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On the (Non)Removability of Spectral Parameters in $\mathbb{Z}_2$-Graded Zero-Curvature Representations and Its Applications

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Abstract We generalise to the $\mathbb{Z}_2$-graded set-up a practical method for inspecting the (non)removability of parameters in zero-curvature representations for partial differential equations (PDEs) under the action of smooth families of gauge transformations. We illustrate the generation and elimination of parameters in the flat structures over $\mathbb{Z}_2$-graded PDEs by analysing the link between deformation of zero-curvature representations via infinitesimal gauge transformations and, on the other hand, propagation of linear coverings over PDEs using the Frölicher–Nijenhuis bracket.

Keywords Zero-curvature representation · Spectral parameter · Removability · Supersymmetry · Korteweg–de Vries equation · Gardner's deformation · Frölicher–Nijenhuis bracket

Mathematics Subject Classification (2010) 35Q53 · 37K25 · 58J72 · 58A50

1 Introduction

The aim of the present paper is to provide a tool for inspection of (non)removability of parameters in matrix Lie super-algebra valued zero-curvature representation for $\mathbb{Z}_2$-graded partial differential equations, e.g., for supersymmetric equations of Korteweg–de Vries (KdV)

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type. This work concludes the cycle of papers [1–3] in which the Gardner deformation problem [4, 5] for the triplet $a \in \{-2, 1, 4\}$ of P. Mathieu's $N = 2$ supersymmetric KdV equations [1, 6] was addressed by using the various geometric, analytic, and computational techniques. In the paper [2] (joint with Hussin and Wolf) we presented our first solution of the Open Problem 2 from [7], constructing the hierarchy of Hamiltonian super-functionals for the $N = 2$ super-KdV with $a = 4$ on the basis of a classical Gardner's deformation for the Kaup–Boussinesq hierarchy of bosonic limit of that super-system. Equally well applicable to many other supersymmetric or just $\mathbb{Z}_2$-graded completely integrable systems, this line of reasoning was furthered in [8]. The alternative solution of Gardner's deformation problem, which we proposed in [3], see also [9, § 5.3], allows to address that problem from a different perspective, namely, by putting it in the context of finding Lie super-algebra valued zero-curvature representations for PDEs or, in broader terms, finding the parametric families of non-local structures over such systems (see [10, 11], also [12–14]). The key idea is that the *deformation* parameter, the presence of which yields an explicit recurrence relations between the hierarchy of integrals of motion for the PDE under study, is an avatar of the *spectral* parameter in the inverse scattering problem [15, 16]. This correspondence, itself referring to the task of switching between the different realisations of Lie algebras [17–21], necessitates the inspection of relevance vs removability of the parameters in families of non-local structures.

We resolve this issue of (non)removability by approaching the problem from two different directions. First, we formulate a ready-to-use procedure for inspecting the (non)removability of parameters in Lie super-algebra valued zero-curvature representations for PDEs under the action of smooth families of gauge transformations.[1] This practical algorithm is the $\mathbb{Z}_2$-graded generalisation of Marvan's technique [22, 23], which was designed initially for purely bosonic systems; a technique for solution of the same problem was developed independently by Sakovich in [24, 25]. Extending Marvan's approach in Sect. 3 to, e.g., the geometry of Mathieu's $N = 2$ supersymmetric Korteweg–de Vries equations [1], we now prove for example that one cannot remove in that way the parameter which is contained in the structure found by Das et al. in [26] and which was used later in [3] within our alternative solution to Mathieu's problem of deforming his $N = 2$, $a = 4$ super-KdV equation.

Another powerful technique for generating the parametric families of non-local structures over PDEs is based on the use of Frölicher–Nijenhuis bracket. This approach was elaborated by Krasil'shchik et al. in [27–29], cf. [30]; basic facts and ideas from this formalism are recalled in Sect. 4 (see also Sect. 2). In this geometric context, we explore the nature and properties of the parameters found by Gardner [4] for the classical Korteweg–de Vries equation (see Example 5) and by Das et al. [26] for the $N = 2$ supersymmetric $a = 4$ Korteweg–de Vries equation. Furthermore, using the results described in Sect. 4 and techniques from [31], we prove the integrability of new fifth order $N = 1$ supersymmetric evolution equation (9a)–(9b) found by Tian and Liu (see [32, 33]): in Example 7 we construct its $\mathfrak{sl}(9|8)$-valued zero-curvature representation with nonremovable parameter.

The main instrument of our study is the notion of prolongation structures over partial differential equations; this concept goes back to Wahlquist and Estabrook [10, 11, 34–36]. Let $x^1, \ldots, x^n$ be the independent variables in a given PDE $\mathcal{E}$ and $\bar{D}_{x^i}$ be the restrictions of the

[1] Zero-curvature representations for partial differential equations are the input data for realisation of the inverse scattering method. For PDEs with unknown functions in two independent variables, the most interesting zero-curvature representations are those which contain a non-removable spectral parameter; in that case the system of PDEs can be kinematically integrable [15, 16].

respective total derivatives to (the infinite prolongation $\mathcal{E}^\infty$ of) the equation $\mathcal{E}$. The prolongation structure over $\mathcal{E}^\infty$ is described by the Maurer–Cartan equation that holds restricted to $\mathcal{E}^\infty$ (which is denoted by $\doteq$),

$$[\bar{D}_{x^i} + A_i, \bar{D}_{x^j} + A_j] \doteq 0, \quad 1 \leqslant i < j \leqslant n.$$

In the case of zero-curvature representations, the objects A_i and A_j are Lie (super-) algebra valued functions on (the infinite prolongation $\mathcal{E}^\infty$ of) the equation $\mathcal{E}$. In the case of linear coverings, the objects A_i and A_j are vertical vector fields along the fibres W in a (vector) bundle over $\mathcal{E}^\infty$. This establishes the correspondence between zero-curvature representations and linear coverings. Indeed, each zero-curvature representation with coefficients belonging to a Lie (super-)algebra determines a linear covering, whereas each covering with fibre W can be regarded as a zero-curvature representation the coefficients of which take values in the Lie algebra of vector fields on W. This correspondence very often allows one to transfer the results from one geometry to the other. Lemma 4 and Proposition 5 in Sect. 4 confirm this general principle; similar results were considered in [27].

Finally, we analyse the link between the two deformation methods in the case of $\mathbb{Z}_2$-graded PDEs. In particular, in Sect. 5 we illustrate that link by switching between the realisations of Lie super-algebras in zero-curvature representations for Mathieu's $N = 2$, $a = 4$ super-KdV and other integrable systems of Korteweg–de Vries type.

This paper is structured as follows. First, in Sect. 2 we recall basic facts from the local and non-local geometry of $\mathbb{Z}_2$-graded partial differential equations and we fix some notation. In Sect. 3 we generalise—to the case of $\mathbb{Z}_2$-graded partial differential equations—Marvan's approach to proving the (non)removability of parameters in zero-curvature representations. In Sect. 4 we analyse the construction of families of coverings by using the Frölicher–Nijenhuis bracket. In Sect. 5 we explore a relation between the two ways to obtain parametric families of those geometric structures, namely, zero-curvature representations and coverings. In conclusion we summarise the result and list open problems. Two appendices supersede the main text; in particular, a technical proof is contained in Appendix A. In Appendix B, which is found at the end of this paper, we discuss the "unconventional" strategies for elimination of parameters—e.g., those which cannot be removed by using smooth families of gauge transformations. For example, we then re-consider Sasaki's result [37] in the context of [27–29].

We refer to the books [38, 39] for definitions and basic concepts from supergeometry. Let us emphasise that in parallel with supergeometry, many objects in the geometry of PDEs can be approached by viewing them as the spectra of given rings of functions (that is, as the ringed spaces), see [40, 41]. Not only is the definition of supermanifold natural in this context (indeed, it is a topological space equipped with a sheaf of smooth functions taking values in some Grassmann algebra) but also the construction of infinite prolongations $\mathcal{E}^\infty$ for partial differential equations $\mathcal{E}$ becomes handy; for unlike the kth order prolongations $\mathcal{E}^{(k)}$ the dimension of which stays finite, the loci $\mathcal{E}^\infty$ are usually infinite-dimensional (see [40–44]). Therefore, it is this algebraic formalism (cf. [45]) which now ensures an almost automatic extension of practical techniques within the formal geometry of PDEs to the $\mathbb{Z}_2$-graded, supergeometric set-up.

2 Preliminaries: $\mathbb{Z}_2$-Graded Infinite Jet Bundles

In this section we recall necessary definitions from supergeometry (we refer to [38, 39, 46] and [44, 47–49] for further details); this material is standard.

2.1 Jet Spaces

Let M^n be an n-dimensional oriented smooth real manifold. Let us consider two vector bundles over the same base M^n, namely, $\pi^0\colon E^{m_0+n} \to M^n$ and $\pi^1\colon \Sigma^{m_1+n} \to M^n$ with fibre dimensions m_0 and m_1, respectively. (In particular, we let $n = 2$ so that the independent variables are $x^1 = x$ and $x^2 = t$; we have that $m_0 = 1$, $m_1 = 0$ for the Korteweg–de Vries equation, $m_0 = 2$, $m_1 = 0$ for the hierarchy of the Kaup–Boussinesq equation, and $m_0 = 2$, $m_1 = 2$ for the $N = 2$ supersymmetric KdV equation, see [1, 7, 38, 39].)

Let $\pi^{\bar 1} = \Pi\pi^1$ be the odd neighbour of the vector bundle π^1 (Π denotes the reversion of parity). By definition, this neighbour is the vector bundle $\pi^{\bar 1}\colon \Pi\Sigma^{n+m_1} \to M^n$ over the same base and with the same vector space $\mathbb{R}^{m_1}$ take as prototype for the fibres. The coordinates $\xi^1, \dots, \xi^{m_1}$ along the fibres $(\pi^{\bar 1})^{-1}(x) \simeq \mathbb{R}^{m_1}$ over $x \in M^n$ are proclaimed $\mathbb{Z}_2$-parity odd and coordinates $u^1, \dots, u^{m_0}$ along the fibre $(\pi^{\bar 0})^{-1}(x) \simeq \mathbb{R}^{m_0}$ are proclaimed $\mathbb{Z}_2$-parity even, i.e. we introduce the $\mathbb{Z}_2$-grading $\mathsf{p}\colon x^i \mapsto \bar 0$, $\xi^k \mapsto \bar 1$, $u^j \mapsto \bar 0$ for the generators of the ring of smooth $\mathbb{R}$-valued functions on the total space $\Pi\Sigma^{m_1+n}$ of the superbundle. We have that $C^\infty(\Pi\Sigma^{m_1+n}) \simeq \Gamma(\bigwedge^\bullet(\Sigma^{m_1+n})^*)$, where $(\Sigma^{m_1+n})^*$ denotes the space of fibrewise-linear functions on Σ^{m_1+n}. Finally, let us construct the Whitney sum $\pi = \pi^{\bar 0} \times_{M^n} \pi^{\bar 1}$ of the bundles $\pi^{\bar 0} = \pi^0$ and $\pi^{\bar 1}$ over the base M^n. We put $\Gamma(\pi) = \Gamma^{\bar 0}(\pi) \oplus \Gamma^{\bar 1}(\pi)$, where $\Gamma^{\bar 0}(\pi) = \Gamma(\pi^{\bar 0})$ and $\Gamma^{\bar 1}(\pi) = \Pi(\Gamma(\Pi\pi^{\bar 1}))$; this construction of $\Gamma^{\bar 1}(\pi)$ in $\Gamma(\pi)$ will be referred to in the definition of Cartan's distribution $\mathcal{C}$ (see below).

Consider the jet space $J^\infty(\pi)$ of sections of the superbundle π. Namely, for the superbundle π the infinite jet superbundle $\pi_\infty\colon J^\infty(\pi) \to M^n$ is defined as follows: we let $(\pi_\infty)^{\bar 0} = (\pi^{\bar 0})_\infty$, $(\pi_\infty)^{\bar 1} = \Pi((\pi^1)_\infty)$ (see [40] and [50] for details). The set of variables describing $J^\infty(\pi)$ is composed by

- even coordinates x^i on M^n,
- even coordinates u^j and parity-odd coordinates ξ^k along the fibres of π; these objects themselves are elements of the set of
- even variables u^j_σ and parity-odd variables ξ^k_σ for the fibres of the infinite jet bundle $\pi_\infty\colon J^\infty(\pi) \to M^n$.

In the above notation we let σ be the multi-index that labels partial derivatives of the unknowns u^j and ξ^k w.r.t. the even variables x^i; by convention, $u^j_\varnothing \equiv u^j$ and $\xi^k_\varnothing \equiv \xi^k$. The parity function p acts *only* on homogeneous elements of $C^\infty(J^\infty(\pi))$ by extending its value from the generators,

$$\begin{aligned} &\mathsf{p}(x^i) = \bar 0, \qquad \mathsf{p}(u^j) = \bar 0, \qquad \mathsf{p}(\xi^k) = \bar 1, \\ &\qquad\qquad\quad\; \mathsf{p}(u^j_\sigma) = \bar 0, \qquad \mathsf{p}(\xi^k_\sigma) = \bar 1, \qquad |\sigma| > 0. \end{aligned}$$

By construction, the parity satisfies the rules

$$\mathsf{p}(a \cdot b) = \mathsf{p}(a) + \mathsf{p}(b),$$
$$\mathsf{p}(a + b) = \mathsf{p}(a) = \mathsf{p}(b) \quad \text{iff} \quad \mathsf{p}(a) = \mathsf{p}(b),$$

where $a, b \in C^\infty(J^\infty(\pi))$.

Every fibrewise linear function $f \in C^\infty_{\mathrm{lin}}(J^\infty)$ can be identified naturally with a linear differential operator $\Delta_f\colon \Gamma(\pi) \to C^\infty(M)$ by using the formula $\Delta_f(s)(x) = f(j_\infty(s)(x))$, where $j_\infty(s)(x)$ is the infinite jet of a section $s \in \Gamma(\pi)$ at $x \in M$. The infinite jet bundle π_∞

admits a natural flat connection such that the lift $\hat{X}$ of a vector field X on M is uniquely defined by the condition $\Delta_{\hat{X}(f)} = X \circ \Delta_f$ for $f \in C^\infty_{\text{lin}}(J^\infty(\pi))$. The lifts $D_{x^i} = \widehat{\partial/\partial x^i}$ of $\partial/\partial x^i$ are called *the total derivatives* on $J^\infty(\pi)$; at every i, they are expressed by the formula

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^{m_0} \sum_{\sigma_{\bar{0}}} u^j_{\sigma_{\bar{0}}+1_i} \frac{\partial}{\partial u^j_{\sigma_{\bar{0}}}} + \sum_{k=1}^{m_1} \sum_{\sigma_{\bar{1}}} \xi^k_{\sigma_{\bar{1}}+1_i} \frac{\vec{\partial}}{\partial \xi^k_{\sigma_{\bar{1}}}},$$

where $\vec{\partial}/\partial \xi^k$ denotes the left derivative. These vector fields commute (in a usual sense, even though the operators D_{x^i} contain directed derivations). By definition, we put $D_\tau = D^{\tau_1}_{x^1} \circ \cdots \circ D^{\tau_n}_{x^n}$. Vector fields of the form $\hat{X}$ generate an n-dimensional distribution on $J^\infty(\pi)$; it is called the Cartan distribution and it is denoted by $\mathcal{C}$.

2.2 Differential Equations

In this $\mathbb{Z}_2$-graded set-up, let a system $\mathcal{E}$ of partial differential equations be given. By definition, the geometric object $\mathcal{E}$ is described by (many equivalent) systems of relations between the unknowns' derivatives with respect to the n independent directions along the base M^n. In local coordinates we have that

$$\mathcal{E} = \big\{ F^\ell\big(x^i, u^j, \dots, u^j_{\sigma_{\bar{0}}}, \xi^k, \dots, \xi^k_{\sigma_{\bar{1}}}\big) = 0,\ \ell = 1, \dots, r \big\}.$$

In fact, not every object $\mathcal{E}$ determined this way would be interesting from either geometric or physical points of view. To get rid of irrelevant cases, from now on we consider only ($\mathbb{Z}_2$-graded) partial differential (super-)equations which are *formally integrable* in the sense of Goldschmidt [51, 52]. Still let us emphasise that at the moment when this paper is written, the expert community has not yet reached a consensus on the proper $\mathbb{Z}_2$-*graded* extension of Goldschmidt's classical result on integrability. It is quite paradoxical that even if such extension is almost as straightforward as the generalisation of Marvan's approach to kinematically integrable systems, that work has not yet been done.

In view of what has been said above, we accept that each partial differential equation to study must possess the non-empty infinite prolongation $\mathcal{E}^\infty$ formed by all the differential consequences[2] $D_\tau(F^\ell) = 0$ with $|\tau| \geqslant 0$; the locus[3] $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ is required to project back onto $\mathcal{E}$ and onto all the lower-order jet (super-)spaces—so that the Cauchy problem for $\mathcal{E}$ is (formally) solvable in the class of formal power series for all Cauchy data.

[2]An obvious logical and geometric distinction between the locus $\mathcal{E}^\infty$ and its algebraic description by using the C^∞-smooth left-hand sides in the system $D_\tau(F^\ell) = 0$ is that the latter are always defined yet they can describe the *empty set*. For instance, consider the overdetermined partial differential equation $\mathcal{E} = \{u_{xx} = 1, u_y = x^2\}$ for which $(u_{xx})_y = 0 \neq 2 = (u_y)_{xx}$. Likewise, the equation $\mathcal{E} = \{v_x = u, v_y = u\}$ can be solved only if the compatibility condition $v_{xy} = v_{yx}$ is satisfied, thus $u_x = u_y$ is the constraint due to which the projection of $\mathcal{E}^\infty$ down to $\mathcal{E}$ is not surjective.

[3]Neither the set $\mathcal{E} \subseteq J^k(\pi)$ nor its prolongation $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ may be expected to be submanifolds in the respective jet spaces. For example, consider the differential equation $\mathcal{E} = \{u_x^2 = u^2\} \subset J^1(\pi)$ which cuts the diagonal cross (i.e. already not a submanifold) in the coordinate plane Ouu_x within $J^1(\pi)$. (It is clear also that the set of solutions to the differential equation $(\frac{\mathrm{d}}{\mathrm{d}x}|_{x_0 - 0} u)^2 = (u(x_0))^2$ on $M^1 = \mathbb{R} \ni x_0$ is immense, compared with the solution sets for the equations $u_x = u$ and $u_x = -u$.) Moreover, should there be *two* independent variables, x and t, so that $\mathcal{E} = \{u_x^2 = u^2\}$ is a *partial* differential equation, then it is readily seen that, parameterised by using infinitely many variables x, t, u, u_t, $u_{tt}, \dots, u_{t\cdots t}, \dots$, the locus $\mathcal{E}^\infty$ is not a submanifold in $J^\infty(\pi)$ as well.

Strange though it may seem, the social request for a $\mathbb{Z}_2$-graded generalisation of Marvan's removability inspection method offers us the most restrictive requirement for the class of PDEs to be studied: they must possess Lie (super-)algebra valued zero-curvature representations—and even parametric families α_λ of such structures.

2.3 Differential Forms

Let us denote by $\bar{D}_{x^i}$ the restrictions of total derivatives D_{x^i} to $\mathcal{E}^\infty \subseteq J^\infty(\pi)$. At every point $\theta^\infty \in \mathcal{E}^\infty$ the tangent space $T_{\theta^\infty}\mathcal{E}^\infty$ splits in a direct sum of two subspaces. The one which is spanned by the Cartan distribution $\mathcal{E}^\infty$ is *horizontal* and the other is *vertical*: $T_{\theta^\infty}\mathcal{E}^\infty = \mathcal{C}_{\theta^\infty} \oplus V_{\theta^\infty}\mathcal{E}^\infty$. We denote by $\Lambda^{1,0}(\mathcal{E}^\infty) = \operatorname{Ann}\mathcal{C}$ and $\Lambda^{0,1}(\mathcal{E}^\infty) = \operatorname{Ann} V\mathcal{E}^\infty$ the $C^\infty(\mathcal{E}^\infty)$-modules of contact and horizontal one-forms which vanish on $\mathcal{C}$ and $V\mathcal{E}^\infty$, respectively. Denote further by $\Lambda^r(\mathcal{E}^\infty)$ the $C^\infty(\mathcal{E}^\infty)$-module of r-forms on $\mathcal{E}^\infty$. There is a natural decomposition $\Lambda^r(\mathcal{E}^\infty) = \bigoplus_{q+p=r} \Lambda^{p,q}(\mathcal{E}^\infty)$, where $\Lambda^{p,q}(\mathcal{E}^\infty) = \bigwedge^p \Lambda^{1,0}(\mathcal{E}^\infty) \wedge \bigwedge^q \Lambda^{0,1}(\mathcal{E}^\infty)$. This implies that the de Rham differential $\bar{\mathrm{d}}$ on $\mathcal{E}^\infty$ is subjected to the decomposition $\bar{\mathrm{d}} = \bar{\mathrm{d}}_h + \bar{\mathrm{d}}_{\mathcal{C}}$, where $\bar{\mathrm{d}}_h \colon \Lambda^{p,q}(\mathcal{E}^\infty) \to \Lambda^{p,q+1}(\mathcal{E}^\infty)$ is the horizontal differential and $\bar{\mathrm{d}}_{\mathcal{C}} \colon \Lambda^{p,q}(\mathcal{E}^\infty) \to \Lambda^{p+1,q}(\mathcal{E}^\infty)$ is the vertical differential.

The differential $\bar{\mathrm{d}}_h$ can be expressed in coordinates by inspection of its action on elements of $C^\infty(\mathcal{E}^\infty) = \Lambda^{0,0}(\mathcal{E}^\infty)$: for any function ϕ we have that

$$\bar{\mathrm{d}}_h\phi = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \bar{D}_{x^i}(\phi), \qquad \bar{\mathrm{d}}_{\mathcal{C}}\phi = \sum_{j=1}^{m_0}\sum_{\sigma_{\bar{0}}} \omega^j_{\sigma_{\bar{0}}} \wedge \frac{\partial\phi}{\partial u^j_{\sigma_{\bar{0}}}} + \sum_{k=1}^{m_1}\sum_{\sigma_{\bar{1}}} \zeta^k_{\sigma_{\bar{1}}} \wedge \frac{\vec{\partial}\phi}{\partial \xi^j_{\sigma_{\bar{1}}}},$$

where we put

$$\omega^j_{\sigma_{\bar{0}}} = \mathrm{d}u^j_{\sigma_{\bar{0}}} - \sum_{j=1}^{n} u^j_{\sigma_{\bar{0}}+1_i}\mathrm{d}x^i, \qquad \zeta^j_{\sigma_{\bar{1}}} = \mathrm{d}\xi^j_{\sigma_{\bar{1}}} - \sum_{i=1}^{n} \xi^j_{\sigma_{\bar{1}}+1_i}\mathrm{d}x^i.$$

The horizontal differential $\bar{\mathrm{d}}_h$ acts on the spaces $\Lambda^{p,q}(\mathcal{E}^\infty)$ of differential forms via the graded Leibniz rule; its application to Cartan's forms $\bar{\mathrm{d}}_{\mathcal{C}}(u^j_\sigma)$ is deduced from the identity $\bar{\mathrm{d}}^2 = 0$ for the de Rham differential $\bar{\mathrm{d}} = \bar{\mathrm{d}}_h + \bar{\mathrm{d}}_{\mathcal{C}}$ on $\mathcal{E}^\infty$. Specifically, from $\bar{\mathrm{d}}_h^2 = \bar{\mathrm{d}}_h \circ \bar{\mathrm{d}}_{\mathcal{C}} + \bar{\mathrm{d}}_{\mathcal{C}} \circ \bar{\mathrm{d}}_h = \bar{\mathrm{d}}_{\mathcal{C}}^2 = 0$ one infers that $\bar{\mathrm{d}}_h \circ \bar{\mathrm{d}}_{\mathcal{C}} = -\bar{\mathrm{d}}_{\mathcal{C}} \circ \bar{\mathrm{d}}_h$, thus reducing the action of $\bar{\mathrm{d}}_h$ to the case where it has already been defined. The formula $\bar{\mathrm{d}}_h = \sum_i \mathrm{d}x^i \wedge \bar{D}_{x^i}$ now means that the vector fields $\bar{D}_{x^i}$ proceed by the Leibniz rule over the argument's wedge factors, acting on each factor—pushed leftmost—via the Lie derivative.

We note further that $\mathrm{d}x^i$, $\mathrm{d}u^j_{\sigma_{\bar{0}}}$, and $\mathrm{d}\xi^k_{\sigma_{\bar{1}}}$ satisfy the following commutation relations:

$$\begin{aligned} &\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i, \quad &&\mathrm{d}x^i \wedge \mathrm{d}u^j_{\sigma_{\bar{0}}} = -\mathrm{d}u^j_{\sigma_{\bar{0}}} \wedge \mathrm{d}x^i, \quad &&\mathrm{d}x^i \wedge \mathrm{d}\xi^k_{\sigma_{\bar{1}}} = -\mathrm{d}\xi^k_{\sigma_{\bar{1}}} \wedge \mathrm{d}x^i, \\ &\mathrm{d}u^j_{\sigma_{\bar{0}}} \wedge \mathrm{d}u^k_{\tau_{\bar{0}}} = -\mathrm{d}u^k_{\tau_{\bar{0}}} \wedge \mathrm{d}u^j_{\sigma_{\bar{0}}}, \quad &&\mathrm{d}\xi^k_{\sigma_{\bar{1}}} \wedge \mathrm{d}u^j_{\tau_{\bar{0}}} = -\mathrm{d}u^j_{\tau_{\bar{0}}} \wedge \mathrm{d}\xi^k_{\sigma_{\bar{1}}}, \quad &&\mathrm{d}\xi^k_{\sigma_{\bar{1}}} \wedge \mathrm{d}\xi^j_{\tau_{\bar{1}}} = +\mathrm{d}\xi^j_{\tau_{\bar{1}}} \wedge \mathrm{d}\xi^k_{\sigma_{\bar{1}}}; \end{aligned}$$

we refer to [53, 54] for the geometric theory of variations in the frames of which one discovers why differential one-forms should anticommute in the $\mathbb{Z}$-graded sense.

The *substitution* of a $\mathbb{Z}_2$-graded vector field X into a $\mathbb{Z}_2$-graded differential form ω is defined by the formula $\mathrm{i}_X(\omega) = (-1)^{\mathrm{p}(X)\cdot\mathrm{p}(\omega)}\omega(X)$, provided that X and ω are both homogeneous with respect to the $\mathbb{Z}_2$-grading. We have that

$$\mathrm{i}_{\bar{D}_{x^i}}\left(\omega^j_{\sigma_{\bar{0}}}\right) = \mathrm{i}_{\bar{D}_{x^i}}\left(\zeta^k_{\sigma_{\bar{1}}}\right) = 0 \quad \text{for all } i, j, k \text{ and } |\sigma| \geqslant 0.$$

These equalities mean that the Cartan distribution can be described equivalently in terms of the Cartan forms $\omega^j_{\sigma_{\bar{0}}}$ and $\zeta^k_{\sigma_{\bar{1}}}$.

2.4 Coverings over Differential Equations

The restriction of Cartan's distribution from $J^\infty(\pi)$ onto $\mathcal{E}^\infty$ is horizontal with respect to the projection $\pi_\infty|_{\mathcal{E}^\infty}: \mathcal{E}^\infty \to M^n$. This determines the connection $\mathcal{C}_{\mathcal{E}^\infty}: \Gamma(TM^n) \to \Gamma(T\mathcal{E}^\infty)$, where $\Gamma(TM^n)$ and $\Gamma(T\mathcal{E}^\infty)$ are the $C^\infty(M^n)$- and $C^\infty(\mathcal{E}^\infty)$-modules of vector fields on M^n and $\mathcal{E}^\infty$, respectively. We denote by $\Gamma T(\Lambda^1(\mathcal{E}^\infty))$ the $C^\infty(\mathcal{E}^\infty)$-module of derivations $C^\infty(\mathcal{E}^\infty) \to \Lambda^1(\mathcal{E}^\infty)$ taking values in the $C^\infty(\mathcal{E}^\infty)$-module of one-forms on $\mathcal{E}^\infty$. The connection form $U_{\mathcal{E}^\infty} \in \Gamma T(\Lambda^1(\mathcal{E}^\infty))$ of $\mathcal{C}_{\mathcal{E}^\infty}$ is called the *structural element* of the equation $\mathcal{E}^\infty$, see (11).

Definition 1 ([44, 55]) A *covering* (or *differential covering*) over a given partial differential (super-)equation $\mathcal{E}$ is another (usually, larger) system of partial differential equations $\tilde{\mathcal{E}}$ endowed with the n-dimensional Cartan distribution $\tilde{\mathcal{C}}$ and such that there is a mapping $\tau: \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ for which at each point $\theta \in \tilde{\mathcal{E}}$, the tangent map $\tau_{*,\theta}$ is an isomorphism of the plane $\tilde{\mathcal{C}}_\theta$ to the Cartan plane $\mathcal{C}_{\tau(\theta)}$ at the point $\tau(\theta)$ in $\mathcal{E}^\infty$.

The construction of a covering over $\mathcal{E}$ means the introduction of new variables such that their compatibility conditions lie inside the initial system $\mathcal{E}^\infty$. In practice (see [48] and references therein), it is the rules to differentiate the new variable(s) which are specified in a consistent way; this implies that those new variables acquire the nature of nonlocalities if their derivatives are local but the variables themselves are not (e.g., consider the potential $\mathfrak{v} = \int u\,\mathrm{d}x$ satisfying $\mathfrak{v}_x = u$ and $\mathfrak{v}_t = -u_{xx} - 3u^2$ for the KdV equation $u_t + u_{xxx} + 6uu_x = 0$). Whenever the covering $\tau: \tilde{\mathcal{E}} \to \mathcal{E}$ is realised as a fibre bundle, the forgetful map τ discards the nonlocalities.

3 (Non)Removability of Parameters in $\mathbb{Z}_2$-Graded Zero-Curvature Representations

In this section we describe an algorithm for inspection of (non)removability of parameters in zero-curvature representations under the action of smooth families of gauge transformations. This technique (and its domain of applicability), which we formulate here for the $\mathbb{Z}_2$-graded set-up of super-equations and Lie super-algebras, patterns upon M. Marvan's approach for the purely bosonic case [22, 23]. We recall that the latter works under the assumption of local analyticity for all the objects and structures involved. We now formulate the most essential half of Marvan's *criterion* of (non)removability; to this end, we explicitly postulate that the admissible families of gauge transformations depend on the parameter in a smooth way. This covers the situations one typically encounters in mathematical physics; the case of smooth families of zero-curvature representations such that the parameter contained in them is removed by using the families of gauge transformations that are *not smooth* is henceforth put aside.

Remark 1 Some agreement on the smoothness class of gauge transformations is always built into the concept of principal fibre bundles and gauge connection one-forms (e.g., those forms which satisfy the Maurer–Cartan flatness equation). Even though the transformations of the wave function Ψ by elements S of the structure Lie (super-) group G are defined pointwise over the base manifold $M^n \ni \boldsymbol{x}$ and therefore, they can be performed discontinuously with respect to the points $\boldsymbol{x}$, this extent of generality is usually not the case—indeed, the gauge set-up is studied only under much more restrictive postulates. In particular, the introduction of gauge connection one-forms requires that both the wave function Ψ and gauge transformations S be (piecewise-) continuously differentiable.

Whenever the principal fibre bundles are towered over partial differential equations—in the context of Lie (super-)algebra $\mathfrak{g}$-valued zero-curvature representations α and the inverse scattering—the smoothness classes of such structures and their gauge transformations are determined from the smoothness classes of equations' solutions $u^i = s^i(\boldsymbol{x})$ in the course of restriction of all the objects at hand to the jets $j_\infty(s^i)(\boldsymbol{x})$ of solutions; the object $\bar{\mathrm{d}}_h S$ in (2) below would be undefined otherwise.

In conclusion, the choice of families $S(\boldsymbol{x}) \in G$ of gauge transformations for the fields $\alpha \in \mathfrak{g} \otimes_{\Bbbk} \Lambda^1(M^n)$ at $\boldsymbol{x} \in M^n$ always refers, either explicitly or tacitly, to some *ad hoc* assumptions on these families' and fields' smoothness. Our technical agreement that the families S_λ and α_λ of *such* structures both depend smoothly on a given parameter $\lambda \in \mathcal{I} \subseteq \mathbb{C}$ fits into the general picture.

Let $\mathcal{E}$ be a partial differential (super-)equation whose infinite prolongation $\mathcal{E}^\infty$ is contained in the infinite jet (super-)bundle $J^\infty(\pi)$; we refer to Sect. 2 for a recollection of concepts and structures that arise in the geometry of jet (super-)spaces (e.g., we refer to that preliminaries chapter for the notions of the space $\bar{\Lambda}(\mathcal{E}^\infty)$ of horizontal differential forms on $\mathcal{E}^\infty$ and the horizontal differential $\bar{\mathrm{d}}_h \colon \bar{\Lambda}(\mathcal{E}^\infty) \to \bar{\Lambda}(\mathcal{E}^\infty)$).

Let G be a finite-dimensional matrix Lie supergroup (i.e., G is a Lie supersubgroup of $GL(k_0|k_1)$ for certain non-negative integers k_0 and k_1). Let $\mathfrak{g}$ be its (matrix) Lie superalgebra. Consider its tensor product $\mathfrak{g} \otimes_{\mathbb{R}} \bar{\Lambda}(\mathcal{E}^\infty)$ with the exterior algebra $\bar{\Lambda}(\mathcal{E}^\infty) = \bigoplus_i \Lambda^{0,i}(\mathcal{E}^\infty)$; by definition, elements of $\mathfrak{g} \otimes C^\infty(\mathcal{E}^\infty)$ are called $\mathfrak{g}$-*(super)matrices* [22].

The product is endowed with the bracket

$$\underline{[}A \otimes \mu, B \otimes \nu\underline{]} = (-1)^{\mathsf{p}(B)\mathsf{p}(\mu)} \underline{[}A, B\underline{]} \otimes \mu \wedge \nu$$

for $\mu, \nu \in \bar{\Lambda}(\mathcal{E}^\infty)$ and $A, B \in \mathfrak{g}$. Define the operator $\bar{\mathrm{d}}_h$ that acts on elements of $\mathfrak{g} \otimes \bar{\Lambda}(\mathcal{E}^\infty)$ by the rule

$$\bar{\mathrm{d}}_h(A \otimes \mu) = A \otimes \bar{\mathrm{d}}_h \mu,$$

where $\bar{\mathrm{d}}_h$ in the right-hand side is the horizontal differential. The tensor product $\mathfrak{g} \otimes \bar{\Lambda}(\mathcal{E}^\infty)$ is a differential graded associative algebra with respect to the multiplication $(A \otimes \mu) \cdot (B \otimes \nu) = (-1)^{\mathsf{p}(B)\mathsf{p}(\mu)} (A \cdot B) \otimes \mu \wedge \nu$ induced by the ordinary matrix multiplication so that

$$\underline{[}\rho, \sigma\underline{]} = \rho \cdot \sigma - (-1)^{rs} (-1)^{\mathsf{p}(\rho)\mathsf{p}(\sigma)} \sigma \cdot \rho,$$

$$\bar{\mathrm{d}}_h(\rho \cdot \sigma) = \bar{\mathrm{d}}_h \rho \cdot \sigma + (-1)^r \rho \cdot \bar{\mathrm{d}}_h \sigma$$

for $\rho \in \mathfrak{g} \otimes \bar{\Lambda}^r(\mathcal{E}^\infty)$ and $\sigma \in \mathfrak{g} \otimes \bar{\Lambda}^s(\mathcal{E}^\infty)$.

Definition 2 ([22, 23, 56]) A horizontal 1-form $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E}^\infty)$ is called a $\mathfrak{g}$-valued *zero-curvature representation* for the equation $\mathcal{E}$ if the Maurer–Cartan condition,

$$\bar{\mathrm{d}}_h \alpha \doteq \frac{1}{2} \underline{[}\alpha, \alpha\underline{]}, \tag{1}$$

holds whenever both sides are restricted to $\mathcal{E}$ and its differential consequences (such restriction is denoted by $\doteq$).

Recall that $\mathfrak{g}$ is the Lie superalgebra of a given Lie supergroup G. Elements of the (pre-)sheaf $C^\infty(\mathcal{E}^\infty, G)$ of G-valued functions on the equation $\mathcal{E}^\infty$ are called *G-matrices*. One can represent elements of $C^\infty(\mathcal{E}^\infty, G)$ as block matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A is a $(k_0 \times k_0)$-size matrix whose entries are even elements of $C^\infty(\mathcal{E}^\infty)$, B is a $(k_0 \times k_1)$-size matrix whose entries are odd elements of $C^\infty(\mathcal{E}^\infty)$, C is a $(k_1 \times k_0)$-size matrix whose entries are odd elements of $C^\infty(\mathcal{E}^\infty)$, and D is a $(k_1 \times k_1)$-size matrix whose entries are even elements of $C^\infty(\mathcal{E}^\infty)$.

Definition 3 Let α and α' be $\mathfrak{g}$-valued zero-curvature representations. Then α and α' are called *gauge-equivalent* if there exists $S \in C^\infty(\mathcal{E}^\infty, G)$ such that

$$\alpha' = \bar{\mathrm{d}}_h S \cdot S^{-1} + S \cdot \alpha \cdot S^{-1} =: \alpha^S. \tag{2}$$

Let α_λ be a family of zero-curvature representations smoothly depending on a parameter $\lambda \in \mathcal{I} \subseteq \mathbb{R}$ (or $\subseteq \mathbb{C}$), where $\mathcal{I}$ is a connected subset. The parameter λ is *removable* under the action of a smooth family of gauge transformations if there exists $\lambda_0 \in \mathcal{I}$ and there is a family of gauge transformation S_λ smoothly depending on λ such that $\alpha_\lambda = \alpha_{\lambda_0}^{S_\lambda}$ for all $\lambda \in \mathcal{I}$.

Remark 2 There are other approaches to the idea of parameters' (non)removability, e.g., under transformations which are not necessarily gauge (this is in contrast to the above definition). It turns out that a given parameter in a family of zero-curvature representations can be nonremovable with respect to the class of smooth gauge transformations but, at the same time, it can be eliminated by using transformations from a wider group. For example, Sasaki showed in [37] that the parameter in the standard Lax pair for the Korteweg–de Vries equation cannot be gauged out but it can be *eliminated* by using the scaling symmetry of KdV (see Appendix B in this paper). We stress that Sasaki's transformation is not gauge and therefore it acts across the gauge group orbits; that parameter is *non-removable* in the sense of Definition 3 because there is no smooth family of gauge transformation which would remove it. We refer to [23] for a discussion about parameters that can be removed by gauge transformation depending on the parameter λ not (only) in a smooth way. We note that in the most interesting examples the nonremovable parameter could be eliminated, see Appendix B and [28] for examples.

Proposition 1 (Cf. [22]) *Let $\mathfrak{g}$ be a complex matrix Lie (super-)algebra. For a connected subset $\mathcal{I} \subseteq \mathbb{C}$, consider a family α_λ, depending smoothly on a parameter $\lambda \in \mathcal{I}$, of $\mathfrak{g}$-valued zero-curvature representations for an equation $\mathcal{E}$. If for each $\lambda \in \mathcal{I}$ there is a $\mathfrak{g}$-matrix Q_λ such that its parity is $\mathsf{p}(Q_\lambda) = \bar{0}$ and*

$$\frac{\partial}{\partial \lambda} \alpha_\lambda = \bar{\mathrm{d}}_h Q_\lambda - [\alpha_\lambda, Q_\lambda], \tag{3}$$

then the parameter λ is removable under the action of a smooth family of gauge transformations.

Proof Suppose that $\dot{\alpha}_\lambda = \bar{\mathrm{d}}_h Q_\lambda - [\alpha_\lambda, Q_\lambda]$ for some $Q_\lambda \in \mathfrak{g} \otimes C^\infty(\mathcal{E}^\infty)$. To begin with, fix a constant $\lambda_0 \in \mathcal{I}$. Let $S_\lambda \in C^\infty(\mathcal{E}^\infty, G)$ be a solution of the matrix equation[4] $\partial S_\lambda / \partial \lambda =$

[4]We recall that $\mathfrak{g}$ is the matrix Lie (super-)algebra of a given matrix Lie (super-)group G, whence the multiplication $Q_\lambda \cdot S_\lambda$ is induced by the ordinary multiplication of (super-)matrices.

$Q_\lambda \cdot S_\lambda$ with the initial datum $S_{\lambda_0} = E$, where E is the identity matrix $\left(\begin{smallmatrix}\mathbf{1} & 0\\ 0 & \mathbf{1}\end{smallmatrix}\right) \in \mathrm{Mat}(k_0|k_1)$. Consider the expression $Z_\lambda = \bar{\mathrm{d}}_h S_\lambda + S_\lambda \alpha_{\lambda_0} - \alpha_\lambda S_\lambda = (\alpha_{\lambda_0}^{S_\lambda} - \alpha_\lambda) S_\lambda$. We have that

$$
\begin{aligned}
\frac{\partial}{\partial\lambda} Z_\lambda &= \frac{\partial}{\partial\lambda}(\bar{\mathrm{d}}_h S_\lambda + S_\lambda \alpha_{\lambda_0} - \alpha_\lambda S_\lambda)\\
&= \bar{\mathrm{d}}_h(\dot{S}_\lambda) + \dot{S}_\lambda \alpha_{\lambda_0} - \dot{\alpha}_\lambda S_\lambda - \alpha_\lambda \dot{S}_\lambda\\
&= \bar{\mathrm{d}}_h(Q_\lambda S_\lambda) + Q_\lambda S_\lambda \alpha_{\lambda_0} - \dot{\alpha}_\lambda S_\lambda - \alpha_\lambda Q_\lambda S_\lambda =\\
&= \bar{\mathrm{d}}_h Q_\lambda S_\lambda + Q_\lambda \bar{\mathrm{d}}_h S_\lambda + Q_\lambda S_\lambda \alpha_{\lambda_0} - \dot{\alpha}_\lambda S_\lambda - \alpha_\lambda Q_\lambda S_\lambda + (Q_\lambda \alpha_\lambda S_\lambda - Q_\lambda \alpha_\lambda S_\lambda)\\
&= (\underbrace{\bar{\mathrm{d}}_h Q_\lambda - \alpha_\lambda Q_\lambda + Q_\lambda \alpha_\lambda}_{\dot{\alpha}_\lambda} - \dot{\alpha}_\lambda) S_\lambda + Q_\lambda(\bar{\mathrm{d}}_h S_\lambda + S_\lambda \alpha_{\lambda_0} - \alpha_\lambda S_\lambda)\\
&= Q_\lambda Z_\lambda.
\end{aligned}
$$

It is obvious that $Z_{\lambda_0} = 0$, whence $\alpha_{\lambda_0}^{S_\lambda} - \alpha_\lambda = 0$. Therefore, the parameter λ is removable by using the explicitly given family S_λ of gauge transformations. □

The above proposition and its proof for *parity-even* $\mathfrak{g}$-matrices Q_λ are a straightforward $\mathbb{Z}_2$-graded generalisation of Marvan's original idea for non-graded PDE systems [22, 23]; the commutator $[\cdot,\cdot]$ in a Lie algebra is now replaced by the graded commutator $[\![\cdot,\cdot]\!]$ in the Lie superalgebra. Note that whenever $\mathsf{p}(Q_\lambda) = \bar{0}$, the parity $\mathsf{p}(Q_\lambda) + \mathsf{p}(S_\lambda)$ of $\dot{S}_\lambda$ in the right-hand side of the equation $\dot{S}_\lambda = Q_\lambda S_\lambda$ is the same as that of S_λ, which agrees with $\mathsf{p}(\lambda) = \bar{0}$.

Remark 3 It is readily seen now *what* could obstruct the converse to be true, which would otherwise convert Proposition 1 into the criterion of (non)removability. Unfortunately, the family S_λ of gauge transformations may not necessarily be (piecewise-)smooth in λ even if the family α_λ is.

Suppose still that the parameter λ in α_λ is removable by a smooth family of G-matrices S_λ (so that the derivatives $\dot{S}_\lambda$ with respect to λ are well defined). This means that for any fixed λ_0 there exists a G-matrix S_λ such that $\alpha_{\lambda_0}^{S_\lambda} = \alpha_\lambda$ with $S_{\lambda_0} = E \in G$, which is viewed here as the set of constant G-valued functions on $\mathcal{E}^\infty$. The matrix $\dot{S}_{\lambda_0} = \partial/\partial\lambda|_{\lambda=\lambda_0} S_\lambda$ belongs to the tangent space at the unit of G, i.e. to the matrix Lie (super-)algebra $\mathfrak{g}$. We have that

$$
\begin{aligned}
0 &= \frac{\partial}{\partial\lambda}\Big|_{\lambda=\lambda_0} \alpha_{\lambda_0} = \frac{\partial}{\partial\lambda}\Big|_{\lambda=\lambda_0} \alpha_\lambda^{S_\lambda^{-1}} = \frac{\partial}{\partial\lambda}\Big|_{\lambda=\lambda_0} \big(\bar{\mathrm{d}}_h\big(S_\lambda^{-1}\big) S_\lambda + S_\lambda^{-1} \alpha_\lambda S_\lambda\big)\\
&= \frac{\partial}{\partial\lambda}\Big|_{\lambda=\lambda_0} \big(-S_\lambda^{-1} \bar{\mathrm{d}}_h S_\lambda + S_\lambda^{-1} \alpha_\lambda S_\lambda\big)\\
&= -\frac{\partial}{\partial\lambda}\big(S_{\lambda_0}^{-1}\big) \bar{\mathrm{d}}_h(S_{\lambda_0}) - S_{\lambda_0}^{-1} d\dot{S}_{\lambda_0} - S_{\lambda_0}^{-1} \dot{S}_{\lambda_0} S_{\lambda_0}^{-1} \alpha_{\lambda_0} S_{\lambda_0}\\
&\quad + S_{\lambda_0}^{-1} \dot{\alpha}_{\lambda_0} S_{\lambda_0} + S_{\lambda_0}^{-1} \alpha_{\lambda_0} \dot{S}_{\lambda_0} = -\bar{\mathrm{d}}_h \dot{S}_{\lambda_0} - \dot{S}_{\lambda_0} \alpha_{\lambda_0} + \alpha_{\lambda_0} \dot{S}_{\lambda_0} + \dot{\alpha}_{\lambda_0}.
\end{aligned}
$$

This implies that $\dot{\alpha}_{\lambda_0} = \bar{\mathrm{d}}_h \dot{S}_{\lambda_0} - [\![\alpha_{\lambda_0}, \dot{S}_{\lambda_0}]\!]$ for all λ_0 in $\mathcal{I}$, where $\dot{S} \in \mathfrak{g} \otimes \bar{\Lambda}^0(\mathcal{E}^\infty)$.

Let us illustrate the technique now offered by Proposition 1 by proving that the parameter in the Das zero-curvature representation for the Mathieu $N = 2$, $a = 4$ super-KdV equation is *essential*.

Consider P. Mathieu's (2|2)-dimensional generalisation of the Korteweg–de Vries equation [1], namely, the $N=2$ supersymmetric KdV equation (SKdV),

$$\boldsymbol{u}_t = -\boldsymbol{u}_{xxx} + 3(\boldsymbol{u}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})_x + \frac{a-1}{2}\big(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}^2\big)_x + 3a\boldsymbol{u}^2\boldsymbol{u}_x, \qquad \mathcal{D}_i = \frac{\vec{\partial}}{\partial\theta_i} + \theta_i\cdot\bar{D}_x, \quad (4)$$

where

$$\boldsymbol{u}(x,t;\theta_1,\theta_2) = u_0(x,t) + \theta_1\cdot u_1(x,t) + \theta_2\cdot u_2(x,t) + \theta_1\theta_2\cdot u_{12}(x,t) \quad (5)$$

is the complex bosonic super-field, θ_1,θ_2 are Grassmann variables such that $\theta_1^2=\theta_2^2=\theta_1\theta_2+\theta_2\theta_1=0$, the two fields u_0 and u_{12} are bosonic ($\mathsf{p}(u_0)=\mathsf{p}(u_{12})=\bar{0}$), and the fields u_1 and u_2 are fermionic ($\mathsf{p}(u_1)=\mathsf{p}(u_2)=\bar{1}$). Expansion (5) converts (4) to the four-component system

$$u_{0;t} = -u_{0;xxx} + \big(au_0^3 - (a+2)u_0u_{12} + (a-1)u_1u_2\big)_x, \quad (6a)$$

$$u_{1;t} = -u_{1;xxx} + \big((a+2)u_0u_{2;x} + (a-1)u_{0;x}u_2 - 3u_1u_{12} + 3au_0^2u_1\big)_x, \quad (6b)$$

$$u_{2;t} = -u_{2;xxx} + \big(-(a+2)u_0u_{1;x} - (a-1)u_{0;x}u_1 - 3u_2u_{12} + 3au_0^2u_2\big)_x, \quad (6c)$$

$$\underline{u_{12;t}} = \underline{-u_{12;xxx} - 6u_{12}u_{12;x}} + 3au_{0;x}u_{0;xx} + (a+2)u_0u_{0;xxx} + 3u_1u_{1;xx} + 3u_2u_{2;xx} + 3a\big(u_0^2u_{12} - 2u_0u_1u_2\big)_x. \quad (6d)$$

The Korteweg–de Vries equation upon u_{12}, see (14) below, is underlined in (6d). The SKdV equation is the most interesting (in particular, bi-Hamiltonian, whence completely integrable) if $a\in\{-2,1,4\}$, see [1, 2, 57]. Let us consider also the bosonic limit $u_1=u_2=0$ of system (6a)–(6d): by setting $a=-2$ we obtain the triangular system which consists of the modified KdV equation for u_0 and an equation of KdV-type for u_{12}; in the case $a=1$ we obtain the Krasil'shchik–Kersten system [8]; for $a=4$, we obtain the third equation in the Kaup–Boussinesq hierarchy. In what follows we consider the case $a=4$.

Example 1 The $N=2$ supersymmetric $a=4$-KdV equation (6a)–(6d) admits the $\mathfrak{sl}(2|1)$-valued zero-curvature representation $\alpha^{N=2}(\varepsilon) = A(\varepsilon)\mathrm{d}x + B(\varepsilon)\mathrm{d}t$, where

$$A = \begin{pmatrix} -\boldsymbol{i}u_0 & \varepsilon^{-1}(u_0^2+u_{12}) - \boldsymbol{i}\varepsilon^{-2}u_0 & -\varepsilon^{-1}(u_2+\boldsymbol{i}u_1) \\ -\varepsilon & -\boldsymbol{i}u_0 - \varepsilon^{-1} & 0 \\ 0 & \boldsymbol{i}u_1 - u_2 & -2\boldsymbol{i}u_0 - \varepsilon^{-1} \end{pmatrix}, \quad \varepsilon > 0. \quad (7)$$

The elements of the $\mathfrak{sl}(2|1)$-matrix B,

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

are as follows (see Remark 4 below),

$$b_{11} = 4\boldsymbol{i}u_0^3 - 6\boldsymbol{i}u_0u_{12} + 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} - u_{12;x} - 4\boldsymbol{i}u_2u_1 + \varepsilon^{-1}\big(2u_0^2 - u_{12} - \boldsymbol{i}u_{0;x}\big) - \boldsymbol{i}\varepsilon^{-2}u_0,$$

$$
\begin{aligned}
b_{12} &= \varepsilon^{-1}\big(4u_0^4 + 2u_0^2u_{12} + 4u_0u_{0;xx} - 2u_{12}^2 + 4u_{0;x}^2 - u_{12;xx} + u_2u_{2;x} + 8u_2u_1u_0 + u_1u_{1;x}\big) \\
&\quad + \varepsilon^{-2}\big(2\boldsymbol{i}u_0^3 - 4\boldsymbol{i}u_0u_{12} + 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} - u_{12;x} - 2\boldsymbol{i}u_2u_1\big) \\
&\quad + \varepsilon^{-3}\big(u_0^2 - u_{12} - \boldsymbol{i}u_{0;x}\big) - \boldsymbol{i}\varepsilon^{-4}u_0, \\
b_{13} &= \varepsilon^{-1}\big(-5\boldsymbol{i}u_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} + \boldsymbol{i}u_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} - 4\boldsymbol{i}u_2u_{0;x} - 8\boldsymbol{i}u_1u_0^2 \\
&\quad + 2\boldsymbol{i}u_1u_{12} - 4u_1u_{0;x}\big) + \varepsilon^{-2}(-u_{2;x} + \boldsymbol{i}u_{1;x} - 3\boldsymbol{i}u_2u_0 - 3u_1u_0) + \varepsilon^{-3}(-u_2 + \boldsymbol{i}u_1), \\
b_{21} &= 2\varepsilon\big(-2u_0^2 + u_{12}\big) + 2\boldsymbol{i}u_0 + \varepsilon^{-1}, \\
b_{22} &= 4\boldsymbol{i}u_0^3 - 6\boldsymbol{i}u_0u_{12} - 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} + u_{12;x} - 4\boldsymbol{i}u_2u_1 + \varepsilon^{-1}\big(-2u_0^2 + u_{12} + \boldsymbol{i}u_{0;x}\big) \\
&\quad + \boldsymbol{i}\varepsilon^{-1}u_0 + \varepsilon^{-3}, \\
b_{23} &= u_{2;x} - \boldsymbol{i}u_{1;x} + 4\boldsymbol{i}u_2u_0 + 4u_1u_0 + \varepsilon^{-1}(u_2 - \boldsymbol{i}u_1), \\
b_{31} &= \varepsilon(-u_{2;x} - \boldsymbol{i}u_{1;x} + 4\boldsymbol{i}u_2u_0 - 4u_1u_0) + u_2 + \boldsymbol{i}u_1, \\
b_{32} &= 5\boldsymbol{i}u_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} - \boldsymbol{i}u_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} + 4\boldsymbol{i}u_2u_{0;x} + 8\boldsymbol{i}u_1u_0^2 \\
&\quad - 2\boldsymbol{i}u_1u_{12} - 4u_1u_{0;x} + \varepsilon^{-1}u_0(\boldsymbol{i}u_2 - u_1), \\
b_{33} &= 2\big(4\boldsymbol{i}u_0^3 - 6\boldsymbol{i}u_0u_{12} - \boldsymbol{i}u_{0;xx} - 4\boldsymbol{i}u_2u_1\big) + \varepsilon^{-3}.
\end{aligned}
$$

Proposition 2 *There is no* $\mathfrak{sl}(2|1)$*-matrix*

$$
Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{11} + q_{22} \end{pmatrix}
$$

which would depend on ε *and satisfy the equalities*

$$
\bar{D}_x(Q) = \frac{\partial}{\partial\varepsilon}A + \underline{[}A, Q\underline{]}, \tag{8a}
$$

$$
\bar{D}_t(Q) = \frac{\partial}{\partial\varepsilon}B + \underline{[}B, Q\underline{]}. \tag{8b}
$$

In other words, the parameter ε *in* $\alpha^{N=2}(\varepsilon)$ *cannot be removed by using a smooth family of gauge transformations.*

An analytic proof of Proposition 2 is contained in Appendix A.

Remark 4 This zero-curvature representation $\alpha^{N=2}(\varepsilon)$ is not equal identically but it is gauge-equivalent to the respective formula in Das et al. [26]. The transformation between these objects contains the imaginary unit $\boldsymbol{i}$. Our choice of normalisation, which is the same as in [3], is due to the following argument: all the structures under study contain the Gardner deformation (15a)–(15b) of the Korteweg–de Vries equation (14) (so that the structures retract to Gardner's deformation [4, 5] under suitable reductions).

We note further that the zero-curvature representation $\alpha^{N=2}$ can be used for construction of a solution, which is an alternative to the first solution reported in [2], of Gardner's deformation problem [1, 7] for the $N = 2$, $a = 4$ SKdV equation (we refer to [3] for details). The parameter ε which we use here is the parameter in the classical Gardner deformation of the KdV equation [4]. This is why we denote this parameter by ε instead of λ.

Example 2 Consider now another $\mathfrak{sl}(2|1)$-valued zero-curvature representation $\beta = A\mathrm{d}x + B\mathrm{d}t$ for the $N=2$, $a=4$-SKdV equation: we let

$$A = \begin{pmatrix} \lambda - \boldsymbol{i}u_0 & -\lambda^2 - (u_0^2 + u_{12}) & -\boldsymbol{i}u_1 - u_2 \\ 1 & -\lambda - \boldsymbol{i}u_0 & 0 \\ 0 & u_2 - \boldsymbol{i}u_1 & -2\boldsymbol{i}u_0 \end{pmatrix}.$$

The elements of the $\mathfrak{sl}(2|1)$-matrix B,

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

are given by the formulas

$$\begin{aligned}
b_{11} &= 2\lambda\big(2u_0^2 - u_{12}\big) - 4\boldsymbol{i}u_0^3 + 6\boldsymbol{i}u_0u_{12} + 4u_0u_{0;x} + \boldsymbol{i}u_{0;xx} - u_{12;x} + 4\boldsymbol{i}u_2u_1,\\
b_{12} &= 2\lambda^2\big(-2u_0^2 + u_{12}\big) + 2\lambda(-4u_0u_{0;x} + u_{12;x}) - 4u_0^4 - 2u_0^2u_{12} - 4u_0u_{0;xx} + 2u_{12}^2\\
&\quad - 4u_{0;x}^2 + u_{12;xx} - u_2u_{2;x} - 8u_2u_1u_0 - u_1u_{1;x},\\
b_{13} &= \lambda(u_{2;x} + \boldsymbol{i}u_{1;x} - 4\boldsymbol{i}u_2u_0 + 4u_1u_0) - 5\boldsymbol{i}u_0u_{2;x} + 5u_0u_{1;x} + u_{2;xx} + \boldsymbol{i}u_{1;xx}\\
&\quad - 8u_2u_0^2 + 2u_2u_{12} - 4\boldsymbol{i}u_2u_{0;x} - 8\boldsymbol{i}u_1u_0^2 + 2\boldsymbol{i}u_1u_{12} + 4u_1u_{0;x},\\
b_{21} &= 2\big(2u_0^2 - u_{12}\big),\\
b_{22} &= 2\lambda\big(-2u_0^2 + u_{12}\big) - 4\boldsymbol{i}u_0^3 + 6\boldsymbol{i}u_0u_{12} - 4u_0u_{0;x} + \boldsymbol{i}u_{0;xx} + u_{12;x} + 4\boldsymbol{i}u_2u_1,\\
b_{23} &= u_{2;x} + \boldsymbol{i}u_{1;x} - 4\boldsymbol{i}u_2u_0 + 4u_1u_0,\\
b_{31} &= u_{2;x} - \boldsymbol{i}u_{1;x} + 4\boldsymbol{i}u_2u_0 + 4u_1u_0,\\
b_{32} &= \lambda(-u_{2;x} + \boldsymbol{i}u_{1;x} - 4\boldsymbol{i}u_2u_0 - 4u_1u_0) - 5\boldsymbol{i}u_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} + \boldsymbol{i}u_{1;xx} + 8u_2u_0^2\\
&\quad - 2u_2u_{12} - 4\boldsymbol{i}u_2u_{0;x} - 8\boldsymbol{i}u_1u_0^2 + 2\boldsymbol{i}u_1u_{12} - 4u_1u_{0;x},\\
b_{33} &= 2\boldsymbol{i}\big(-4u_0^3 + 6u_0u_{12} + u_{0;xx} + 4u_2u_1\big).
\end{aligned}$$

The $\mathfrak{sl}(2|1)$-matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfies the equations

$$\frac{\partial}{\partial\lambda}A = \bar{D}_x(Q) - \underline{[}A, Q\underline{]}, \qquad \frac{\partial}{\partial\lambda}B = \bar{D}_t(Q) - \underline{[}B, Q\underline{]}.$$

Solving the Cauchy problem

$$\frac{\partial}{\partial\lambda}S = QS, \quad S|_{\lambda=0} = \mathbf{1},$$

we obtain the $SL(2|1)$-matrix

$$S = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix S defines the—obviously, smooth with respect to λ—family of gauge transformations that remove the parameter λ from the zero-curvature representation β, i.e. $(\beta)^{S^{-1}} = \beta|_{\lambda=0}$. Consequently, the parameter λ in β is removable.

Remark 5 Marvan's computational and moreover, cohomological techniques from [22, 23] seem to be working really fine in the $\mathbb{Z}_2$-graded set-up.[5] However, let us say a word of caution.

Lemma 3 *Let* $\alpha = \alpha^{\bar{0}} + \alpha^{\bar{1}}$ *be a* $\underline{\mathfrak{g}}$*-valued zero-curvature representation of a given* $\mathbb{Z}_2$*-graded equation* $\mathcal{E}$ *such that* $\mathsf{p}(\alpha^{\bar{0}}) = \bar{0}$ *and* $\mathsf{p}(\alpha^{\bar{1}}) = \bar{1}$. *Then Marvan's operator* $\bar{\boldsymbol{\partial}}_\alpha = \bar{\mathrm{d}}_h - \underline{[}\alpha, \cdot\underline{]}$ *is not necessarily a differential.*

We refer to Eq. (19) below and to the papers [22, 50] for more details on the nature and use of the mapping $\bar{\boldsymbol{\partial}}_\alpha$.

Proof Let $\beta \in \mathfrak{g} \otimes \bar{\Lambda}^0(\mathcal{E}^\infty)$ so that $\beta = \beta^{\bar{0}} + \beta^{\bar{1}}$ and consider $\alpha = \alpha^{\bar{0}} + \alpha^{\bar{1}}$, where $\mathsf{p}(\alpha^{\bar{0}}) = \mathsf{p}(\beta^{\bar{0}}) = \bar{0}$ and $\mathsf{p}(\alpha^{\bar{1}}) = \mathsf{p}(\beta^{\bar{1}}) = \bar{1}$. Then we have that

$$
\begin{aligned}
&\bar{\boldsymbol{\partial}}_\alpha \circ \bar{\boldsymbol{\partial}}_\alpha(\beta)\\
&\quad = \bar{\boldsymbol{\partial}}_\alpha\big(\bar{\mathrm{d}}_h\beta - \underline{[}\alpha, \beta\underline{]}\big) = \bar{\mathrm{d}}_h \circ \bar{\mathrm{d}}_h\beta - \bar{\mathrm{d}}_h\big(\underline{[}\alpha, \beta\underline{]}\big) - \underline{\big[}\alpha, \bar{\mathrm{d}}_h\beta - \underline{[}\alpha, \beta\underline{]}\,\underline{\big]}\\
&\quad = -\underline{[}\bar{\mathrm{d}}_h\alpha, \beta\underline{]} + \underline{[}\alpha, \bar{\mathrm{d}}_h\beta\underline{]} - \underline{[}\alpha, \bar{\mathrm{d}}_h\beta\underline{]} + \underline{\big[}\alpha, \underline{[}\alpha, \beta\underline{]}\,\underline{\big]} = \underline{\big[}\alpha, \underline{[}\alpha, \beta\underline{]}\,\underline{\big]} - \frac{1}{2}\underline{\big[}\underline{[}\alpha, \alpha\underline{]}, \beta\underline{\big]}\\
&\quad = \underline{\big[}\alpha, \alpha^{\bar{0}}\beta^{\bar{0}} - \beta^{\bar{0}}\alpha^{\bar{0}} + \alpha^{\bar{0}}\beta^{\bar{1}} - \beta^{\bar{1}}\alpha^{\bar{0}} + \alpha^{\bar{1}}\beta^{\bar{0}} - \beta^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\beta^{\bar{1}} + \beta^{\bar{1}}\alpha^{\bar{1}}\underline{\big]}\\
&\qquad - \underline{\big[}\alpha^{\bar{0}}\alpha^{\bar{0}} + \alpha^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{0}}, \beta\underline{\big]}\\
&\quad = \alpha^{\bar{0}}\alpha^{\bar{0}}\beta^{\bar{0}} + \alpha^{\bar{0}}\beta^{\bar{0}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\beta^{\bar{0}}\alpha^{\bar{0}} - \beta^{\bar{0}}\alpha^{\bar{0}}\alpha^{\bar{0}} + \alpha^{\bar{0}}\alpha^{\bar{0}}\beta^{\bar{1}} + \alpha^{\bar{0}}\beta^{\bar{1}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\beta^{\bar{1}}\alpha^{\bar{0}} - \beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{0}}\\
&\qquad + \alpha^{\bar{0}}\alpha^{\bar{1}}\beta^{\bar{0}} + \alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\beta^{\bar{0}}\alpha^{\bar{1}} - \beta^{\bar{0}}\alpha^{\bar{1}}\alpha^{\bar{0}} + \alpha^{\bar{0}}\alpha^{\bar{1}}\beta^{\bar{1}} + \alpha^{\bar{1}}\beta^{\bar{1}}\alpha^{\bar{0}} + \alpha^{\bar{1}}\alpha^{\bar{0}}\beta^{\bar{0}} + \alpha^{\bar{0}}\beta^{\bar{0}}\alpha^{\bar{1}}\\
&\qquad - \alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{0}} - \beta^{\bar{0}}\alpha^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{0}}\beta^{\bar{1}} - \alpha^{\bar{0}}\beta^{\bar{1}}\alpha^{\bar{1}} - \alpha^{\bar{1}}\beta^{\bar{1}}\alpha^{\bar{0}} - \beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{1}}\beta^{\bar{0}} - \alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{1}}\\
&\qquad - \alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{1}} + \beta^{\bar{0}}\alpha^{\bar{1}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{1}}\beta^{\bar{1}} + \alpha^{\bar{1}}\beta^{\bar{1}}\alpha^{\bar{1}} - \alpha^{\bar{0}}\alpha^{\bar{0}}\beta^{\bar{0}} + \beta^{\bar{0}}\alpha^{\bar{0}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\alpha^{\bar{0}}\beta^{\bar{1}} + \beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{0}}\\
&\qquad - \alpha^{\bar{1}}\alpha^{\bar{0}}\beta^{\bar{0}} + \beta^{\bar{0}}\alpha^{\bar{1}}\alpha^{\bar{0}} - \alpha^{\bar{1}}\alpha^{\bar{0}}\beta^{\bar{1}} - \beta^{\bar{1}}\alpha^{\bar{1}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\alpha^{\bar{1}}\beta^{\bar{0}} + \beta^{\bar{0}}\alpha^{\bar{0}}\alpha^{\bar{1}} - \alpha^{\bar{0}}\alpha^{\bar{1}}\beta^{\bar{1}} - \beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{1}}\\
&\quad = -\alpha^{\bar{0}}\beta^{\bar{1}}\alpha^{\bar{1}} - 2\beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{1}}\beta^{\bar{0}} - 2\alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{1}} + \beta^{\bar{0}}\alpha^{\bar{1}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{1}}\beta^{\bar{1}} + \alpha^{\bar{1}}\beta^{\bar{1}}\alpha^{\bar{1}} - \beta^{\bar{1}}\alpha^{\bar{1}}\alpha^{\bar{0}}\\
&\quad \neq 0.
\end{aligned}
$$

This argument shows that only for *parity-even* zero-curvature representations (which are constrained by $\alpha^{\bar{1}} = 0$) is the operator $\bar{\boldsymbol{\partial}}_\alpha$ always a differential and does Marvan's horizontal cohomology interpretation [22] work in the $\mathbb{Z}_2$-graded set-up. □

[5] The *horizontal cohomology* groups introduced by Marvan in [22, 23] are informative for the algebraic approach to kinematic integrability, yet they may be hard to compute (in fact, this has not been attempted industrially). It is the removability of "fake" parameters in the zero-curvature representations which must be focused on first; whenever it is established that a parameter cannot be removed in a smooth way from a smooth family, the integration of PDE under study by using the inverse scattering [15, 16] should be attempted as the proper next step (or a nontrivial Gardner deformation of that system be derived from the family of zero-curvature representations, and the integrals of motion be constructed).

Remark 6 We have not seen any nontrivial example of zero-curvature representations with nonzero odd part (i.e. such that $\alpha^{\bar{1}} \neq 0$). It would be interesting to either find such example or prove that it cannot exist.

Let us remember that Proposition 1 and Lemma 3 can be used to construct parametric families α_λ of zero-curvature representations for an equation $\mathcal{E}$. In [23], the horizontal gauge cohomology complex $\bar{H}^q_{\alpha_\lambda}(\mathcal{E}, \mathfrak{g})$ was associated with every such family. It is standard that the first horizontal gauge cohomology group $\bar{H}^1_{\alpha_\lambda}(\mathcal{E}, \mathfrak{g})$ contains the obstructions to removability of a parameter λ (cf. Sect. 4 above and [27, 28]).

Example 3 Using the technique described in [23], let us examine an $\mathfrak{sl}(2|1)$-valued zero-curvature representation for the equation which found by Tian and Liu (Case F in [32], see also [33]):

$$u_t = u_{5x} + 2auu_{xxx} + 4au_xu_{xx} + \frac{6a^2}{5}u^2u_x - a\xi_{xxx}\xi_x + \frac{3a^2}{5}u\xi_{xx}\xi + \frac{3a^2}{5}u_x\xi_x\xi, \quad \text{(9a)}$$

$$\xi_t = \xi_{5x} + 2au\xi_{xxx} + 3au_x\xi_{xx} + au_{xx}\xi_x + \frac{3a^2}{5}u^2\xi_x + \frac{3a^2}{5}uu_x\xi, \quad (a = 5); \quad \text{(9b)}$$

the parities are $\mathsf{p}(u) = \bar{0}$ and $\mathsf{p}(\xi) = \bar{1}$.

We start from the non-parametric zero-curvature representation $\alpha_0^{\text{5ord}} = A_0\mathrm{d}x + B_0\mathrm{d}t$ for system (9a)–(9b):

$$A_0 = \begin{pmatrix} 0 & -u & \xi \\ 1 & 0 & 0 \\ 0 & -\xi & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 6uu_x + u_{xxx} + 3\xi\xi_{xxx} & b_{12} & b_{13} \\ 6u^2 + 2u_{xx} + 6\xi\xi_x & -6uu_x - u_{xxx} - 3\xi\xi_{xx} & b_{23} \\ 9u\xi_x + \xi_{xxx} - 3\xi u_x & -9u\xi_{xx} - \xi_{4x} - 6\xi_xu_x - 6\xi u^2 + \xi u_{xx} & 0 \end{pmatrix},$$

where

$$b_{12} = -6u^3 - 8uu_{xx} - u_{4x} - 6u_x^2 + 3\xi_{xx}\xi_x + 3\xi u\xi_x - 2\xi\xi_{xxx},$$
$$b_{13} = 9u\xi_{xx} + \xi_{4x} + 6\xi_xu_x + 6\xi u^2 - \xi u_{xx},$$
$$b_{23} = 9u\xi_x + \xi_{xxx} - 3\xi u_x.$$

This $\mathfrak{sl}(2|1)$-valued zero-curvature representation can easily be obtained by using two ideas: (*i*) it should reduce to the standard $\mathfrak{sl}(2)$-valued zero-curvature representation of the Korteweg–de Vries equation under a reduction of (9a) to its higher symmetry; (*ii*) the sought-for structure is expected to be homogeneous with respect to the tuple of scaling weights (see [2, 58] in this context).

Let us recall from [23] that for a given $\mathfrak{g}$-valued zero-curvature representation $\alpha_0^{\text{5ord}} = A_0\mathrm{d}x + B_0\mathrm{d}t$, the $\mathfrak{g}$-valued 1-form $A_1\mathrm{d}x + B_1\mathrm{d}t$ is a $\bar{\boldsymbol{\partial}}_\alpha$-cocycle if and only if the block matrices

$$A^{[1]} = \begin{pmatrix} A_0 & 0 \\ A_1 & A_0 \end{pmatrix}, \qquad B^{[1]} = \begin{pmatrix} B_0 & 0 \\ B_1 & B_0 \end{pmatrix}$$

constitute a zero-curvature representation, i.e.

$$\bar{D}_t A^{[1]} - \bar{D}_x B^{[1]} + \big[A^{[1]}, B^{[1]}\big] = 0.$$

Moreover, two cocyles differ by a coboundary if and only if the respective zero-curvature representations are gauge equivalent with respect to a gauge matrix of the form

$$S^{[1]} = \begin{pmatrix} E & 0 \\ S & E \end{pmatrix},$$

with the unit matrix E at the diagonal and some matrix S.

Using this condition and the analytic software `SsTools` [58], we found the cocycle $A_1 \mathrm{d}x + B_1 \mathrm{d}t$, where

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10a}$$

$$B_1 = \begin{pmatrix} 6u^2 + 2u_{xx} + 6\xi\xi_x & -12uu_x - 2u_{xxx} - 6\xi\xi_{xx} & 9u\xi_x + \xi_{xxx} - 3\xi u_x \\ 0 & -6u^2 - 2u_{xx} - 6\xi\xi_x & 0 \\ 0 & -9u\xi_x - \xi_{xxx} + 3\xi u_x & 0 \end{pmatrix}, \tag{10b}$$

for the non-parametric $\mathfrak{sl}(2|1)$-valued zero-curvature representation α_0^{5ord} of system (9a)–(9b). Thus we extend the zero-curvature representation α_0^{5ord} to the $\mathfrak{sl}(2|1)$-valued zero-curvature representation $\alpha^{\text{5ord}} = A\mathrm{d}x + B\mathrm{d}t$ with a parameter λ, here

$$A = \begin{pmatrix} \lambda & -u - \lambda^2 & \xi \\ 1 & -\lambda & 0 \\ 0 & -\xi & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & -b_{11} & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix},$$

where

$$b_{11} = 2\lambda\big(3u^2 + u_{xx} + 3\xi\xi_x\big) + 6uu_x + u_{xxx} + 3\xi\xi_{xx},$$

$$b_{12} = 2\lambda^2\big(-3u^2 - u_{xx} - 3\xi\xi_x\big) + 2\lambda(-6uu_x - u_{xxx} - 3\xi\xi_{xx}) - 6u^3 - 8uu_{xx} - u_{4x} - 6u_x^2$$
$$+ 3\xi_{xx}\xi_x + 3\xi u\xi_x - 2\xi\xi_{xxx},$$

$$b_{13} = \lambda(9u\xi_x + \xi_{xxx} - 3\xi u_x) + 9u\xi_{xx} + \xi_{4x} + 6\xi_x u_x + 6\xi u^2 - \xi u_{xx},$$

$$b_{21} = 2\big(3u^2 + u_{xx} + 3\xi\xi_x\big),$$

$$b_{23} = 9u\xi_x + \xi_{xxx} - 3\xi u_x,$$

$$b_{31} = 9u\xi_x + \xi_{xxx} - 3\xi u_x,$$

$$b_{32} = \lambda(-9u\xi_x - \xi_{xxx} + 3\xi u_x) - 9u\xi_{xx} - \xi_{4x} - 6\xi_x u_x - 6\xi u^2 + \xi u_{xx}.$$

But is this parameter λ (non)removable?

It is easy to see that the gauge transformation $S^{[1]}$ given by

$$S^{[1]} = \begin{pmatrix} E & 0 \\ S & E \end{pmatrix}, \quad \text{where } S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

trivialises cocycle (10a)–(10b)! Namely, we have that

$$\left(A^{[1]}\right)^{S^{[1]}} = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix}, \qquad \left(B^{[1]}\right)^{S^{[1]}} = \begin{pmatrix} B_0 & 0 \\ 0 & B_0 \end{pmatrix}.$$

Therefore, cocycle (10a)–(10b) cannot be used to insert a nonremovable parameter in the zero-curvature representation $\alpha_0^{5\text{ord}}$.

We deduce that the above family of zero-curvature representations $\alpha^{5\text{ord}}$ does not manifest that Eq. (9a)–(9b) indeed is integrable.[6] This is why in Example 7 below we shall consider much larger, $\mathfrak{sl}(9|8)$-valued zero-curvature representations for (9a)–(9b). We argue that the parameter λ in that family is not removable using gauge transformations.

4 Families of Coverings and the Frölicher–Nijenhuis Bracket

In this section we recall another mechanism for deforming nonlocal structures over partial differential (super-)equations $\mathcal{E}$. Referring to the notion of Frölicher–Nijenhuis bracket, this concept is better known in the context of Bäcklund (auto)transformations between PDEs, see [28, 30, 59]. The two deformation strategies from the preceding and present sections will be brought together in what follows.

4.1 The Structure Element of a Covering

Consider a $(k_0|k_1)$-dimensional covering $\tau\colon \tilde{\mathcal{E}} = W \times \mathcal{E}^\infty \to \mathcal{E}^\infty$ with even nonlocal coordinates $w^1, \ldots, w^{k_0}$ and odd nonlocal coordinates $f^1, \ldots, f^{k_1}$ on the $(k_0|k_1)$-dimensional fibre superspace W, see Definition 1. The prolongations $\tilde{D}_{x^i}$ of the total derivatives $\bar{D}_{x^i}$ to the covering equation $\tilde{\mathcal{E}}$ are given by the formulas [44, 47]

$$\tilde{D}_{x^i} = \bar{D}_{x^i} + w^p_{x^i} \frac{\partial}{\partial w^p} + f^q_{x^i} \frac{\vec{\partial}}{\partial f^q}, \qquad 1 \leqslant i \leqslant n.$$

These total derivatives $\tilde{D}_{x^i}$ determine the Cartan distribution $\mathcal{C}(\tilde{\mathcal{E}})$ on the covering equation $\tilde{\mathcal{E}}$. In turn, the Cartan distribution $\mathcal{C}(\tilde{\mathcal{E}})$ yields the Cartan connection $\mathcal{C}_{\tilde{\mathcal{E}}}\colon \Gamma(TM) \to \Gamma(T\tilde{\mathcal{E}})$ such that $\mathcal{C}_{\tilde{\mathcal{E}}}\colon \partial/\partial x^i \mapsto \tilde{D}_{x^i}$. Using those differential one-forms from $\Lambda^1(\tilde{\mathcal{E}})$ which annihilate the horizontal n-dimensional planes of the Cartan distribution $\mathcal{C}(\tilde{\mathcal{E}})$, we obtain the linear $\Gamma(T\tilde{\mathcal{E}})$-valued connection form $U_{\tilde{\mathcal{E}}} \in \Gamma T(\Lambda^1(\tilde{\mathcal{E}}))$, also called the *structure element* of the covering τ in this setting.

Specifically, for $\mathcal{A} := C^\infty(M)$, for the properly understood inductive limit $\mathcal{B} := C^\infty(\tilde{\mathcal{E}})$ of algebras filtered by the jet orders, and embedding homomorphism $\varphi = (\pi \circ \tau)^*\colon$

[6] At the same time, we do not claim that no non-removable parameter can be inserted into the $\mathfrak{sl}(2|1)$-valued zero-curvature representation $\alpha_0^{5\text{ord}}$. Indeed, to establish that one must prove the vanishing of the respective gauge cohomology group.

$\mathcal{A} \hookrightarrow \mathcal{B}$, it is readily seen that the composition $\mathrm{d}_{\mathrm{dR}} \circ \varphi \colon \mathcal{A} \to \Lambda^1(\mathcal{B})$ is a derivation. Consequently, we can consider the derivation $\mathcal{C}_{\tilde{\mathcal{E}}}(\mathrm{d}_{\mathrm{dR}} \circ \varphi) \in \Gamma T(\Lambda^1(\tilde{\mathcal{E}}))$. The structure element $U_{\tilde{\mathcal{E}}} \in \Gamma T(\Lambda^1(\tilde{\mathcal{E}}))$ of the covering τ is defined by the formula $U_{\tilde{\mathcal{E}}} = -\mathcal{C}_{\tilde{\mathcal{E}}}(\mathrm{d}_{\mathrm{dR}} \circ \varphi) + \mathrm{d}_{\mathrm{dR}}$ (see [27–29] and references therein). In coordinates, we have that[7]

$$U_{\tilde{\mathcal{E}}} = \bar{\mathrm{d}}_{\mathcal{C}}\big(u^k_{\sigma_{\bar{0}}}\big)\frac{\partial}{\partial u^k_{\sigma_{\bar{0}}}} + \bar{\mathrm{d}}_{\mathcal{C}}\big(\xi^a_{\sigma_{\bar{1}}}\big)\frac{\vec{\partial}}{\partial \xi^a_{\sigma_{\bar{1}}}} + \big(\mathrm{d}w^p - w^p_{x^i}\,\mathrm{d}x^i\big)\frac{\partial}{\partial w^p} + \big(\mathrm{d}f^q - f^q_{x^i}\,\mathrm{d}x^i\big)\frac{\vec{\partial}}{\partial f^q}. \quad (11)$$

The Cartan connection on $\tilde{\mathcal{E}}$ is flat:

$$[U_{\tilde{\mathcal{E}}}, U_{\tilde{\mathcal{E}}}]^{\mathrm{FN}} = 2 \times \text{curvature} = 0;$$

we recall that the Frölicher–Nijenhuis bracket $[\cdot,\cdot]^{\mathrm{FN}}$ on the space $\Gamma T(\Lambda^*(\tilde{\mathcal{E}}))$ of vector-valued differential forms is defined by the formula [47]

$$[\Omega, \Theta]^{\mathrm{FN}}(g) = \mathrm{L}_{\Omega}\big(\Theta(g)\big) - (-1)^{rs + \mathrm{p}(\Omega)\mathrm{p}(\Theta)}\,\mathrm{L}_{\Theta}\big(\Omega(g)\big)$$

for $\Omega \in \Gamma T(\Lambda^r(\tilde{\mathcal{E}}))$, $\Theta \in \Gamma T(\Lambda^s(\tilde{\mathcal{E}}))$, and $g \in C^\infty(\tilde{\mathcal{E}})$. Here $\mathrm{L}_{\Omega} = \mathrm{i}_{\Omega} \circ \mathrm{d} + \mathrm{d} \circ \mathrm{i}_{\Omega}$ is the Lie derivative.

Let $\tau_\lambda \colon \tilde{\mathcal{E}}_\lambda = W_\lambda \times \mathcal{E}^\infty \to \mathcal{E}^\infty$ be a smooth family of coverings over $\mathcal{E}^\infty$ depending on a parameter $\lambda \in \mathbb{C}$ and U_λ be the corresponding structure element of τ_λ. In agreement with [28], we assume that the distributions $\mathcal{C}(\tilde{\mathcal{E}}_\lambda)$ are diffeomorphic to each other at different values of λ under a smooth family of diffeomorphisms of the manifolds $\tilde{\mathcal{E}}_\lambda$. The evolution of U_λ with respect to λ is described by the equation [27, 28]

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} U_\lambda = [X, U_\lambda]^{\mathrm{FN}}, \quad (12)$$

where $X \in \Gamma(T\tilde{\mathcal{E}})$ is some vector field on $\tilde{\mathcal{E}}_\lambda$.

4.2 Two Realisations of Lie Superalgebras

In the covering τ which has been considered so far, the superdimension of the fibre W is $(k_0|k_1)$; the covering is realised in terms of vector fields on the fibres. To narrow the class of vector field subalgebras at hand—in particular, to force such vector fields belong to a given Lie algebra $\mathfrak{g}$ that appeared in Sect. 3 in the construction of $\mathfrak{g}$-valued zero-curvature representations—let us recall two techniques of matrix vs (non)linear vector field realisations of Lie algebras. We have that elements $\rho(g)$ of matrix representation for $g \in \mathfrak{g}$ act by endomorphisms on a vector superspace of superdimension $(k_0 + 1|k_1)$. The excess of parity-even dimension allows us to view the respective graded Cartesian coordinates as homogeneous coordinates on projective superspaces W such that vector fields $\varrho(g) \in \Gamma(TW)$ on them belong to the other representation, $\varrho \colon \mathfrak{g} \to \mathrm{Vect}(W; \mathrm{poly})$, of the Lie algebra $\mathfrak{g}$.

Specifically, let $\mathfrak{g} \subseteq \mathfrak{gl}(k_0 + 1|k_1)$ be a finite-dimensional Lie superalgebra with basis e_i, here $k_0, k_1 \geqslant 0$ and the index i runs from 1 to the dimension of $\mathfrak{g}$. We consider two representation of $\mathfrak{g}$:

[7] The notion of Cartan differential $\mathrm{d}_{\mathcal{C}}$ and its restriction to equations $\mathcal{E}^\infty$ is recalled in §2.3.

(1) $\rho\colon \mathfrak{g} \to \mathrm{Mat}(k_0+1, k_1)$, that is, a matrix representation;
(2) $\boldsymbol{\varrho}\colon \mathfrak{g} \to \mathrm{Vect}(W; \mathrm{poly})$, which is the representation in the space of vector fields with polynomial coefficients on the $(k_0|k_1)$-dimensional supermanifold W with local parity-even coordinates $w^1, \ldots, w^{k_0}$ and $f^1, \ldots, f^{k_1}$ of odd parity.

Let us recall an explicit construction of such representations $\boldsymbol{\varrho}$; it will be essential in what follows.

Example 4 (Nonlinear realisations of Lie algebras in the spaces of vector fields via the projective substitution [20]) Let N be a $(k_0+1|k_1)$-dimensional manifold. Because the reasoning is local, consider a chart $\mathcal{U} \subseteq N$ equipped with a (k_0+k_1+1)-tuple of rectifying coordinates $\boldsymbol{v} = (v^0, \ldots, v^{k_0+k_1})$, where $v^0, \ldots, v^{k_0}$ are even coordinates and $v^{k_0+1}, \ldots, v^{k_0+k_1}$ are odd coordinates. By definition, put

$$\partial_{\boldsymbol{v}} = (\vec{\partial}_{v^0}, \ldots, \vec{\partial}_{v^{k_0+k_1}})^{\mathrm{t}}.$$

For the matrix Lie superalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(k_0+1|k_1)$ at hand, take any matrix $g \in \mathfrak{g}$ and represent it in the space of *linear* vector fields on the domain in $\mathbb{R}^{k_0+1|k_1}$ by using the formula

$$g \longmapsto V_g = \boldsymbol{v} g \partial_{\boldsymbol{v}}.$$

By construction, the linear vector field representation $g \mapsto V_g$ of the matrix Lie algebra $\mathfrak{g}$ preserves all the commutation relations in it,

$$[V_g, V_h] = [\boldsymbol{v} g \partial_{\boldsymbol{v}}, \boldsymbol{v} h \partial_{\boldsymbol{v}}] = \boldsymbol{v} \underline{[}g, h\underline{]} \partial_{\boldsymbol{v}} = V_{\underline{[}g,h\underline{]}}, \quad \forall h, g \in \mathfrak{g}.$$

The problem we are solving is the realisation of matrix Lie superalgebra $\mathfrak{g}$ by using vector fields with (non)linear coefficients. Consider a point $\boldsymbol{x} \in \mathbb{R}^{k_0+1|k_1}$—originally, from the chart $\mathcal{U} \subseteq N$—with nonzero coordinate $v^0(\boldsymbol{x}) =: \mu \neq 0$. By construction, $\boldsymbol{v} = (v^0, \ldots, v^{k_0+k_1})$ is the tuple of Cartesian coordinates on the image of $\mathcal{U}$ under the coordinate mapping. Consider the locally defined mapping $p\colon \boldsymbol{v}(\mathcal{U}) \subseteq \mathbb{R}^{k_0+1|k_1} \to \mathbb{R}^{k_0|k_1}$ that takes every point $\boldsymbol{v} = (v^0, \ldots, v^{k_0+k_1})$ from the domain at hand to the point $(w^1, \ldots, w^{k_0+k_1}) \in \mathbb{R}^{k_0|k_1}$, where

$$w^i = \frac{\mu v^i}{v^0}, \quad 1 \leqslant i \leqslant k_0 + k_1.$$

The differential $\mathrm{d}p_{\boldsymbol{v}}\colon T_{\boldsymbol{v}}\mathbb{R}^{k_0+1|k_1} \to T_{p(\boldsymbol{v})}\mathbb{R}^{k_0|k_1}$ at the point $\boldsymbol{v} \in \mathbb{R}^{k_0+1|k_1}$ acts on the basic vectors from the (k_0+k_1+1)-tuple $\partial_{\boldsymbol{v}}$ as follows,

$$\mathrm{d}p_{\boldsymbol{v}}\left(\frac{\partial}{\partial v^0}\right) = \sum_{j=1}^{k_0+k_1} \frac{\partial w^j}{\partial v^0} \frac{\partial}{\partial w^j} = \sum_{j=1}^{k_0+k_1} -\frac{\mu v^j}{(v^0)^2} \frac{\partial}{\partial w^j},$$

$$\mathrm{d}p_{\boldsymbol{v}}\left(\frac{\partial}{\partial v^i}\right) = \sum_{j=1}^{k_0+k_1} \frac{\partial w^j}{\partial v^i} \frac{\partial}{\partial w^j} = \frac{\mu}{v^0} \frac{\partial}{\partial w^i}, \quad 1 \leqslant i \leqslant k_0 + k_1.$$

By definition, put

$$\boldsymbol{w} = \big(\mu, w^1, \ldots, w^{k_0+k_1}\big), \qquad \partial_{\boldsymbol{w}} = \left(-\frac{1}{\mu} \sum_{j=1}^{k_0+k_1} w^j \frac{\vec{\partial}}{\partial w^j}, \frac{\vec{\partial}}{\partial w^1}, \ldots, \frac{\vec{\partial}}{\partial w^{k_0+k_1}}\right)^{\mathrm{t}}.$$

Now it is readily seen that the vector field $X_g = \mathrm{d}p(V_g)$ is expressed by the formula

$$X_g = \boldsymbol{w} g \partial_{\boldsymbol{w}}. \tag{13}$$

Generally speaking, the vector field X_g on the respective subset of the target space $\mathbb{R}^{k_0|k_1}$ is nonlinear with respect to the variables $w^0, \ldots, w^{k_0+k_1}$. Nevertheless, the commutation relations between vector fields of such type are inherited from the relations in Lie algebra $\mathfrak{g} \ni g, h$:

$$[X_g, X_h] = \big[\mathrm{d}p(V_g), \mathrm{d}p(V_h)\big] = \mathrm{d}p\big([V_g, V_h]\big) = \mathrm{d}p(V_{\underline{[}g,h\underline{]}}) = X_{\underline{[}g,h\underline{]}}.$$

Take X_g for the representation $\boldsymbol{\varrho}(g)$ of elements g of Lie superalgebra $\mathfrak{g}$.

Notation Whenever $\mathfrak{g} \subseteq \mathfrak{gl}(n_{\bar{0}}|n_{\bar{1}})$ is a finite-dimensional Lie superalgebra, it admits the tautological matrix representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(n_{\bar{0}}|n_{\bar{1}})$ and realisation $\boldsymbol{\varrho}$ in spaces of vector fields from Example 4. We denote by $\nabla\colon \rho \rightleftarrows \boldsymbol{\varrho}$ the switch between these representations.

We refer to [18, 19] for other examples of realisations of Lie algebras by using vector fields.

4.3 Examples of the Structure Element Deformation

Example 4 yields a regular procedure that takes Lie super-algebra $\mathfrak{g}$-valued zero-curvature representations to certain $(k_0|k_1)$-dimensional coverings over the $\mathbb{Z}_2$-graded PDE at hand; the coefficients of Maurer–Cartan's horizontal one-form α determine the rules to differentiate the nonlocal variables with respect to the independent coordinates such as x and t. Let us see what the deformation mechanism of Frölicher–Nijenhuis bracket can then do for such coverings—and let us inspect what the vector fields X in (12) mean in terms of the PDE under study.

Example 5 Consider the $N = 2$, $a = 4$ SKdV equation (6a)–(6d) and the family of coverings over it derived from the zero-curvature representation which we addressed in Example 1. Let us find the vector field X corresponding to that family. We are interested in finding those solutions of (12) which do not degenerate under the reduction of (6a)–(6d) to its bosonic limit. In turn, for that system of KdV-type we are interested in finding only those solutions of (12) which stem from the deformation generators X for the Korteweg–de Vries equation for u_{12}; we recall that the KdV equation is contained in the bosonic limit of (6a)–(6d) by Mathieu's construction.

This allows us to analyse the structural element's deformation problem "inside-out" in three steps: first, we do it for the Korteweg–de Vries equation; we proceed with the Kaup–Boussinesq hierarchy and finally, we recover the full (2|2)-dimensional supergeometry of the $N = 2$ supersymmetric $a = 4$-equation (6a)–(6d).

We begin with the Korteweg–de Vries equation

$$u_{12;t} = -u_{12;xxx} - 6u_{12}u_{12;x}. \tag{14}$$

Over it, we consider the covering derived from the Gardner deformation [4],

$$w_x = \frac{1}{\varepsilon}(w - u_{12}) - \varepsilon w^2, \tag{15a}$$

$$w_t = \frac{1}{\varepsilon}\big(u_{12;xx} + 2u_{12}^2\big) + \frac{1}{\varepsilon^2}u_{12;x} + \frac{1}{\varepsilon^3}u_{12} + \left(-2u_{12;x} - \frac{2}{\varepsilon}u_{12} - \frac{1}{\varepsilon^3}\right)w + \left(2\varepsilon u_{12} + \frac{1}{\varepsilon}\right)w^2. \tag{15b}$$

The Cartan structure element for the covering (15a)–(15b) is as follows,

$$U_{\tilde{\mathcal{E}}} = \bar{\mathrm{d}}_{\mathcal{C}}(u_{12;\sigma})\frac{\partial}{\partial u_{12;\sigma}} + (\mathrm{d}w - w_x\mathrm{d}x - w_t\mathrm{d}t)\frac{\partial}{\partial w},$$

where w_x and w_t are given by (15a)–(15b). The solutions of Eq. (12) for this case are

$$X_1 = \varepsilon^{-2}(-x\partial/\partial x - 3t\partial/\partial t + 2u_{12}\partial/\partial u_{12} + \cdots + 2w\partial/\partial w),$$
$$X_2 = -2\varepsilon(6t\partial/\partial x + \partial/\partial u_{12} + \cdots) - \partial/\partial w.$$

Let us show that the vector field X_1 can be lifted to the corresponding covering over the higher symmetry,

$$u_{0;t} = -u_{0;xxx} + \big(4u_0^3 - 6u_0u_{12}\big)_x, \tag{16a}$$

$$u_{12;t} = -u_{12;xxx} - 6u_{12}u_{12;x} + 12u_{0;x}u_{0;xx} + 6u_0u_{0;xxx} + 12\big(u_0^2u_{12}\big)_x, \tag{16b}$$

of the Kaup–Boussinesq equation [60]

$$u_{0;s} = \big(-u_{12} + 2u_0^2\big)_x, \qquad u_{12;s} = (u_{0;xx} + 4u_0u_{12})_x.$$

We recall that system (16a)–(16b) is the bosonic limit of (6a)–(6d) with $a = 4$ under setting $u_1 = u_2 = 0$.

A family of coverings over Eq. (16a)–(16b) is determined by the formulas[8]

$$\underline{w_x} = \underline{-\varepsilon w^2} + \varepsilon^{-1}\big(\underline{w - u_{12}} - u_0^2\big) + \boldsymbol{i}\varepsilon^{-2}u_0, \tag{17a}$$

$$\underline{w_t} = 2\varepsilon w^2\big(-2u_0^2 + \underline{u_{12}}\big) + 2w(-\boldsymbol{i}wu_0 + 4u_0u_{0;x}\underline{-u_{12;x}}) + \varepsilon^{-1}\big(\underline{w^2 - 2wu_{12} + 2u_{12}^2} \underline{+u_{12;xx}} + 2\boldsymbol{i}wu_{0;x} - 4u_0^4 - 2u_0^2u_{12} - 4u_0u_{0;xx} + 4wu_0^2 - 4u_{0;x}^2\big) + \varepsilon^{-2}\big(2\boldsymbol{i}wu_0 + 2\boldsymbol{i}u_0^3 - 4\boldsymbol{i}u_0u_{12} - 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} + \underline{u_{12;x}}\big) + \varepsilon^{-3}\big(\underline{u_{12} - w} - u_0^2 - \boldsymbol{i}u_{0;x}\big) - \boldsymbol{i}\varepsilon^{-4}u_0. \tag{17b}$$

The Cartan structure element for the covering (17a)–(17b) is as follows,

$$U_{\tilde{\mathcal{E}}} = \bar{\mathrm{d}}_{\mathcal{C}}(u_{0;\sigma})\frac{\partial}{\partial u_{0;\sigma}} + \bar{\mathrm{d}}_{\mathcal{C}}(u_{12;\sigma})\frac{\partial}{\partial u_{12;\sigma}} + (\mathrm{d}w - w_x\mathrm{d}x - w_t\mathrm{d}t)\frac{\partial}{\partial w},$$

[8] Here and in what follows we underline the covering that encodes Gardner's deformation (15a)–(15b) for the classical KdV equation (14).

where the derivatives w_x and w_t are given by formulas (17a)–(17b) and σ is a summation index. At every $\varepsilon \neq 0$ such coverings are constructed by the change of Lie algebra realisation ∇ in the zero-curvature representation for (16a)–(16b). In turn, that representation[9] is obtained by using the reduction $u_1 = u_2 = 0$ in the zero-curvature representation $\alpha^{N=2}$ for the $N = 2$, $a = 4$ SKdV equation (6a)–(6d) (see [3, 26] and Example 1).

For this family of coverings over system (16a)–(16b), the solution of Eq. (12) is given by the vector field

$$X = \varepsilon^{-1}(-x\partial/\partial x - 3t\partial/\partial t + u_0\partial/\partial u_0 + 2u_{12}\partial/\partial u_{12} + \cdots + 2w\partial/\partial w).$$

We note that, the same as it is in the case of KdV equation (14), we obtain the vector field corresponding to the scaling symmetry (see Appendix B; we refer to diagram (29) in particular).

Finally, let us consider the full $N = 2$, $a = 4$ SKdV equation (6a)–(6d) and over it, let us construct a $(1|1)$-dimensional covering by switching to a different realisation of the Lie superalgebra in the $\mathfrak{sl}(2|1)$-valued zero-curvature representation α, which was considered in Example 1. Using the realisation $\boldsymbol{\varrho}$ from Example 4, we obtain the $(1|1)$-dimensional covering over $N = 2$, $a = 4$ SKdV equation (6a)–(6d):

$$\begin{aligned}
\underline{w_x} &= \underline{-\varepsilon w^2} + (f_2 u_2 - \boldsymbol{i} f_2 u_1) + \varepsilon^{-1}\big(\underline{w - u_{12}} - u_0^2\big) + \varepsilon^{-2}\boldsymbol{i} u_0,\\
f_x &= -\varepsilon w f_2 + \boldsymbol{i} u_0 f_2 + \varepsilon^{-1}(f_2 + u_2 + \boldsymbol{i} u_1),\\
\underline{w_t} &= \varepsilon\big(-4w^2u_0^2 + \underline{2w^2u_{12}} + f_2 w u_{2;x} - \boldsymbol{i} f_2 w u_{1;x} + 4\boldsymbol{i} f_2 u_2 w u_0 + 4 f_2 u_1 w u_0\big) - 2\boldsymbol{i} w^2 u_0\\
&\quad + 8wu_0u_{0;x} \underline{-2wu_{12;x}} - 5\boldsymbol{i} f_2 u_0 u_{2;x} - 5 f_2 u_0 u_{1;x} - f_2 u_{2;xx} + \boldsymbol{i} f_2 u_{1;xx} - f_2 u_2 w\\
&\quad + 8 f_2 u_2 u_0^2 - 2 f_2 u_2 u_{12} - 4\boldsymbol{i} f_2 u_2 u_{0;x} + \boldsymbol{i} f_2 u_1 w - 8\boldsymbol{i} f_2 u_1 u_0^2 + 2\boldsymbol{i} f_2 u_1 u_{12} - 4 f_2 u_1 u_{0;x}\\
&\quad + \varepsilon^{-1}\big(\underline{w^2 + 2u_{12}^2 - 2wu_{12} + u_{12;xx}} + 4wu_0^2 + 2\boldsymbol{i} w u_{0;x} - 4u_0^4 - 2u_0^2u_{12} - 4u_0u_{0;xx}\\
&\quad - 4u_{0;x}^2 - \boldsymbol{i} f_2 u_2 u_0 - f_2 u_1 u_0 - u_2 u_{2;x} - 8u_2u_1u_0 - u_1u_{1;x}\big) + \varepsilon^{-2}\big(2\boldsymbol{i} w u_0 + 2\boldsymbol{i} u_0^3\\
&\quad - 4\boldsymbol{i} u_0 u_{12} - 4u_0u_{0;x} - \boldsymbol{i} u_{0;xx} + \underline{u_{12;x}} - 2\boldsymbol{i} u_2 u_1\big) + \varepsilon^{-3}\big(\underline{u_{12} - w} - u_0^2 - \boldsymbol{i} u_{0;x}\big)\\
&\quad - \varepsilon^{-4}\boldsymbol{i} u_0,\\
f_t &= 2\varepsilon w\big(-2f_2u_0^2 + f_2u_{12}\big) + wu_{2;x} + \boldsymbol{i} w u_{1;x} - 2\boldsymbol{i} f_2 w u_0 + 4\boldsymbol{i} f_2 u_0^3 - 6\boldsymbol{i} f_2 u_0 u_{12}\\
&\quad + 4 f_2 u_0 u_{0;x} - \boldsymbol{i} f_2 u_{0;xx} - f_2 u_{12;x} - 4\boldsymbol{i} f_2 u_2 u_1 - 4\boldsymbol{i} u_2 w u_0 + 4u_1 w u_0\\
&\quad + \varepsilon^{-1}\big(5\boldsymbol{i} u_0 u_{2;x} - 5u_0u_{1;x} - u_{2;xx} - \boldsymbol{i} u_{1;xx} + f_2 w + 2 f_2 u_0^2 - f_2 u_{12} + \boldsymbol{i} f_2 u_{0;x} + u_2 w\\
&\quad + 8u_2u_0^2 - 2u_2u_{12} + 4\boldsymbol{i} u_2 u_{0;x} + \boldsymbol{i} u_1 w + 8\boldsymbol{i} u_1 u_0^2 - 2\boldsymbol{i} u_1 u_{12} - 4u_1u_{0;x}\big)\\
&\quad + \varepsilon^{-2}(-u_{2;x} - \boldsymbol{i} u_{1;x} + \boldsymbol{i} f_2 u_0 + 3\boldsymbol{i} u_2 u_0 - 3u_1u_0) - \varepsilon^{-3}(f_2 + u_2 + \boldsymbol{i} u_1).
\end{aligned}$$

[9] Remarkably, that zero-curvature representation for (16a)–(16b) was re-discovered in [61] not in the context of super-system (6a)–(6d).

The Cartan structure element is, therefore,

$$U_{\tilde{\mathcal{E}}} = \bar{\mathrm{d}}_{\mathcal{C}}(u_{0;\sigma})\frac{\partial}{\partial u_{0;\sigma}} + \bar{\mathrm{d}}_{\mathcal{C}}(u_{1;\sigma})\frac{\partial}{\partial u_{1;\sigma}} + \bar{\mathrm{d}}_{\mathcal{C}}(u_{2;\sigma})\frac{\partial}{\partial u_{2;\sigma}} + \bar{\mathrm{d}}_{\mathcal{C}}(u_{12;\sigma})\frac{\partial}{\partial u_{12;\sigma}}$$
$$+ (\mathrm{d}f - f_x\mathrm{d}x - f_t\mathrm{d}t)\frac{\partial}{\partial f} + (\mathrm{d}w - w_x\mathrm{d}x - w_t\mathrm{d}t)\frac{\partial}{\partial w}.$$

The reduction $u_1 = u_2 = 0$ maps this covering over $N = 2$, $a = 4$ SKdV equation to the covering over higher symmetry (16a)–(16b) of the Kaup–Boussinesq equation, see above. We lift the solution of Eq. (12) for the covering over higher symmetry (16a)–(16b) of the Kaup–Boussinesq equation to the covering over the $N = 2$, $a = 4$ SKdV equation. That solution of Eq. (12) is the vector field

$$X = \varepsilon^{-1}\Big(-x\partial/\partial x - 3t\partial/\partial t + u_0\partial/\partial u_0 + \frac{3}{2}u_1\partial/\partial u_1 + \frac{3}{2}u_2\partial/\partial u_2$$
$$+ 2u_{12}\partial/\partial u_{12} + \cdots + 2w\partial/\partial w + \frac{3}{2}f\partial/\partial f\Big).$$

It has been computed by solving Eq. (12) explicitly with the help of analytic software [58].

We note again that—as we had it in the above two reductions of the $N = 2$, $a = 4$-SKdV—we obtain the vector field corresponding to the scaling symmetry of the underlying equation.

Remark 7 For scaling-invariant families of coverings depending on a parameter the homogeneity weight of which is not equal to zero, one could always try to find a solution of Eq. (12) by properly extending the scaling symmetry of the underlying PDE, cf. [28, 30].

5 Zero-Curvature Representations and Coverings

In this section we bring together the gauge geometry of zero-curvature representations from Sect. 3 and the deformation of coverings as described in Sect. 4. In particular, we study the relation between the construction of parametric families of zero-curvature representations and deformation of the corresponding nonlocalities by using the Frölicher–Nijenhuis bracket. We let the geometry stay in the context of (super-)KdV type systems; therefore, we let $x^1 = x$ and $x^2 = t$ be the two independent variables.

Let $\alpha = a^i\rho(e_i)\mathrm{d}x + b^j\rho(e_j)\mathrm{d}t$ be a $\mathfrak{g}$-valued zero-curvature representation for the system $\mathcal{E}$. Over its infinite prolongation $\mathcal{E}^\infty$, construct a $(k_0|k_1)$-dimensional covering with the k_0 parity-even and k_1 parity-odd nonlocal fibre variables w^ℓ such that $\mathsf{p}(w^\ell) = \bar{0}$ if $1 \leqslant \ell \leqslant k_0$ and $\mathsf{p}(w^\ell) = \bar{1}$ if $\ell > k_0$ and such that

$$w^\ell_x = -a^i\boldsymbol{\varrho}(e_i)\lrcorner\,\mathrm{d}w^\ell, \tag{18a}$$
$$w^\ell_t = -b^j\boldsymbol{\varrho}(e_j)\lrcorner\,\mathrm{d}w^\ell, \quad \ell = 1\ldots(k_0+k_1). \tag{18b}$$

Consider two mappings:

$$\boldsymbol{\partial}_\alpha = \bar{\mathrm{d}}_h - [\alpha, \cdot]\colon \mathfrak{g}\otimes\Lambda^0(\mathcal{E}^\infty) \to \mathfrak{g}\otimes\Lambda^1(\mathcal{E}^\infty), \tag{19}$$

see [22] and [50], and

$$\partial_U = [\cdot, U_\lambda]^{\mathrm{FN}} \colon \varGamma T\big(\varLambda^0(\tilde{\mathcal{E}})\big) \to \varGamma T\big(\varLambda^1(\tilde{\mathcal{E}})\big),$$

see [28, 29]. We recall that the mappings $\boldsymbol{\partial}_\alpha$ and ∂_U yield the horizontal [22] and Cartan [28] cohomologies, respectively. However, we claim that in the geometry at hand one of these two differentials is a particular instance of the other by virtue of the switch $\rho \rightleftarrows \boldsymbol{\varrho}$ between the Lie superalgebra representations.

Lemma 4 *The following diagram is commutative*:

$$\begin{array}{ccccc}
\mathfrak{g}\otimes\varLambda^0(\mathcal{E}^\infty) & \xrightarrow{\ \rho\ } & \mathrm{Mat}(k_0+1|k_1)\otimes\varLambda^0(\mathcal{E}^\infty) & \xrightarrow{\ \boldsymbol{\partial}_\alpha\ } & \mathrm{Mat}(k_0+1|k_1)\otimes\varLambda^1(\mathcal{E}^\infty) \\
\Big\| & & \Big\downarrow{\scriptstyle\nabla} & & \Big\downarrow{\scriptstyle\nabla} \\
\mathfrak{g}\otimes\varLambda^0(\mathcal{E}^\infty) & \xrightarrow{\ \boldsymbol{\varrho}\ } & \varGamma T(\varLambda^0(\tilde{\mathcal{E}})) & \xrightarrow{\ \partial_U\ } & \varGamma T(\varLambda^1(\tilde{\mathcal{E}})),
\end{array}$$

where $\nabla = \boldsymbol{\varrho}\circ\rho^{-1}$ is a switch from the representation ρ to the representation $\boldsymbol{\varrho}$ for the Lie superalgebra $\mathfrak{g}$.

Proof Consider any $\gamma = q^k\cdot e_k \in \mathfrak{g}\otimes\varLambda^0(\mathcal{E}^\infty)$ and put $\boldsymbol{\varrho}(\gamma) = X \in \varGamma T(\varLambda^0(\tilde{\mathcal{E}}))$ and $\rho(\gamma) = Q \in \mathrm{Mat}(k_0+1|k_1)\otimes\varLambda^0(\mathcal{E}^\infty)$. A direct calculation shows that

$$\begin{aligned}
(\nabla\circ\boldsymbol{\partial}_\alpha\circ\rho)(\gamma) &= (\nabla\circ\boldsymbol{\partial}_\alpha)(Q) = \nabla\big(\bar{\mathrm{d}}_h Q - \underline{[}\alpha, Q\underline{]}\big) \\
&= \nabla(\mathrm{d}x\big(\bar{D}_x\big(q^k\big)\rho(e_k) - \underline{\big[}a^i\rho(e_i), q^k\rho(e_k)\underline{\big]}\big) + \mathrm{d}t\big(\bar{D}_t\big(q^k\big)\rho(e_k) \\
&\quad - \underline{\big[}b^j\rho(e_j), q^k\rho(e_k)\underline{\big]}\big) \\
&= \mathrm{d}x\big(\bar{D}_x\big(q^k\big)\boldsymbol{\varrho}(e_k) + \big[-a^i\boldsymbol{\varrho}(e_i), q^k\boldsymbol{\varrho}(e_k)\big]\big) + \mathrm{d}t\big(\bar{D}_t\big(q^k\big)\boldsymbol{\varrho}(e_k) \\
&\quad + \big[-b^i\boldsymbol{\varrho}(e_i), q^k\boldsymbol{\varrho}(e_k)\big]\big).
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
(\partial_U\circ\boldsymbol{\varrho})(\gamma) &= \partial_U X = [X, U_\lambda]^{\mathrm{FN}} \\
&= \bigg[\mathrm{d}x\bigg(\tilde{D}_x\big(X\lrcorner\,\mathrm{d}w^\ell\big) - \big(X\lrcorner\,\mathrm{d}w^s\big)\frac{\partial w_x^\ell}{\partial w^s}\bigg) \\
&\quad + \mathrm{d}t\bigg(\tilde{D}_t\big(X\lrcorner\,\mathrm{d}w^\ell\big) - \big(X\lrcorner\,\mathrm{d}w^s\big)\frac{\partial w_t^\ell}{\partial w^s}\bigg)\bigg]\otimes\frac{\partial}{\partial w^\ell}.
\end{aligned}$$

By using the formula $\tilde{D}_x(X\lrcorner\,\mathrm{d}w^\ell) = \bar{D}_x(X\lrcorner\,\mathrm{d}w^\ell) + w_x^s\frac{\partial}{\partial w^s}(X\lrcorner\,\mathrm{d}w^\ell)$, we continue the equality and obtain that

$$\begin{aligned}
&= \bigg[\mathrm{d}x\bigg(\bar{D}_x\big(X\lrcorner\,\mathrm{d}w^\ell\big) + w_x^s\frac{\partial}{\partial w^s}\big(X\lrcorner\,\mathrm{d}w^\ell\big) - \big(X\lrcorner\,\mathrm{d}w^s\big)\frac{\partial w_x^\ell}{\partial w^s}\bigg) \\
&\quad + \mathrm{d}t\bigg(\bar{D}_t\big(X\lrcorner\,\mathrm{d}w^\ell\big) + w_t^s\frac{\partial}{\partial w^s}\big(X\lrcorner\,\mathrm{d}w^\ell\big) - \big(X\lrcorner\,\mathrm{d}w^s\big)\frac{\partial w_t^\ell}{\partial w^s}\bigg)\bigg]\otimes\frac{\partial}{\partial w^\ell}
\end{aligned}$$

$$= \Big[\mathrm{d}x \Big(\bar{D}_x(X) \lrcorner \mathrm{d}w^\ell + \Big[w_x^s \frac{\partial}{\partial w^s}, X \Big] \lrcorner \mathrm{d}w^\ell \Big) + \mathrm{d}t \Big(\bar{D}_t(X) \lrcorner \mathrm{d}w^\ell + \Big[w_t^s \frac{\partial}{\partial w^s}, X \Big] \lrcorner \mathrm{d}w^\ell \Big) \Big] \otimes \frac{\partial}{\partial w^\ell}.$$

Recall that the covering at hand is obtained by switching between Lie algebra's representations. From formulas (18a)–(18b) we infer that the equality continues,

$$= \big[\mathrm{d}x \big(\bar{D}_x(q^k) \varrho(e_k) + [-a^i \varrho(e_i), q^k \varrho(e_k)] \big) \lrcorner \mathrm{d}w^\ell + \mathrm{d}t \big(\bar{D}_t(q^k) \varrho(e_k) + [-b^i \varrho(e_i), q^k \varrho(e_k)] \big) \lrcorner \mathrm{d}w^\ell \big] \otimes \frac{\partial}{\partial w^\ell}$$
$$= \mathrm{d}x \big(\bar{D}_x(q^k) \varrho(e_k) + [-a^i \varrho(e_i), q^k \varrho(e_k)] \big) + \mathrm{d}t \big(\bar{D}_t(q^k) \varrho(e_k) + [-b^i \varrho(e_i), q^k \varrho(e_k)] \big).$$

We finally obtain that $(\nabla \circ \boldsymbol{\partial}_\alpha \circ \rho)(\gamma) = (\partial_U \circ \varrho)(\gamma)$ for all γ, which proves our claim. □

Now let us study in more detail the case of *removable* parameters. Let $\alpha(\lambda) = a^i \rho(e_i)\mathrm{d}x + b^j \rho(e_j)\mathrm{d}t$ be a smooth family of $\mathfrak{g}$-valued zero-curvature representations for the system $\mathcal{E}$ but let the parameter $\lambda \in \mathcal{I} \subseteq \mathbb{C}$ be removable by using a smooth family of gauge transformations S_λ. By Remark 3, there are $\mathfrak{g}$-matrices $Q = q^k \rho(e_k)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\alpha = \bar{\mathrm{d}}_h Q - [\alpha, Q].$$

In components, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(a^i)\rho(e_i) = \bar{D}_x(q^k)\rho(e_k) - a^i q^k \rho([e_i, e_k]), \tag{20a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(b^j)\rho(e_i) = \bar{D}_t(q^k)\rho(e_k) - b^j q^k \rho([e_j, e_k]). \tag{20b}$$

By virtue of the representation ϱ, at every λ the $\mathfrak{g}$-matrix $Q = q^k \rho(e_k)$ determines the vector field $X = q^k \varrho(e_k)$ on $\tilde{\mathcal{E}}$.

The following proposition is a regular generator of solutions for Eq. (12) in the case of coverings derived from zero-curvature representations with removable parameters. It was remarked in [27] that the formalism of zero-curvature representations and their parametric families can be viewed as a special case of the Frölicher–Nijenhuis bracket formalism for deformations of coverings of generic nature; we thus substantiate that claim from *loc. cit.* by giving an explicit proof.

Proposition 5 *The vector field $X = q^k \varrho(e_k)$ satisfies structure equation* (12).

Proof From Lemma 4 we infer that

$$[X, U_\lambda]^{\mathrm{FN}} = \big[\mathrm{d}x \big(\bar{D}_x(q^k) \varrho(e_k) + [-a^i \varrho(e_i), q^k \varrho(e_k)] \big) \lrcorner \mathrm{d}w^\ell + \mathrm{d}t \big(\bar{D}_t(q^k) \varrho(e_k) + [-b^i \varrho(e_i), q^k \varrho(e_k)] \big) \lrcorner \mathrm{d}w^\ell \big] \otimes \frac{\partial}{\partial w^\ell}.$$

Using (20a)–(20b), we obtain that

$$\left[\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}\lambda}\big(a^i\big)\big(\varrho(e_i)\,\lrcorner\,\mathrm{d}w^\ell\big)+\mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}\lambda}\big(b^i\big)\big(\varrho(e_i)\,\lrcorner\,\mathrm{d}w^\ell\big)\right]\otimes\frac{\partial}{\partial w^\ell}$$
$$=\left[-\frac{\mathrm{d}}{\mathrm{d}\lambda}\big(w^\ell_x\big)\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{d}\lambda}\big(w^\ell_t\big)\mathrm{d}t\right]\otimes\frac{\partial}{\partial w^\ell}=\frac{\mathrm{d}}{\mathrm{d}\lambda}U_\lambda.$$

This proves that the vector field X is a solution of Eq. (12). □

Remark 8 This proof can easily be extended to the case of any finite n.

We refer to [62] for discussion of the set-up when a symmetry $X\in\Gamma(T\mathcal{E}^\infty)$ of an equation $\mathcal{E}$ can be used to extend a given zero-curvature representation for $\mathcal{E}$ to a "nontrivial" family of zero-curvature representations. We expect that this result must have a straightforward generalisation to the $\mathbb{Z}_2$-graded case.

Corollary 6 *Let $\tau\colon\tilde{\mathcal{E}}_\lambda\to\mathcal{E}$ be a family of coverings for the family of $\mathfrak{g}$-valued zero-curvature representations α_λ (smoothly depending on the parameter λ). If Eq. (12) has no τ-vertical solutions, then the parameter λ in α_λ is not removable by gauge transformations.*

Example 6 Let us illustrate the claim of Proposition 5. Namely, let us construct a (1|1)-dimensional covering over the $N=2$, $a=4$ SKdV equation (6a)–(6d) by taking the $\mathfrak{sl}(2|1)$-valued zero-curvature representation β from Example 2. Using representation ϱ from Example 4, we obtain

$$\begin{aligned}
w_x&=\lambda^2+2\lambda w+w^2+u_0^2+u_{12}-f_2u_2+\boldsymbol{i}f_2u_1,\\
f_x&=\lambda f_2+f_2w+\boldsymbol{i}f_2u_0+u_2+\boldsymbol{i}u_1,\\
w_t&=2\lambda^2\big(2u_0^2-u_{12}\big)+\lambda\big(8wu_0^2-4wu_{12}+8u_0u_{0;x}-2u_{12;x}+fu_{2;x}-\boldsymbol{i}fu_{1;x}\\
&\quad+4\boldsymbol{i}fu_2u_0+4fu_1u_0\big)+4w^2u_0^2-2w^2u_{12}+8wu_0u_{0;x}-2wu_{12;x}+4u_0^4+2u_0^2u_{12}\\
&\quad+4u_0u_{0;xx}-2u_{12}^2+4u_{0;x}^2-u_{12;xx}+fwu_{2;x}-\boldsymbol{i}fwu_{1;x}+5\boldsymbol{i}fu_0u_{2;x}+5fu_0u_{1;x}\\
&\quad+fu_{2;xx}-\boldsymbol{i}fu_{1;xx}+4\boldsymbol{i}fu_2wu_0-8fu_2u_0^2+2fu_2u_{12}+4\boldsymbol{i}fu_2u_{0;x}+4fu_1wu_0\\
&\quad+8\boldsymbol{i}fu_1u_0^2-2\boldsymbol{i}fu_1u_{12}+4fu_1u_{0;x}+u_2u_{2;x}+8u_2u_1u_0+u_1u_{1;x},\\
f_t&=\lambda\big(-u_{2;x}-\boldsymbol{i}u_{1;x}+4fu_0^2-2fu_{12}+4\boldsymbol{i}u_2u_0-4u_1u_0\big)-wu_{2;x}-\boldsymbol{i}wu_{1;x}+5\boldsymbol{i}u_0u_{2;x}\\
&\quad-5u_0u_{1;x}-u_{2;xx}-\boldsymbol{i}u_{1;xx}+4fwu_0^2-2fwu_{12}+4\boldsymbol{i}fu_0^3-6\boldsymbol{i}fu_0u_{12}+4fu_0u_{0;x}\\
&\quad-fu_{0;xx}\boldsymbol{i}-fu_{12;x}-4fu_2u_1\boldsymbol{i}+4u_2wu_0\boldsymbol{i}+8u_2u_0^2-2u_2u_{12}+4\boldsymbol{i}u_2u_{0;x}-4u_1wu_0\\
&\quad+8\boldsymbol{i}u_1u_0^2-2\boldsymbol{i}u_1u_{12}-4u_1u_{0;x}.
\end{aligned}$$

In agreement with Proposition 5, we find the solution $X=\partial/\partial w$ of Eq. (12): indeed, this field is obtained from the $\mathfrak{sl}(2|1)$-matrix Q which we introduced in Example 2. Let us finally note that this example of vector field X is *not* the infinitesimal generator of a scaling.

Example 7 A recursion operator R has been obtained for fifth-order evolution superequation (9a)–(9b) in components in [63].[10] Denote by λ the parameter under study. A known relation between the inverse recursion operators $(R-\lambda\mathrm{Id})^{-1}$ and zero-curvature representations (see [31] for details) yields a new family of $\mathfrak{sl}(9|8)$-valued zero-curvature representations for (9a)–(9b). We now realise this family of zero-curvature representations as a family of $(9|8)$-dimensional linear coverings over (9a)–(9b). Seventeen new nonlocalities are introduced; the variables S, W_1, W_2, W_3, $V_1,\ldots,V_5$ are parity-even and the variables Q_1, Q_2, Q_3, $O_1,\ldots,O_5$ are parity-odd. At every λ, the derivatives of the new variables are given by the formulas

$$
\begin{aligned}
W_{3;x} &= -O_3\xi - 3O_1\xi u - 9Q_2\xi u + 3Qu\xi_x + 2V_2u + S\big(6u^2 + 3\xi\xi_x\big),\\
W_{2;x} &= 2O_1\xi + 2Su,\\
W_{1;x} &= S,\quad S_x = V_1,\quad V_{1;x} = V_2,\quad V_{2;x} = V_3,\quad V_{3;x} = V_4,\quad V_{4;x} = V_5,\\
V_{5;x} &= 6O_4\xi_x + O_3(3\xi_{xx} - 18\xi u) + O_2(6u\xi_x - 4\xi_{xxx} - 29\xi u_x) + O_1\big(6u\xi_{xx} - 4\xi_{4x}\\
&\quad + 11\xi_x u_x - 36\xi u^2 - 15\xi u_{xx}\big) + (-3Q_3\xi_x)/2 + Q_2\big(21u\xi_{xx} + 3\xi_{4x} + 21\xi_x u_x\\
&\quad + 9\xi u^2\big) + Q\big(51u^2\xi_x + 12u\xi_{xxx} - 2\xi_{5x} + 26\xi_{xx}u_x + 22\xi_x u_{xx} - 18\xi uu_x\\
&\quad + 2\xi u_{xxx}\big)/2 - 12V_4u - 30V_3u_x + V_2\big(-48u^2 - 40u_{xx} + 15\xi\xi_x\big)\\
&\quad + V_1(-140uu_x - 30u_{xxx} + 27\xi\xi_{xx}) - 2W_3u_x + W_2(-12uu_x - 2u_{xxx} - 3\xi\xi_{xx})\\
&\quad + W_1\big(-60u^2u_x - 20uu_{xxx} - 2u_{5x} - 40u_{xx}u_x + 10\xi_{xxx}\xi_x + 30\xi u\xi_{xx} + 30\xi\xi_x u_x\big)\\
&\quad + S\big(-64u^3 - 96uu_{xx} - 12u_{4x} - 72u_x^2 + 8\xi_{xx}\xi_x\\
&\quad + 90\xi u\xi_x + 15\xi\xi_{xxx} - \lambda\big),\\
Q_{3;x} &= 2O_3u + 7O_1u^2 + Q_2\big(6u^2 - 6\xi\xi_x\big) - 2V_3\xi - 6W_2\xi u + 14Su\xi_x,\\
Q_{2;x} &= Qu + S\xi,\\
Q_x &= O_1,\quad O_{1;x} = O_2,\quad O_{2;x} = O_3,\quad O_{3;x} = O_4,\quad O_{4;x} = O_5,\\
O_{5;x} &= -12O_4u - 24O_3u_x + O_2\big(-27u^2 - 19u_{xx} - 2\xi\xi_x\big) + O_1(-63uu_x - 7u_{xxx})\\
&\quad - 3/2Q_3u + Q_2(-21uu_x - 3u_{xxx}) + Q\big(-35u^3 - 48uu_{xx} - 2u_{4x} - 40u_x^2\\
&\quad + 6\xi_{xx}\xi_x - 4\xi\xi_{xxx} - 2\lambda\big)/2 - 6V_3\xi_x + V_2(-21\xi_{xx} - 21\xi u) + V_1(-50u\xi_x\\
&\quad - 23\xi_{xxx} - 27\xi u_x) - 2W_3\xi_x + W_2(6u\xi_x + \xi_{xxx} + 3\xi u_x) + W_1\big(-30u^2\xi_x\\
&\quad - 20u\xi_{xxx} - 2\xi_{5x} - 30\xi_{xx}u_x - 10\xi_x u_{xx} - 30\xi uu_x\big) + S\big(-57u\xi_{xx} - 11\xi_{4x}\\
&\quad - 52\xi_x u_x - 36\xi u^2 - 15\xi u_{xx}\big),\\
S_t &= -5O_3\xi_x + 15O_2\xi u + O_1(5\xi_{xxx} + 15\xi u_x) + Q(-15u\xi_{xx} - 15\xi_x u_x) + V_5 + 10V_3u\\
&\quad + 20V_2u_x + V_1\big(30u^2 + 20u_{xx} - 15\xi\xi_x\big) + S(60uu_x + 10u_{xxx} - 15\xi\xi_{xx}),\\
Q_t &= O_5 + 10O_3u + 15O_2u_x + O_1\big(15u^2 + 5u_{xx}\big) + 15Quu_x + 5V_2\xi_x\\
&\quad + V_1(15\xi_{xx} + 15\xi u) + S(30u\xi_x + 10\xi_{xxx} + 15\xi u_x),
\end{aligned}
$$

[10] A recursion operator, formulated for (9a)–(9b) in terms of superfields and superderivatives, was conjectured in [32].

$$V_{1;t} = (S_t)_x, \quad V_{2;t} = (V_{1;t})_x, \quad V_{3;t} = (V_{2;t})_x, \quad V_{4;t} = (V_{3;t})_x, \quad V_{5;t} = (V_{4;t})_x,$$

$$O_{1;t} = (Q_t)_x, \quad O_{2;t} = (O_{1;t})_x, \quad O_{3;t} = (O_{2;t})_x, \quad O_{4;t} = (O_{3;t})_x, \quad O_{5;t} = (O_{4;t})_x,$$

$$\begin{aligned}
W_{1;t} &= -5O_2\xi_x + O_1(5\xi_{xx} + 15\xi u) - 15Qu\xi_x + V_4 + 10V_2u + 10V_1u_x \\
&\quad + S\big(30u^2 + 10u_{xx} - 15\xi\xi_x\big),\\
W_{2;t} &= 2O_5\xi - 2O_4\xi_x + O_3(2\xi_{xx} + 20\xi u) + O_2(-30u\xi_x - 2\xi_{xxx} + 30\xi u_x) + O_1\big(30u\xi_{xx} \\
&\quad + 2\xi_{4x} + 60\xi u^2 + 10\xi u_{xx}\big) + Q\big(-30u^2\xi_x + 30\xi uu_x\big) + 2V_4u - 2V_3u_x + V_2\big(20u^2 \\
&\quad + 2u_{xx} - 10\xi\xi_x\big) + V_1(-2u_{xxx} - 30\xi\xi_{xx}) + S\big(60u^3 + 40uu_{xx} + 2u_{4x} - 30\xi_{xx}\xi_x \\
&\quad - 120\xi u\xi_x - 20\xi\xi_{xxx}\big),\\
Q_{2;t} &= O_4u - O_3u_x + O_2\big(10u^2 + u_{xx} + 5\xi\xi_x\big) + O_1(-5uu_x - u_{xxx} - 5\xi\xi_{xx}) + Q\big(15u^3 \\
&\quad + 10uu_{xx} + u_{4x} + 5u_x^2 - 5\xi_{xx}\xi_x\big) + V_4\xi - V_3\xi_x + V_2(\xi_{xx} + 10\xi u) + V_1(-5u\xi_x \\
&\quad - \xi_{xxx} + 10\xi u_x) + S\big(20u\xi_{xx} + \xi_{4x} - 5\xi_x u_x + 45\xi u^2 + 10\xi u_{xx}\big),\\
W_{3;t} &= O_5(-\xi_{xx} - \xi u) + O_4(6u\xi_x + \xi_{xxx} + 4\xi u_x) + O_3\big(-20u\xi_{xx} - \xi_{4x} + 2\xi_x u_x - 30\xi u^2 \\
&\quad - 5\xi u_{xx}\big) + O_2\big(54u^2\xi_x + 8u\xi_{xxx} - 15\xi_{xx}u_x - 2\xi_x u_{xx} + 11\xi uu_x + 4\xi u_{xxx}\big) \\
&\quad + O_1\big(-57u^2\xi_{xx} - 4u\xi_{4x} + 4u\xi_x u_x - \xi_{xxx}u_x - 2\xi_{xx}u_{xx} - 2\xi_x u_{xxx} - 89\xi u^3 \\
&\quad - 9\xi uu_{xx} - 4\xi u_x^2 - 3\xi\xi_{xx}\xi_x + \xi\lambda\big) + Q_3(-9u\xi_x + 3\xi u_x)/2 + Q_2\big(-48u^2\xi_{xx} \\
&\quad - 3u\xi_{4x} + 66u\xi_x u_x + 9\xi_{xxx}u_x - 9\xi_{xx}u_{xx} + 6\xi_x u_{xxx} - 108\xi u^3 - 69\xi uu_{xx} - 6\xi u_{4x} \\
&\quad - 24\xi u_x^2 + 45\xi\xi_{xx}\xi_x\big) + Q\big(157u^3\xi_x + 42u^2\xi_{xxx} + 2u\xi_{5x} - 32u\xi_{xx}u_x + 38u\xi_x u_{xx} \\
&\quad - 6\xi_{4x}u_x + 6\xi_{xxx}u_{xx} - 6\xi_{xx}u_{xxx} + 4\xi_x u_{4x} + 44\xi_x u_x^2 - 2\xi_x\lambda + 111\xi u^2u_x \\
&\quad + 28\xi uu_{xxx} + 2\xi u_{5x} + 38\xi u_{xx}u_x - 14\xi\xi_{xxx}\xi_x\big)/2 - 2V_5u_x + V_4\big(2u^2 + 2u_{xx} \\
&\quad + 2\xi\xi_x\big) + V_3(-32uu_x - 2u_{xxx} - 5\xi\xi_{xx}) + V_2\big(24u^3 + 32uu_{xx} + 2u_{4x} \\
&\quad - 28u_x^2 + \xi_{xx}\xi_x + 15\xi u\xi_x + 4\xi\xi_{xxx}\big) + V_1\big(-100u^2u_x - 12uu_{xxx} - 16u_{xx}u_x \\
&\quad - 7\xi_{xxx}\xi_x - 50\xi u\xi_{xx} - 2\xi\xi_{4x} + 18\xi\xi_x u_x\big) + W_3(-4uu_x - 2\xi\xi_{xx}) \\
&\quad + W_2\big(-24u^2u_x - 4uu_{xxx} + \xi_{xxx}\xi_x + \xi\xi_{4x} + 12\xi\xi_x u_x\big) + W_1\big(-120u^3u_x \\
&\quad - 40u^2u_{xxx} - 4uu_{5x} - 80uu_{xx}u_x - 2\xi_{5x}\xi_x - 30\xi_{xx}\xi_x u_x + 30\xi u^2\xi_{xx} - 20\xi u\xi_{4x} \\
&\quad - 60\xi u\xi_x u_x - 2\xi\xi_{6x} - 50\xi\xi_{xxx}u_x - 40\xi\xi_{xx}u_{xx} \\
&\quad - 10\xi\xi_x u_{xxx}\big) + S\big(52u^4 + 108u^2u_{xx} + 8uu_{4x} - 144uu_x^2 + 97u\xi_{xx}\xi_x - 2u\lambda \\
&\quad - 32u_{xxx}u_x + 16u_{xx}^2 + 5\xi_{4x}\xi_x - 16\xi_{xxx}\xi_{xx} + 192\xi u^2\xi_x + 33\xi u\xi_{xxx} - 46\xi\xi_{xx}u_x \\
&\quad + 19\xi\xi_x u_{xx}\big),\\
Q_{3;t} &= O_5\big(3u^2 + 2u_{xx} + 2\xi\xi_x\big) + O_4(-12uu_x - 2u_{xxx} - 2\xi\xi_{xx}) + O_3\big(46u^3 + 38uu_{xx} \\
&\quad + 2u_{4x} + 12u_x^2 - 14\xi_{xx}\xi_x + 6\xi u\xi_x - 2\xi\xi_{xxx}\big) + O_2\big(-59u^2u_x - 16uu_{xxx} \\
&\quad + 26u_{xx}u_x + 2\xi_{xxx}\xi_x - 10\xi u\xi_{xx} + 4\xi\xi_{4x} + 66\xi\xi_x u_x\big) + O_1\big(109u^4 + 121u^2u_{xx}
\end{aligned}$$

$$
\begin{aligned}
&+ 8uu_{4x} - 12uu_x^2 - 12u\xi_{xx}\xi_x - 2u\lambda - 10u_{xxx}u_x + 12u_{xx}^2 + 2\xi_{4x}\xi_x - 10\xi_{xxx}\xi_{xx} \\
&- 78\xi u^2\xi_x - 18\xi u\xi_{xxx} - 54\xi\xi_{xx}u_x + 14\xi\xi_x u_{xx}\big) - 3Q_3\xi\xi_{xx} + Q_2\big(72u^4 \\
&+ 78u^2u_{xx} + 6uu_{4x} - 138u\xi_{xx}\xi_x - 6u_{xxx}u_x + 6u_{xx}^2 - 6\xi_{4x}\xi_x + 12\xi_{xxx}\xi_{xx} \\
&- 198\xi u^2\xi_x - 18\xi u\xi_{xxx} - 6\xi\xi_{xx}u_x + 12\xi\xi_x u_{xx}\big) + Q\big(-67u^3u_x - 30u^2u_{xxx} \\
&- 2uu_{5x} + 4uu_{xx}u_x + 10u\xi_{xxx}\xi_x + 2u_{4x}u_x + 12u_x^3 + 2u_x\lambda - 2\xi_{5x}\xi_x - 64\xi_{xx}\xi_x u_x \\
&+ 153\xi u^2\xi_{xx} - 10\xi u\xi_{4x} + 66\xi u\xi_x u_x - 2\xi\xi_{6x} - 54\xi\xi_{xxx}u_x - 36\xi\xi_{xx}u_{xx} \\
&- 10\xi\xi_x u_{xxx}\big) + V_5(-2\xi_{xx} + 4\xi u) + V_4(8u\xi_x + 2\xi_{xxx} + 4\xi u_x) + V_3\big(-38u\xi_{xx} \\
&- 2\xi_{4x} - 12\xi_x u_x + 24\xi u^2\big) + V_2\big(79u^2\xi_x + 16u\xi_{xxx} - 40\xi_{xx}u_x + 14\xi_x u_{xx} \\
&+ 120\xi uu_x\big) + V_1\big(-123u^2\xi_{xx} - 12u\xi_{4x} - 52u\xi_x u_x + 24\xi_{xxx}u_x - 12\xi_{xx}u_{xx} \\
&- 14\xi_x u_{xxx} + 101\xi u^3 + 142\xi uu_{xx} + 4\xi u_{4x} + 64\xi u_x^2 - 82\xi\xi_{xx}\xi_x + 2\xi\lambda\big) \\
&+ W_3(-4u\xi_{xx} + 4\xi u_{xx}) + W_2\big(-48u^2\xi_{xx} - 4u\xi_{4x} + 12u\xi_x u_x + 4\xi_{xxx}u_x - 6\xi_{xx}u_{xx} \\
&+ 2\xi_x u_{xxx} - 72\xi u^3 - 30\xi uu_{xx} - 2\xi u_{4x} - 12\xi u_x^2 + 18\xi\xi_{xx}\xi_x\big) + W_1\big(-60u^3\xi_{xx} \\
&- 40u^2\xi_{4x} - 240u^2\xi_x u_x - 4u\xi_{6x} - 60u\xi_{xxx}u_x - 80u\xi_{xx}u_{xx} - 60u\xi_x u_{xxx} + 4\xi_{5x}u_x \\
&+ 60\xi_{xx}u_x^2 - 4\xi_x u_{5x} - 60\xi_x u_{xx}u_x + 60\xi u^2u_{xx} + 40\xi uu_{4x} + 240\xi uu_x^2 \\
&+ 120\xi u\xi_{xx}\xi_x + 4\xi u_{6x} + 120\xi u_{xxx}u_x + 80\xi u_{xx}^2 - 20\xi\xi_{4x}\xi_x - 20\xi\xi_{xxx}\xi_{xx}\big) \\
&+ S\big(358u^3\xi_x + 138u^2\xi_{xxx} + 8u\xi_{5x} - 192u\xi_{xx}u_x + 244u\xi_x u_{xx} - 12\xi_{4x}u_x \\
&+ 34\xi_{xxx}u_{xx} - 34\xi_{xx}u_{xxx} + 10\xi_x u_{4x} - 12\xi_x u_x^2 - 2\xi_x\lambda + 549\xi u^2u_x + 102\xi uu_{xxx} \\
&+ 8\xi u_{5x} + 224\xi u_{xx}u_x - 48\xi\xi_{xxx}\xi_x\big),
\end{aligned}
$$

For this family of coverings over (9a)–(9b), the solution of Eq. (12) is given by the vector field

$$
\begin{aligned}
X = \lambda^{-1}\Bigg(& x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - \frac{3}{2}\xi\frac{\partial}{\partial \xi} + \cdots - \frac{9}{2}S\frac{\partial}{\partial S} + 5Q\frac{\partial}{\partial Q} + \frac{11}{2}W_1\frac{\partial}{\partial W_1} \\
& + \frac{7}{2}W_2\frac{\partial}{\partial W_2} + \frac{3}{2}W_3\frac{\partial}{\partial W_3} + 4Q_2\frac{\partial}{\partial Q_2} + Q_3\frac{\partial}{\partial Q_3} + \frac{7}{2}V_1\frac{\partial}{\partial V_1} + \frac{5}{2}V_2\frac{\partial}{\partial V_2} + \frac{3}{2}V_3\frac{\partial}{\partial V_3} \\
& + \frac{1}{2}V_4\frac{\partial}{\partial V_4} - \frac{1}{2}V_5\frac{\partial}{\partial V_5} + 4O_1\frac{\partial}{\partial O_1} + 3O_2\frac{\partial}{\partial O_2} + 2O_3\frac{\partial}{\partial O_3} + O_4\frac{\partial}{\partial O_4}\Bigg).
\end{aligned} \tag{21}
$$

It is clear that vector field (21) is not τ-verticalisable.

Claim 7 *In the covering τ constructed in Example 7 for super-equation* (9a)–(9b), *deformation equation* (12) *does not admit* any *τ-vertical solutions.*

Sketch of the proof Equation (12), viewed for this covering as a differential equation with respect to components of the vector field X, is a system of linear inhomogeneous equations in total derivatives. The corresponding *homogeneous* system describes symmetries of this covering. Therefore, the general solution X_{gen} of (12) has the form $X_{\text{gen}} = X + X_{\text{sym}}$, where X is given by (21) and a solution X_{sym} of the homogeneous system is a symmetry of the covering. The τ-horizontal part of (21) is the scaling symmetry of (9a)–(9b). By a straightforward calculation we have established that the scaling symmetry of (9a)–(9b) cannot be lifted to a symmetry of the covering τ. Consequently, all solutions of (12) have a nonzero τ-horizontal part, hence none of them is τ-verticalisable. □

The above claim and Corollary 6 combined yield that the parameter λ in the covering in Example 7 is not removable by gauge transformations, and therefore, Eq. (9a)–(9b) is integrable.

6 Conclusion

We extended—to the $\mathbb{Z}_2$-graded case—Marvan's method for inspecting the (non)removability of a parameter under the action of a smooth family of gauge transformations on a given family of zero-curvature representations. This generalisation of the standard technique can be used further for solution of Gardner's deformation problems for the $N = 2$-supersymmetric KdV equations and other $\mathbb{Z}_2$-graded completely integrable systems. At the same time, we confirmed that a switch between the representations of Lie (super-)algebras establishes a link between the two classes of nonlocal geometries and also between the arising differentials. In particular, by analysing this relation in the case of zero-curvature representations with removable parameters λ, we explicitly described the equivalence classes of τ_λ-shadows that determine, by virtue of structure equation (12), the evolution of Cartan's structural elements in families of coverings τ_λ.

We remember that the technique for calculation of the horizontal gauge cohomology groups (see Lemma 3) constitutes another result of the original papers [22, 23]. Namely, suppose that α_λ is a family, depending on a parameter λ, of $\mathfrak{g}$-valued zero-curvature representation for an equation $\mathcal{E}$. In [22], the horizontal gauge cohomology complex $\bar{H}^q_{\alpha_\lambda}(\mathcal{E}, \mathfrak{g})$ was associated[11] with such family α_λ. It is standard that the first horizontal gauge cohomology group $\bar{H}^1_{\alpha_\lambda}(\mathcal{E}, \mathfrak{g})$ contains the obstructions to removability of the parameter λ (cf. Sect. 5 above and [27, 28]). However, calculating the cohomology group $\bar{H}^1_{\alpha_\lambda}(\mathcal{E}^\infty, \mathfrak{g})$ is, in general, harder than solving Eq. (3) from Proposition 1 whenever the (non)removability of a given parameter λ is examined.

Marvan's technique for calculation of the first horizontal gauge cohomology group $\bar{H}^1_{\alpha_\lambda}(\mathcal{E}, \mathfrak{g})$ is based on finding $\mathfrak{g} \times \mathfrak{g}$-valued zero-curvature representations for the system $\mathcal{E}$. Let us keep in mind that an efficient approach to finding zero-curvature representations α for purely bosonic PDEs $\mathcal{E}$ was known from [64, 65]: it involves the use of such auxiliary structures as the characteristic elements $\chi(\alpha)$ and then, consideration of the Jordan normal forms for the $\mathfrak{g}$-matrices contained therein. Consequently, a proper $\mathbb{Z}_2$-graded generalisation of the concept of Jordan normal forms was indispensable, to make that method work in the larger set-up. Such generalisation has become available from the extended edition [66, §D7.2] of [38], see also [67].

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[11] This cohomology theory is helpful for solution of a different problem, namely, *construction* of parametric families of zero-curvature representations with nonremovable parameters (see [23] for details).

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Appendix A: Proof of Proposition 2

To verify the claim, let us first introduce some helpful notation. The number k is called the differential order of a differential function $f(x, t, [u^j, \xi^l])$ with respect to the variable u^j if following conditions hold:

(1) the function f essentially depends on the kth order derivative of u^j with respect to x:

$$\frac{\partial f}{\partial u^j_\sigma} \neq 0, \quad \sigma = (x \dots x), \ |\sigma| = k;$$

(2) the function f does not depend on derivatives of u^j of order higher than k with respect to x:

$$\forall p > k, \quad \frac{\partial f}{\partial u^j_\sigma} = 0, \quad \sigma = (x \dots x), \ |\sigma| = p.$$

We denote by $\mathrm{dord}_x^{u^j}(f)$ the differential order of a given function f with respect to u^j. In the same manner we define the differential order $\mathrm{dord}_x^{\xi^l}(f)$ of differential function $f(x, t, [u^j, \xi^l])$ with respect to the odd variables ξ^l.

For $N = 2$, $a = 4$-SKdV equation (6a)–(6d) we have that $u^1 = u_0$, $u^2 = u_{12}$, $\xi^1 = u_1$, $\xi^2 = u_2$. The maximum of four numbers $\mathrm{dord}_x^{u_0}(f)$, $\mathrm{dord}_x^{u_1}(f)$, $\mathrm{dord}_x^{u_2}(f)$, and $\mathrm{dord}_x^{u_{12}}(f)$ is called the *differential order* of the function $f(x, t, [u_0, u_1, u_2, u_{12}])$, denoted by $\mathrm{dord}_x(f)$. The maximum of $\mathrm{dord}_x(a_{ij})$, where a_{ij}'s are the entries of a given matrix A with differential-functional coefficients, is the differential order of the matrix A; it is denoted by $\mathrm{dord}_x(A)$. Obviously, the following formulas hold:

$$0 \leqslant \mathrm{dord}_x(f + g) \leqslant \max\{\mathrm{dord}(f), \mathrm{dord}(g)\},$$
$$0 \leqslant \mathrm{dord}_x(f \cdot g) \leqslant \max\{\mathrm{dord}(f), \mathrm{dord}(g)\},$$
$$\mathrm{dord}_x(\bar{D}_x f) = \begin{cases} \mathrm{dord}_x(f) + 1 & \text{if } f = f(x, t, [u_0, u_1, u_2, u_{12}]), \\ 0 & \text{if } f = f(x, t). \end{cases}$$

Now, for the matrix (7) we have that

$$\mathrm{dord}_x A = 0, \qquad \mathrm{dord}_x \frac{\partial}{\partial \varepsilon} A = 0.$$

Let us calculate the maximal differential order of left-hand side of Eq. (8a),

$$\mathrm{dord}_x \left(\frac{\partial}{\partial \varepsilon} A + [\underline{A}, Q\underline{]} \right) \leqslant \mathrm{dord}_x(Q).$$

The differential orders of the right-hand side and the left-hand side of Eq. (8a) must coincide. Therefore, we have that

$$\mathrm{dord}_x(\bar{D}_x Q) = \mathrm{dord}_x(Q).$$

This equality holds only in the case when all entries of the matrix Q do not depend on u_0, u_1, u_2, and u_{12}. This implies that the total derivative $\bar{D}_x$ in Eq. (8a) amounts to the partial derivative $\partial/\partial x$. We finally obtain the following system of equations for the entries q_{ij} of the matrix Q:

$$\frac{\partial}{\partial x}q_{11} = -u_2 q_{31}\varepsilon^{-1} - \boldsymbol{i}u_1 q_{31}\varepsilon^{-1} + u_0^2 q_{21}\varepsilon^{-1} - \boldsymbol{i}u_0 q_{21}\varepsilon^{-2} + u_{12}q_{21}\varepsilon^{-1} + q_{12}\varepsilon, \tag{22a}$$

$$\begin{aligned}\frac{\partial}{\partial x}q_{12} &= -u_2(q_{32} + q_{13}\varepsilon)\varepsilon^{-1} + \boldsymbol{i}u_1(-q_{32} + q_{13}\varepsilon)\varepsilon^{-1} + u_0^2(-\varepsilon q_{11} + \varepsilon q_{22} - 1)\varepsilon^{-2} \\ &\quad + \boldsymbol{i}u_0(\varepsilon q_{11} - \varepsilon q_{22} + 2)\varepsilon^{-3} + u_{12}(-\varepsilon q_{11} + \varepsilon q_{22} - 1)\varepsilon^{-2} + q_{12}\varepsilon^{-1},\end{aligned} \tag{22b}$$

$$\begin{aligned}\frac{\partial}{\partial x}q_{13} &= u_2(-\varepsilon q_{22} + 1)\varepsilon^{-2} + \boldsymbol{i}u_1(-\varepsilon q_{22} + 1)\varepsilon^{-2} + u_0^2 q_{23}\varepsilon^{-1} + \boldsymbol{i}u_0\big(-q_{23} + q_{13}\varepsilon^2\big)\varepsilon^{-2} \\ &\quad + u_{12}q_{23}\varepsilon^{-1} + q_{13}\varepsilon^{-1},\end{aligned} \tag{22c}$$

$$\frac{\partial}{\partial x}q_{21} = -q_{11}\varepsilon + q_{22}\varepsilon - 1 - q_{21}\varepsilon^{-1}, \tag{22d}$$

$$\frac{\partial}{\partial x}q_{22} = -u_2 q_{23} + \boldsymbol{i}u_1 q_{23} - u_0^2 q_{21}\varepsilon^{-1} + \boldsymbol{i}u_0 q_{21}\varepsilon^{-2} - u_{12}q_{21}\varepsilon^{-1} - \varepsilon q_{12} + \varepsilon^{-2}, \tag{22e}$$

$$\frac{\partial}{\partial x}q_{23} = q_{21}u_2\varepsilon^{-1} + \boldsymbol{i}q_{21}u_1\varepsilon^{-1} + \boldsymbol{i}u_0 q_{23} - \varepsilon q_{13}, \tag{22f}$$

$$\frac{\partial}{\partial x}q_{31} = -q_{21}u_2 + \boldsymbol{i}q_{21}u_1 - \boldsymbol{i}u_0 q_{31} + q_{32}\varepsilon - q_{31}\varepsilon^{-1}, \tag{22g}$$

$$\frac{\partial}{\partial x}q_{32} = q_{11}u_2 - \boldsymbol{i}q_{11}u_1 - u_0^2 q_{31}\varepsilon^{-1} + \boldsymbol{i}u_0\big(-q_{32}\varepsilon^2 + q_{31}\big)\varepsilon^{-2} - u_{12}q_{31}\varepsilon^{-1}. \tag{22h}$$

Since every q_{ij} does not depend on u_0, u_1, u_2, and u_{12}, it follows that the coefficients of nonzero powers of u_0, u_1, u_2, and u_{12} in (22a)–(22h) are equal to zero. We obtain the system

$$\begin{aligned} q_{31} &= 0, & q_{32} + q_{13}\varepsilon &= 0, & -q_{32} + q_{13}\varepsilon &= 0, \\ q_{23} &= 0, & -q_{11}\varepsilon + q_{22}\varepsilon - 1 &= 0, & -q_{32}\varepsilon^2 + q_{31} &= 0, \\ q_{11}\varepsilon - q_{22}\varepsilon + 2 &= 0, & -q_{22}\varepsilon + 1 &= 0, & q_{11} &= 0. \end{aligned} \tag{23}$$

By adding the last equation in the first column, $q_{11}\varepsilon - q_{22}\varepsilon + 2 = 0$, to the second equation in the other column, $-q_{11}\varepsilon + q_{22}\varepsilon - 1 = 0$, we obtain the contradiction $1 = 0$. Therefore, system (23) is not compatible. This proves Proposition 2: there is no $\mathfrak{sl}(2|1)$-matrix Q satisfying Eqs. (8a)–(8b) at $\varepsilon > 0$.

Appendix B: Two Descriptions of One Elimination Procedure: An Example

We analyse the following tautological construction: by re-addressing Sasaki,[12] see [37], we first track how the scaling symmetry of KdV equation (14) acts on its standard matrix Lax

[12] A parameter-dependent zero-curvature representation for Burgers' equation was considered in [68] in the same context of pseudospherical surfaces as in Sasaki's paper [37]. We refer to [22] for the analysis of removability of the parameter in that zero-curvature representation for Burgers' equation [68].

pair; on the other hand, we reveal how these objects are phrased in the language of coverings.

B.1 The Sasaki Construction

Recall that the Korteweg–de Vries equation is

$$\mathcal{E} = \{u_t = -u_{xxx} - 6uu_x\}. \tag{14}$$

Consider the family of coverings $\tau_\eta \colon \tilde{\mathcal{E}}_\eta \to \mathcal{E}$ over it,

$$v_x = 2v\eta - \big(v^2 + u\big), \tag{24a}$$

$$v_t = -8\eta^3 v + 4\eta^2\big(v^2 + u\big) + 2\eta(-2vu + u_x) + 2v^2u - 2vu_x + 2u^2 + u_{xx}; \tag{24b}$$

these formulas are obtained from the $\mathfrak{sl}_2$-valued zero-curvature representation (see [37]),

$$\alpha_\eta = \begin{pmatrix} \eta & u \\ -1 & -\eta \end{pmatrix} \mathrm{d}x + \begin{pmatrix} -(4\eta^3 + 2\eta u + u_x) & -(u_{xx} + 2\eta u_x + 4\eta^2 u + 2u^2) \\ 4\eta^2 + 2u & 4\eta^3 + 2\eta u + u_x \end{pmatrix} \mathrm{d}t.$$

Let us recall that the parameter η cannot be removed from the zero-curvature representations α_η by using gauge transformations. However, it can be *eliminated* by using a wider class of transformations. Namely, consider the scaling symmetry of Eq. (14),

$$x \mapsto \eta x, \quad t \mapsto \eta^3 t, \quad u \mapsto \eta^{-2}u, \qquad \eta \in \mathbb{R}.$$

Using it, one transforms the zero-curvature representation α_η into

$$\alpha'_\eta = \begin{pmatrix} 1 & \eta u \\ -\eta^{-1} & -1 \end{pmatrix} \mathrm{d}x + \begin{pmatrix} -(4 + 2u + u_x) & -\eta(u_{xx} + 2u_x + 4u + 2u^2) \\ \eta^{-1}(4 + 2u) & 4 + 2u + u_x \end{pmatrix} \mathrm{d}t.$$

The parameter η in α'_η is removable under the gauge transformation

$$g = \begin{pmatrix} \eta^{-1/2} & 0 \\ 0 & \eta^{1/2} \end{pmatrix} \in C^\infty\big(\mathcal{E}^\infty, GL_2(\mathbb{C})\big),$$

that is, we have that $(\alpha'_\eta)^g = \alpha'_\eta|_{\eta=1} = \alpha_\eta|_{\eta=1}$.

B.2 How the Elimination Works in Terms of the Structure Element

Let us now address the removability of parameter η in coverings (24a)–(24b) in terms of the formalism of Cartan's structural element.

For a vector field

$$X = a \otimes \frac{\partial}{\partial x} + b \otimes \frac{\partial}{\partial t} + \omega_\sigma \otimes \frac{\partial}{\partial u_\sigma} + \varphi \otimes \frac{\partial}{\partial v},$$

the equation for evolution of Cartan's structural element,

$$\frac{\mathrm{d}}{\mathrm{d}\eta} U_\eta = [X, U_\eta]^{\mathrm{FN}}, \tag{12}$$

splits into the system

$$-\frac{\mathrm{d}}{\mathrm{d}\eta}v_x = \tilde{D}_x\varphi - \varphi\frac{\partial v_x}{\partial v} - \omega_\sigma\frac{\partial v_x}{\partial u_\sigma} + b\left(\frac{\partial v_x}{\partial u_\sigma}u_{\sigma t} + \frac{\partial v_x}{\partial v}v_t - \tilde{D}_x v_t\right) - v_t\frac{\partial b}{x} + a\left(-\tilde{D}_x v_x + \frac{\partial v_x}{\partial u_\sigma}u_{\sigma x} + \frac{\partial v_x}{\partial v}v_x\right) - v_x\frac{\partial a}{\partial x}, \tag{25a}$$

$$-\frac{\mathrm{d}}{\mathrm{d}\eta}v_t = \tilde{D}_t\varphi - \varphi\frac{\partial v_t}{\partial v} - \omega_\sigma\frac{\partial v_t}{\partial u_\sigma} + b\left(\frac{\partial v_t}{\partial u_\sigma}u_{\sigma t} + \frac{\partial v_t}{\partial v}v_t - \tilde{D}_t v_t\right) - v_t\frac{\partial b}{\partial t} + a\left(-\tilde{D}_t v_x + \frac{\partial v_t}{\partial u_\sigma}u_{\sigma x} + \frac{\partial v_t}{\partial v}v_x\right) - v_x\frac{\partial a}{\partial t}, \tag{25b}$$

$$\omega_{\sigma x} = \tilde{D}_x\omega_\sigma - u_{\sigma t}\frac{\partial b}{\partial x} - u_{\sigma x}\frac{\partial a}{\partial x}, \tag{25c}$$

$$\omega_{\sigma t} = \tilde{D}_t\omega_\sigma - u_{\sigma t}\frac{\partial b}{\partial t} - u_{\sigma x}\frac{\partial a}{\partial t}. \tag{25d}$$

Suppose now that the vector field is vertical: $X^{\mathrm{v}} = \omega_\sigma^{\mathrm{v}} \otimes \partial/\partial u_\sigma + \varphi^{\mathrm{v}} \otimes \partial/\partial v$. This simplifies system (25a)–(25d); it then becomes

$$-\frac{\mathrm{d}}{\mathrm{d}\eta}v_x = \tilde{D}_x\varphi^{\mathrm{v}} - \varphi^{\mathrm{v}}\frac{\partial v_x}{\partial v} - \omega_\sigma^{\mathrm{v}}\frac{\partial v_x}{\partial u_\sigma}, \tag{26a}$$

$$-\frac{\mathrm{d}}{\mathrm{d}\eta}v_t = \tilde{D}_t\varphi^{\mathrm{v}} - \varphi^{\mathrm{v}}\frac{\partial v_t}{\partial v} - \omega_\sigma^{\mathrm{v}}\frac{\partial v_t}{\partial u_\sigma}, \tag{26b}$$

$$\omega_{\sigma x}^{\mathrm{v}} = \tilde{D}_x\omega_\sigma^{\mathrm{v}}, \tag{26c}$$

$$\omega_{\sigma t}^{\mathrm{v}} = \tilde{D}_t\omega_\sigma^{\mathrm{v}}. \tag{26d}$$

Let us use the Ansatz

$$\omega^{\mathrm{v}} = \omega - au_x - bu_t, \qquad \varphi^{\mathrm{v}} = \varphi - av_x - bu_t,$$

assuming that $a = a(x,t,\eta)$, $b = b(x,t,\eta)$, $\varphi = \varphi(\eta,u,v)$, and $\omega = \omega(\eta,u,v,u_x,u_{xx})$. By construction, the unknowns ω^{v} and φ^{v} satisfy system (26a)–(26d). Using the analytic software `Jets` [69] and `Crack` [58], we find the solution

$$a = 24c_4t\eta^3 + 2c_4x\eta + \frac{1}{\eta}(c_6 + x),$$

$$b = 6c_4t\eta + \frac{1}{\eta}(-c_7 + 3t),$$

$$\omega = 4c_4\eta^3 - 4c_4u\eta + u_xc_4 + \frac{1}{\eta}\left(-\frac{1}{2}u_xc_3 - 2u\right) + \frac{1}{2\eta^2}u_x,$$

$$\varphi = 2c_4\eta^2 - c_4v^2 - c_3v - c_4u + \frac{c_3}{2\eta}\left(v^2 + u\right) - \frac{1}{2\eta^2}\left(v^2 + u\right),$$

which contains four arbitrary constants c_3, c_4, c_6, and c_7.

Let us set $c_3 = 0$, $c_4 = -1/(2\eta^2)$ at $\eta \neq 0$, $c_6 = 0$, and $c_7 = 0$. This determines the solution which corresponds to the lift of Galilean symmetry of (14):

$$X_2 = -2\eta(6t\partial/\partial x + \partial/\partial u + \cdots) - \partial/\partial v.$$

On the other hand, set $c_3 = 1/\eta$ if $\eta \neq 0$ and let $c_4 = 0$, $c_6 = 0$, and $c_7 = 0$. This yields the solution which corresponds to the lift of scaling symmetry of (14); namely, we have that

$$X_1 = \eta^{-2}(-x\partial/\partial x - 3t\partial/\partial t + 2u\partial/\partial u + \cdots + v\partial/\partial v). \tag{27}$$

The integral curves of vector field (27) encode the transformation

$$x \mapsto \eta x, \qquad t \mapsto \eta^3 t, \qquad u \mapsto \eta^{-2} u, \qquad v \mapsto \eta^{-1} v. \tag{28}$$

Its action on the covering τ_η in (24a)–(24b) results in the covering $\tau' = \tau_\eta|_{\eta=1}$, which is described by the formulas

$$\begin{aligned} v_x &= 2v - \left(v^2 + u\right), \\ v_t &= -8v + 4v^2 + 4u - 4vu + 2u_x + 2v^2u - 2vu_x + 2u^2 + u_{xx}. \end{aligned}$$

It is readily seen that the covering τ' is the image of zero-curvature representation $(\alpha'_\eta)^g$ under a swapping of representations for the Lie algebra at hand. This is shown in the following diagram:

$$\begin{array}{ccccc} \alpha_\eta & \xrightarrow{\text{scaling}} & \alpha'_\eta & \xrightarrow{g} & \alpha'_\eta|_{\eta=1} \\ \| & & & & \downarrow{\scriptstyle \nabla} \\ \alpha_\eta & \xrightarrow{\nabla} & \tau & \xrightarrow{(28)} & \tau'. \end{array} \tag{29}$$

We conclude that the problem of finding transformations (which are possibly not gauge) that eliminate the parameter in a given family of zero-curvature representations can be approached via a solution of Eq. (12) in the family of coverings which are the $(\rho \rightleftarrows \varrho)$-avatars of those zero-curvature representations.

B.3 Overview: Taxonomy of the Parameters

Depending on their elimination scenario, "removable" parameters in zero-curvature representations are classified as follows:

(1) There are parameters which are truly removable under the action of smooth families of gauge transformations (see [22, 23] by Marvan and [24, 25] by Sakovich).
(2) There could be zero-curvature representations α_λ which are (piecewise-)smooth in the parameter $\lambda \in \mathcal{I} \subseteq \mathbb{C}$ but such that the families S_λ of gauge transformations removing the parameter are *not* smooth at all points $\lambda \in \mathcal{J} \subseteq \mathcal{I}$, where the set $\mathcal{J}$ is
   (a) finite,
   (b) countable,
   (c) everywhere dense in $\mathcal{I}$ but not amounting to it, or
   (d) equal to the entire set $\mathcal{I}$ of admissible values of the parameter λ.
   This analytic curiosity would be the threshold limit of the preceding case.

(3) Next, there are parameters which cannot be removed by using gauge transformations but which indicate the presence of conserved currents in zero-curvature representations and the reducibility of such representations[13] (see [70] and [48, §12]).
(4) There are parameters which vanish under the action of those symmetries of the underlying differential equation which cannot be lifted to the covering Maurer–Cartan equation (see [37, 71]).
(5) Finally, there are parameters which can be eliminated by the same procedure as in the preceding case but by using *shadows* of nonlocal symmetries in some auxiliary covering over the equation at hand (namely, *not* in the covering which grasps the ZCR geometry but in an extension of the equation's geometry by a set of "nonlocalities"), see [72–74].

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[13]For example, consider a "fake" $\mathfrak{sl}_2$-valued zero-curvature representation $\alpha = \begin{pmatrix} 0 & X_1+\lambda X_2 \\ 0 & 0 \end{pmatrix} \mathrm{d}x + \begin{pmatrix} 0 & T_1+\lambda T_2 \\ 0 & 0 \end{pmatrix} \mathrm{d}t$ for an equation $\mathcal{E}$ possessing two conserved currents $\bar{D}_t X_i = \bar{D}_x T_i$, here $i = 1, 2$.

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