\sum_{l=1}^{j-1} I_l < t \le \sum_{l=1}^{j} I_l\})$, where $1\{A\} = 1$ if A is true, else 0.

The chronology of events that occur during each period is as follows. At the beginning of each period, scheduled receipts arrive for this period, following which the buyer may place a new order in the current period. Consequently, a demand is generated for the product during the period. At the end of the period, the inventory position and level are recorded and the holding cost for this period is calculated.

An ordering policy stipulates the quantity of the stock to be ordered for each period of the cycle. Therefore, optimization entails formulating an ordering policy that minimizes the total in-transit cost and holding costs per cycle of length T:

$$\min_{q_1,\ldots,q_T} c\sum_{t=1}^{T}(ip_t - il_t) + h\sum_{t=1}^{T} il_t \tag{1}$$

$$s.t.\ ip_t = ip_{t-1} + q_t - D_t, \qquad \forall t = 1\ldots T \tag{2}$$

$$il_t = ip_t - \sum_{i=1}^{t}(q_i \cdot 1\{\Lambda_i > t - i\}), \qquad \forall t = 1\ldots T \tag{3}$$

$$il_t \ge 0, \qquad \forall t = 1\ldots T \tag{4}$$

$$il_{L_1} = 0. \tag{5}$$

Eq. (1) shows that the main decision variables $q_1, \ldots, q_T$ are determined to minimize the sum of the cycle holding costs, where in-transit costs are charged over the pipeline inventory per period and holding costs are charged over the inventory level per period. Eq. (2) provides a standard balance equation relating to the inventory position. Eq. (3) indicates that the difference between the inventory level and the inventory position is the amount of already issued but not yet received orders. If orders placed in periods 1 to t have not yet been received, then their ordering time i, and lead time, Λ_i, will exceed the current time t. The inventory level cannot be allowed to become negative, as shown in Eq. (4). Eq. (5) subsequently fixes the starting inventory level in the period just before an order with the smallest lead time in the system can arrive at zero. We are able to prove that this setting can be established without any loss of generality.

Table 1a
Ordering policy for alternating lead times wherein $L_1 < L_2$.

Ordering interval		Order size $q_t = \sum_{n=a1}^{a2} D_{n \bmod T}$ with $a1$ and $a2$:	
From $o1$	To $o2$	From $a1$	To $a2$
1	$I_1 - 1$	$t + L_1$	$t + L_1$
I_1	I_1	$I_1 + L_1$	$\min\{T + L_1, I_1 + L_2\}$
$I_1 + 1$	$(L_1 - L_2) \bmod T$	$t + L_2$	$t + L_2$

4.1.1. Base case model with two alternating lead times

First, we analyze a base case that includes alternating lead times. For this case, we assume that one cycle comprises two intervals. During the first interval I_1, a lead-time length of L_1 is applied, and during the subsequent interval of length $I_2 = T - I_1$, a lead-time length of L_2 is applied, where $L_1 < L_2$. We assume that the interval lengths and the lead times are deterministic Table 1a.

Table 1 presents the structure of the cyclical ordering policy. During each period t, q_t is ordered in the interval $[o1, o2]$, as specified at the far right of Table 1. For all other periods, $q_t = 0$. If $o2 < o1$, the interval will be empty, and the specified order size will not be ordered. For a positive value of m, as per the definition of the modulo operator, $-m \bmod T \equiv (T - m) \bmod T$.

As Table 1 shows, because the lead time is short during interval $[1, I_1 - 1]$, the buyer should order the precise amount of the product that is required for the period when the order is scheduled to arrive. However, during period I_1, because $L_1 < L_2$, the following two scenarios should be considered:

(a) If the first order, for which there is a long lead time in the current cycle, is received before the first order associated with a short lead time in the next cycle, it becomes necessary for the buyer to order an amount that is sufficient to cover the demand that arises during the interval $[I_1 + L_1, I_1 + L_2]$.

(b) If the first order associated with a long lead time in the current cycle is received after the arrival of the first order associated with a short lead time in the next cycle, the buyer will need to order an amount that covers the demand generated during the interval $[I_1 + L_1, T + L_1]$.

When $t + L_2 \le T + L_1$, then the buyer will need to avail of the long lead time to fulfill the demand before the arrival of the first order with the short lead time. Accordingly, two possible outcomes derived from Table 1 are obtained:

i. $L_1 \bmod T > (I_1 + L_2) \bmod T$,
ii. $L_1 \bmod T \le (I_1 + L_2) \bmod T$.

The first outcome indicates that in period I_1, the firm only needs to order a sufficient amount of stock to meet the demand in the interval $[I_1 + L_1, I_1 + L_2]$. The second outcome indicates that the time of arrival of the first order with a long lead time in the current cycle occurs after the time of arrival of the first order with a short lead time in the next cycle. Consequently, no orders associated with such a long lead time should be submitted.

To give an illustrative example of Table 1, we construct the following numerical example entailing 12 monthly periods, using a case study of a tobacco sourcing problem described by Riezebos [7]. In this example, a short lead time of one month exists for four months (periods 1 to 4). An ordering option with a long lead time of three months, entailing several production stages, exists during the remaining interval length of eight months (periods 5 to 12) in the year. The demand per period is 100 units and the initial inventory level is zero. The results are as follows. During periods 1 to 3, in which the lead time is short, 100 units per period are ordered. During period 4, which is the last period when the short lead time is available, 300 units are ordered. During periods 5 to 10, which entails a long lead time, 100 units per period are order. Finally, no order is placed during periods 11 to 12, as it is considered preferable to postpone the order and wait for the short lead-time option to become available again.

As indicated in Table 1, an order-up-to policy is optimal. Even if a backorder cost is incorporated into this model, the order-up-to policy remains optimal because of the convexity of the total cost function.

The following insights are derived from his base model. During periods when the lead time is short, the exact amount of product required for that period should be ordered with the exception of the last period of the interval. During this last period, an order should be placed in advance of subsequent periods, thereby availing of the shorter lead time and preventing the occurrence of stock-outs caused by the longer lead time during these periods. Although ordering in advance increases the holding cost, this is necessary to fulfil the demand. For the periods in the interval with the long lead time, it is necessary to decide whether to order with a long lead time or to postpone the order to the next cycle, availing of the short lead time. Postponement reduces the in-transit cost, while the holding cost remains the same. Depending on the length of the long lead time, if the time of arrival of the current order is no later than the time of arrival of the first order of the next cycle, orders should be placed during the current period. Otherwise, it is cost-efficient to postpone ordering.

4.1.2. Extension with multiple lead times

The optimal ordering quantity for the model with multiple lead times (i.e., a cycle comprising N seasons) can now be computed. First, a forward procedure is applied to determine whether the arrival period can be covered directly. If there is an ordering period whose lead time enables the order to arrive during that arrival period, then the arrival period can be covered directly. If there are more ordering periods that cover that arrival period, then the one with the smallest lead time is selected in order to reduce the in-transit cost, which results in a decrease in the total cost.

Algorithm: *Multiple lead times*

Step 1 (Forward loop).

Define $M = 1 + \max_{j=1}^{N} L_j$

For $t = 1, \dots, T$, calculate the best season for placing an order that will directly cover the demand during t:

$$j* = \underset{j=1,\dots,N}{\arg\max} \left(\left(M - L_j\right) \cdot 1 \left\{ \sum_{k=1}^{j-1} I_k < 1 + \left(t - L_j - 1\right) \bmod T \leq \sum_{k=1}^{j} I_k \right\} \right).$$

If the maximum exceeds 0, that is, one or more feasible ordering times exist, then the one with the smallest lead time L_{j*} (i.e., the one that maximizes $M - L_j$) is preferable, enabling $q_{1+(t-L_{j*}-1) \bmod T} = D_t$ and the demand in the arrival period t to be covered directly by the order in period $1 + (t - L_{j*} - 1) \bmod T$. The indicator variable is set to $C_t = 1$ to cover this arrival period.

If the maximum is 0, then there is no ordering moment that directly coincides with the arrival moment t, hence $C_t = 0$. Step 2 is performed to determine from what ordering moment that arrival moment will be served indirectly.

Step 2 (Backward loop).

For $t = T, \dots, 1$ do if $C_t = 0$:

$t_u := t$

repeat $t := t - 1$ until $C_t = 1$

$t_l := t$

$$j* = \underset{j=1,\dots,N}{\arg\max} \left(\left(M - L_j\right) \cdot 1 \left\{ \sum_{k=1}^{j-1} I_k < 1 + \left(t_l - L_j - 1\right) \bmod T \leq \sum_{k=1}^{j} I_k \right\} \right)$$

$$q_{1+\left(t_l - L_{j*} - 1\right) \bmod T} = \sum_{n=t_l}^{t_u} D_n$$

Theorem 1. *The above algorithm provides an optimal solution for Eq. (1).*

Proof. For any given period t_u, if $1\{\sum_{k=1}^{j-1} I_k < 1 + (t_u - L_j - 1) \bmod T \leq \sum_{k=1}^{j} I_k\}$ is equal to 1, then the demand in period t can be fulfilled by the order in period $1 + (t_u - L_j - 1) \bmod T$ with a lead-time length of L_j. Because the calculation of the in-transit cost is based on the pipeline inventory, the order with the shortest lead time is always preferable for minimizing the in-transit cost. On the one hand, when the maximum is positive ($C_{t_u} = 1$), the order with the shortest lead time L_{j*}, required to fulfil the demand in period t_u is obtained. On the other hand, if a feasible lead-time option is not available for period t ($C_{t_u} = 0$), then it is necessary to take a step backwards to find the closest period t_l for which at least one lead-time option available. The shortest lead-time option available for period t_l,enabling the demand from period t_l to period t_u to be satisfied, is then used. Thus, we have $q_{1+(t_l - L_{j*} - 1) \bmod T} = \sum_{n=t_l}^{t_u} D_n$. This completes the proof.

Next, we consider a model with three lead times. As shown in Tables 2 and 3, there are two different scenarios for three intervals per season: $L_1 < L_2 < L_3$ and $L_1 < L_3 < L_2$.

The following four cases can be derived from the seasonal ordering policy shown in Table 2:

i. $L_1 \bmod T < (I_1 + L_2) \bmod T$,
ii. $(I_1 + L_2) \bmod T \leq L_1 \bmod T < (I_1 + I_2 + L_2) \bmod T$,
iii. $(I_1 + I_2 + L_2) \bmod T \leq L_1 \bmod T < (I_1 + I_2 + L_3) \bmod T$, and
iv. $(I_1 + I_2 + L_3) \bmod T \leq L_1 \bmod T$.

In case (i), the order is placed immediately prior to the commencement of a lead-time change and covers the demand from the current period to the period prior to the time of arrival of the first order with the changed lead time. In case (iv), the firm never orders after period I_1 because of the increase of L_2 and L_3.

In a situation where $L_1 < L_3 < L_2$, order crossover wherein replenishment orders are not received in the sequence in which they are ordered may occur. For instance, if we place an order with L_2, because $L_3 < L_2$, that order may arrive after an order placed during a later period associated with L_3, which results in order crossover.

The following four cases can be derived from the seasonal ordering policy shown in Table 3:

i. $L_1 \bmod T \leq (I_1 + I_2 + L_3) \bmod T \quad \wedge \quad L_1 \bmod T \leq (I_1 + L_2) \bmod T$,

Table 2
Seasonal ordering policy when $L_1 < L_2 < L_3$.

Ordering interval		Order size $q_t = \sum_{n=a1}^{a2} D_{n \bmod T}$ with $a1$ and $a2$:	
From $o1$	To $o2$	From $a1$	To $a2$
1	$I_1 - 1$	$t + L_1$	$t + L_1$
I_1	I_1	$I_1 + L_1$	$\min\{T + L_1,\ I_1 + L_2\}$
$I_1 + 1$	$\min\begin{Bmatrix}(L_1 - L_2) \bmod T \\ I_1 + I_2 - 1\end{Bmatrix}$	$t + L_2$	$t + L_2$
$I_1 + I_2$	$I_1 + I_2$	$I_1 + I_2 + L_2$	$\min\{T + L_1,\ I_1 + I_2 + L_3\}$
$I_1 + I_2 + 1$	$(L_1 - L_3) \bmod T$	$t + L_3$	$t + L_3$

Table 3
Seasonal ordering policy when $L_1 < L_3 < L_2$.

Ordering interval		Order size $q_t = \sum_{n=a1}^{a2} D_{n \bmod T}$ with $a1, a2$:	
From $o1$	To $o2$	From $a1$	To $a2$
1	$I_1 - 1$	$t + L_1$	$t + L_1$
I_1	I_1	$I_1 + L_1$	$\min\begin{Bmatrix}T + L_1, \\ I_1 + L_2, \\ I_1 + I_2 + L_3\end{Bmatrix}$
$I_1 + 1$	$\min\begin{Bmatrix}(L_1 - L_2) \bmod T \\ (I_1 + I_2 + L_3 - L_2) \bmod T\end{Bmatrix}$	$t + L_2$	$t + L_2$
$I_1 + I_2 + 1$	$(L_1 - L_3) \bmod T$	$t + L_3$	$t + L_3$

ii. $L_1 \bmod T \leq (I_1 + I_2 + L_3) \bmod T \quad \wedge \quad L_1 \bmod T \geq (I_1 + L_2) \bmod T$,

iii. $(I_1 + I_2 + L_3) \bmod T \leq L_1 \bmod T \quad \wedge \quad (I_1 + I_2 + L_3) \bmod T \leq (I_1 + L_2) \bmod T$,

iv. $(I_1 + I_2 + L_3) \bmod T \leq L_1 \bmod T \quad \wedge \quad (I_1 + I_2 + L_3) \bmod T \geq (I_1 + L_2) \bmod T$.

In case (i), the firm only places orders during the period with the shortest lead time. However, in case (iv), the firm places orders when all three lead times occur. Case (iii) reveals an interesting situation in which the firm does not place any orders at all when the lead time is L_2, and instead postpones the ordering process until a shorter lead time, L_3, becomes available, as these orders will arrive earlier. Case (ii) reveals a contrasting situation in which orders are placed when lead time L_2, but not lead time L_3 occurs. Thus, whereas orders are always placed when lead time L_1 is available, the decision on whether or not orders should be placed when the lead time are L_2 or L_3 depends on the size of $I_1, I_2, L_2,$ and L_3.

In sum, this extension of the base case to cover a more complex situation entailing multiple lead-time changes per cycle reveals that the fundamental insight relating to the base case, namely that the decision on whether or not to order is strongly influenced by the seasonal pattern of lead-time changes during the cycle, extends to the case of $L_1 < L_2 < L_3$.

This extension yields an additional fundamental insight relating to the case in which $L_1 < L_3 < L_2$: anticipation of crossovers caused by the lead-time seasonality may result in the postponement of orders. It is cost-efficient to postpone some orders during the second interval by taking advantage of the shorter lead time in the third interval preventing the occurrence of order crossovers. Assuming that demand is deterministic, an optimal policy is one in which the order is always fulfilled during the interval with the shortest lead time, thereby avoiding crossovers.

4.2. Fixed length and random timing of seasonal lead times

Next, we relax the assumption of deterministic interval lengths for the base model with two alternating lead times per season. Instead, we assume that the length of the first interval, entailing a short lead time, is stochastic, while the sum of both intervals remains a constant T, that is, the first interval occurs at the same time during every season. When modelling the uncertain length of the first interval, we assume that if it continues at the start of period t, the probability that it will conclude at the end of period t is p_t. To ensure that the length of the first interval would not exceed the cycle length T, the value of p_T is set as 1. Intuitively, if the length of the first interval approaches T, then the likelihood of the termination of the first interval will be higher. Thus, the probability of a transition to the other lead-time length (i.e., closure of the first interval) should not decrease over the course of time (e.g., $p_t = \frac{t}{T}$). The objective function is to minimize the expected in-transit, holding and backordering costs. Accordingly, the in-transit costs are calculated on the basis of the pipeline inventory, the holding costs are calculated on the basis of the inventory level, and the backordering costs are calculated on the basis of the inventory level at the time of arrival. We denote b as the unit backordering cost. Here, we assume that the in-transit cost is calculated at the time of arrival of an order. The results remain the same when the in-transit costs are calculated as the time of ordering or for each period.

We define $V_t(x, i)$ as the minimal expected total cost from period t to the end of the cycle T, given that the initial inventory level in period t is x and the number of periods during which lead time L_1 has been available for ordering up to t is i (including the current period). Define $Q_{t,j}$ as the amount arriving at the beginning of period t when the lead time is L_j.

$$V_t(x,i) = \min_{Q_{t,1},Q_{t,2}} \begin{Bmatrix} c\left(\sum_{j=1}^{2} L_j Q_{t,j}\right) + h\left(x + \sum_{j=1}^{2} Q_{t,j} - D_t\right)^+ \\ +b\left(D_t - x - \sum_{j=1}^{2} Q_{t,j}\right)^+ \\ +p_t V_{t+1}\left(x + \sum_{j=1}^{2} Q_{t,j} - D_t, i\right) \\ +(1-p_t)V_{t+1}\left(x + \sum_{j=1}^{2} Q_{t,j} - D_t, i+1\right) \end{Bmatrix};$$

$$t = 1, \ldots, T-1$$

$$V_T(x,i) = \min_{Q_{T,1},Q_{T,2}} \begin{Bmatrix} c\left(\sum_{j=1}^{2} L_j Q_{T,j}\right) + h\left(x + \sum_{j=1}^{2} Q_{T,j} - D_T\right)^+ \\ +b\left(D_T - x - \sum_{j=1}^{2} Q_{T,j}\right)^+ \\ +V_{T+1}\left(x + \sum_{j=1}^{2} Q_{T,j} - D_T, i\right) \end{Bmatrix}.$$

Without loss of generality, we assume that $V_{T+1} = 0$.

The following lemmas constitute the building blocks for our solution, demonstrating that if i is equal to t, then interval I_1 continues; otherwise, I_1 ends and I_2 begins.

Lemma 1. *If $i = t,\ t \in I_1$. Otherwise, if $i \neq t,\ t \in I_2$.*

Proof. Because I_1 always begins in period $t = 1$ and i is the length of I_1, it is clear that if $i = t$, then $t \in I_1$. Otherwise, if $t > i$, it means that I_1 ends before t. Therefore, t belongs to I_2.

Lemma 2. *For $t > i$ we have $p_t = 1$.*

Proof. By Lemma 1, we have t belongs to I_2 for all $t > i$. Since the state cannot revert back to I_1 once it entersI_2, we obtain $p_t = 1$.

The optimal order quantity that is determined for the time of arrival is transformed into the corresponding ordering time by calculating i. Because the optimal solution depends on the length of the lead time, we analyze two cases: $L_1 < L_2$ and $L_2 < L_1$. The structure of the optimal policy can be characterized as follows.

Proposition 1. *For $L_1 < L_2$ and $L_1 + 1 \le t \le T,\ Q_{t,2} = 0$ for $i \le t \le i + L_2$, and $Q_{t,1} = 0$ for $i + L_1 < t$.*

Proof. For $L_1 < L_2$, we need to consider two cases.

Case 1: $L_2 + 1 \le t \le T$.

If $i \le t \le i + L_1$, then the option to order with lead time L_1 is available during period $t - L_1$. Because the in-transit cost increases with the length of the lead time, an order associated with the short lead time is a preferred option for minimizing the cost. Thus we have $Q_{t,2} = 0$.

If $i + L_1 < t \le i + L_2$, then L_1 is unavailable during period $t - L_1$ and L_2 is unavailable during period $t - L_2$. Therefore, $Q_{t,2} = Q_{t,1} = 0$.

If $t \ge i + L_2$, then L_1 is unavailable during period $t - L_1$, and $Q_{t,1} = 0$.

Case 2: $L_1 + 1 \le t \le L_2$.

Following a similar logic to that in Case 1, if $i \le t \le i + L_1$, then $Q_{t,2} = 0$;if $i + L_1 < t$, as L_1 is unavailable during period $t - L_1$, therefore, $Q_{t,1} = 0$.

This completes the proof.

It is noteworthy that L_1 is always available during the first period. Because the in-transit cost increases with the length of the lead time, an order with a short lead time is preferable for minimizing the cost. The firm should therefore order the demand stock during period t associated with a short lead time L_1. Thus, $Q_{t,2} = 0$.

To obtain the optimal solution from period 1 to period L_1, the information on the number of periods when the lead time L_1 is available during the previous cycle is required. This can be denoted as i^0, where $1 \le i^0 \le T$.

Proposition 2. *For $L_1 < L_2$ and $1 \le t \le L_1$,*

- *when $i^0 = T$, then $Q_{t,2} = 0$; and*
- *when $i^0 < T$, then $Q_{t,2} = 0\, \forall t \le i^0 + L_2 - T$ and $Q_{t,1} = 0\, \forall t \ge i^0 + L_1 - T$.*

Proof. During the preceding cycle, the lead time L_1 is, by definition, available from period 1 to period i^0 while L_2 is available from period $i^0 + 1$ to period T. For $i^0 = T,\ L_2$ is unavailable. Hence, $Q_{t,2} = 0$.

For $i^0 < T$, the following three scenarios should be considered.

If $t \le i^0 + L_1 - T$, then lead time L_1 is available at the time of ordering, that is, $T + t - L_1$. A shorter lead time leads to a lower in-transit cost. Therefore, $Q_{t,2} = 0$.

If $i^0 + L_1 - T < t < i^0 + L_2 - T$, then L_1 is unavailable during period $T + t - L_1$ and L_2 is unavailable during period $T + t - L_2$. Consequently, $Q_{t,1} = Q_{t,2} = 0$.

If $t \ge i^0 + L_2 - T$, then the lead time L_1 is unavailable at the ordering time $T + t - L_1$. Hence, $Q_{t,1} = 0$.

This completes the proof.

Under the assumption of $L_1 < L_2$, Proposition 1 indicates that orders should be placed in advance by taking advantage of the short lead time. Proposition 2 demonstrates orders associated with a long lead time should be postponed to the next cycle if the order with the short lead time arrives earlier than the current order with the long lead time. Under the assumption of $L_1 > L_2$, we provide similar propositions, omitting their proofs, which are similar to the previous proofs.

Proposition 3. *For $L_1 > L_2$ and $L_1 + 1 \le t \le T,\ Q_{t,2} = 0\ \forall i \le t \le i + L_2$ and $Q_{t,1} = 0\ \forall i + L_2 < t$.*

Proposition 4. *For $L_1 > L_2$ and $1 \le t \le L_1$,*

- *when $i^0 = T,\ Q_{t,2} = 0$;*
- *when $i^0 < T$, if $t \le i^0 + L_2 - T$, then $Q_{t,2} = 0$;and if $t > i^0 + L_2 - T$,then $Q_{t,1} = 0$.*

Propositions 3 and 4 indicate that orders associated with a long lead time should not be placed, thereby avoiding order crossovers when $L_1 > L_2$. According to the optimality equations, we can prove that the cost function is jointly convex in the order decision for any given state. Based on the convexity, the structure of the optimal policy can be characterized as a state-dependent order-up-to policy.

Given the above-described characteristics of the optimal policy, we offer the following managerial recommendations. First, when making decisions relating to orders, a manager should compare the current inventory level with the order-up-to level. If the former is below the latter, then the manager should consider raising the inventory level to the order-up-to level by availing of the shortest lead-time option. Second, the last period for which a short lead time is available for orders should be used to cover the demand for several successive periods. Thus, orders are placed during the last period of the interval. Finally, orders associated with a long lead time should be postponed to the next cycle if another order with a short lead time arrives before the current order with a long lead time.

The propositions presented in this section extend the application of building blocks discussed in Section 4.1 to the solution for seasonal lead-time problems in which the timing of seasons is stochastic. The extended building blocks balance in-transit, holding, and backordering costs by availing of the known lead-time differences and accounting for the probability of a lead-time change occurring during the cycle. It should be noted that under the optimal policy for fixed length and random timing of seasonal lead times, order crossovers never occur. However, in case of stochastic lead-time lengths, the optimal policy may anticipate order crossovers. This is discussed in Section 5.

5. Seasonality where lead-time lengths are stochastic

In the case of seasonal patterns in stochastic lead-time lengths, reconsideration of the building blocks presented in Section 4 becomes necessary. Because the exact time of arrival of issued orders cannot be controlled, some safety measures are required to avoid excessive costs associated with moving, storing, and backordering inventory. Backordering costs are generally much higher than the costs of moving and holding inventory. This trade-off between backordering and inventory costs traditionally results in some positive safety stock. However, in the case of non-stationary lead times, these safety stocks also become non-stationary, and insights on how to set safety stock levels in this context are required. Safety lead time can also be introduced by splitting the end-of-interval order quantity (which generally differs from the regular order quantity; see Tables 1–3). Splitting large orders and placing partial orders in advance during some periods increases the probability of an on-time arrival of some items, thereby avoiding backorders. Hence, the inclusion of some safety time could be beneficial. However, the literature does not provide insights on how to set these safety time levels.

5.1. Setting-up of numerical experiments

Our numerical experiments have to reveal the impact of safety stock and safety lead time on the total cost per cycle as compared to using dedicated methods that account for seasonality in stochastic lead times. In the benchmark scenario, seasonal patterns are neglected and variations in lead times are considered to arise from a general stochastic process.

We introduce the following additional notation:

b Backordering costs per period per item short
Λ_t^a Expected lead time of order arriving in period t
SS_t Safety stock required for orders arriving in period t
S_t Order-up-to level when ordering in period t
ST Safty time = number of periods over which large orders are evenly split
k Service factor

We now assume that L_j (i.e. the lead-time length during interval I_j) is a stochastic variable that is independently and identically distributed within the interval period. However, immediately following the commencement of the new interval, the lead times will be drawn from another distribution.

In this section we present the simplest case to illustrate the effects of safety factor and safety time in seasonal inventory control. This case entails just two different lead-time distributions or parameter sets per cycle, and equal (and hence known) lengths of the two intervals per cycle. We assume: $L_1 \sim \mathrm{N}(\mu_1, \sigma_1)\text{and}L_2 \sim \mathrm{N}(\mu_2, \sigma_2)$, where$\sigma_j = 0.4\mu_j\text{for}j = 1, 2$. Hence, following the commencement of another season, the lead-time distribution remains normal and the coefficient of variation remains 0.4, so only both means change, resulting in different standard deviations. This assumption makes the benchmark situation more realistic as well, because the seasonal pattern yields a mixture of two normal distributions that might not easily be distinguishable from a unimodal normal distribution with mean $(\mu_1 + \mu_2)/2$ when testing for normality. The following parameters are used in the numerical experiments: $\mu_1 = 15$; $\mu_2 = 25$; and for the benchmark:$\mu = 20$; $\sigma = 9.6\,\text{weeks}$.

The standard deviation for the benchmark distribution is determined through repeated applications of the Anderson-Darling test for normality on samples obtained from the time series data of our lead-time distributions. As expected, the coefficient of variation of the benchmark distribution is slightly greater than the original distributions (0.48 versus 0.4). The average p-value of the normality tests exceeds 0.1. Therefore, the benchmark distribution is found to be normal based on the results of the statistical tests, whereas in reality its normal distribution is bimodal rather than unimodal.

In the numerical experiments, the values of the backordering, holding, and in-transit costs are 50, 1, and 0.8, respectively. When determining the service factor k, we have to compensate for early arrivals because of the occurrence of order crossovers in these situations. We use the bounds on the difference in the lead-time variance for normally distributed lead times provided by [31]. The service factor k is obtained by dividing the required safety factor in cases where no order crossovers occur by a factor ranging between $\sqrt{\sigma\sqrt{\pi}}$and $\sigma\sqrt{1/2}$ to compensate for the effective lead times. In our example, where $\sigma = 9.6$, it is necessary to divide the required safety factor by at least 4.125. We therefore experiment with service factors $k = 0.4$ and $k = 0.5$, which corresponds to the lower bounds of the cycle service levels (i.e., the probabilities of no stock-out at the time of arrival of an order), ranging between 95% and 98% if the benchmark is the actual lead-time distribution. Moreover, we assume that cycle length $T = 52$ weeks, demand per week D is constant and equal to 1, and interval lengths $I_1 = I_2 = 26$ weeks. The simulation is performed with one hundred replications. Each replication comprises one thousand cycles. The performance measure is the average total cost per cycle. In light of the results of a statistical analysis, the warm-up period is set at 150 cycles to eliminate the impact of the initial inventory available during the first cycle.

5.2. Lead times and safety buffers in the numerical experiments

We assume that the lead time of an issued order is not known until it arrives. However, at the time of ordering, the distribution of the lead time length is known as we have known interval lengths. This assumption allows for situations wherein orders that are issued later arrive before other orders that have been issued earlier. As ordering costs are not considered, we expect frequent ordering to occur and hence frequent instances of order crossovers.

We will now consider the safety stock requirements over time in the case of seasonal stochastic lead-time lengths. The decision not to order at the current time of ordering will only be made if sufficient stock is anticipated during the next possible time of arrival of an order. Hence, to determine whether or not to order in periodic ordering systems, we calculate the required safety stocks at the first point of arrival, t, following the time of arrival of the current order. If the expected stock is lower than the safety stock level during that period, we need to submit an order immediately. The size of the required safety stock at the next time of arrival is dependent on the uncertainty of the total demand up to this time of arrival. The safety stock at the time of arrival, t, is equal to $SS_t = kD\sigma_j$, where σ_j denotes the standard deviation of the lead time of the order that is expected to arrive during period t. It should be noted that it is not necessary to include the lengths of the interval and review period in the safety stock calculation, as no uncertainty is faced regarding the lengths of these periods.

The safety stock at t has to be determined prior to deciding on the order for the previous time of arrival. Hence, we have to search backwards to find the time of ordering for which this safety stock needs to be included. We assume that the last order that arrives before t does arrive at period $t - i$, and denote the expected lead time of that order as Λ_{t-i}^a. The time of placing that order is $t_o = t - i - \Lambda_{t-i}^a$. For every expected time of arrival t, we are able to identify the time of ordering at which the safety stock at t has to be accounted for. The values of $i\text{and}t_o$ are calculated based on the information presented in Table 1 and applying our multiple lead-time algorithm.

5.3. Non-stationary base stock policy

We will now formulate the non-stationary base stock policy. If the expected stock at t is below the required safety stock level at t, the expected stock should be raised to the safety stock level by ordering the difference at the current ordering time t_o. Therefore, at time t_o, both the future t and the expected stock at this time need to be estimated along with the required safety stock at t. If we assume that all already issued orders and the current inventory level will be used to cover the demand up to t, then the expected stock level at time t will be equal to the current inventory position minus all of the demand arising up to t, that is, during $t - t_o$ periods. Hence, at the time of ordering t_o, the inventory position is raised to the order-up-to level $S_{t_o} = D(t - t_o) + SS_t$. This order-up-to level is non-stationary, as both the safety stock and the length of time

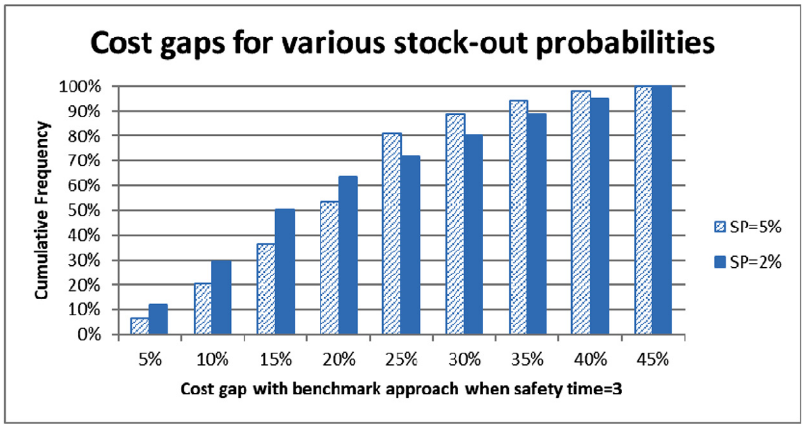

Fig. 4. Cumulative gap-size frequencies for stock-out probabilities SP of 2 and 5%.

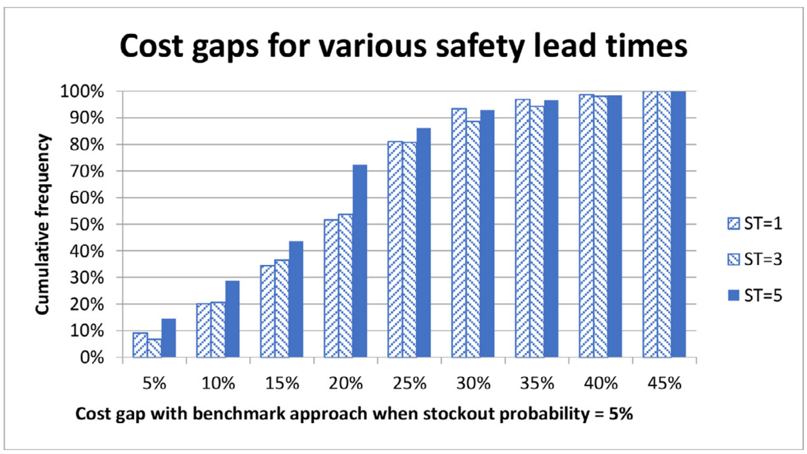

Fig. 5. Cumulative gap-size frequencies for different safety lead-time lengths (ST).

$t - t_o = t - i - \Lambda^a_{t-i}$ for covering the demand change over time. The expected value of i is normally equal to 1. However, in the case of an interval change from I_1 to I_2 (caused by a lead-time increase), the expected value of i becomes $E(i) = \mu_2 - \mu_1$. At some point in time, ordering with lead time L_2 should be stopped, as waiting until the short lead-time option L_1 becomes available again is a more profitable option. However, in terms of the time of arrival, t, of the next order, L_2 is simply reverting to L_1, which reduces the order-up-to level and constrains ordering until the difference between the expected stock and the required safety stock becomes positive again (see [23]) for a detailed analysis of this effect).

If a safety time is applied to anticipate a change affecting a season with a longer expected lead time (i.e., the approaching end of interval I_1), the large end-of-interval order needs to be placed in advance, spread over several periods. Therefore, when a safety time $ST > 1$, wherein the large order is spread evenly over a few periods, our order-up-to policy at t_o takes account of whether $I_1 - ST < t_0 < I_1$. If this is the case, the inventory position is raised to the order-up-to level $S_{t_o} = D(t - t_o) + SS_t + D(\mu_2 - \mu_1)(ST - I_1 + t_0)/(ST)$.

5.4. Results of the numerical analysis

The results of the numerical analysis reveal the costs incurred by ignoring the seasonal pattern and using the benchmark distribution. Ignoring seasonality leads to higher costs. The cost gap is the difference between the average cycle cost of the model that ignores seasonality and that of our model, divided by the cost of the model that ignores seasonality. This gap is expected to be higher if service levels are lower, as a lower service level results in more frequent stock outs. Moreover, the gap is expected to be higher if the safety time is lower, mainly due to the higher probability of facing a long waiting time before the first order in the season with the longer lead times arrives.

Fig. 4 depicts the results of our experiments using different stock-out probabilities. The distribution of the cost gap with the benchmark solution that ignores seasonality is shown for stock-out probabilities SP=2% and SP=5%, with the safety time, ST, covering three periods. On average, a 2% stock-out probability results in an average cost gap of 19.6%, while a 5% stock-out probability results in an even higher average cost gap of 23.1%. The distribution of the cost gaps shows a substantial long tail, indicating that high cost gaps ranging from 25% to 45% are encountered in almost 30% (SP=2%) or 20% (SP=5%) of the experiments.

Fig. 5 shows the effect of varying the safety time factor on the cost gap in case of a stock-out probability of 5%. Compared to the previous experiment where we spread the large order equally over three weeks (ST=3), we include both a lower (ST=1) and higher (ST=5) safety time factor. ST=1 means that we do not include additional safety time at all, and the results of the gap distribution show that the difference with the benchmark solution increases to on average 26.4%, while nearly 50% of the cases showing a gap of 20% or more. An increase of the safety time factor to 5 periods decreases the average cost gap to 18.9%. At this large safety time factor, the stock-out costs in the benchmark solution are very low, while inventory costs have increased. The impact of a further increase of the safety time is marginal.

5.5. Conclusion regarding the outcomes of numerical experiments

We conclude that ignoring seasonal patterns in the length of randomly distributed lead times may lead to a substantial cost increase compared with the cost performance of our solution approach. Cost advantages range on average between 18.9% and 26.4%, depending on the length of the safety time. The median value, which is almost 20%, is indicative of a long tail with more extensive cost savings. The incorporation of more extensive safety measures, such as higher safety stocks and longer safety lead times, reduces the average cost gaps from 23.1% to 19.6% and 18,9%, which are still huge cost improvements for using the seasonality approach. Moreover, these measures do not shorten the long tails of the gap distributions. Therefore, we conclude that our solution that takes account of seasonal patterns in stochastic lead-time lengths avoids these large cost gaps and is the preferred approach.

6. Conclusions and future research directions

Whereas the issue of lead-time seasonality is frequently encountered in practice, mainly in relation to problems associated with transportation and suppliers, it has largely been ignored within the literature on inventory control. To the best of our knowledge, this is the first study that has examined various types of seasonality in relation to lead times. Important insights on how to contend with lead-time seasonality can thus be of value for both the theory and practice of inventory control. We have proposed a framework that accounts for the seasonality of lead times and can be applied to characterize inventory control problems. This framework distinguishes between seasonality relating to the dimensions of time and magnitude as well as the deterministic or stochastic nature of seasonal variations, resulting in four families of seasonal inventory control problems: ‘all fixed’ (i.e., purely deterministic problems), ‘fixed length, random timing’, ‘fixed timing, random length’, and ‘all random’ (stochasticity in both the magnitude and time dimensions). For each of these families, we have offered insights and recommendations on how to cope with seasonality.

Our elementary models enable us to develop generic insights based on an analysis of various families of seasonal inventory problems. For deterministic magnitude dimension, we focus on cases entailing two or three different lead-time lengths. For the all fixed class, which is encountered, for example, in tobacco sourcing, the intervals when these lead times occur are known in advance. We have developed an optimal control policy that minimizes costs by seeking a balance between ordering precisely what is needed, ordering in advance, and postponing orders, to avail of benefits resulting from a future reduction of lead time. In cases entailing more than two different lead times, the optimal solution may also include order postponement because of anticipated order crossovers. These order crossovers are avoided to save costs. In light of these findings derived from the use of elementary deterministic models, we suggest that managers should attempt to save costs by either postponing orders or by ordering in advance in accordance with known lead-time changes and order crossovers.

These results have been extended to the stochastic time dimension, while the magnitude remains deterministic (entailing a fixed length and random timing). Such conditions are encountered in the fashion business and timber transport, and are reflected in the non-availability of transportation over several seasons, revealing the salience of this class of problems. We examined a specific problem in which the length of the first interval is not known in advance. The optimal solution strategy for this stochastic optimization problem still entails ordering in advance as well as order postponement as a result of anticipated order crossovers and lead-time changes. However, the optimal policy now takes into account backordering costs as well as the probability of a lead-time change occurring during the cycle. Hence, our advice to managers who face this type of seasonal variability remains similar: apply a policy that enables benefits to be obtained from the known changes and anticipated order crossovers, as this policy saves costs while ensuring high customer service levels.

There are many examples of extensions in the stochastic magnitude dimension within the literature, such as make-to-order furniture production, craft production, and coffee supply. It is difficult to find optimal solutions for these problems. Precise control over the times of arrival of issued orders is not possible in the case of stochastic lead-time lengths. The recommendation in the existing literature relating to variable/stochastic lead times (see [1,14,19,38]) is to use safety stocks and/or safety time to mitigate the height and variance in inventory levels. However, our insights indicate that accounting for the non-stationarity of the seasonal lead times is advisable. Managers should anticipate on seasonal shifts of lead-time lengths and use smaller safety buffers, which yields a significant cost reduction. As shown in numerical experiments, the cost reductions of applying our solution range between 18.9% and 26.4% on average, entailing substantial long tails of even higher cost reductions. We therefore conclude that the current practice of ignoring seasonal patterns in the lengths of randomly distributed lead-time lengths is a costly strategy. Accordingly, our recommendation to managers who face such seasonal variability is to apply our solution instead of relying on traditional buffering mechanisms such as safety stocks and time.

Future research could focus on developing models for application within the four families of seasonal lead-time problems. Firstly, we have assumed that the unit purchasing cost is identical for both lead times. In a future study, we will prove that an order-up-to optimal policy remains valid if we assume that the long lead-time option has a lower unit purchasing cost than the short lead-time option. Secondly, the impacts of fixed ordering costs on the design of the optimal policy merit an investigation. Finally, researchers could experiment with other lead-time length distributions, more diverse interval lengths, more relevant models for assessing interval change probabilities, and a wider variety of pattern changes that occur during a cycle. Such studies would advance the theory of inventory control in relation to seasonal lead-time patterns.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.omega.2019.102162.

References

[1] Liu H, Zhang J, Zhou C, Ru Y. Optimal purchase and inventory retrieval policies for perishable seasonal agricultural products. Omega (Westport) 2018;79:133–45. doi:10.1016/J.OMEGA.2017.08.006.

[2] Hylleberg S. Seasonality in regression. Orlando, USA: Academic Press; 1986.

[3] de Treville S, Bicer I, Chavez-Demoulin V, Hagspiel V, Schürhoff N, Tasserit C, et al. Valuing lead time. J Oper Manag 2014;32:337–46. doi:10.1016/j.jom.2014.06.002.

[4] Epstein R, Rönnqvist M, Weintraub A. Forest transportation. In: Weintraub A, Romero C, Bjørndal T, Epstein R, Miranda J, editors. Handb. oper. res. nat. resour. se - 20, 99. US: Springer; 2007. p. 391–403. doi:10.1007/978-0-387-71815-6_20.

[5] Schweighofer J. The impact of extreme weather and climate change on inland waterway transport. Nat Hazards 2013:1–18. doi:10.1007/s11069-012-0541-6.
[6] Minner S. Multiple-supplier inventory models in supply chain management: a review. Int J Prod Econ 2003;81–82:265–79. doi:10.1016/S0925-5273(02)00288-8.
[7] Riezebos J. Inventory order crossovers. Int J Prod Econ 2006;104:666–75. doi:10.1016/j.ijpe.2004.11.011.
[8] Hadley GF, Whitin TM. Analysis of inventory systems. Englewood Cliffs: Prentice-Hall; 1963.
[9] Silver EA, Pyke DF, Peterson R. Inventory management and production planning and scheduling. 3rd editor. New York: John Wiley & Sons; 1998.
[10] Vollmann TE, Berry LB, Whybark DC, Jacobs FR. Manufacturing planning and control systems for supply chain management. New York: McGraw-Hill; 2005.
[11] Zipkin PH. Foundations of inventory management. Boston, MA: McGraw-Hill; 2000.
[12] Boute RN, Van Mieghem JA. Global dual sourcing and order smoothing: the impact of capacity and lead times. Manage Sci 2015;61:2080–99. doi:10.1287/mnsc.2014.1992.
[13] Mihalache M, Mihalache OR. A decisional framework of offshoring: integrating insights from 25 years of research to provide direction for future*. Decis Sci 2016;47:1103–49. doi:10.1111/deci.12206.
[14] Disney SM, Maltz A, Wang X, Warburton RDH. Inventory management for stochastic lead times with order crossovers. Eur J Oper Res 2016;248:473–86. doi:10.1016/j.ejor.2015.07.047.
[15] Song J-S, Zipkin PH. Inventory control with information about supply conditions. Manage Sci 1996;42:1409–19. doi:10.1287/mnsc.42.10.1409.
[16] Silver EA, Zufferey N. Inventory control of raw materials under stochastic and seasonal lead times. Int J Prod Res 2005;43:5161–79. doi:10.1080/00207540500219866.
[17] Wright GP. Optimal policies for a multi-product inventory system with negotiable lead times. Nav Res Logist Q 1968;15:375–401. doi:10.1002/nav.3800150304.
[18] Moinzadeh K, Nahmias S. A continuous review model for an inventory system with two supply modes. Manage Sci 1988;34:761–73. doi:10.1287/mnsc.34.6.761.
[19] Riezebos J, Zhu SX. MRP planned orders in a multiple-supplier environment with differing lead times. Prod Oper Manag 2015;24:883–95. doi:10.1111/poms.12318.
[20] Tagaras G, Vlachos D. A periodic review inventory system with emergency replenishments. Manage Sci 2001;47:415–29.
[21] Wang Y, Gilland W, Tomlin B. Mitigating supply risk: dual sourcing or process improvement? Manuf Serv Oper Manag 2010;12:489–510. doi:10.1287/msom.1090.0279.
[22] Jansen MM, Kok AGde, Fransoo JC. Lead time anticipation in supply chain operations planning. OR Spectr 2011;35:251–90. doi:10.1007/s00291-011-0267-y.
[23] Axsäter S. Inventory control when the lead-time changes. Prod Oper Manag 2011;20:72–80. doi:10.1111/j.1937-5956.2010.01125.x.
[24] Zalkind D. Order-level inventory systems with independent stochastic lead-times. Manage Sci 1978;24:1384–92.
[25] Nevison CH, Burstein MC. The dynamic lot-size model with stochastic lead times. Manage Sci 1984;30:100–9.
[26] Zipkin PH. Stochastic leadtimes in continuous-time inventory models. Nav Res Logist 1986;33:763–74.
[27] Huang K, Küçükyavuz S. On stochastic lot-sizing problems with random lead times. Oper Res Lett 2008;36:303–8. doi:10.1016/j.orl.2007.10.009.
[28] Ramasesh RV, Ord JK, Hayya JC, Pan A. Sole versus dual sourcing in stochastic lead-time (s,Q) inventory models. Manage Sci 1991;37:428–43.
[29] Zhao L, Lau H. Reducing inventory costs and choosing suppliers with order splitting. J Oper Res Soc 1992;43:1003–8.
[30] Ernst A, Jiang H, Krishnamoorthy M. Exact solutions to task allocation problems. Manage Sci 2006;52:1634–46. doi:10.1287/mnsc.1060.0578.
[31] Hayya JC, Bagchi U, Kim J, Sun D. On static stochastic order crossover. Int J Prod Econ 2008;114:404–13. doi:10.1016/j.ijpe.2008.03.007.
[32] Harvey AC, Snyder RD. Structural time series models in inventory control. Int J Forecast 1990;6:187–98. doi:10.1016/0169-2070(90)90004-U.
[33] Proietti T. Comparing seasonal components for structural time series models. Int J Forecast 2000;16:247–60. doi:10.1016/S0169-2070(00)00037-6.
[34] De Gooijer JG, Hyndman RJ. 25 Years of time series forecasting. Int J Forecast 2006;22:443–73. doi:10.1016/j.ijforecast.2006.01.001.
[35] Nielsen P, Michna Z, Sørensen BB, Dung NDA. Lead times - Their Behavior and the impact on planning and control in supply chains. Manag Prod Eng Rev 2017;8:30–40. doi:10.1515/mper-2017-0015.
[36] Wang X, Disney SM. Mitigating variance amplification under stochastic lead-time: the proportional control approach. Eur J Oper Res 2017;256:151–62. doi:10.1016/j.ejor.2016.06.010.
[37] Wang Y, Tomlin BT. To wait or not to wait: optimal ordering under lead time uncertainty and forecast updating. Nav Res Logist 2009;56:766–79. doi:10.1002/nav.
[38] Hayya JC, Harrison TP, He XJ. The impact of stochastic lead time reduction on inventory cost under order crossover. Eur J Oper Res 2011;211:274–81. doi:10.1016/j.ejor.2010.11.025.
[39] Srinivasan M, Novack R, Thomas DJ. Optimal and approximate policies for inventory systems with order crossover. J Bus Logist 2011;32:180–93.
[40] Fang X, Zhang C, Robb DJ, Blackburn JD. Decision support for lead time and demand variability reduction. Omega (Westport) 2012:1–7. doi:10.1016/j.omega.2012.03.005.
[41] Şen A. The us fashion industry: a supply chain review. Int J Prod Econ 2008;114:571–93. doi:10.1016/j.ijpe.2007.05.022.
[42] Cachon GP, Swinney R. The value of fast fashion: quick response, enhanced design, and strategic consumer behavior. Manage Sci 2011;57:778–95. doi:10.1287/mnsc.1100.1303.
[43] Bhardwaj V, Fairhurst A. Fast fashion: response to changes in the fashion industry. Int Rev Retail Distrib Consum Res 2010;20:165–73. doi:10.1080/09593960903498300.
[44] Törnlund E, Östlund L. Floating timber in northern sweden: the construction of floatways and transformation of rivers. Environ Hist Camb 2002;8:85–106. doi:10.3197/096734002129342611.
[45] Shukla V, Naim MM. Sensing endogenous seasonality in the case of a coffee supply chain. Int J Logist Res Appl 2018;21:279–99. doi:10.1080/13675567.2017.1395829.