

University of Groningen

## Well-posedness of the fractional Ginzburg-Landau equation

Gu, Xian-Ming; Shi, Lin; Liu, Tianhua

*Published in:*  
Applicable Analysis

*DOI:*  
[10.1080/00036811.2018.1466281](https://doi.org/10.1080/00036811.2018.1466281)

**IMPORTANT NOTE:** You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

*Document Version*  
Publisher's PDF, also known as Version of record

*Publication date:*  
2019

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Gu, X-M., Shi, L., & Liu, T. (2019). Well-posedness of the fractional Ginzburg-Landau equation. *Applicable Analysis*, 98(14), 2545-2558. <https://doi.org/10.1080/00036811.2018.1466281>

### Copyright

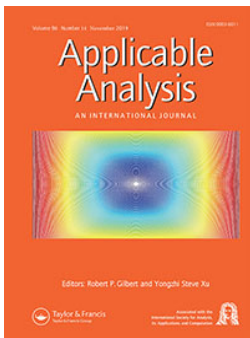
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

### Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

*Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.*



# Well-posedness of the fractional Ginzburg–Landau equation

Xian-Ming Gu, Lin Shi & Tianhua Liu

To cite this article: Xian-Ming Gu, Lin Shi & Tianhua Liu (2019) Well-posedness of the fractional Ginzburg–Landau equation, *Applicable Analysis*, 98:14, 2545-2558, DOI: [10.1080/00036811.2018.1466281](https://doi.org/10.1080/00036811.2018.1466281)

To link to this article: <https://doi.org/10.1080/00036811.2018.1466281>



Published online: 30 May 2018.



Submit your article to this journal [↗](#)



Article views: 174



View related articles [↗](#)



View Crossmark data [↗](#)



Citing articles: 2 View citing articles [↗](#)



# Well-posedness of the fractional Ginzburg–Landau equation

Xian-Ming Gu<sup>a,b</sup>, Lin Shi<sup>c</sup> and Tianhua Liu<sup>d</sup>

<sup>a</sup>School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu, P.R. China;

<sup>b</sup>Johann Bernoulli Institute of Mathematics and Computer Science, University of Groningen, Groningen, The Netherlands; <sup>c</sup>School of Mathematics, Southwest Jiaotong University, Chengdu, P.R. China; <sup>d</sup>School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, P.R. China

## ABSTRACT

In this paper, we investigate the well-posedness of the real fractional Ginzburg–Landau equation in several different function spaces, which have been used to deal with the Burgers’ equation, the semilinear heat equation, the Navier–Stokes equations, etc. The long time asymptotic behavior of the nonnegative global solutions is also studied in details.

## ARTICLE HISTORY

Received 18 October 2017  
Accepted 11 April 2018

## COMMUNICATED BY

G. N’Guerekata

## KEYWORDS

Fractional Ginzburg–Landau equation; Well-posedness; Sobolev space; Asymptotic behavior

## AMS SUBJECT

## CLASSIFICATIONS

35K55; 35Q92; 35Q35; 92C17

## 1. Introduction

In this paper, we are interested in the Cauchy problem for the Ginzburg–Landau equation with fractional Laplacian

$$\begin{cases} u_t = -\Lambda^{2\alpha} u + u - |u|^{2\sigma} u, & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}^n, \end{cases} \quad (1.1)$$

where  $\alpha \in (0, 1]$ ,  $\sigma \geq \frac{1}{2}$ , and the square root of the Laplacian,  $\Lambda = (-\Delta)^{\frac{1}{2}}$ , is a pseudo-differential operator defined by the Fourier transform  $\mathcal{F}(\Lambda^{2\alpha} f)(\xi) = |\xi|^{2\alpha} \mathcal{F}(f)(\xi)$ , which means that, unlike the conventional differential operators, the operator  $\Lambda^{2\alpha}$  with  $\alpha \in (0, 1)$  is nonlocal, that is,  $\Lambda^{2\alpha} u(x)$  depends not just on  $u(y)$  for  $y$  near  $x$ , but on  $u(y)$  for all  $y$ . Equation (1.1) with  $\alpha = 1$  is the classical Ginzburg–Landau equation

$$u_t = \Delta u + u - |u|^{2\sigma} u,$$

which was intensively studied in the past decades; refer to [1–3] and references therein.

In the current study, we pay our special attention to the fractional case  $\alpha \in (0, 1)$ . The fractional partial differential equations, which appear in mathematical physics such as chaotic dynamics [4], material science [5], cosmic-rays propagation [6], and long-range dissipation [7], now attract the growing interests of many researchers. They are not just another way of presenting old stories, but a powerful framework which is of use for many complex systems. In particular, the fractional Ginzburg–Landau equations can capture some long-range interactions of a system which can not be captured

by traditional integer order differential equations; refer, e.g. to [8–13] for a discussion of these issues. As far as (1.1) is considered, it is the real form of the complex fractional Ginzburg–Landau equation

$$u_t = -(1 + ai)\Lambda^{2\alpha} u + u - (1 + bi)|u|^{2\sigma} u, \tag{1.2}$$

which plays a fundamental role in the understanding of the dynamical processes in continuums with fractal dispersion and the media with fractal mass dimension [9–11,14–16].

For Equation (1.1), Tarasov [17] considered the case  $n = 1$  and studied the Psi-series solutions, the leading order behavior of solution of arbitrary singularity as well as their resonance structure. For the case  $\sigma = 1$  and  $n \geq 1$ , Li and Xia [18,19] obtained the well-posedness result for initial data in  $L^p$  or  $W^{s\alpha, \frac{3}{2}}$ . Recently, Pu and Guo [15] studied the global existence of weak and strong solutions to Equation (1.2) posed on the periodic box. In this paper, we will investigate the well-posedness of Equation (1.1) with general  $\sigma$  and initial data in several different function spaces, which have been used to deal with the Burgers’s equation, the semilinear heat equation, the Navier–Stokes equations etc.

We now state the function spaces, which will be used in the next sections. The first one is the standard  $L^p$  space. The local existence and uniqueness of mild solution for the initial value  $u_0 \in L^p$  will be established.

The second one is the homogeneous Sobolev space  $\dot{W}^{s,p}$ , which consists of all  $\theta$  such that

$$(-\Delta)^{\frac{s}{2}}\theta \in L^q, \quad s \in \mathbb{R}, \quad 1 \leq q < \infty,$$

with the standard norm given by

$$\|\theta\|_{\dot{W}^{s,q}} := \|(-\Delta)^{\frac{s}{2}}\theta\|_{L^q}.$$

We are interested in the case of non-positive index  $s \leq 0$ . More precisely, we will show the well-posedness of the initial-value problem (1.1) with initial data  $u_0 \in \dot{W}^{r,p}$  when  $r$  and  $p$  satisfy

$$1 < p < \infty, \quad \frac{\alpha}{(2\sigma + 1)\sigma} < \frac{n}{p} \leq \frac{\alpha}{\sigma}, \quad \text{and} \quad r := \frac{n}{p} - \frac{\alpha}{\sigma} \leq 0.$$

Here, to detect the index  $r$ , the dimensional analysis might be employed. Indeed, we only need to notice that if  $u(x, t)$  is a solution to (1.1) without the zero-order term  $u$ , then for any  $\lambda > 0$ ,  $u_\lambda := \lambda^{\frac{\alpha}{\sigma}} u(\lambda x, \lambda^{2\alpha} t)$  solves the same problem and

$$\|u_\lambda(\cdot, t)\|_{\dot{W}^{r,p}} = \lambda^{r - (\frac{n}{p} - \frac{\alpha}{\sigma})} \|u(\cdot, \lambda^{2\alpha} t)\|_{\dot{W}^{r,p}}.$$

The third one is a weighted Banach space. We will show that if the initial data  $u_0 \in W^{s,p}$ , which is the inhomogeneous Sobolev space consisting of all  $\theta$  such that

$$\|\theta\|_{W^{s,p}} \equiv \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{\theta}(\xi)\|_{L^p} < \infty,$$

then Equation (1.1) admits a solution  $u \in BC_s((0, T], W^{r,p})$  with  $s \leq r, s, r \in \mathbb{R}, T > 0$ , where the space  $BC_s((0, T], W^{r,p})$  denotes the class of all functions

$$\theta \in C([0, T], W^{s,p}) \cap C((0, T], W^{r,p})$$

with the norm given by

$$\|\theta\|_{BC_s((0,T],W^{r,p})} := \sup_{t \in [0,T]} \left\| (1 + |\xi|^2)^s \left(1 + |\xi|^2 t^{\frac{1}{2\alpha}}\right)^{r-s} \widehat{\theta}(\xi, t) \right\|_{L^p} < \infty. \tag{1.3}$$

Here  $\hat{\theta}$  denotes the Fourier transform of  $\theta$ . More precisely, we will prove that if  $u_0 \in W^{s\alpha, 1 + \frac{1}{2\sigma}}$  with  $s$  satisfying  $s > \frac{n}{(2\sigma+1)\alpha} - \frac{1}{\sigma}$ , then the initial-value problem of (1.1) is locally well-posed and the solution  $u \in BC_{s\alpha}((0, T], W^{r\alpha, 1 + \frac{1}{2\sigma}})$  for some  $T > 0$  and all  $r \geq s$ .

To establish the above well-posedness results, we will use a basic transform  $v = e^{-t}u$  to transfer (1.1) into a nonlocal parabolic equation with absorption

$$\begin{cases} v_t = -\Lambda^{2\alpha} v - e^{2\sigma t} |v|^{2\sigma} v, & \text{in } \mathbb{R}^n \times (0, +\infty), \\ v(x, 0) = v_0(x), & \text{in } \mathbb{R}^n, \end{cases} \tag{1.4}$$

where  $v_0(x) = u_0(x)$ . Then we analyze the boundedness of the solutions to the corresponding linear equation

$$\begin{cases} w_t = -\Lambda^{2\alpha} w, & \text{in } \mathbb{R}^n \times (0, +\infty), \\ w(x, 0) = v_0(x), & \text{in } \mathbb{R}^n, \end{cases} \tag{1.5}$$

and estimate the nonlinear term in the above function spaces. Such an analysis is based on some basic properties the convolution operator  $e^{-\Lambda^{2\alpha} t}$  with kernel  $g_\alpha(x, t)$ . The above results are also valid for  $\mathbb{T}^n$ .

At the end of this paper, we will also investigate the long time asymptotic behavior of the nonnegative global solutions to Equation (1.4) with initial data  $v_0 \in L^1(\mathbb{R}^n)$  in the case  $0 < \sigma < \frac{\alpha}{n-2\alpha}$ .

The rest of this paper is organized as follows. In Section 2, we describe some definitions and recall some preliminary results about the fractional calculus. In Section 3, we will show the local well-posedness of Equation (1.4) with  $v_0 \in L^p, \dot{W}^{r,p}$  and  $W^{s\alpha, 1 + \frac{1}{2\sigma}}$ . Finally, in Section 4, we study the long-time asymptotic behavior of the nonnegative global solutions.

## 2. Preliminaries

In this section, we give some preliminaries, which is needed to prove our well-posedness result. We first consider the linear Equation (1.5) and investigate its explicit solution

$$w(x, t) = e^{-\Lambda^{2\alpha} t} v_0(x) = g_\alpha(\cdot, t) * v_0(\cdot)(x),$$

where  $e^{-\Lambda^{2\alpha} t}$  is a convolution operator with its kernel  $g_\alpha(x, t)$  being defined through the Fourier transform

$$\widehat{g}_\alpha(\xi, t) = e^{-|\xi|^{2\alpha} t}.$$

When  $\alpha = \frac{1}{2}$  and  $\alpha = 1$ , the kernel  $g_\alpha(x, t)$  is the classical Poisson kernel and heat kernel, respectively. When  $\alpha \in (0, 1], g_\alpha(x, t)$  is a nonnegative and non-increasing radial function, and satisfies the dilation relation

$$g_\alpha(x, t) = t^{-\frac{n}{2\alpha}} g_\alpha(xt^{-\frac{1}{2\alpha}}, 1).$$

Furthermore, both  $g_\alpha$  and  $\nabla g_\alpha$  are bounded linear operators from  $L^p$  to  $L^q$ . That is, the following lemma holds.

**Lemma 2.1:** *Let  $1 \leq p \leq q \leq \infty$ . For any  $t > 0$  and  $v_0 \in L^p(\mathbb{R}^n)$ , we have*

$$\|g_\alpha(\cdot, t) * v_0\|_{L^q(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2\alpha}(\frac{1}{p} - \frac{1}{q})} \|v_0\|_{L^p(\mathbb{R}^n)}$$

and

$$\|\nabla g_\alpha(\cdot, t) * v_0\|_{L^q(\mathbb{R}^n)} \leq Ct^{-(\frac{1}{2\alpha} + \frac{n}{2\alpha}(\frac{1}{p} - \frac{1}{q}))} \|v_0\|_{L^p(\mathbb{R}^n)},$$

where  $C$  is a constant depending on  $\alpha, p$ , and  $q$  only.

**Proof:** The proof can be completed by following the idea of Wu [20], where the author proved the same conclusion for  $n = 2$ . □

Next, we define the spaces of weighted continuous functions in time, which were introduced by Kato–Ponce [21] and others in solving the Navier–Stokes equations and then employed by Wu [22] in dealing with semilinear heat equation.

**Definition 2.1:** Suppose  $T > 0$  and  $\alpha \geq 0$  are real numbers. The space  $C_{\alpha,s,q}$  is defined as

$$C_{\alpha,s,q} = \{f \in C((0, T), \dot{W}^{s,q}); \|f\|_{C_{\alpha,s,q}} < \infty\},$$

where the norm is given by

$$\|f\|_{C_{\alpha,s,q}} = \sup_{t \in [0, T]} t^\alpha \|f\|_{\dot{W}^{s,q}}.$$

Then  $\dot{C}_{\alpha,s,q}$  is a subspace of  $C_{\alpha,s,q}$  defined by

$$\dot{C}_{\alpha,s,q} = \left\{ f \in C_{\alpha,s,q}, \lim_{t \rightarrow 0} t^\alpha \|f\|_{\dot{W}^{s,q}} = 0 \right\}.$$

When  $\alpha = 0$ , we use  $\bar{C}_{s,q}$  to denote  $BC([0, T], \dot{W}^{s,p})$ .

From Lemma 2.1, we have the following strongly continuous semigroup property.

**Lemma 2.2 [18,19]:** For any  $T > 0$ , we denote by  $w(x, t) := g_\alpha(\cdot, t) * v_0(x)$  in  $\mathbb{R}^n \times [0, T]$ .

- (i) Let  $v_0 \in L^p(\mathbb{R}^n)$ , ( $1 \leq p < \infty$ ), then  $w \in C([0, T]; L^p(\mathbb{R}^n))$  and  $\lim_{t \rightarrow 0} \|w(x, t) - v_0(x)\|_{L^p(\mathbb{R}^n)} = 0$ .
- (ii) Let  $v_0(x) \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ , then  $w \in C([0, T]; L^\infty(\mathbb{R}^n))$  and  $\lim_{t \rightarrow 0} \|w(x, t) - v_0(x)\|_{L^\infty(\mathbb{R}^n)} = 0$ .
- (iii) Let  $v_0 \in \dot{W}^{s,q}(\mathbb{R}^n)$ , ( $s \in \mathbb{R}, q \in [1, \infty)$ ), then  $w \in C((0, T); \dot{W}^{s,q}(\mathbb{R}^n))$  and  $\lim_{t \rightarrow 0} \|w(x, t) - v_0(x)\|_{\dot{W}^{s,q}(\mathbb{R}^n)} = 0$ .
- (iv) Assume that  $s_1 \leq s_2, q_1 \leq q_2$ , and  $\lambda_2 = \frac{1}{2\alpha}(s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2})$ , then  $v_0 \rightarrow w$  is a continuous mapping from  $\dot{W}^{s_1, q_1}$  into  $\dot{C}_{\lambda_2, s_2, q_2}$ . When  $\lambda_2 = 0$ ,  $\dot{C}_{\lambda_2, s_2, q_2}$  should be replaced by  $\bar{C}_{s_2, q_2}$ .

We now turn to consider the following nonlinear problem

$$\begin{cases} v_t = Lv + N(v), \\ v(x, 0) = v_0(x), \end{cases} \tag{2.1}$$

where  $L$  is the infinitesimal generator of a strongly continuous semigroup  $S(t)$  on Banach space  $X$ . If  $v(x, t)$  is a classical or strong solution of (2.1), then  $v(x, t)$  satisfies the integral equation

$$v(x, t) = S(t)v_0(x) + \int_0^t S(t - \tau)N(v(x, \tau))d\tau. \tag{2.2}$$

It is clear that a solution given by (2.2) is weaker than the classical solution. Thus, we introduce the following definition.

**Definition 2.2 [23]:** Let  $X$  be a Banach space,  $v_0 \in X$  and  $N \in L^1(0, T; X)$ , then a function  $v \in C([0, T]; X)$  given by (2.2) is called a mild solution of problem (2.1) on  $[0, T]$ .

The next lemma shows the local existence of mild solution to nonlinear problem (2.1).

**Lemma 2.3 [2]:** Let  $X, Y$  and  $Z$  be Banach spaces such that the semigroup  $S(t)$  is strongly continuous when acting on  $X$ , and  $S(t)\omega \in Y$  for every  $\omega \in X$  or  $Z$ . Furthermore,  $S(t)$  satisfies

$$\|S(t)\omega\|_Y \leq Ct^{-\beta_1} \|\omega\|_X \quad \forall \omega \in X, \quad \text{and} \quad \|S(t)\omega\|_Y \leq Ct^{-\beta_2} \|\omega\|_Z \quad \forall \omega \in Z,$$

where the constants  $\beta_1$  and  $\beta_2$  satisfy

$$0 \leq \beta_2 < 1, \quad 0 \leq \beta_1 < \frac{1}{2\sigma + 1}, \quad \text{and} \quad \beta_2 + 2\sigma\beta_1 < 1.$$

Let  $N$  be locally Lipschitz from  $Y$  to  $Z$  and  $N(0) = 0$ , that is,

$$\|N(v_1) - N(v_2)\|_Z \leq C (\|v_1\|_Y^{2\sigma} + \|v_2\|_Y^{2\sigma}) \|v_1 - v_2\|_Y \quad \forall v_1, v_2 \in Y$$

for some  $C > 0$ . Then for every  $\rho > 0$ , there exists a  $T = T(\rho) > 0$ , such that for any  $v_0 \in X$ ,  $\|v_0\|_X \leq \rho$ , the nonlinear problem (2.1) admits a unique mild solution

$$v(x, t) \in C([0, T]; X) \cap C((0, T]; Y).$$

Furthermore, the mapping  $v_0 \rightarrow v : X \rightarrow C([0, T]; X)$  is locally Lipschitz.

Then for the linear operator  $G$  defined by

$$Gg(t) := \int_0^t e^{-\Lambda^{2\alpha}(t-\tau)} g(\tau) d\tau, \tag{2.3}$$

we have the following continuous property.

**Lemma 2.4:** If  $q_1, q_2, \lambda_1, \lambda_2, s_1, s_2$  satisfy

$$q_1 \leq q_2, \quad \lambda_1 < 1, \quad \lambda_2 = \lambda_1 - 1 + \frac{1}{2\alpha} \left( s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2} \right), \quad 0 \leq s_2 - s_1 < 2\alpha - n \left( \frac{1}{q_1} - \frac{1}{q_2} \right),$$

then  $G$  maps continuously from  $\dot{C}_{\lambda_1, s_1, q_1}$  to  $\dot{C}_{\lambda_2, s_2, q_2}$ .

**Proof:** The proof of this lemma is quite similar to that of Proposition 2.2 in [19]. We omit the details. □

### 3. Well-posedness

In this section, we study the well-posedness of Cauchy problem (1.4). We first consider the local well-posedness in  $L^p$  and then show the local well-posedness in  $W^{r,p}$ . Next, the well-posedness in  $W^{s\alpha, 1 + \frac{1}{2\sigma}}$  will be proved.

#### 3.1. Local well-posedness in $L^p$

In this subsection, we establish the local well-posedness of Cauchy problem (1.4) in  $L^p$ . We have the following conclusion.

**Theorem 3.1:** Let  $v_0 \in L^p(\mathbb{R}^n)$  with  $\frac{n\sigma}{\alpha} < p < +\infty$ , then there exists a  $T > 0$  such that Cauchy problem (1.4) possesses a unique mild solution

$$v(x, t) \in C([0, T]; L^p(\mathbb{R}^n)) \cap C((0, T]; L^{(2\sigma+1)p}(\mathbb{R}^n)).$$

Furthermore, the mapping  $v_0 \rightarrow v$  is locally Lipschitz. The same conclusion holds if we instead  $\mathbb{R}^n$  by  $\mathbb{T}^n$ .

**Proof:** We will use Lemma 2.3 to prove our conclusion. For this purpose, we take  $X = Z = L^p(\mathbb{R}^n)$  and  $Y = L^{(2\sigma+1)p}(\mathbb{R}^n)$ . Then by setting  $Lv = -\Lambda^{2\sigma}v$  and  $N(v) = -e^{2\sigma t}|v|^{2\sigma}v$ , we can define the semigroup  $S(t)$  by

$$S(t)v = g_\alpha(\cdot, t) * v(\cdot, t).$$

By Lemma 2.2, we have that  $S(t)$  is a  $C_0$  semigroup on  $L^p(\mathbb{R}^n)$ . Moreover, if we take

$$\beta_1 = \beta_2 = \frac{n\sigma}{p\alpha(2\sigma + 1)},$$

which satisfy

$$0 \leq \beta_2 < 1, \quad 0 \leq \beta_1 < \frac{1}{2\sigma + 1}, \quad \text{and} \quad \beta_2 + 2\sigma\beta_1 < 1$$

by the assumption  $p > \frac{n\sigma}{\alpha}$ , then we have the estimate

$$\|g_\alpha(\cdot, t) * \omega\|_{L^{(2\sigma+1)p}(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2\alpha}\left(\frac{1}{p} - \frac{1}{(2\sigma+1)p}\right)} \|\omega\|_{L^p(\mathbb{R}^n)} = Ct^{-\beta_1} \|\omega\|_{L^p(\mathbb{R}^n)} = Ct^{-\beta_2} \|\omega\|_{L^p(\mathbb{R}^n)}$$

by Lemma 2.1.

On the other hand, it follows from Young’s inequality, Hölder’s inequality and Minkowski’s inequality [13] that

$$\begin{aligned} \|N(v_1) - N(v_2)\|_{L^p(\mathbb{R}^n)} &\leq Ce^{2\sigma T} \left( \int_{\mathbb{R}^n} |v_1 - v_2|^p (|v_1|^{2\sigma} + |v_2|^{2\sigma})^p dx \right)^{\frac{1}{p}} \\ &\leq Ce^{2\sigma T} \left( \int_{\mathbb{R}^n} |v_1 - v_2|^{(2\sigma+1)p} dx \right)^{\frac{1}{(2\sigma+1)p}} \\ &\quad \times \left( \int_{\mathbb{R}^n} (|v_1|^{2\sigma} + |v_2|^{2\sigma})^{\frac{(2\sigma+1)p}{2\sigma}} dx \right)^{\frac{2\sigma}{(2\sigma+1)p}} \\ &\leq Ce^{2\sigma T} \left( \|v_1\|_{L^{(2\sigma+1)p}(\mathbb{R}^n)}^{2\sigma} + \|v_2\|_{L^{(2\sigma+1)p}(\mathbb{R}^n)}^{2\sigma} \right) \|v_1 - v_2\|_{L^{(2\sigma+1)p}(\mathbb{R}^n)}. \end{aligned}$$

Then we can use Lemma 2.3 to obtain the desired local well-posedness.

The case for  $\mathbb{T}^n$  can be similarly dealt with. □

### 3.2. Local well-posedness in $\dot{W}^{r,p}$

In this subsection, we establish the well-posedness of Cauchy problem (1.4) in  $\dot{W}^{r,p}$  with  $r \leq 0$ . For  $r = 0$ , in particular, we extend the result of the previous subsection to the case  $v_0 \in L^p$  with  $p = \frac{n\sigma}{\alpha}$ . Our proposed method is the use of integral equation and contraction mapping arguments, which has been extensively used by Kato, Ponce and other researchers to prove the well-posedness of the Navier–Stokes equations in various types of functional spaces [13,21,24,25]. The main result is stated as follows.

**Theorem 3.2:** Let  $v_0 \in \dot{W}^{r,p}(\mathbb{R}^n)$  with  $r$  and  $p$  satisfying

$$1 < p < \infty, \quad \frac{\alpha}{(2\sigma + 1)\sigma} < \frac{n}{p} \leq \frac{\alpha}{\sigma}, \quad \text{and} \quad r := \frac{n}{p} - \frac{\alpha}{\sigma} \leq 0. \tag{3.1}$$

Then there exists  $T = T(v_0) > 0$  such that the Cauchy problem (1.4) admits a unique solution  $v$  satisfying

$$v \in Y_T \equiv \left( \bigcap_{p \leq q < \infty} \dot{C}^{\frac{n}{q} - \frac{\alpha}{\sigma}, q} \right) \cap \left( \bigcap_{p \leq q < \infty} \bigcap_{s > \frac{n}{q} - \frac{\alpha}{\sigma}} \dot{C}^{\frac{n}{q} - \frac{\alpha}{\sigma}, s, q} \right). \tag{3.2}$$



In particular, (3.2) implies that

$$v \in BC([0, T], \dot{W}^{r,p}) \cap (\cap_{s>r} C((0, T), \dot{W}^{s,p})).$$

Moreover, the mapping  $v_0 \rightarrow v$  is locally Lipschitz. The same conclusion holds if we instead  $\mathbb{R}^n$  by  $\mathbb{T}^n$ .

**Remark 3.1:** It follows from the proof of the theorem that  $T = \infty$  if  $\|v_0\|_{\dot{W}^{r,p}}$  is sufficiently small. Also, the homogeneous spaces  $\dot{W}^{s,q}$  can be replaced by inhomogeneous spaces  $W^{s,q}$ .

**Proof of Theorem 3.2:** We first rewrite (1.4) in the integral form

$$v(x, t) = S(t)v_0 + G(e^{2\sigma t}|v|^{2\sigma}v) \equiv e^{-\Lambda^{2\alpha}t}v_0 + \int_0^t e^{-\Lambda^{2\alpha}(t-\tau)}(e^{2\sigma\tau}|v|^{2\sigma}v)(\tau)d\tau$$

and seek a solution  $v$  for this integral equation. For this purpose, we divide the assumption  $r \leq 0$  into two cases:  $r < 0$  and  $r = 0$ .

For the case  $r < 0$ , we define

$$X = \bar{C}_{r,p} \cap \dot{C}_{-\frac{r}{2\alpha},0,p}$$

with norm for  $v \in X$  given by

$$\|v\|_X = \|v\|_{C_{0,r,p}} + \|v\|_{C_{-\frac{r}{2\alpha},0,p}}$$

and the complete metric space  $X_R$  to be the closed ball in  $X$  of radius  $R$ . Then we consider the operator  $A : X_R \rightarrow X$  defined by

$$A(v, v_0)(t) = S(t)v_0 + G(e^{2\sigma t}|v|^{2\sigma}v)(t), \quad 0 \leq t < T.$$

By using Lemma 2.2 with

$$s_1 = r, \quad s_2 = 0, \quad q_1 = q_2 = p, \quad \lambda_2 = -\frac{r}{2\alpha},$$

we find that  $S(t)v_0 \in X_R$  if  $T > 0$  is small enough. Then we estimate  $G$ . Noticing that the restrictions on  $p$  and in  $r$  in (3.1), we can use Lemma 2.4 with

$$q_1 = \frac{p}{2\sigma + 1}, \quad q_2 = p, \quad \lambda_1 = -\frac{(2\sigma + 1)r}{2\alpha}, \quad \lambda_2 = \frac{l}{2\sigma + 1}, \quad s_1 = 0, \quad s_2 = \frac{2\alpha l}{2\sigma + 1} + r,$$

to obtain

$$\|G(e^{2\sigma t}|v|^{2\sigma}v)\|_{C_{\frac{l}{2\sigma+1}, \frac{2\alpha l}{2\sigma+1} + r, p}} \leq Ce^{2\sigma T} \| |v|^{2\sigma}v \|_{C_{-\frac{2\sigma+1}{2\alpha}r, 0, \frac{p}{2\sigma+1}}} \leq Ce^{2\sigma T} \|v\|_{C_{-\frac{r}{2\alpha}, 0, p}}^{2\sigma+1} \leq Ce^{2\sigma T} R^{2\sigma+1}$$

for all  $l \in [0, -\frac{(2\sigma+1)^2}{2\alpha}r)$ . In particular, we can take  $l = 0$  and  $l = -\frac{2\sigma+1}{2\alpha}r \in [0, -\frac{(2\sigma+1)^2}{2\alpha}r)$  to deduce that

$$\|G(e^{2\sigma t}|v|^{2\sigma}v)(t)\|_X \leq Ce^{2\sigma T} R^{2\sigma+1}.$$

Next, we consider the contraction property. A basic computation yields

$$\begin{aligned} \|A(v, v_0) - A(\tilde{v}, v_0)\|_X &\leq \|G(e^{2\sigma t}|v|^{2\sigma}v) - G(e^{2\sigma t}|\tilde{v}|^{2\sigma}\tilde{v})\|_X \\ &\leq \|G(e^{2\sigma t}|v - \tilde{v}|^{2\sigma}v)\|_X + \|G(e^{2\sigma t}|v - \tilde{v}||\tilde{v}|^{2\sigma})\|_X. \end{aligned}$$

Then by using Lemma 2.4 again, we have

$$\begin{aligned} \|A(v, v_0) - A(\tilde{v}, v_0)\|_X &\leq Ce^{2\sigma T} \| |v - \tilde{v}|^{2\sigma} v \|_{C_{-\frac{2\sigma+1}{2\alpha}r, 0, \frac{p}{2\sigma+1}}} + Ce^{2\sigma T} \| |v - \tilde{v}| |\tilde{v}|^{2\sigma} \|_{C_{-\frac{2\sigma+1}{2\alpha}r, 0, \frac{p}{2\sigma+1}}} \\ &\leq Ce^{2\sigma T} \|v\|_X \|v - \tilde{v}\|_X^{2\sigma} + Ce^{2\sigma T} \|v - \tilde{v}\|_X \|\tilde{v}\|_X^{2\sigma}. \end{aligned}$$

Thus if we choose  $T$  to be small and  $R$  properly, then  $A(v, v_0)$  maps  $X_R$  into itself and is a contraction map. Consequently, there exists a unique fixed point  $v \in X_R$  satisfying  $v = A(v, v_0)$ . It is easy to see from the above estimates that the uniqueness can be extended to  $X_{R'}$ , for all  $R'$  by reducing the time interval and thus to the whole  $X$ .

To show that  $v$  is in the class of  $Y_T$  defined by (3.2), we notice that

$$v(x, t) = A(v, v_0)(t) \equiv S(t)v_0 + G(e^{2\sigma t}|v|^{2\sigma}v)(t), \quad t \in [0, T].$$

We apply Lemma 2.2 twice to  $S(t)v_0$  to have that

$$S(t)v_0 \in \bar{C}_{\frac{n}{q} - \frac{\alpha}{\sigma}, q} \cap \dot{C}_{\frac{s - \frac{n}{q} + \frac{\alpha}{\sigma}}{\frac{2\alpha}{s, q}}}$$

for any  $p \leq q < \infty$  and  $s > \frac{n}{q} - \frac{\alpha}{\sigma}$ . On the other hand, to show

$$G(e^{2\sigma t}|v|^{2\sigma}v) \in \bar{C}_{\frac{n}{q} - \frac{\alpha}{\sigma}, q}, \quad \forall p \leq q < \infty,$$

we use Lemma 2.4 with

$$q_1 = \frac{p}{2\sigma + 1}, \quad q_2 = q, \quad s_1 = 0, \quad s_2 = \frac{n}{q} - \frac{\alpha}{\sigma}, \quad \lambda_1 = -\frac{(2\sigma + 1)r}{2\alpha}, \quad \lambda_2 = 0,$$

and then obtain

$$\|G(e^{2\sigma t}|v|^{2\sigma}v)\|_{C_{0, \frac{n}{q} - \frac{\alpha}{\sigma}, q}} \leq Ce^{2\sigma T} \| |v|^{2\sigma} v \|_{C_{-\frac{(2\sigma+1)}{2\alpha}r, 0, \frac{p}{2\sigma+1}}} \leq Ce^{2\sigma T} \|v\|_{C_{-\frac{r}{2\alpha}, 0, p}}^{2\sigma+1}.$$

Again, by using Lemma 2.4 with

$$q_1 = \frac{p}{2\sigma + 1}, \quad q_2 = q, \quad s_1 = 0, \quad s_2 = s, \quad \lambda_1 = -\frac{(2\sigma + 1)r}{2\alpha}, \quad \lambda_2 = \frac{s - \frac{n}{q} + \frac{\alpha}{\sigma}}{2\alpha},$$

we see that

$$G(e^{2\sigma t}|v|^{2\sigma}v) \in \dot{C}_{\frac{s - \frac{n}{q} + \frac{\alpha}{\sigma}}{\frac{2\alpha}{s, q}}}, \quad s > \frac{n}{q} - \frac{\alpha}{\sigma}, \tag{3.3}$$

but  $s$  should also satisfy

$$s < 2\alpha - \left( \frac{2\sigma + 1}{p} n - \frac{n}{q} \right),$$

which is required by Lemma 2.4. For large  $s$ , (3.3) can be shown by an induction process. The proof of the local Lipschitz continuity is standard and thus we omitted the details here.

In the case  $r = 0$ , we define

$$X = \bar{C}_{0, p} \cap \dot{C}_{\frac{1}{8\sigma}, 0, \frac{4p}{3}},$$

with the norm

$$\|v\|_X = \|v\|_{C_{0, 0, p}} + \|v\|_{C_{\frac{1}{8\sigma}, 0, \frac{4p}{3}}},$$

and  $X_R$  is again the closed ball in  $X$  of radius  $R$ . The proof is parallel to the case  $r < 0$  except that we need to use Lemma 2.4 to estimate

$$\begin{aligned} \|G(e^{2\sigma t}|v|^{2\sigma}v)\|_X &= \|G(e^{2\sigma t}|v|^{2\sigma}v)\|_{C_{0,0,p}} + \|G(e^{2\sigma t}|v|^{2\sigma}v)\|_{C^{\frac{1}{8\sigma},0,\frac{4p}{3}}} \\ &\leq Ce^{2\sigma T}\|v\|^{2\sigma+1} \|C^{\frac{2\sigma+1}{8\sigma},0,\frac{4p}{3(2\sigma+1)}}\| \leq Ce^{2\sigma T}\|v\|_{C^{\frac{1}{8\sigma},0,\frac{4p}{3}}}^{2\sigma+1} \\ &\leq Ce^{2\sigma T}R^{2\sigma+1}. \end{aligned}$$

The previous proof also holds for the case  $\mathbb{T}^n$ . This completes the proof of Theorem 3.2. □

### 3.3. Well-posedness in $W^{s\alpha,1+\frac{1}{2\sigma}}$

In this subsection, we construct a new working space  $BC_s((0, T], W^{r,p})$  (refer to (1.3)) to prove the well-posedness of (1.4). Such a space with  $\alpha = 1$  has been used to solve the nonlinear Burgers' equations in [26]. We have the following local well-posedness result.

**Theorem 3.3:** Assume  $v_0 \in W^{s\alpha,1+\frac{1}{2\sigma}}$  with  $s$  and  $\alpha$  satisfying

$$s > -\frac{2}{2\sigma + 1} \quad \text{and} \quad s > \frac{n}{(2\sigma + 1)\alpha} - \frac{1}{\sigma}.$$

Then for some  $T = T(v_0) > 0$ , there is a unique solution  $v(x, t)$  to Cauchy problem (1.4) on the time interval  $[0, T]$  satisfying

$$v \in BC_{s\alpha} \left( (0, T], W^{r\alpha,1+\frac{1}{2\sigma}} \right)$$

for any  $s \leq r$  with  $r \geq 0$ . Moreover, the mapping  $v_0 \rightarrow v : W^{s\alpha,1+\frac{1}{2\sigma}} \rightarrow BC_{s\alpha}((0, T], W^{r\alpha,1+\frac{1}{2\sigma}})$  is locally Lipschitz.

The proof of this theorem is again based on the contraction mapping principle. Recall that we can rewrite (1.4) in the integral form

$$v(x, t) = S(t)v_0 + G(e^{2\sigma t}|v|^{2\sigma}v) \equiv e^{-\Lambda^{2\alpha}t}v_0 + \int_0^t e^{-\Lambda^{2\alpha}(t-\tau)}(e^{2\sigma\tau}|v|^{2\sigma}v)(\tau)d\tau.$$

Then we need to estimate the operators  $S$  and  $G$  on  $BC_{s\alpha}([0, T], W^{r\alpha,1+\frac{1}{2\sigma}})$ . We first give the estimate for  $S$ .

**Proposition 3.1:** Let  $T > 0$ ,  $s \in \mathbb{R}$  and  $v_0 \in W^{s\alpha,1+\frac{1}{2\sigma}}$ , then for all  $r \geq s$ , we have  $S(t)v_0 \in BC_{s\alpha}((0, T], W^{r\alpha,1+\frac{1}{2\sigma}})$  and

$$\|S(t)v_0\|_{BC_{s\alpha}((0,T],W^{r\alpha,1+\frac{1}{2\sigma}})} \leq C_\Delta \|v_0\|_{BC_{s\alpha}((0,T],W^{s\alpha,1+\frac{1}{2\sigma}})}$$

where  $C_\Delta = \left\| (1 + |\xi|^2)^{\frac{r-s}{2}} \exp(-|\xi|^2) \right\|_{L^\infty}$  is constant.

**Proof:** The proof involves merely the definition of  $\|v\|_{BC_{s\alpha}([0,T],W^{r\alpha,1+\frac{1}{2\sigma}})}$  and is exactly similar to the case  $\alpha = 1$  in [26]. We omit the details here. □

To estimate the operator  $G$ , we need the following lemma, whose proof is similar to that of Lemma 2 in [20].

**Lemma 3.1:** Let  $\gamma \geq 0$  be a real number. If  $1 \leq i \leq m$  and  $1 < p_i < \infty$  satisfy

$$\frac{1}{p_1} + \frac{2}{p_2} + \frac{3}{p_3} + \dots + \frac{1}{p_m} = m - 1,$$

then for any  $g_i \in W^{\gamma, p_i}$  we have

$$\|\omega(\xi a)^\gamma \widehat{g_1 g_2 \cdots g_m}(\xi)\|_{L^\infty} \leq C \|\omega(\xi a)^\gamma \widehat{g_1}(\xi)\|_{L^{p_1}} \cdots \|\omega(\xi a)^\gamma \widehat{g_m}(\xi)\|_{L^{p_m}}$$

where  $C$  is a constant and  $\omega(\xi) = (1 + |\xi|^2)^{\frac{1}{2}}$ .

Now we can estimate the operator  $G$  as follows.

**Proposition 3.2:** Let  $0 < T \leq 1$  and  $s > \frac{\sigma n}{(2\sigma+1)\alpha} - \frac{1}{\sigma}$ . Assume that  $r \geq 0$  and  $v \in BC_{s\alpha}((0, T), W^{r\alpha, 1+\frac{1}{2\sigma}})$ . Then for any  $q$  satisfying

$$s \leq q < r + 2 - \frac{2\sigma n}{(2\sigma + 1)\alpha},$$

the function  $G(e^{2\sigma t}|v|^{2\sigma}v) \in BC_{s\alpha}((0, T), W^{q\alpha, 1+\frac{1}{2\sigma}})$  and

$$\|G(e^{2\sigma t}|v|^{2\sigma}v)\|_{BC_{s\alpha}((0, T), W^{q\alpha, 1+\frac{1}{2\sigma}})} \leq CT^{\sigma s+1-\frac{\sigma n}{(2\sigma+1)\alpha}} \|v\|_{BC_{s\alpha}((0, T), W^{r\alpha, 1+\frac{1}{2\sigma}})}^{2\sigma+1},$$

where  $C$  is constant. Moreover, we have

$$\lim_{t \rightarrow 0} \|G(e^{2\sigma t}|u|^{2\sigma+1}v)\|_{BC_{s\alpha}((0, T), W^{q\alpha, 1+\frac{1}{2\sigma}})} = 0.$$

**Proof:** We first estimate  $\|G(e^{2\sigma t}|v|^{2\sigma}v)\|_{BC_{s\alpha}((0, T), W^{q\alpha, 1+\frac{1}{2\sigma}})}$ . It is only necessary to prove for the case  $r \leq q \leq r + 2 - \frac{2\sigma n}{(2\sigma+1)\alpha}$  since the norm is a nondecreasing function of  $q$ . It is easy to check that for  $0 < T \leq 1, -\frac{2}{2\sigma+1} \leq s \leq 0 \leq r$ ,

$$t^{\frac{|s|}{2\alpha}} \leq \omega(s)\omega(\xi t^{\frac{1}{2\alpha}})^{-s} \leq |\xi|^s + t^{\frac{|s|}{2\alpha}}$$

and it then follows that  $\|v\|_{BC_{s\alpha}((0, T), W^{r\alpha, 1+\frac{1}{2\sigma}})}$  is bounded by

$$\begin{aligned} & \sup_{0 < t \leq T} t^{\frac{|s|}{2}} \left\| \int_0^t \omega(\xi t^{\frac{1}{2\alpha}}) e^{-|\xi|^{2\alpha}(t-\tau)} \widehat{|v|^{2\sigma}v}(\xi, t) d\tau \right\|_{L^{1+\frac{1}{2\sigma}}} \\ & \leq \|G(e^{2\sigma t}|v|^{2\sigma}v)\|_{BC_{s\alpha}((0, T), W^{q\alpha, 1+\frac{1}{2\sigma}})} \\ & \leq e^{2\sigma T} \sup_{0 < t \leq T} t^{\frac{|s|}{2}} \left\| \int_0^t \omega(\xi t^{\frac{1}{2\alpha}})^{q\alpha} e^{-|\xi|^{2\alpha}(t-\tau)} \widehat{|v|^{2\sigma}v}(\xi, t) d\tau \right\|_{L^{1+\frac{1}{2\sigma}}} \\ & \quad + e^{2\sigma T} \sup_{0 < t \leq T} \left\| \int_0^t |\xi|^s \omega(\xi t^{\frac{1}{2\alpha}})^{q\alpha} e^{-|\xi|^{2\alpha}(t-\tau)} \widehat{|v|^{2\sigma}v}(\xi, t) d\tau \right\|_{L^{1+\frac{1}{2\sigma}}} \\ & := I + II. \end{aligned}$$

We first estimate  $I$ . For all  $\xi \in \mathbb{R}^n$  and  $0 \leq \tau \leq t$ , it is clear that

$$\omega(\xi t^{\frac{1}{2\alpha}}) \leq C\omega(\xi(t-\tau)^{\frac{1}{2\alpha}})\omega(\xi t^{\frac{1}{2\alpha}}).$$

Therefore, we have

$$\begin{aligned}
 I &\leq e^{2\sigma T} t^{\frac{|s|}{2}} \int_0^t \left\| \omega(\xi t^{\frac{1}{2\alpha}})^{(q-r)\alpha} \omega(\xi(t-\tau)^{\frac{1}{2\alpha}})^{r\alpha} e^{-|\xi|^{2\alpha}(t-\tau)} \right\|_{L^{1+\frac{1}{2\sigma}}} \left\| \omega(\xi \tau^{\frac{1}{2\alpha}})^{r\alpha} \widehat{|v|^{2\sigma} v}(\xi, t) \right\|_{L^\infty} d\tau \\
 &\leq C e^{2\sigma T} t^{\frac{|s|}{2}} \|v\|_{BC_{s\alpha}((0,T), W^{r\alpha, 1+\frac{1}{2\sigma}})} \int_0^t \frac{\left\| \omega(\xi t^{\frac{1}{2\alpha}})^{(q-r)\alpha} \omega(\xi \tau^{\frac{1}{2\alpha}})^{r\alpha} e^{-|\xi|^{2\alpha}\tau} \right\|_{L^{1+\frac{1}{2\sigma}}}}{(t-\tau)^{\frac{2\sigma+1}{2}|s|}} d\tau \\
 &\leq C e^{2\sigma T} t^{-|s|\sigma+1-\frac{\sigma n}{(2\sigma+1)\alpha}} \|v\|_{BC_{s\alpha}((0,T), W^{r\alpha, 1+\frac{1}{2\sigma}})}^{2\sigma+1} \int_0^1 \frac{\left\| \omega(\xi \theta^{-\frac{1}{2\alpha}})^{(q-r)\alpha} \omega(\xi)^{r\alpha} e^{-|\xi|^{2\alpha}} \right\|_{L^{1+\frac{1}{2\sigma}}}}{(1-\theta)^{\frac{2\sigma+1}{2}|s|} \theta^{\frac{\sigma n}{(2\sigma+1)\alpha}}} d\theta.
 \end{aligned}$$

It shows that

$$\int_0^1 \frac{\left\| \omega(\xi \theta^{-\frac{1}{2\alpha}})^{(q-r)\alpha} \omega(\xi)^{r\alpha} e^{-|\xi|^{2\alpha}} \right\|_{L^{1+\frac{1}{2\sigma}}}}{(1-\theta)^{\frac{2\sigma+1}{2}|s|} \theta^{\frac{\sigma n}{(2\sigma+1)\alpha}}} d\theta \leq C$$

for some  $C > 0$ . Indeed, since  $0 \leq q - r < 2 - \frac{2\sigma n}{(2\sigma+1)\alpha}$  and thus  $0 \leq \frac{q-r}{2} \leq 1$ , we can deduce the desired result by

$$\omega(\xi \theta^{-\frac{1}{2}})^{q-r} = (1 + |\xi|^2 \theta^{-1})^{\frac{q-r}{2}} \leq 1 + |\xi|^{q-r} \theta^{-\frac{q-r}{2}}$$

and the fact for  $a > 0, b > 0$  the Beta function

$$B(a, b) = \int_0^1 (1-x)^{a-1} x^{b-1} dx$$

is finite. The term  $II$  can be estimated in a quite similar way and the final result is the same as that of  $I$  apart from that the constant  $C$  may be different. □

We now turn to prove Theorem 3.3.

**Proof of Theorem 3.3:** Let  $r \geq 0$  be any real number,  $X_T = BC_{s\alpha}((0, T), W^{r\alpha, 1+\frac{1}{2\sigma}})$  and  $X_{T,R}$  be the closed ball centered at zero of radius  $R$ , where  $T$  and  $R$  are yet to be determined. Define the nonlinear map  $\Phi$  on  $X_{T,R}$ :

$$\Phi(v) = S(t)v_0 + G(e^{2\sigma t}|v|^{2\sigma}v).$$

By using Propositions 3.1 and 3.2, we have

$$\|\Phi(v)\|_{X_T} \leq C_\Delta \|v_0\|_{W^{s\alpha, 1+\frac{1}{2\sigma}}} + CT^{\sigma s+1-\frac{\sigma n}{(2\sigma+1)\alpha}} \|v\|_{X_T}^{2\sigma+1}.$$

For  $v, \tilde{v} \in X_{T,R}$ , by similar procedure as proof of Proposition 3.2, we have

$$\|\Phi(v) - \Phi(\tilde{v})\|_{X_T} \leq CT^{\sigma s+1-\frac{\sigma n}{(2\sigma+1)\alpha}} \left\| |v|^{2\sigma} + |\tilde{v}|^{2\sigma} \right\|_{X_T} \|v - \tilde{v}\|_{X_T}$$

It is not hard to check that  $\Phi$  maps  $X_{T,R}$  into  $X_{T,R}$  and is a contraction map for some properly-chosen  $T$  and  $R$ . Thus there is a unique fixed point  $v = \Phi(v)$  in  $X_{T,R}$ . It is clear that by reducing the time interval  $(0, T)$ , we can extend the existence and uniqueness to  $X_{T,R'}$  for any  $R'$  and thus to the whole space of  $X_T$ . The Lipschitz continuity is easily obtain by using the fact that map is a contraction map. This finishes the proof of this theorem. □

### 4. Asymptotic analysis

In this section, we deal with the decay of the ‘mass’. Note that for any nonnegative initial data  $v_0 \in L^1(\mathbb{R}^n)$ , we have a nonnegative global  $L^1$ -solution  $v$  to (1.4) satisfying

$$v \in L^\infty([0, \infty), L^1(\mathbb{R}^n)) \quad \text{and} \quad e^{2\sigma t} v^{2\sigma+1} \in L^1(\mathbb{R}^n \times (0, \infty))$$

(see [27]). Then integrating Equation (1.4) with respect to  $x$  and  $t$ , we have

$$M(t) = \int_{\mathbb{R}^n} v(x, t) dx = \int_{\mathbb{R}^n} v_0(x, t) dx - \int_0^t \int_{\mathbb{R}^n} e^{2\sigma\tau} v(x, \tau)^{2\sigma+1} dx d\tau \tag{4.1}$$

We will show that in the remaining rang of  $\sigma$ , the mass  $M(t)$  converges to zero and this phenomena can be interpreted as the domination of nonlinear effects in the large time asymptotic of solutions to (1.4). Note here that the mass  $M(t) = \int_{\mathbb{R}^n} v(x, t) dx$  of every solution to the linear equation  $v_t + \Lambda^{2\alpha} v = 0$  is constant in time.

**Theorem 4.1:** *Assume that  $v = v(x, t)$  is a non-negative solution of problem (1.4) with  $0 < \sigma < \frac{\alpha}{n-2\alpha}$ . Then  $\lim_{t \rightarrow \infty} M(t) = 0$ .*

**Proof:** We will adapt the so-called rescaled test function method used in [28] to prove our conclusion. Let us define the function  $\varphi(x, t) = \varphi_1^l(x)\varphi_2^l(t)$ , where

$$l = \frac{4\sigma + 1}{2\sigma}, \quad \varphi_1(x) = \psi\left(\frac{|x|}{R}\right), \quad \varphi_2(t) = \psi\left(\frac{t}{R^{2\alpha}}\right)$$

with  $R > 1$ . Here  $\psi$  is a smooth non-increasing function on  $[0, \infty)$  such that

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

In the following, we denote by  $\Omega_1$  and  $\Omega_2$  the supports of  $\varphi_1$  and  $\varphi_2$ , respectively:

$$\Omega_1 = \{x \in \mathbb{R}^n : |x| \leq 2R\}, \quad \Omega_2 = \{t \in [0, \infty) : t \leq 2R^{2\alpha}\}.$$

Now, we multiply Equation (1.4)<sub>1</sub> by  $\varphi(x, t)$  and integrate with respect to  $x$  and  $t$  to obtain

$$\begin{aligned} & \int_{\Omega_1} v_0(x)\varphi(x, 0) dx - \int_{\Omega_2} \int_{\Omega_1} e^{2\sigma t} v^{2\sigma+1}(x, t)\varphi(x, t) dx dt \\ &= \int_{\Omega_2} \int_{\Omega_1} v(x, t)\varphi_2^l(t)\Lambda^{2\alpha}\varphi_1^l(x) dx dt - \int_{\Omega_2} \int_{\Omega_1} v(x, t)\varphi_1^l(x)\partial_t\varphi_2^l(t) dx dt \\ &\leq l \int_{\Omega_2} \int_{\Omega_1} v(x, t)\varphi_2^l(t)\varphi_1^{l-1}(x)\Lambda^{2\alpha}\varphi_1(x) dx dt - l \int_{\Omega_2} \int_{\Omega_1} v(x, t)\varphi_1^l(x)\varphi_2^{l-1}(t)\partial_t\varphi_2(t) dx dt. \end{aligned} \tag{4.2}$$

Here, we have used the inequality  $\Lambda^{2\alpha}\varphi_1^l \leq l\varphi_1^{l-1}\Lambda^{2\alpha}\varphi_1$ , which is valid for all  $\alpha \in (0, 1], l \geq 1$ , and any sufficiently regular, non-negative, decaying at infinity function  $\varphi_1$  (see [29,30] for the corresponding proof).

By the  $\epsilon$ -Young inequality  $ab \leq \epsilon a^{2\sigma+1} + C(\epsilon)b^{l-1}$  with  $\frac{1}{2\sigma+1} + \frac{1}{l-1} = 1$  and  $\epsilon > 0$ , we deduce from (4.2) that

$$\begin{aligned} & \int_{\Omega_1} v_0(x)\varphi(x, 0) dx - (1 + 2l\epsilon) \int_{\Omega_2} \int_{\Omega_1} e^{2\sigma t} v^{2\sigma+1}(x, t)\varphi(x, t) dx dt \\ & \leq C(\epsilon)l \left\{ \int_{\Omega_2} \int_{\Omega_1} e^{-2\sigma t \frac{l-1}{2\sigma+1}} \varphi_1(x)\varphi_2^l(t) |\Lambda^{2\alpha}\varphi_1|^{l-1} dx dt \right. \\ & \quad \left. + \int_{\Omega_2} \int_{\Omega_1} e^{-2\sigma t \frac{l-1}{2\sigma+1}} \varphi_1^l\varphi_2|\partial_t\varphi_2|^{l-1} dx dt \right\}. \end{aligned} \tag{4.3}$$

Recall now that the functions  $\varphi_1$  and  $\varphi_2$  depend on  $R > 0$ . Hence changing the variables  $\xi = R^{-1}x$  and  $\tau = R^{-2\alpha}t$ , we easily obtain from (4.3) the following estimate

$$\int_{\Omega_1} v_0(x)\varphi(x,0)dx - (1 + 2l\epsilon) \int_{\Omega_2} \int_{\Omega_1} e^{2\sigma t} v^{2\sigma+1}(x,t)\varphi(x,t)dxdt \leq CR^{n-2\alpha(l-1)}, \quad (4.4)$$

where the constant  $C$  is independent of  $R$ .

Note that  $\sigma < \frac{\alpha}{n-2\alpha}$  if and only if  $n - 2\alpha(l - 1) < 0$ . Computing the limit  $R \rightarrow \infty$  in (4.4) and using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{t \rightarrow \infty} M(t) = \int_{\mathbb{R}^n} v_0(x)dx - \int_0^\infty \int_{\mathbb{R}^n} e^{2\sigma t} v^{2\sigma+1}(x,t)dxdt \leq 2l\epsilon \int_0^\infty \int_{\mathbb{R}^n} e^{2\sigma t} v^{2\sigma+1}(x,t)dxdt. \quad (4.5)$$

Since  $v \in e^{2\sigma t} v^{2\sigma+1} \in L^1(\mathbb{R}^n \times (0, \infty))$  and  $\epsilon > 0$  can be chosen arbitrary small, we immediately obtain that  $\lim_{t \rightarrow \infty} M(t) = 0$ .  $\square$

## Acknowledgements

The authors would like to thank the anonymous referees and the editor Prof. Yongzhi Steve Xu for their helpful comments that improved the presentation of this paper. The authors are also grateful to Prof. Ting-Zhu Huang and Prof. Zhao-Yin Xiang for carefully reading earlier version of the current paper.

## Disclosure statement

No potential conflict of interest was reported by the authors.

## Funding

This research is supported by NSFC [grant number 61702083], [grant number 11501085], [grant number 11701467], [grant number 11701475], [grant number 11601365]; the Fundamental Research Funds for the Central Universities [grant number ZYGX2016]132], [grant number ZYGX2016]138], [grant number 2682017CX068].

## References

- [1] Cazenave T, Dickstein F, Weissler FB. Finite-time blowup for a complex Ginzburg-Landau equation. *SIAM J Math Anal.* **2013**;45:244–266.
- [2] Guo B, Huang H, Jiang M. Ginzburg-Landau equation. Chinese ed. Beijing: Science Press; **2002**.
- [3] Mitidieri E, Pokhozhaev SI. Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on  $\mathbb{R}^N$ . *J Evol Equ.* **2001**;1:189–220.
- [4] Zaslavsky G. Hamiltonian chaos and fractional dynamics. Oxford: Oxford University Press; **2005**.
- [5] Tarasov V. Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media. Berlin: Springer-Verlag, jointly with Higher Education Press, Beijing; **2011**.
- [6] Wilk G, Wlodarczyk Z. Do we observe Levy flights in cosmic rays? *Nucl Phys.* **1999**;75:191–193.
- [7] Mainardi F, Gorenflo R. On Mittag-Leffler-type functions in fractional evolution processes. *J Comput Appl Math.* **2000**;118:283–299.
- [8] Metzler R, Klafter J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys Rep.* **2000**;339:1–77.
- [9] Weitzner H, Zaslavsky G. Some applications of fractional derivatives. *Commun Nonlinear Sci Numer Simul.* **2003**;8:273–281.
- [10] Tarasov V, Zaslavsky G. Fractional dynamics of coupled oscillators with long-range interaction. *Chaos.* **2006**;16:23–110.
- [11] Nec Y, Nepomnyashchy A, Golovin A. Oscillatory instability in super-diffusive reaction diffusion systems: fractional amplitude and phase diffusion equations. *Phys Rev E.* **2008**;78:60–102.
- [12] Li M, Huang C, Wang N. Galerkin finite element method for the nonlinear fractional Ginzburg-Landau equation. *Appl Numer Math.* **2017**;118:131–149.
- [13] Mitidieri E, Pokhozhaev SI. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Proc Steklov Inst Math.* **2001**;234:1–362.

- [14] Milovanov A, Rasmussen J. Fractional generalization of the Ginzburg-Landau equation: an unconventional approach to critical phenomena in complex media. *Phys Lett A*. 2005;337:75–80.
- [15] Pu X, Guo B. Well-posedness and dynamics for the fractional Ginzburg-Landau equation. *Appl Anal*. 2013;92:318–334.
- [16] Tarasov V, Zaslavsky G. Fractional Ginzburg-Landau equation for fractal media. *Physica A*. 2005;354:249–261.
- [17] Tarasov V. Psi-series solution of fractional Ginzburg-Landau equation. *J Phys A Math Gen*. 2006;39:8395–8407.
- [18] Li J, Xia L. Well-posedness of fractional Ginzburg-Landau equation in sobolev spaces. *Appl Anal*. 2013;5:1074–1084.
- [19] Li J, Xia L. The fractional Ginzburg-Landau equation with distributional initial data. *Commun Pure Appl Anal*. 2013;5:2173–2187.
- [20] Wu J. Dissipative quasi-geostrophic equations with  $L^p$  data. *Electron J Differ Equ*. 2001;2001:1–13.
- [21] Kato T, Ponce G. The Navier-Stokes equations with weak initial data. *Int Math Res Notices*. 1994;10:435–444.
- [22] Wu J. Well-posedness of a semi-linear heat equation with weak initial data. *J Fourier Anal Appl*. 1998;4:629–642.
- [23] Pazy A. Semigroups of linear operators and applications to partial differential equations. New York (NY): Springer-Verlag; 1983.
- [24] Kato T. Strong  $L^p$  solutions of the Navier-Stokes equation in  $\mathbb{R}^m$  with applications to weak solutions. *Math Z*. 1984;187:471–480.
- [25] Kato T. The Navier-Stokes solutions for an incompressible fluid in  $\mathbb{R}^2$  with measure as the initial vorticity. *Differ Integr Equ*. 1994;7:949–966.
- [26] Dix DB. Nonuniqueness and uniqueness in the initial value problem for Burgers' equation. *SIAM J Math Anal*. 1996;27:708–724.
- [27] Taylor M. Remarks on fractional diffusion equations. Lecture Notes. Department of Mathematics, University of North Carolina. [cited 2007 Oct 3]:[44 p]. Available from: <http://www.unc.edu/math/Faculty/met/fdif.pdf>.
- [28] Fino A, Karch G. Decay of mass for nonlinear equation with fractional Laplacian. *Monatsh Math*. 2010;160:375–384.
- [29] Cordoba A, Cordoba D. A maximum principle applied to quasi-geostrophic equations. *Commun Math Phys*. 2004;249:511–528.
- [30] Ju N. The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations. *Commun Math Phys*. 2005;255:161–181.