## University of Groningen

## Limit-point/limit-circle classification for Hain-Lust type equations

Hassi, Seppo; Moller, Manfred; de Snoo, Henk

Published in:
Mathematische Nachrichten

DOI:
10.1002/mana. 201600254

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2018

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
Hassi, S., Moller, M., \& de Snoo, H. (2018). Limit-point/limit-circle classification for Hain-Lust type equations. Mathematische Nachrichten, 291(4), 652-668. https://doi.org/10.1002/mana.201600254

## Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25 fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: https://www.rug.nl/library/open-access/self-archiving-pure/taverneamendment.

## Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

# Limit-point/limit-circle classification for Hain-Lüst type equations 

Seppo Hassi ${ }^{1}$ | Manfred Möller ${ }^{2}$ | Henk de Snoo ${ }^{3}$

${ }^{1}$ Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, 65101 Vaasa, Finland
${ }^{2}$ The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, Wits, 2050, South Africa
${ }^{3}$ Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, Nederland

## Correspondence

Professor Henk de Snoo, Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, Nederland
Email: hsvdesnoo@gmail.com

## Funding Information

Finnish Academy of Science and Letters; NRF of South Africa, Grant No. 69659.


#### Abstract

Hain-Lüst equations appear in magnetohydrodynamics. They are Sturm-Liouville equations with coefficients depending rationally on the eigenvalue parameter. In this paper such equations are connected with a $2 \times 2$ system of differential equations, where the dependence on the eigenvalue parameter is linear. By means of this connection Weyl's fundamental limit-point/limit-circle classification is extended to a general setting of Hain-Lüst-type equations.


## KEYWORDS

Hain-Lüst equation, mixed-order differential system, Sturm-Liouville problem, Weyl's limit-point/limitcircle classification

MSC (2010)
Primary: 34B20; Secondary: 34A30, 34B07, 34B24

## 1 | INTRODUCTION

Let $-D p D+q$ be a Sturm-Liouville expression on the interval $[0, \infty$, where the coefficient functions $p$ and $q$ are real-valued and measurable with $p \neq 0$ a. e. and $D$ denotes differentiation with respect to the single variable. In [48] it was shown by Weyl that for this expression there is the so-called limit-point/limit-circle alternative. This means firstly that for every nonreal $\lambda$, the set of solutions of $(-D p D+q) y=\lambda y$ belonging to $L^{2}[0, \infty)$ is a vector space of dimension 2 or 1 , and secondly that either for each $\lambda \in \mathbb{C}$ this solution space has dimension at most 1 or for each $\lambda \in \mathbb{C}$ its dimension is 2 . Years later Weyl came back to this theme in [49, Equation (13)], in which he gave a similar treatment for a first-order system of differential equations where the eigenvalue parameter appears in the coefficients in a linear-fractional way.

In the present paper a Sturm-Liouville expression will be considered on $[0, \infty)$ whose coefficient functions depend rationally on the eigenvalue parameter. In fact, this Sturm-Liouville expression is closely connected to the differential expression

$$
\mathbb{L}=\left(\begin{array}{cc}
-D p D+q & -D c+a  \tag{1.1}\\
c D+a & r
\end{array}\right)
$$

which is a $2 \times 2$ system of mixed-order formal differential expressions considered on the interval $[0, \infty)$, where the coefficient functions $p, q, c, r, a$ are real-valued and measurable with $p \neq 0$ a. e. Appropriate additional assumptions on the coefficient functions will be given later. It is Weyl's classical approach that will be generalized to the system $\mathbb{L}$. To this end it is important to observe the following. For any solution $y=\left(y_{1}, y_{2}\right)^{\top}$ of the equation $(\mathbb{L}-\lambda) y=0$ the first component $y_{1}$ is a solution of the (generalized) Sturm-Liouville equation

$$
\begin{equation*}
-\left(\omega(\cdot, \lambda) y_{1}^{\prime}\right)^{\prime}+(\tilde{q}(\cdot, \lambda)-\lambda) y_{1}=0 \tag{1.2}
\end{equation*}
$$

where the coefficients are given by

$$
\begin{equation*}
\omega(\cdot, \lambda)=p+\frac{c^{2}}{\lambda-r}, \quad \tilde{q}(\cdot, \lambda)=q-\left(\frac{c a}{\lambda-r}\right)^{\prime}+\frac{a^{2}}{\lambda-r} \tag{1.3}
\end{equation*}
$$

while the second component $y_{2}$ is related to the first component $y_{1}$ via

$$
\begin{equation*}
y_{2}=\frac{c}{\lambda-r} y_{1}^{\prime}+\frac{a}{\lambda-r} y_{1} . \tag{1.4}
\end{equation*}
$$

Conversely, it is easily seen that any solution $y_{1}$ of (1.2) with $y_{2}$ defined by (1.4) gives a solution $y=\left(y_{1}, y_{2}\right)^{\top}$ of $(\mathbb{L}-\lambda) y=0$. Note that the coefficient functions in (1.3) depend rationally on the eigenvalue parameter. Of course, restrictions on $\lambda$ have to be imposed such that (1.3) and (1.4) are well-defined. Although the differential equation (1.2) for $y_{1}$ can be considered in its own right, $y_{2}$ given by (1.4) is needed to find the appropriate regularity condition for the solutions, i. e., $y \in\left(L^{2}[0, \infty)\right)^{2}$ rather than $y_{1} \in L^{2}[0, \infty)$ turns out to be the adequate criterion for the limit-point/limit-circle classification.

The first part of the Weyl approach, the pointwise limit-point/limit-circle alternative for $\lambda \in \mathbb{C} \backslash \mathbb{R}$, is based on Green's formula for $\mathbb{L}$, and as such is a generalization of the results obtained in [13] for the case $a=0$. Subject to a suitable definition of the quasi-derivative, the proof is identical to that in the classical case. For the second part, Equation (1.2) will be used, and the classical variation of constants approach will be employed. This requires the components $y_{1}$ and $y_{2}$ of all solutions $y$ involved, although originally only the component $y_{1}$ occurs in (1.2). The occurrence of the function $y_{2}$ is due to the fact that integration by parts has to be used in order to obtain useful estimates. Indeed, taking two linearly independent solutions $\varphi_{1}$ and $\psi_{1}$ of (1.2) with $\lambda=\lambda_{0}$ and a solution $\chi_{1}$ of (1.2), the variation of constants formula already involves $\varphi_{2}, \psi_{2}$ and $\chi_{2}$, so that they must be taken into account. This integration by parts requires certain regularity conditions on the coefficient functions.

The motivation to consider expressions as in (1.1) or equations as in (1.2) is provided by their occurrence in magnetohydrodynamics. After separation of variables, a cylindrical plasma configuration can be described by a $3 \times 3$ system, where one component is a second-order ordinary differential equation, whereas all other components are first-order differential equations or multiplications, see [11, (8)]; see also [10, (9.28)], [33]. Eliminating two of the three components of the system leads to the celebrated Hain-Lüst equation, see [11, (9)] or [10, (9.31)], which is a second-order ordinary differential equation depending rationally on the spectral parameter. More precisely, the $3 \times 3$ system is a $2 \times 2$ block system whose block diagonal elements are a second-order differential operator and a multiplication operator, respectively, whereas the off-diagonal blocks are first-order differential operators. An abstract theory of $2 \times 2$ blocks of unbounded operators has been initiated in [4,12]; the above $3 \times 3$ system was shown to be a particular example, where the Hain-Lüst equation is the Schur complement of the multiplication block.

Various aspects related to such blocks of differential operators of different order or corresponding differential operators depending rationally on the eigenvalue parameter, such as selfadjoint realizations, spectral functions, and essential spectra have been dealt with in the literature. Results on the essential spectrum can be found in $[1,4,8,9,12,13,19,25,29,35,36,43]$; see also [47]. Titchmarsh-Weyl theory for second-order differential equations depending rationally on the eigenvalue parameter was considered in $[1,3,6,13,28,31,46]$, where apart from [3] the denominator of the rational coefficients depends linearly on the eigenvalue parameter, that is, the problem is of the form (1.2). The simplest form of such a $2 \times 2$ block system of differential operators is when the system itself is $2 \times 2$, and if such a system is additionally formally selfadjoint, then it is of the form (1.1). Most of the publications above have dealt with this case, in particular when $c=0$ (and $p=1$ ).

As indicated above in the present setting it is the original approach of Weyl which is being adapted to treat the delicate interaction between systems of the form (1.1) and differential expressions in (1.2) with coefficients of the form (1.3). The results in the present paper provide a suitable framework for an operator-theoretic and spectral-theoretic treatment of boundary value problems for Hain-Lüst-type differential expressions generated by the system of differential equations in (1.1) along the lines of the particular case treated in [13]. In particular, the limit-point/limit-circle classification will play a fundamental role when associating appropriate boundary triplets to the differential expression $\mathbb{L}$; see also the remarks at the end of the paper.

An outline of this paper follows. In Section 2, Green's formula for the differential expression is derived. The second-order differential expression generated by $\mathbb{L}$ is considered in Section 3, which also contains an existence and uniqueness result for an initial value problems associated with the equation $(\mathbb{L}-\lambda) y=0$. In Section 4 and Section 5 the limit-point/limit-circle alternative will be shown for the system (1.1) under mild conditions on the coefficient functions; see Theorem 4.1 and Corollary 5.4. Furthermore, conditions on $\lambda_{0}, \lambda \in \mathbb{C}$ will be given such that limit-circle at $\lambda_{0}$ implies limit-circle at $\lambda$ when $\lambda_{0}$ and $\lambda$ do not belong to the essential range of the functions $r$ and $c^{2} / p$; see Theorem 5.3. These results generalize the classical result known for Sturm-Liouville expressions of the form $-D p D+q$; see Corollary 6.1. In addition, in Section 6 the effect of the coefficients
$a$ and $c$, determining the off-diagonal entries in (1.1), on the limit-point/limit-circle classification is discussed by means of some examples.

For a measurable set $\Lambda \subset \mathbb{R}$ and a measurable function $f: \Lambda \rightarrow \mathbb{C}$ the spectrum of the multiplication operator induced by $f$ is denoted by

$$
\sigma(f)=\left\{\lambda \in \mathbb{C}: \operatorname{ess}_{\inf }^{x \in \Lambda} ⿵ ⺆|f(x)-\lambda|=0\right\}
$$

Note that $\sigma(f)$ coincides with the essential range of $f$.
In order not to make the formulation unnecessarily lengthy, notations like almost everywhere are dropped or essential limit inferior is shortened to limit inferior, which imposes no loss of generality if the functions involved are suitably changed on a set of Lebesgue measure zero. In this sense, the support of $f$ defined on $[0, \infty), \operatorname{supp}(f)$, is the closure of $\{x \in[0, \infty): f(x) \neq 0\}$.

## 2 | GREEN'S FORMULA FOR THE DIFFERENTIAL EXPRESSION $\mathbb{L}$

Some formal properties of the differential expression $\mathbb{L}$ in (1.1), which is a $2 \times 2$ system of mixed-order formal differential expressions, will be presented. The following set of conditions on the coefficients guarantees that the differential expression $\mathbb{L}$ is well-defined.

Assumption 2.1. The functions $p, q, c, r, a:[0, \infty) \rightarrow \mathbb{R}$ are measurable and satisfy the following conditions:
(i) $p(x) \neq 0$ for all $x \in[0, \infty)$;
(ii) $1 / p$ and $q$ belong to $L_{\mathrm{loc}}^{1}[0, \infty)$;
(iii) a belongs to $L_{\text {loc }}^{2}[0, \infty)$;
(iv) $c^{2} / p$ and $r$ belong to $L_{\mathrm{loc}}^{\infty}[0, \infty)$.

In the rest of this paper the following notation

$$
\begin{equation*}
\gamma=c^{2} / p \tag{2.1}
\end{equation*}
$$

will be useful; by assumption the function $\gamma$ belongs to $L_{\text {loc }}^{\infty}[0, \infty)$. Observe that Assumption 2.1 has as a direct consequence: $(c / p)^{2}=(1 / p) \gamma \in L_{\mathrm{loc}}^{1}[0, \infty)$ or

$$
\begin{equation*}
c / p \in L_{\mathrm{loc}}^{2}[0, \infty) \tag{2.2}
\end{equation*}
$$

Let $A C_{\text {loc }}[0, \infty)$ denote the set of all functions which are absolutely continuous on every compact subset of $[0, \infty)$. For a 2-vector function $y \in\left(L_{\mathrm{loc}}^{2}[0, \infty)\right)^{2}$ whose first component $y_{1}$ belongs to $A C_{\mathrm{loc}}[0, \infty)$ define the following quasi-derivative

$$
\begin{equation*}
y^{[1]}=p y_{1}^{\prime}+c y_{2} . \tag{2.3}
\end{equation*}
$$

The next result describes a natural domain on which the differential expression in $\mathbb{L}$ in (1.1) is meaningful. This interpretation of $\mathbb{L}$ will be used in the rest of this paper.

Proposition 2.2. Let the domain of $\mathbb{L}$ be defined by

$$
\begin{equation*}
D(\mathbb{L})=\left\{y \in\left(L_{\mathrm{loc}}^{2}[0, \infty)\right)^{2}: y_{1}, y^{[1]} \in A C_{\mathrm{loc}}[0, \infty)\right\} . \tag{2.4}
\end{equation*}
$$

Then $\mathbb{L}$ maps $D(\mathbb{L})$ into $L_{\mathrm{loc}}^{1}[0, \infty) \times L_{\mathrm{loc}}^{2}[0, \infty)$ and

$$
\begin{equation*}
\mathbb{L} y=\binom{-D y^{[1]}+q y_{1}+a y_{2}}{c D y_{1}+a y_{1}+r y_{2}} \tag{2.5}
\end{equation*}
$$

for all $y \in D(\mathbb{L})$.

Proof. It follows from (1.1) that for $y \in D(\mathbb{\mathbb { L }})$,

$$
\mathbb{L y =}\left(\begin{array}{cc}
-D p D+q & -D c+a \\
c D+a & r
\end{array}\right)\binom{y_{1}}{y_{2}}=\binom{-D p D y_{1}+q y_{1}-D c y_{2}+a y_{2}}{c D y_{1}+a y_{1}+r y_{2}}
$$

where $D p D y_{1}+D c y_{2}$ has to be understood as meaning $D\left(p D y_{1}+c y_{2}\right)=D y^{[1]}$. Therefore $\mathbb{\square} y$ is well-defined for $y \in D(\mathbb{L})$ and it has the representation (2.5). It is clear from Assumption 2.1 that the first component of $\llbracket y$ belongs to $L_{\mathrm{loc}}^{1}[0, \infty)$ and also that $a y_{1}+r y_{2}$ belongs to $L_{\mathrm{loc}}^{2}[0, \infty)$. Furthermore,

$$
D y_{1}=\frac{1}{p}\left(p D y_{1}+c y_{2}\right)-\frac{c}{p} y_{2}
$$

so that by definition of $D(\mathbb{L})$ and by Assumption 2.1 and its consequence (2.2) it follows that

$$
c D y_{1}=\frac{c}{p} y^{[1]}-\frac{c^{2}}{p} y_{2} \in L_{\mathrm{loc}}^{2}[0, \infty)
$$

Hence the second component of $\mathbb{L y}$ belongs to $L_{\mathrm{loc}}^{2}[0, \infty)$, which completes the proof.
For $u, v \in D(\mathbb{\mathbb { }})$ define the Lagrange bracket of $u$ and $v$ by

$$
\begin{equation*}
[u(t), v(t)]=u_{1}(t) \overline{v^{[1]}(t)}-u^{[1]}(t) \overline{v_{1}(t)}, \quad t \in[0, \infty) \tag{2.6}
\end{equation*}
$$

Clearly, the Lagrange bracket is locally absolutely continuous. Furthermore, one sees that $(\mathbb{L} u)^{\top} \bar{v} \in L_{\text {loc }}^{1}[0, \infty)$ by Proposition 2.2, and therefore

$$
(u, v)_{b}=\int_{0}^{b}(\mathbb{L} u)^{\top}(t) \overline{v(t)} d t
$$

is well-defined for all $b \in(0, \infty)$. Green's formula for $\mathbb{L}$ can now be formulated and shown.
Lemma 2.3. Let $b>0$ and let $u, v \in D(\mathbb{L})$. Then

$$
\begin{equation*}
(u, v)_{b}-\overline{(v, u)_{b}}=[u(b), v(b)]-[u(0), v(0)] . \tag{2.7}
\end{equation*}
$$

Proof. By Proposition 2.2,

$$
\begin{aligned}
(u, v)_{b}= & -\int_{0}^{b}\left(u^{[1]}\right)^{\prime}(t) \overline{v_{1}(t)} d t+\int_{0}^{b}\left(q(t) u_{1}(t)+a(t) u_{2}(t)\right) \overline{v_{1}(t)} d t \\
& +\int_{0}^{b}\left(c(t) u_{1}^{\prime}(t)+a(t) u_{1}(t)+r(t) u_{2}(t)\right) \overline{v_{2}(t)} d t
\end{aligned}
$$

Integration by parts leads to

$$
\begin{aligned}
(u, v)_{b}= & -\left.\left(u^{[1]}(t) \overline{v_{1}(t)}\right)\right|_{0} ^{b}+\int_{0}^{b} u^{[1]}(t) \overline{v_{1}^{\prime}(t)} d t+\int_{0}^{b}\left(q(t) u_{1}(t)+a(t) u_{2}(t)\right) \overline{v_{1}(t)} d t \\
& +\int_{0}^{b}\left(c(t) u_{1}^{\prime}(t)+a(t) u_{1}(t)+r(t) u_{2}(t)\right) \overline{v_{2}(t)} d t
\end{aligned}
$$

Hence, by symmetry,

$$
\begin{aligned}
\overline{(v, u)_{b}}= & -\left.\left(u_{1}(t) \overline{v^{[1]}(t)}\right)\right|_{0} ^{b}+\int_{0}^{b} u_{1}^{\prime}(t) \overline{v^{[1]}(t)} d t+\int_{0}^{b}\left(q(t) \overline{v_{1}(t)}+a(t) \overline{v_{2}(t)}\right) u_{1}(t) d t \\
& +\int_{0}^{b}\left(c(t) \overline{v_{1}^{\prime}(t)}+a(t) \overline{v_{1}(t)}+r(t) \overline{v_{2}(t)}\right) u_{2}(t) d t
\end{aligned}
$$

It follows that

$$
(u, v)_{b}-\overline{(v, u)_{b}}=[u(b), v(b)]-[u(0), v(0)]+\int_{0}^{b}\left\{u^{[1]}(t) \overline{v_{1}^{\prime}(t)}+c(t) u_{1}^{\prime}(t) \overline{v_{2}(t)}-u_{1}^{\prime}(t) \overline{v^{[1]}(t)}-c(t) \overline{v_{1}^{\prime}(t)} u_{2}(t)\right\} d t
$$

where the integrand on the right-hand side is zero in view of

$$
u^{[1]} \overline{v_{1}^{\prime}}-u_{1}^{\prime} \overline{v^{[1]}}=\left(p u_{1}^{\prime}+c u_{2}\right) \overline{v_{1}^{\prime}}-u_{1}^{\prime}\left(p \overline{v_{1}^{\prime}}+c \overline{v_{2}}\right)=c u_{2} \overline{v_{1}^{\prime}}-u_{1}^{\prime} c \overline{v_{2}} .
$$

Therefore (2.7) has been shown.
The following consequence of Green's formula is now immediate.
Lemma 2.4. Let $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \bar{\mu}$, let $u(\cdot, \lambda) \in D(\mathbb{C})$ be a solution of $(\mathbb{L}-\lambda) y=0$, and let $v(\cdot, \mu) \in D(\mathbb{L})$ be a solution of $(\mathbb{L}-\mu) y=0$. Then

$$
\begin{equation*}
\int_{0}^{b} u(t, \lambda)^{\top} \overline{v(t, \mu)} d t=\frac{[u(b, \lambda), v(b, \mu)]-[u(0, \lambda), v(0, \mu)]}{\lambda-\bar{\mu}} \tag{2.8}
\end{equation*}
$$

holds for all $b \in(0, \infty)$.
Proof. Observing that

$$
(\lambda-\bar{\mu}) \int_{0}^{b} u(t, \lambda)^{\bar{\top}} \overline{v(t, \mu)} d t=(u(\cdot, \lambda), v(\cdot, \mu))_{b}-\overline{(v(\cdot, \mu), u(\cdot, \lambda))_{b}},
$$

(2.8) follows immediately from (2.7).

## 3 | THE HOMOGENEOUS DIFFERENTIAL EQUATION

The solutions of the equation $(\mathbb{L}-\lambda) y=0$ with $y \in D(\mathbb{L})$ will now be connected to the solutions of the equations (1.2) and (1.4); moreover, an existence and uniqueness theorem will be presented for the corresponding initial value problems. In order to give an appropriate meaning to formulas like (1.2) and (1.4) one needs to restrict the parameter $\lambda \in \mathbb{C}$.

Definition 3.1. The sets $\Omega$ and $\Omega^{\prime}$ of $\llbracket$-admissible numbers $\lambda \in \mathbb{C}$ are defined by

$$
\Omega=\mathbb{C} \backslash \sigma(r), \quad \Omega^{\prime}=\mathbb{C} \backslash(\sigma(r) \cup \sigma(r-\gamma))
$$

respectively, where $\gamma$ is as in (2.1).
Now the conditions in the following Assumption 3.2 will be assumed; they are stronger than the ones in Assumption 2.1.
Assumption 3.2. The functions $p, q, c, r, a:[0, \infty) \rightarrow \mathbb{R}$ are measurable, satisfy the conditions (i)-(iv) from Assumption 2.1, and satisfy in addition:
(v) the functions $a$ and $c$ are locally absolutely continuous on $[0, \infty)$;
(vi) for all $d_{1}>0, r$ is absolutely continuous on some open neighborhood of $\operatorname{supp}(c a) \cap\left[0, d_{1}\right)$ in $[0, \infty)$.

Observe that it immediately follows from Assumption 3.2, (v)-(vi) that for $\lambda \in \Omega$ one has

$$
\begin{equation*}
\frac{c a}{\lambda-r} \in A C_{\mathrm{loc}}[0, \infty) \tag{3.1}
\end{equation*}
$$

Proposition 3.3. Let $\lambda \in \Omega$ and assume that $(\mathbb{L}-\lambda) y=0$ with $y \in D(\mathbb{L})$. Then, with the quasi-derivative $y^{[1]}$ defined as in (2.3), one has

$$
\begin{equation*}
\omega(\cdot, \lambda) y_{1}^{\prime}=y^{[1]}-\frac{c a}{\lambda-r} y_{1} \tag{3.2}
\end{equation*}
$$

and $\omega(\cdot, \lambda) y_{1}^{\prime} \in A C_{\mathrm{loc}}[0, \infty)$. Moreover, (1.2) and (1.4) are satisfied.

Proof. Recall from (2.4) that $y \in D(\mathbb{L})$ means that $y_{1}, y_{2} \in L_{\mathrm{loc}}^{2}[0, \infty)$ and that $y_{1}, y^{[1]} \in A C_{\mathrm{loc}}[0, \infty)$. Observing the representation (2.5) of $\mathbb{L} y$ one sees from the second component of $(\mathbb{L}-\lambda) y=0$ that $y_{2}$ can be expressed by $y_{1}$ as in (1.4). It then follows that

$$
y^{[1]}-\frac{c a}{\lambda-r} y_{1}=p y_{1}^{\prime}+c y_{2}-\frac{c a}{\lambda-r} y_{1}=p y_{1}^{\prime}+\frac{c^{2}}{\lambda-r} y_{1}^{\prime}+\frac{c a}{\lambda-r} y_{1}-\frac{c a}{\lambda-r} y_{1}=\omega(\cdot, \lambda) y_{1}^{\prime},
$$

which is (3.2).
As $y_{1}, y^{[1]} \in A C_{\mathrm{loc}}[0, \infty)$, it follows from (3.1) and (3.2) that $\omega(\cdot, \lambda) y_{1}^{\prime} \in A C_{\mathrm{loc}}[0, \infty)$. It remains to show that (1.2) holds. By substituting $y_{2}$ as in (1.4) to the first component of $(\mathbb{L}-\lambda) y=0$ and applying (3.2) one obtains

$$
\begin{aligned}
0 & =-D y^{[1]}+\frac{c a}{\lambda-r} y_{1}^{\prime}+\frac{a^{2}}{\lambda-r} y_{1}+(q-\lambda) y_{1} \\
& =-\left(\omega(\cdot, \lambda) y_{1}^{\prime}\right)^{\prime}-\left(\frac{c a}{\lambda-r}\right)^{\prime} y_{1}+\frac{a^{2}}{\lambda-r} y_{1}+(q-\lambda) y_{1} \\
& =-\left(\omega(\cdot, \lambda) y_{1}^{\prime}\right)^{\prime}+(\tilde{q}(\cdot, \lambda)-\lambda) y_{1} .
\end{aligned}
$$

Thus also (1.2) is satisfied.
The above proof shows that the left-hand side of (1.2) is the Schur complement in $\mathbb{L}-\lambda$ of its lower right entry $r-\lambda$.
Lemma 3.4. Let $\lambda \in \Omega$ and let $u, v \in D(\mathbb{L})$ be solutions of $(\mathbb{L}-\lambda) y=0$. Then

$$
\begin{equation*}
u^{[1]} v_{1}-u_{1} v^{[1]}=\omega(\cdot, \lambda) u_{1}^{\prime} v_{1}-u_{1} \omega(\cdot, \lambda) v_{1}^{\prime} \tag{3.3}
\end{equation*}
$$

is constant.
Proof. In view of Proposition 3.3, (3.2) and (1.2) hold. Thus the identity (3.3) follows. Differentiating this identity and using (1.2) lead to

$$
\left(u^{[1]} v_{1}-u_{1} v^{[1]}\right)^{\prime}=\left(\omega(\cdot, \lambda) u_{1}^{\prime}\right)^{\prime} v_{1}-u_{1}\left(\omega(\cdot, \lambda) v_{1}^{\prime}\right)^{\prime}=(\tilde{q}(\cdot, \lambda)-\lambda) u_{1} v_{1}-u_{1}(\tilde{q}(\cdot, \lambda)-\lambda) v_{1}=0
$$

The further study of the solutions of the equation $(\mathbb{L}-\lambda) y=0$ requires a restriction to the set $\Omega^{\prime} \subset \Omega$ of $\mathbb{L}$-admissible numbers; cf. Definition 3.1.

Lemma 3.5. Let $\lambda \in \Omega^{\prime}$. Then

$$
1 / \omega(\cdot, \lambda) \in L_{\mathrm{loc}}^{1}[0, \infty), \quad \tilde{q}(\cdot, \lambda) \in L_{\mathrm{loc}}^{1}[0, \infty)
$$

Proof. The first statement immediately follows from (1.3), (2.1), Assumption 2.1, and

$$
\begin{equation*}
\omega(\cdot, \lambda)=\frac{p}{\lambda-r}(\lambda-r+\gamma) . \tag{3.4}
\end{equation*}
$$

The second statement is clear by Assumption 3.2 and by (3.1).
The next result is a converse to Proposition 3.3.
Proposition 3.6. Let $\lambda \in \Omega^{\prime}$, let $y_{1} \in A C_{\mathrm{loc}}[0, \infty)$, and assume that $\omega(\cdot, \lambda) y_{1}^{\prime} \in A C_{\mathrm{loc}}[0, \infty)$. Then

$$
\begin{equation*}
y_{2}=\frac{c}{\lambda-r} y_{1}^{\prime}+\frac{a}{\lambda-r} y_{1} \in L_{\mathrm{loc}}^{2}[0, \infty) \tag{3.5}
\end{equation*}
$$

and $y=\left(y_{1}, y_{2}\right)^{\top} \in D(\mathbb{L})$. Moreover, if $y_{1}$ satisfies $(1.2)$ then $(\mathbb{L}-\lambda) y=0$.
Proof. In order to check (3.5) observe that (3.4), (2.2), the definition of $\Omega^{\prime}$, and the present assumptions show that

$$
\frac{c}{\lambda-r} y_{1}^{\prime}=\frac{c}{\lambda-r} \frac{1}{\omega(\cdot, \lambda)} \omega(\cdot, \lambda) y_{1}^{\prime}=\frac{c}{p} \frac{1}{\lambda-(r-\gamma)} \omega(\cdot, \lambda) y_{1}^{\prime} \in L_{\mathrm{loc}}^{2}[0, \infty) .
$$

By Assumption (2.1) and $\lambda \in \Omega^{\prime}$ one sees that $a(\lambda-r)^{-1} \in L_{\mathrm{loc}}^{2}[0, \infty)$, so that $a(\lambda-r)^{-1} y_{1} \in L_{\mathrm{loc}}^{2}[0, \infty)$. This proves (3.5).

By the definitions (2.3) of $y^{[1]}$ and (3.5) of $y_{2}$ it is now clear by direct substitution that

$$
y^{[1]}=p y_{1}^{\prime}+c y_{2}=\omega(\cdot, \lambda) y_{1}^{\prime}+\frac{c a}{\lambda-r} y_{1} .
$$

Thus by (3.1) and the present assumptions it follows that $y^{[1]} \in A C_{\mathrm{loc}}[0, \infty)$. Hence $y \in D(\mathbb{C})$, and the second component of $(\mathbb{L}-\lambda) y$ is zero. One sees by direct substitution that

$$
\begin{aligned}
-D y^{[1]}+q y_{1}+a y_{2}-\lambda y_{1} & =-\left(\omega(\cdot, \lambda) y_{1}^{\prime}+\frac{c a}{\lambda-r} y_{1}\right)^{\prime}+\left(\frac{c a}{\lambda-r} y_{1}^{\prime}+\frac{a^{2}}{\lambda-r} y_{1}\right)+(q-\lambda) y_{1} \\
& =-\left(\omega(\cdot, \lambda) y_{1}^{\prime}\right)^{\prime}+(\tilde{q}(\cdot, \lambda)-\lambda) y_{1} .
\end{aligned}
$$

Hence, if $y_{1}$ satisfies (1.2) then also the first component of $(\mathbb{L}-\lambda) y$ is zero.
Together, Proposition 3.3 and Proposition 3.6 show the following correspondence. Recall that Assumption 3.2 is used throughout.

Corollary 3.7. For $\lambda \in \Omega^{\prime}$ the equation $(\mathbb{L}-\lambda) y=0$ is equivalent to the pair of equations (1.2), (1.4). In particular, there is $a$ one-to-one relation between the solutions $y$ of $(\mathbb{L}-\lambda) y=0$ and the solutions $y_{1}$ of (1.2).

Theorem 3.8. Let $\lambda \in \Omega^{\prime}$ and let $c_{1}, c_{2} \in \mathbb{C}$. Then the initial value problem

$$
(\mathbb{L}-\lambda) y=0, \quad y_{1}(0)=c_{1}, \quad y^{[1]}(0)=c_{2},
$$

has a unique solution $y=y(\cdot, \lambda) \in D(\mathbb{L})$. For a fixed $x \in[0, \infty)$ the functions $\lambda \mapsto y_{1}(x, \lambda)$ and $\lambda \mapsto y^{[1]}(x, \lambda)$ are holomorphic on $\Omega^{\prime}$.

Proof. Uniqueness. Let $y \in D(\mathbb{L})$ be a solution of $(\mathbb{L}-\lambda) y=0$ with $y_{1}(0)=c_{1}$ and $y^{[1]}(0)=c_{2}$. From (3.2) it follows that

$$
\begin{equation*}
y_{1}^{\prime}=-\frac{c a}{(\lambda-r) \omega(\cdot, \lambda)} y_{1}+\frac{1}{\omega(\cdot, \lambda)} y^{[1]} . \tag{3.6}
\end{equation*}
$$

According to Proposition 3.3 one has (1.2) with (1.3). Thus differentiating (3.2) and using (3.6) leads to

$$
\begin{align*}
D y^{[1]} & =\left(\frac{c a}{\lambda-r}\right)^{\prime} y_{1}+\frac{c a}{\lambda-r} y_{1}^{\prime}+\left(\omega(\cdot, \lambda) y_{1}^{\prime}\right)^{\prime} \\
& =\left(\frac{c a}{\lambda-r}\right)^{\prime} y_{1}+\frac{c a}{\lambda-r} y_{1}^{\prime}+\left(q-\left(\frac{c a}{\lambda-r}\right)^{\prime}+\frac{a^{2}}{\lambda-r}-\lambda\right) y_{1} \\
& =\left(-\frac{c^{2} a^{2}}{(\lambda-r)^{2} \omega(\cdot, \lambda)}+q+\frac{a^{2}}{\lambda-r}-\lambda\right) y_{1}+\frac{c a}{(\lambda-r) \omega(\cdot, \lambda)} y^{[1]} \tag{3.7}
\end{align*}
$$

Hence by (3.6) and (3.7) one has the first-order system of differential equations

$$
\begin{equation*}
\binom{y_{1}}{y^{[1]}}^{\prime}=A(\cdot, \lambda)\binom{y_{1}}{y^{[1]}}, \tag{3.8}
\end{equation*}
$$

where $A(\cdot, \lambda)$ is a $2 \times 2$ matrix function whose entries belong to $L_{\mathrm{loc}}^{1}[0, \infty)$ in view of Assumption 3.2, (3.1), and Lemma 3.5 and depend holomorphically on $\lambda$ in $\Omega^{\prime}$. In view of [34, Theorem 2.5.3], the system (3.8) has a unique fundamental matrix $Y(\cdot, \lambda)$ with $Y(0, \lambda)$ being the identity matrix, and the entries of $Y(\cdot, \lambda)$ are locally absolutely continuous on $[0, \infty)$ and depend holomorphically on $\lambda$ in $\Omega^{\prime}$. Then $y_{1}(\cdot, \lambda)$ and $y^{[1]}(\cdot, \lambda)$ are the components of $Y(\cdot, \lambda)\left(c_{1}, c_{2}\right)^{\top}$, which shows that for a fixed $x \in[0, \infty)$ the functions $\lambda \mapsto y_{1}(x, \lambda)$ and $\lambda \mapsto y^{[1]}(x, \lambda)$ are holomorphic on $\Omega^{\prime}$. Furthermore, $\left(y_{1}(\cdot, \lambda), y^{[1]}(\cdot, \lambda)\right)^{\top}$ are uniquely determined by the initial conditions. In view of Proposition 3.3, $y_{2}$ has the representation (1.4), and substituting (3.6) gives

$$
\begin{equation*}
y_{2}=\frac{c}{(\lambda-r) \omega(\cdot, \lambda)} y^{[1]}+\left(\frac{a}{\lambda-r}-\frac{c^{2} a}{(\lambda-r)^{2} \omega(\cdot, \lambda)}\right) y_{1} . \tag{3.9}
\end{equation*}
$$

This shows that the solution of the initial value problem $(\mathbb{L}-\lambda) y=0, y_{1}(0)=c_{1}, y^{[1]}(0)=c_{2}$ is unique.

Existence. Let $c_{1}, c_{2} \in \mathbb{C}$ and define $\left(y_{1}, y^{[1]}\right)^{\top}=Y(\cdot, \lambda)\left(c_{1}, c_{2}\right)^{\top}$, where $Y(\cdot, \lambda)$ is the fundamental matrix of $(3.8)$ with $Y(0, \lambda)$ being the identity matrix. Then $y_{1} \in A C_{\mathrm{loc}}[0, \infty)$ and $y^{[1]} \in A C_{\mathrm{loc}}[0, \infty)$ are solutions of the system (3.6), (3.7). From (3.6) and (3.1), it is easily seen that $\omega(\cdot, \lambda) y_{1}^{\prime} \in A C_{\text {loc }}[0, \infty)$. Substituting $y^{[1]}$ from (3.6) into (3.7) leads to

$$
\begin{aligned}
\left(\omega(\cdot, \lambda) y_{1}^{\prime}\right)^{\prime} & =D y^{[1]}-\left(\frac{c a}{\lambda-r} y_{1}\right)^{\prime} \\
& =\left(-\frac{c^{2} a^{2}}{(\lambda-r)^{2} \omega(\cdot, \lambda)}+q+\frac{a^{2}}{\lambda-r}-\lambda\right) y_{1}+\frac{c a}{\lambda-r} y_{1}^{\prime}+\frac{c^{2} a^{2}}{(\lambda-r)^{2} \omega(\cdot, \lambda)} y_{1}-\left(\frac{c a}{\lambda-r}\right)^{\prime} y_{1}-\left(\frac{c a}{\lambda-r}\right) y_{1}^{\prime} \\
& =\left(q-\left(\frac{c a}{\lambda-r}\right)^{\prime}+\frac{a^{2}}{\lambda-r}-\lambda\right) y_{1}=(\tilde{q}(\cdot, \lambda)-\lambda) y_{1},
\end{aligned}
$$

that is, $y_{1}$ satisfies (1.2). Defining $y_{2}$ by (3.5), it follows that $y^{[1]}=p y_{1}^{\prime}+c y_{2}^{\prime}$; moreover one sees that $y=\left(y_{1}, y_{2}\right) \in D(\mathbb{L})$ and $(\mathbb{L}-\lambda) y=0$ by Proposition 3.6. Hence the initial value problem $(\mathbb{L}-\lambda) y=0, y_{1}(0)=c_{1}, y^{[1]}(0)=c_{2}$ has a solution $y \in D(\mathbb{L})$.
With $\lambda \in \Omega^{\prime}$ it follows from Theorem 3.8 that the mapping $y \mapsto\left(y_{1}(0), y^{[1]}(0)\right)^{\top} \in \mathbb{C}^{2}$ is bijective for solutions $y$ of $(\mathbb{L}-\lambda) y=0$. Hence
Corollary 3.9. Let $\lambda \in \Omega^{\prime}$. The set of solutions $y \in D(\mathbb{L})$ of $(\mathbb{L}-\lambda) y=0$ is a vector space of dimension 2 .
Due to Corollary 3.7, the following result is an obvious consequence of Theorem 3.8.
Corollary 3.10. Let $\lambda \in \Omega^{\prime}$ and let $c_{1}, c_{2} \in \mathbb{C}$. Then the initial value problem (1.2), $y_{1}(0)=c_{1}, y^{[1]}(0)=c_{2}$ with $y_{1}, \omega(\cdot, \lambda) y_{1}^{\prime} \in$ $A C_{\mathrm{loc}}[0, \infty)$ has a unique solution $y_{1}=y_{1}(\cdot, \lambda)$. For fixed $x \in[0, \infty)$ the functions $\lambda \mapsto y_{1}(x, \lambda)$ and $\lambda \mapsto y^{[1]}(x, \lambda)$ are holomorphic on $\Omega^{\prime}$.

Note that according to (3.2) the initial conditions $y_{1}(0)=c_{1}, y^{[1]}(0)=c_{2}$ for the Sturm-Liouville equation (1.2) in Corollary 3.10 stand for

$$
y_{1}(0)=c_{1}, \quad\left[\omega(\cdot, \lambda) y_{1}^{\prime}+\frac{c a}{\lambda-r} y_{1}\right](0)=c_{2},
$$

where the functions $y_{1}, \omega(\cdot, \lambda) y_{1}^{\prime}$, and $\frac{c a}{\lambda-r} y_{1}$ belong to $A C_{\mathrm{loc}}[0, \infty)$.

## 4 | SQUARE-INTEGRABILITY OF SOLUTIONS FOR $\lambda \in \mathbb{C} \backslash \mathbb{R}$

The square-integrability of the solutions of the equation $(\mathbb{L}-\lambda) y=0$ when $\lambda \in \mathbb{C} \backslash \mathbb{R}$ will be considered. The arguments given here are entirely classical once the necessary adaptations have been taken care of.

Let $-\pi / 2<\alpha \leq \pi / 2$. By Corollary 3.9 the eigenvalue problem (1.2) has two linearly independent solutions $\varphi_{1}(\cdot, \lambda)$ and $\psi_{1}(\cdot, \lambda)$, holomorphic in $\lambda \in \Omega^{\prime}$, which are defined by the initial conditions

$$
\begin{cases}\varphi_{1}(0, \lambda)=\sin \alpha, & \psi_{1}(0, \lambda)=\cos \alpha  \tag{4.1}\\ \varphi^{[1]}(0, \lambda)=-\cos \alpha, & \psi^{[1]}(0, \lambda)=\sin \alpha\end{cases}
$$

Note that for a fixed $x \in[0, \infty)$ the functions

$$
\lambda \mapsto \varphi_{1}(x, \lambda), \quad \lambda \mapsto \varphi^{[1]}(x, \lambda), \quad \lambda \mapsto \psi_{1}(x, \lambda), \quad \text { and } \quad \lambda \mapsto \psi^{[1]}(x, \lambda)
$$

are holomorphic on $\Omega^{\prime}$; cf. Corollary 3.10. The vector functions $\varphi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ are defined according to Proposition 3.6.
Theorem 4.1. Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$. Then the set of solutions y of $(\mathbb{L}-\lambda) y=0$ belonging to $\left(L^{2}(0, \infty)\right)^{2}$ is a vector space of dimension 1 or 2.

Proof. The proof follows the usual reasoning in the Sturm-Liouville case, see e.g. [7, Chapter 9, Theorems 2.2 and 2.3], but a number of adaptations is needed. Let $b>0$. Every solution of (1.2) is a linear combination of $\varphi_{1}(\cdot, \lambda)$ and $\psi_{1}(\cdot, \lambda)$. Consider on the interval $[0, b]$ the function $\chi_{1}(\cdot, \lambda, z)$ defined by

$$
\begin{equation*}
\chi_{1}(\cdot, \lambda, z)=\varphi_{1}(\cdot, \lambda)+z \psi_{1}(\cdot, \lambda) \tag{4.2}
\end{equation*}
$$

with a number $z \in \mathbb{C}$. Then at $b$ the function $\chi_{1}(\cdot, \lambda, z)$ satisfies the boundary condition

$$
\begin{equation*}
\cos \beta \chi_{1}(b, \lambda, z)+\sin \beta \chi^{[1]}(b, \lambda, z)=0 \quad \text { for some } \quad \beta \in(-\pi / 2, \pi / 2] \tag{4.3}
\end{equation*}
$$

if and only if $z$ is given by

$$
\begin{equation*}
z=-\frac{A \cot \beta+B}{C \cot \beta+D} \tag{4.4}
\end{equation*}
$$

with the usual interpretation for $\beta=0$, where the numbers $A, B, C$, and $D$ are defined by

$$
A=\varphi_{1}(b, \lambda), \quad B=\varphi^{[1]}(b, \lambda), \quad C=\psi_{1}(b, \lambda), \quad D=\psi^{[1]}(b, \lambda)
$$

By Lemma 3.4 and the initial conditions (4.1),

$$
\begin{equation*}
\varphi_{1}(\cdot, \lambda) \psi^{[1]}(\cdot, \lambda)-\psi_{1}(\cdot, \lambda) \varphi^{[1]}(\cdot, \lambda)=1 \tag{4.5}
\end{equation*}
$$

In other words, with the above definitions one has

$$
A D-B C=1
$$

In terms of the Lagrange bracket notation in (2.6) the above definitions give rise to the following identities:

$$
A \bar{D}-B \bar{C}=[\varphi(b, \lambda), \psi(b, \lambda)], \quad \bar{C} D-C \bar{D}=-[\psi(b, \lambda), \psi(b, \lambda)]
$$

Observing that the initial conditions in (4.1) are real, two particular Lagrange brackets (2.6) have the values

$$
\begin{equation*}
[\varphi(0, \lambda), \varphi(0, \lambda)]=0, \quad[\psi(0, \lambda), \psi(0, \lambda)]=0 \tag{4.6}
\end{equation*}
$$

Since $\lambda \in \mathbb{C} \backslash \mathbb{R}$ it follows from Lemma 2.4 with $\mu=\lambda$ that

$$
\begin{equation*}
\int_{0}^{b}|\psi(t, \lambda)|^{2} d t=\frac{[\psi(b, \lambda), \psi(b, \lambda)]}{2 i \operatorname{Im} \lambda} \tag{4.7}
\end{equation*}
$$

where the following notation is being used: $|d|=\left(d^{\top} \bar{d}\right)^{\frac{1}{2}}$ for $d \in \mathbb{C}^{2}$. In particular, $[\psi(b, \lambda), \psi(b, \lambda)] \neq 0$ for all $b>0$ and thus $\bar{C} D \in \mathbb{C} \backslash \mathbb{R}$. It follows that (4.4) maps the extended real line represented by $\cot \beta, \beta \in(-\pi / 2, \pi / 2]$, onto a circle $C_{b}$ in the $z$-plane. On the other hand, (4.3) means that a nontrivial real linear combination of $\chi_{1}(b, \lambda, z)$ and $\chi^{[1]}(b, \lambda, z)$ is zero, which is equivalent to $\chi_{1}(b, \lambda, z) \overline{\chi^{[1]}(b, \lambda, z)}$ being real. But

$$
[\chi(b, \lambda, z), \chi(b, \lambda, z)]=2 i \operatorname{Im} \chi_{1}(b, \lambda, z) \overline{\chi^{[1]}(b, \lambda, z)}
$$

so that $C_{b}$ is the set of all $z \in \mathbb{C}$ for which $[\chi(b, \lambda, z), \chi(b, \lambda, z)]=0$. Writing this condition in the form

$$
\begin{equation*}
\frac{[\chi(b, \lambda, z), \chi(b, \lambda, z)]}{[\psi(b, \lambda), \psi(b, \lambda)]}=0 \tag{4.8}
\end{equation*}
$$

an expansion of the numerator shows that (4.8) can be written as

$$
\begin{equation*}
z \bar{z}-\frac{\bar{A} D-\bar{B} C}{C \bar{D}-\bar{C} D} z-\frac{B \bar{C}-A \bar{D}}{C \bar{D}-\bar{C} D} \bar{z}+\frac{A \bar{B}-B \bar{A}}{C \bar{D}-\bar{C} D}=0 \tag{4.9}
\end{equation*}
$$

and that the inside of $C_{b}$ corresponds to the left-hand side of (4.9) being $<0$. In particular, the inside of the circle $C_{b}$ is described by those $z \in \mathbb{C}$ with $<$ instead of $=$ in (4.8).

Hence the center $c_{b}$ and the radius $r_{b}$ of the circle $C_{b}$ are given by

$$
c_{b}=\frac{A \bar{D}-B \bar{C}}{\bar{C} D-C \bar{D}}=-\frac{[\varphi(b, \lambda), \psi(b, \lambda)]}{[\psi(b, \lambda), \psi(b, \lambda)]}
$$

$$
\begin{equation*}
r_{b}=\frac{|A D-B C|}{|\bar{C} D-C \bar{D}|}=\frac{1}{|[\psi(b, \lambda), \psi(b, \lambda)]|} . \tag{4.10}
\end{equation*}
$$

Observing that the initial conditions in (4.1) are real, Lemma 2.4, the definition (4.2) of $\chi_{1}(\cdot, \lambda, z)$, (4.5), and (4.6) show that one has

$$
\begin{aligned}
{[\chi(b, \lambda, z), \chi(b, \lambda, z)] } & =2 i \operatorname{Im} \lambda \int_{0}^{b}|\chi(t, \lambda, z)|^{2} d t+[\chi(0, \lambda, z), \chi(0, \lambda, z)] \\
& =2 i \operatorname{Im} \lambda \int_{0}^{b}|\chi(t, \lambda, z)|^{2} d t-2 i \operatorname{Im} z .
\end{aligned}
$$

Hence

$$
\begin{equation*}
\frac{[\chi(b, \lambda, z), \chi(b, \lambda, z)]}{[\psi(b, \lambda), \psi(b, \lambda)]}=\frac{2 i \operatorname{Im} \lambda}{[\psi(b, \lambda), \psi(b, \lambda)]}\left(\int_{0}^{b}|\chi(t, \lambda, z)|^{2} d t-\frac{\operatorname{Im} z}{\operatorname{Im} \lambda}\right), \tag{4.11}
\end{equation*}
$$

and it follows from (4.8) that $z$ in (4.2) is on the circle $C_{b}$ if and only if

$$
\begin{equation*}
\int_{0}^{b}|\chi(t, \lambda, z)|^{2} d t=\frac{\operatorname{Im} z}{\operatorname{Im} \lambda} \tag{4.12}
\end{equation*}
$$

and from (4.7) and (4.11) that $z$ is inside the circle $C_{b}$ if and only if

$$
\begin{equation*}
\int_{0}^{b}|\chi(t, \lambda, z)|^{2} d t<\frac{\operatorname{Im} z}{\operatorname{Im} \lambda} \tag{4.13}
\end{equation*}
$$

Therefore (4.12) and (4.13) imply that $b<b^{\prime}$ leads to the circle $C_{b^{\prime}}$ being inside the circle $C_{b}$.
As $b$ gets larger new circles are contained in previous ones; there is a limit-point when $\lim _{b \rightarrow \infty} r_{b}=0$ and there is a limitcircle when $\lim _{b \rightarrow \infty} r_{b}>0$. Any point $z$ on the limit-circle in the limit-circle case or the limit point $z$ in the limit-point case lies inside $C_{b}$ for any $b>0$ so that the corresponding $\chi(\cdot, \lambda, z) \in\left(L^{2}(0, \infty)\right)^{2}$ as follows from (4.13). Therefore $(\mathbb{L}-\lambda) y=0$ has at least one solution belonging to $\left(L^{2}(0, \infty)\right)^{2}$, which means that the set of solutions of $(\mathbb{L}-\lambda) y=0$ with $y \in\left(L^{2}(0, \infty)\right)^{2}$ forms a vector space of dimension at least one. By Corollary 3.10, this dimension can be at most two. The theorem is proved.

It should be emphasized that the interpretation of Theorem 4.1 for the square-integrability of the solutions $y_{1}$ of the differential equation (1.2) is in conjunction with the square-integrability of the function $y_{2}$ in (1.4). In other words, the square-integrability condition is $y \in\left(L^{2}(0, \infty)\right)^{2}$ rather than just $y_{1} \in L^{2}(0, \infty)$.
In the proof of Theorem 4.1 it was shown that $\chi(\cdot, \lambda, z) \in\left(L^{2}(0, \infty)\right)^{2}$. The identities (4.10) and (4.7) show that in the limitcircle case the solution $\psi(\cdot, \lambda) \in\left(L^{2}(0, \infty)\right)^{2}$ and that in the limit-point case the solution $\psi(\cdot, \lambda) \notin\left(L^{2}(0, \infty)\right)^{2}$. Since $\psi(\cdot, \lambda)$ and $\chi(\cdot, \lambda, z)$ are linearly independent, this leads to the following terminology:
Definition 4.2. The differential expression $\mathbb{L}$ is said to be
(i) limit-point at $\lambda \in \mathbb{C} \backslash \mathbb{R}$ if the set of solutions of $(\mathbb{L}-\lambda) y=0$ which belong to $\left(L^{2}(0, \infty)\right)^{2}$ forms a one-dimensional space;
(ii) limit-circle at $\lambda \in \mathbb{C} \backslash \mathbb{R}$ if all solutions of $(\mathbb{L}-\lambda) y=0$ belong to $\left(L^{2}(0, \infty)\right)^{2}$.

In fact, in the following the notion of limit-circle will be extended. The differential expression $\mathbb{L}$ is said to be limit-circle at any $\lambda \in \Omega^{\prime}$ if all solutions of $(\mathbb{L}-\lambda) y=0$ belong to $\left(L^{2}(0, \infty)\right)^{2}$.

## 5 | LIMIT-POINT/LIMIT-CIRCLE ALTERNATIVE

Assume that at some $\lambda_{0} \in \Omega^{\prime}$ the differential expression $\mathbb{L}$ is limit-circle, i.e., all solutions of $\left(\mathbb{L}-\lambda_{0}\right) y=0$ belong to $\left(L^{2}(0, \infty)\right)^{2}$. The question is now what one can say about the square-integrability of the solutions of $(\mathbb{L}-\lambda) y=0$ at $\lambda \in \Omega^{\prime}$ when $\lambda \neq \lambda_{0}$. In this section it will be shown the the classical limit-point/limit-circle alternative holds on $\Omega^{\prime}$.

Here the coefficient functions of the differential expression $\mathbb{L}$ in (1.1) are assumed to satisfy the following conditions, which are stronger than the ones in Assumption 3.2.

Assumption 5.1. The functions $p, q, c, r, a:[0, \infty) \rightarrow \mathbb{R}$ are measurable, satisfy the conditions (i)-(vi) from Assumption 3.2, and satisfy in addition:
(vii) there exists some $d_{0}>0$ such that for all $d_{1}>d_{0}$, r is absolutely continuous on an open neighborhood of $\operatorname{supp}(c) \cap\left[d_{0}, d_{1}\right]$ in $[0, \infty)$ and $\gamma$ in $(2.1)$ is absolutely continuous on $\left[d_{0}, d_{1}\right]$.

Observe that it immediately follows from Assumption 5.1 that for $\lambda \in \Omega^{\prime}$ one has

$$
\begin{equation*}
\frac{c^{2}}{\omega(\cdot, \lambda)}=\gamma\left(1+\frac{\gamma}{\lambda-r}\right)^{-1} \in A C\left[d_{0}, d_{1}\right] \quad \text { for all } \quad d_{1}>d_{0} \tag{5.1}
\end{equation*}
$$

as follows from (3.4) and (2.1).
In order to formulate a variation of constants formula to express solutions of $(\mathbb{L}-\lambda) y=0$ in terms of solutions of $\left(\mathbb{L}-\lambda_{0}\right) y=$ 0 the function $P\left(\cdot, \lambda_{0}, \lambda\right)$ will be introduced for $\lambda_{0} \in \Omega^{\prime}$ and $\lambda \in \mathbb{C}$ :

$$
\begin{equation*}
P\left(x, \lambda_{0}, \lambda\right)=\frac{\lambda-r(x)+\gamma(x)}{\lambda_{0}-r(x)+\gamma(x)}, \quad x \geq 0 \tag{5.2}
\end{equation*}
$$

Lemma 5.2. Let $\lambda_{0}, \lambda \in \Omega^{\prime}$ be distinct and let $d_{0}$ be as in Assumption 5.1. Let $\varphi, \psi \in D(\mathbb{L})$ be two linearly independent solutions of $\left(\mathbb{L}-\lambda_{0}\right) y=0$ satisfying initial conditions of the form

$$
\begin{cases}\varphi_{1}(0)=\sin \alpha, & \psi_{1}(0)=\cos \alpha  \tag{5.3}\\ \varphi^{[1]}(0)=-\cos \alpha, & \psi^{[1]}(0)=\sin \alpha\end{cases}
$$

Let $\chi \in D(\mathbb{L})$ be a solution of $(\mathbb{L}-\lambda) y=0$. Then for all $x>d>d_{0}$,

$$
\begin{align*}
& \binom{\chi_{1}(x)}{P\left(x, \lambda_{0}, \lambda\right) \chi_{2}(x)}=c_{1}(d) \varphi(x)+c_{2}(d) \psi(x) \\
&  \tag{5.4}\\
& \quad+\left(\lambda-\lambda_{0}\right) \psi(x) \int_{x}^{d} \varphi(t)^{\top} \chi(t) d t+\left(\lambda-\lambda_{0}\right) \varphi(x) \int_{d}^{x} \psi(t)^{\top} \chi(t) d t
\end{align*}
$$

where $c_{1}(d)$ and $c_{2}(d)$ are constants depending on $d$.
Proof. By Assumption 5.1 and Proposition 3.3,

$$
\left(p+\frac{c^{2}}{\lambda_{0}-r}\right) \chi_{1}^{\prime}=\frac{1+\frac{\gamma}{\lambda_{0}-r}}{1+\frac{\gamma}{\lambda-r}}\left(p+\frac{c^{2}}{\lambda-r}\right) \chi_{1}^{\prime}
$$

is locally absolutely continuous on $\left(d_{0}, \infty\right)$. Since $\chi_{1}$ is a solution of $(1.2)$ on $\left(d_{0}, \infty\right)$, it follows that

$$
\begin{aligned}
0 & =-\left(\left[p+\frac{c^{2}}{\lambda-r}\right] \chi_{1}^{\prime}\right)^{\prime}+(\tilde{q}(\cdot, \lambda)-\lambda) \chi_{1} \\
& =-\left(\left[p+\frac{c^{2}}{\lambda_{0}-r}\right] \chi_{1}^{\prime}\right)^{\prime}-\left(\left[\frac{c^{2}}{\lambda-r}-\frac{c^{2}}{\lambda_{0}-r}\right] \chi_{1}^{\prime}\right)^{\prime}+\left(\tilde{q}\left(\cdot, \lambda_{0}\right)-\lambda_{0}\right) \chi_{1}+\left(\tilde{q}(\cdot, \lambda)-\tilde{q}\left(\cdot, \lambda_{0}\right)+\lambda_{0}-\lambda\right) \chi_{1}
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
& -\left(\left[p+\frac{c^{2}}{\lambda_{0}-r}\right] \chi_{1}^{\prime}\right)^{\prime}+\left(\tilde{q}\left(\cdot, \lambda_{0}\right)-\lambda_{0}\right) \chi_{1} \\
& \quad=\left(\lambda-\lambda_{0}\right)\left[-\left(\frac{c^{2}}{(\lambda-r)\left(\lambda_{0}-r\right)} \chi_{1}^{\prime}\right)^{\prime}+\left(1-\left(\frac{c a}{(\lambda-r)\left(\lambda_{0}-r\right)}\right)^{\prime}+\frac{a^{2}}{(\lambda-r)\left(\lambda_{0}-r\right)}\right) \chi_{1}\right] \\
& \quad=\left(\lambda-\lambda_{0}\right)\left[\chi_{1}-\left(\frac{c}{\lambda_{0}-r} \chi_{2}\right)^{\prime}+\frac{a}{\lambda_{0}-r} \chi_{2}\right]
\end{aligned}
$$

where it was used that

$$
\frac{c \chi_{2}}{\lambda_{0}-r}=\frac{c^{2}}{\left(\lambda_{0}-r\right)(\lambda-r) \omega(\cdot, \lambda)} \omega(\cdot, \lambda) \chi_{1}^{\prime}+\frac{c a}{\left(\lambda_{0}-r\right)(\lambda-r)} \chi_{1}
$$

is locally absolutely continuous on $\left(d_{0}, \infty\right)$ by Proposition 3.3, Assumption 5.1, and (5.1). With the solutions $\varphi, \psi \in D(\mathbb{L})$ of $\left(\mathbb{L}-\lambda_{0}\right) y=0$ satisfying the initial conditions (5.3), the variation of constants method, Lemma 3.4, and (4.5) gives, with $d>d_{0}$ and constants $\tilde{c}_{1}, \tilde{c}_{2}$,

$$
\begin{gathered}
\chi_{1}(x)=\tilde{c}_{1} \varphi_{1}(x)+\tilde{c}_{2} \psi_{1}(x)+\left(\lambda-\lambda_{0}\right) \psi_{1}(x) \int_{x}^{d} \varphi_{1}(t)\left(\chi_{1}(t)-\left[\frac{c(t)}{\lambda_{0}-r(t)} \chi_{2}(t)\right]^{\prime}+\frac{a(t)}{\lambda_{0}-r(t)} \chi_{2}(t)\right) d t \\
\quad+\left(\lambda-\lambda_{0}\right) \varphi_{1}(x) \int_{d}^{x} \psi_{1}(t)\left(\chi_{1}(t)-\left[\frac{c(t)}{\lambda_{0}-r(t)} \chi_{2}(t)\right]^{\prime}+\frac{a(t)}{\lambda_{0}-r(t)} \chi_{2}(t)\right) d t
\end{gathered}
$$

Observing that $\varphi_{1}, \psi_{1}, c \chi_{2} /\left(\lambda_{0}-r\right) \in A C_{\mathrm{loc}}[d, \infty)$, integration by parts leads to

$$
\begin{aligned}
\chi_{1}(x)=\tilde{c}_{1} \varphi_{1}(x)+\tilde{c}_{2} \psi_{1}(x)+\left(\lambda-\lambda_{0}\right)\{ & \psi_{1}(x) \int_{x}^{d} \varphi_{1}(t) \chi_{1}(t) d t+\varphi_{1}(x) \int_{d}^{x} \psi_{1}(t) \chi_{1}(t) d t \\
& +\psi_{1}(x) \int_{x}^{d} \varphi_{2}(t) \chi_{2}(t) d t+\varphi_{1}(x) \int_{d}^{x} \psi_{2}(t) \chi_{2}(t) d t \\
& \left.-\psi_{1}(x) \varphi_{1}(d) \frac{c(d)}{\lambda_{0}-r(d)} \chi_{2}(d)+\varphi_{1}(x) \psi_{1}(d) \frac{c(d)}{\lambda_{0}-r(d)} \chi_{2}(d)\right\}
\end{aligned}
$$

so that

$$
\begin{equation*}
\chi_{1}(x)=c_{1}(d) \varphi_{1}(x)+c_{2}(d) \psi_{1}(x)+\left(\lambda-\lambda_{0}\right)\left\{\psi_{1}(x) \int_{x}^{d} \varphi(t)^{\top} \chi(t) d t+\varphi_{1}(x) \int_{d}^{x} \psi(t)^{\top} \chi(t) d t\right\} \tag{5.5}
\end{equation*}
$$

where

$$
\begin{aligned}
& c_{1}(d)=\tilde{c}_{1}+\frac{\lambda-\lambda_{0}}{\lambda_{0}-r(d)} c(d) \psi_{1}(d) \chi_{2}(d) \\
& c_{2}(d)=\tilde{c}_{2}-\frac{\lambda-\lambda_{0}}{\lambda_{0}-r(d)} c(d) \varphi_{1}(d) \chi_{2}(d)
\end{aligned}
$$

This proves the identity (5.4) for the first components. Differentiation of (5.5) leads to

$$
\begin{aligned}
& \frac{\lambda-r(x)}{\lambda_{0}-r(x)} \chi_{2}(x)= c_{1}(d) \varphi_{2}(x)+ \\
& c_{2}(d) \psi_{2}(x) \\
&+\left(\lambda-\lambda_{0}\right)\left\{\psi_{2}(x) \int_{x}^{d} \varphi(t)^{\top} \chi(t) d t+\varphi_{2}(x) \int_{d}^{x} \psi(t)^{\top} \chi(t) d t\right.
\end{aligned} \quad \begin{aligned}
& \left.\quad-\psi_{1}(x) \frac{c(x)}{\lambda_{0}-r(x)} \varphi_{2}(x) \chi_{2}(x)+\varphi_{1}(x) \frac{c(x)}{\lambda_{0}-r(x)} \psi_{2}(x) \chi_{2}(x)\right\} \\
= & c_{1}(d) \varphi_{2}(x)+c_{2}(d) \psi_{2}(x) \\
& +\left(\lambda-\lambda_{0}\right)\left\{\psi_{2}(x) \int_{x}^{d} \varphi(t)^{\top} \chi(t) d t+\varphi_{2}(x) \int_{d}^{x} \psi(t)^{\top} \chi(t) d t+\frac{c^{2}(x)}{\omega\left(x, \lambda_{0}\right)} \frac{\chi_{2}(x)}{\left(\lambda_{0}-r(x)\right)^{2}}\right\}
\end{aligned}
$$

where $\omega\left(\cdot, \lambda_{0}\right) \psi_{1}^{\prime} \varphi_{1}-\omega\left(\cdot, \lambda_{0}\right) \varphi_{1}^{\prime} \psi_{1}=1$ has been used. Rearranging terms, this gives

$$
P\left(x, \lambda_{0}, \lambda\right) \chi_{2}(x)=c_{1}(d) \varphi_{2}(x)+c_{2}(d) \psi_{2}(x)+\left(\lambda-\lambda_{0}\right)\left\{\psi_{2}(x) \int_{x}^{d} \varphi(t)^{\top} \chi(t) d t+\varphi_{2}(x) \int_{d}^{x} \psi(t)^{\top} \chi(t) d t\right\}
$$

This completes the proof of the lemma.

The next result is the main theorem of this paper.
Theorem 5.3. Let $\lambda_{0}, \lambda \in \Omega^{\prime}$ be distinct. Assume that $\mathbb{L}$ is limit-circle at $\lambda_{0}$. Then $\mathbb{L}$ is also limit-circle at $\lambda$.
Proof. Writing $P\left(x, \lambda_{0}, \lambda\right)$ defined by (5.2) in the form

$$
P\left(x, \lambda_{0}, \lambda\right)=1+\frac{\lambda-\lambda_{0}}{\lambda_{0}-r(x)+\gamma(x)}, \quad x \geq 0
$$

one obtains an upper bound, independent of $x \in[0, \infty)$, as follows

$$
\left|P\left(x, \lambda_{0}, \lambda\right)\right| \leq 1+\frac{\left|\lambda-\lambda_{0}\right|}{\inf _{x \in[0, \infty)}\left|\lambda_{0}-r(x)+\gamma(x)\right|}
$$

Interchanging $\lambda$ and $\lambda_{0}$ gives

$$
C_{2}:=\sup _{x \in\left(d_{0}, \infty\right)}\left|P\left(x, \lambda_{0}, \lambda\right)\right|^{-1}<\infty
$$

Let $\varphi, \psi$, and $\chi$ be as in Lemma 5.2, and for $d>d_{0}$ put

$$
M(d):=\max \left\{\left\|\left.\varphi\right|_{(d, \infty)}\right\|,\left\|\left.\psi\right|_{(d, \infty)}\right\|: j=1,2\right\}
$$

where the norm is the norm in $\left(L^{2}(d, \infty)\right)^{2}$. Since $\mathbb{L}$ is limit-circle at $\lambda_{0}$, it follows that $M(d)<\infty$ for all $d>d_{0}, M$ is a decreasing function, and $\lim _{d \rightarrow \infty} M(d)=0$. For $d_{0}<d<x<\infty$ and $j=1,2$ put

$$
N_{j}(d, x):=\left\|\left.\chi_{j}\right|_{(d, x)}\right\|, N(d, x):=\left\|\left.\chi\right|_{(d, x)}\right\|
$$

Applying the Cauchy-Schwarz inequality to the integrals on the right-hand side of (5.4) it follows for all $x>d$ and $j=1,2$ that

$$
\begin{equation*}
\left|\chi_{j}(x)\right| \leq C_{j}\left[\left(\left|c_{1}(d)\right|+\left|c_{2}(d)\right|\right) \max (|\varphi(x)|,|\psi(x)|)+\left|\lambda-\lambda_{0}\right|(|\psi(x)|+|\varphi(x)|) M(d) N(d, x)\right] \tag{5.6}
\end{equation*}
$$

where $C_{1}=1$. Taking $L^{2}$-norms over $(d, b)$ for $d_{0}<d<b$ in (5.6) it follows that

$$
N_{j}(d, b) \leq 2 C_{j}\left[\left(\left|c_{1}(d)\right|+\left|c_{2}(d)\right|\right) M(d)+\left|\lambda-\lambda_{0}\right| M(d)^{2} N(d, b)\right]
$$

Putting $C=2\left(C_{1}+C_{2}\right)$ this gives

$$
N(d, b) \leq C\left[\left(\left|c_{1}(d)\right|+\left|c_{2}(d)\right|\right) M(d)+\left|\lambda-\lambda_{0}\right| M(d)^{2} N(d, b)\right]
$$

Choosing $d$ so large that $C\left|\lambda-\lambda_{0}\right| M(d)^{2}<1$, it follows for all $b>d$ that

$$
N(d, b) \leq C\left(1-C\left|\lambda-\lambda_{0}\right| M(d)^{2}\right)^{-1}\left(\left|c_{1}(d)\right|+\left|c_{2}(d)\right|\right) M(d)
$$

and since the right-hand side is independent of $b$, it follows that $\chi$ belongs to $\left(L^{2}(d, \infty)\right)^{2}$.
Combining Theorem 5.3 with Theorem 4.1 gives the following alternative.
Corollary 5.4. Either all $\lambda \in \mathbb{C} \backslash \mathbb{R}$ are limit-circle or all $\lambda \in \mathbb{C} \backslash \mathbb{R}$ are limit-point.
Proof. If one $\lambda \in \mathbb{C} \backslash \mathbb{R}$ is limit-circle, then all $\lambda \in \mathbb{C} \backslash \mathbb{R}$ are limit-circle by Theorem 5.3. If all $\lambda \in \mathbb{C} \backslash \mathbb{R}$ are not limit-circle, then they must be limit-point by Theorem 4.1.

## 6 | PERTURBATION RESULTS

In the standard Sturm-Liouville case the limit-circle occurs at $\lambda_{0}$ if and only if the limit-circle case occurs at $\lambda$ for any $\lambda_{0}, \lambda \in \mathbb{C}$. This result can be easily deduced as a corollary from Theorem 5.3.

Corollary 6.1. Let $p$ and $q$ be real-valued and measurable functions on $[0, \infty)$ such that $1 / p, q \in L_{\mathrm{loc}}^{1}[0, \infty)$. If $-D p D+q$ is limit-circle at one point in $\mathbb{C}$, then $-D p D+q$ is limit-circle at all points in $\mathbb{C}$.

Proof. Putting $a=c=0$ and $r$ constant, the functions $p, q, a, c, r$ satisfy Assumption 5.1. In particular, $\gamma=0$, and therefore $\Omega^{\prime}=\mathbb{C} \backslash\{r\}$. In view of Theorem 5.3 it follows that the corresponding operator $\mathbb{L}$ is limit-circle at all $\lambda \in \Omega^{\prime}$ if it is limit-circle at one $\lambda \in \Omega^{\prime}$. Since (1.2) becomes $(-D p D+q-\lambda) y_{1}=0$ under the current assumptions, the one-to-one relation between the solutions of (1.2) and $(\mathbb{L}-\lambda) y=0$ established in Propositions 3.3 and 3.6, implies that $-D p D+q$ is limit-circle at all $\lambda \in \Omega^{\prime}$ if it is limit-circle at one $\lambda \in \Omega^{\prime}$. Taking two distinct values for the constant function $r$, it follows that $\lambda \in \Omega^{\prime}$ can be replaced by $\lambda \in \mathbb{C}$.

Hence, in the standard Sturm-Liouville case, either all $\lambda \in \mathbb{C}$ are limit-circle or all $\lambda \in \mathbb{C} \backslash \mathbb{R}$ are limit-point. Now the differential expression (1.2) may be seen as a perturbation of the standard Sturm-Liouville expression. It will be shown as an application of the results in Section 5 that the limit-point/limit-circle alternative is not stable under such perturbations.

In order to substantiate this claim recall the following simple result; see e.g. [22, Example 3.4].
Example 6.2. For $\rho>0$ the differential expression $L_{\rho}:=-D^{2}-\delta x^{\rho}, \delta>0$, on $L^{2}[b, \infty), b \geq 0$, is in the limit-point case if $0<\rho \leq 2$, and in the limit-circle case if $\rho>2$.

In the previous sections, results have been formulated for the interval $[0, \infty)$. Clearly, they are true for any interval $[b, \infty)$, mutatis mutandis. Observe, that the differential expression $\mathbb{L}$ in (1.1) can be interpreted as an, in general, unbounded perturbation of the differential expression

$$
\mathbb{L}^{(0)}=\left(\begin{array}{cc}
-D p D+q & 0  \tag{6.1}\\
0 & r
\end{array}\right),
$$

by the differential expression:

$$
\mathbb{L}^{(1)}=\left(\begin{array}{cc}
0 & -D c+a \\
c D+a & 0
\end{array}\right) .
$$

For $\lambda \in \Omega$ it is clear that $\left(\mathbb{L}^{(0)}-\lambda\right) y=0$ if and only if $(-D p D+q-\lambda) y_{1}=0$ and $y_{2}=0$. Hence, the limit-point/limit-circle alternative for $\mathbb{L}^{(0)}$ is determined by the standard Sturm-Liouville expression $-D p D+q$ appearing in the first diagonal entry in (6.1); cf. Corollary 6.1. Now it will be shown that $\mathbb{L}$ in (1.1) and $\mathbb{L}^{(0)}$ in (6.1) can have different behavior with respect to the limit-point/limit-circle classification depending on the choice of the coefficient functions $a$ and $c$ in the perturbation. Note that even when the perturbation is just off-diagonal multiplication (i.e., when $c=0$ ), this different behaviour may appear. In the first example, $\mathbb{L}^{(0)}$ is limit-circle while $\mathbb{L}$ is limit-point. In the second example, $\mathbb{L}^{(0)}$ is limit-point while $\mathbb{L}$ is limit-circle.

Example 6.3. Consider the differential expression $\mathbb{L}^{(0)}$ in (6.1) on $[b, \infty)$ with the choice $p=1$ and $q=-x^{3}$. It is clear that $L_{3}$ and, therefore, also $\mathbb{L}^{(0)}$ is in the limit-circle case. Now for the differential expression $\mathbb{L}$ in (1.1) different choices of the coefficients $a, c$, and $r$ will be made.
(i) Take $c(x) \equiv 0, r(x) \equiv-1$, and $a(x)=\left(x^{3}-x\right)^{1 / 2}$ for $x \geq b$ with $b=1$. Then it follows from (1.3) that

$$
\omega(x, \lambda)=p(x) \equiv 1, \quad \tilde{q}(x, \lambda)=q(x)+\frac{a^{2}}{\lambda-r}=-x^{3}+\frac{x^{3}-x}{\lambda+1}, \quad x \geq 1
$$

Hence one sees that $\gamma=0$ and that $\Omega=\Omega^{\prime}=\mathbb{C} \backslash\{-1\}$. Furthermore, it is easily checked that Assumption 5.1 is satisfied, so that Theorem 5.3 may be applied. Now consider the point $\lambda=0 \in \Omega^{\prime}$. Then Equation (1.2) takes the form

$$
\left(-D^{2}-x\right) y_{1}=0
$$

and the differential expression in the left-hand side coincides with $L_{1}(\rho=1, \delta=1)$, which by the above discussion is in the limit-point case. In particular, by Corollary $6.1 L_{1}$ cannot be limit-circle at any $\lambda \in \mathbb{C}$. It follows that the differential expression $\mathbb{L}$ cannot be limit-circle at $\lambda=0$. Then by Theorem $5.3 \mathbb{L}$ is limit-point at all $\lambda \in \mathbb{C} \backslash \mathbb{R}$.
(ii) Take $c(x) \equiv 1, r(x) \equiv-1$, and $a(x)=x^{3 / 2}$, for $x \geq b$ with $b=0$. Then it follows from (1.3) that

$$
\omega(x, \lambda)=1+\frac{1}{\lambda+1}, \quad \tilde{q}(x, \lambda)=-x^{3}-\frac{3}{2} \frac{x^{1 / 2}}{\lambda+1}+\frac{x^{3}}{\lambda+1}, \quad x \geq 0
$$

Hence one sees that $\gamma(x) \equiv 1$ and that $\Omega^{\prime}=\mathbb{C} \backslash\{-1,-2\}$. Furthermore, it is easily checked that Assumption 5.1 is satisfied, so that Theorem 5.3 may be applied. For $\lambda=0 \in \Omega^{\prime}$ Equation (1.2) takes the form

$$
2\left(-D^{2}-\frac{3}{4} x^{1 / 2}\right) y_{1}=0
$$

containing the differential expression $L_{1 / 2}(\rho=1 / 2, \delta=3 / 4)$, which by the above discussion is in the limit-point case. The same reasoning as in item (i) shows that the differential expression $\mathbb{L}$ is limit-point at all $\lambda \in \mathbb{C} \backslash \mathbb{R}$.

Example 6.4. Consider the differential expression $\mathbb{L}^{(0)}$ in (6.1) with the choice $p=1$ and $q=1$. The equation $-D^{2} y+(1-\lambda) y=0$ can be explicitly solved and one concludes that $\mathbb{L}^{(0)}$ is in the limit point-case at all $\lambda \in \mathbb{C} \backslash \mathbb{R}$. For the differential expression $\mathbb{L}$ in (1.1) the following choice of the coefficients $a$, $c$, and $r$ will be made: $c=0, a(x)=x^{4}+1$, and $r(x)=x^{4}+2$, for $x \geq b$ with $b=0$. Then it follows from (1.3) that

$$
\omega(x, \lambda)=p(x) \equiv 1, \quad \tilde{q}(x, \lambda)=1+\frac{\left(x^{4}+1\right)^{2}}{\lambda-\left(x^{4}+2\right)}, \quad x \geq 0
$$

Now one sees that $\gamma=0$, and that $\Omega=\Omega^{\prime}=\mathbb{C} \backslash[2, \infty)$. Moreover, Assumption 5.1 is satisfied, so that Theorem 5.3 may be applied. Now consider the point $\lambda=1 \in \Omega^{\prime}$. Then $\tilde{q}(x, 1)=-x^{4}$ and the equation (1.2) takes the form

$$
\left(-D^{2}-x^{4}\right) y_{1}=0
$$

and the differential expression in the left-hand side coincides with $L_{4}(\rho=4, \delta=1)$ and by the above discussion is in the limit-circle case for the first component $y_{1}$. For the second component (1.4) gives

$$
y_{2}(x)=\frac{a(x)}{1-r(x)} y_{1}(x)=-y_{1}(x)
$$

Hence, $y_{1}$ and $y_{2}$ both belong to $L^{2}[0, \infty)$ for each solution $y$ of $(\mathbb{L}-\lambda) y=0$. Therefore, $\mathbb{L}$ is limit-circle at $\lambda=1$ and then according to Theorem $5.3 \mathbb{L}$ is limit-circle at all $\lambda \in \mathbb{C} \backslash[2, \infty)$.

Some historical remarks At the suggestion of an associate editor some historical remarks about the defect indices of squareintegrable solutions of (systems of) differential equations are added. The function-theoretic approach going back to H. Weyl [48] was augmented by different approaches appearing in the books of M. H. Stone (1932) and E. C. Titchmarsh (1946). A bit later Weyl's theory was extended to general even order equations by K. Kodaira [23]. The generalization of Weyl's theory to the setting of so-called singular $S$-hermitian systems, which are formally more general than the first-order canonical systems, but can be reduced to them (see [30]) was initiated in [44,45] in the case of real $S$-hermitian systems and was soon extended to the case of general complex $S$-hermitian systems in [38-40]. In these papers an approach using monotonicity arguments was developed, see also [5] for a simplified treatment based on multivalued selfadjoint limit values of monotone matrix functions. Constancy of defect indices for these systems can be found e.g. in [44, Satz 6.23], [45, Sazt 1.9], [39, p. 655] as Weyl's first theorem, while the generalization of the limit-point/limit-circle classification, called the second Weyl's theorem, is given in [44, Satz 9.1], [45, Sazt 3.17], and [39, Satz 5.9]; cf. also [26, Chapters V-VI]. Further results on constancy of (formal) defect indices and various other results about them for symmetric systems of differential equations have been established in the more widely known paper [24] by V. I. Kogan and F. S. Rofe-Beketov. In the functional-analytic/operator theoretic approach the idea is to associate to a (symmetric) system of differential equations a (symmetric) operator in a suitably constructed Hilbert space and then apply the general theory of abstract (symmetric) operators. This approach then yields e.g. the constancy of (formal) defect indices, in the half-planes $\mathbb{C}_{ \pm}$in case of a symmetric operator [37], or, more generally, in connected components of the points of regular type [27]. For a long time this approach was restricted by the requirement that the (minimal) symmetric operator be densely defined in a suitably chosen Hilbert space. The introduction of linear relations (multivalued operators) meant that this restriction no longer needed to be imposed. In particular, for canonical systems of differential equations this point of view was chosen in [41]; a systematic treatment for 2-dimensional canonical systems was worked out in [20,21], see also [14] for a further exposition of this approach. More recently the framework of symmetric linear relations in Hilbert space has been used in the general setting of symmetric first-order systems in [32], where generalizations for several earlier results and criteria concerning (formal and ordinary) defect indices have been obtained. The connection between the Titchmarsh-Weyl coefficient and the squareintegrable solutions has been investigated in [15-18] for special cases. For singular canonical systems of differential equations the method of boundary triplets has been worked out in [5] offering a functional-analytic/operator theoretic framework to express
square-integrable solutions via the corresponding Weyl functions as an analog of the use of Titchmarsh-Weyl coefficients for the usual Sturm-Liouville expressions.

## ACKNOWLEDGEMENTS

The authors are grateful to Prof. Vadim Adamyan for bringing reference [49] to their attention. The first author was partially supported by a grant from the Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters. The second author was partially supported by a grant from the NRF of South Africa, grant number 69659.

## REFERENCES

[1] V. Adamjan, H. Langer, and M. Langer, A spectral theory for a $\lambda$-rational Sturm-Liouville problem, J. Differential Equations 171 (2001), no. 2, 315-345.
[2] F. V. Atkinson, Discrete and continuous boundary problems, Academic Press, New York, 1964.
[3] F. V. Atkinson, H. Langer, and R. Mennicken, Sturm-Liouville problems with coefficients which depend analytically on the eigenvalue parameter, Acta Sci. Math. (Szeged) 57 (1993), no. 1-4, 25-44.
[4] F. V. Atkinson, H. Langer, R. Mennicken, and A. A. Shkalikov, The essential spectrum of some matrix operators, Math. Nachr. 167 (1994), 5-20.
[5] J. Behrndt, S. Hassi, H. de Snoo, and H. L. Wietsma, Square-integrable solutions and Weyl functions for singular canonical systems, Math. Nachr. 284 (2011), 1334-1384.
[6] B. M. Brown, M. Marletta, S. N. Naboko, and I. G. Wood, Detectable subspaces and inverse problems for Hain-Lüst-type operators, Math. Nachr. 289 (2016), 2108-2132.
[7] E. A. Coddington and N. Levinson, Theory of ordinary differential equations, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
[8] M. Faierman, R. Mennicken, and M. Möller, The essential spectrum of a system of singular ordinary differential operators of mixed order. I. The general problem and an almost regular case, Math. Nachr. 208 (1999), 101-115.
[9] M. Faierman, R. Mennicken, and M. Möller, The essential spectrum of a system of singular ordinary differential operators of mixed order. II. The generalization of Kako's problem, Math. Nachr. 209 (2000), 55-81.
[10] J. P. Goedbloed and S. Poedts, Principles of magnetohydrodynamics: With applications to laboratory and astrophysical plasmas, Cambridge University Press, Cambridge, 2004.
[11] K. Hain and R. Lüst, Zur Stabilität zylindersymmetrischer Plasmakonfigurationen mit Volumenströmen, Z. Naturforsch. A 13 (1959), 936-940.
[12] V. Hardt, R. Mennicken, and S. Naboko, Systems of singular differential operators of mixed order and applications to 1-dimensional MHD problems, Math. Nachr. 205 (1999), 19-68.
[13] S. Hassi, M. Möller, and H. de Snoo, Singular Sturm-Liouville problems whose coefficients depend rationally on the eigenvalue parameter, J. Math. Anal. Appl. 295 (2004), no. 1, 258-275.
[14] S. Hassi, H. S. V. de Snoo, and H. Winkler, Boundary-value problems for two-dimensional canonical systems, Integral Equations Operator Theory 36 (2000), 445-479.
[15] D. B. Hinton and A. Schneider, On the Titchmarsh-Weyl coefficients for singular S-Hermitian systems I, Math. Nachr. 163 (1993), $323-342$.
[16] D. B. Hinton and A. Schneider, On the Titchmarsh-Weyl coefficients for singular S-Hermitian systems II, Math. Nachr. 185 (1997), 67-84.
[17] D. B. Hinton and A. Schneider, Titchmarsh-Weyl coefficients for odd-order linear Hamiltonian systems, J. Spectral Math. Appl. 1 (2006), 1-36.
[18] D. B. Hinton and J. K. Shaw, On Titchmarsh-Weyl m( $\lambda$ )-functions for linear Hamiltonian systems, J. Differential Equations 40 (1981), 316-342.
[19] O. O. Ibrogimov, H. Langer, M. Langer, and C. Tretter, Essential spectrum of systems of singular differential equations, Acta Sci. Math. (Szeged) 79 (2013), no. 3-4, 423-465.
[20] I. S. Kac, On the Hilbert spaces generated by monotone Hermitian matrix functions, Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč 22 (1950/1951), 95-113.
[21] I. S. Kac, Linear relations generated by canonical differential equations, Funct. Anal. Appl. 17 (1983), 86-87 (Russian).
[22] R. M. Kauffman, T. T. Read, and A. Zettl, The deficiency index problem for powers of ordinary differential expressions, Lecture Notes Math., vol. 621, Springer-Verlag, Berlin-New York, 1977.
[23] K. Kodaira, On ordinary differential equations of any even order and the corresponding eigenfunction expansions, Amer. J. Math. 72 (1950), 502-544.
[24] V. Kogan and F. Rofe-Beketov, On square-integrable solutions of symmetric systems of differential equations of arbitrary order, Proc. Roy. Soc. Edinburgh Sect. A 74 (1974/75), 5-40.
[25] A. Konstantinov and R. Mennicken, On the Friedrichs extension of some block operator matrices, Integral Equations Operator Theory 42 (2002), no. 4, 472-481.
[26] A. M. Krall, Hilbert Space, Boundary value problems and orthogonal polynomials, Oper. Theory Adv. Appl., vol. 133, Birkhäuser Verlag, Basel, 2002.
[27] M. G. Kreĭn, M. A. Krasnoselski, and D. P. Milman, On the defect numbers of operators in Banach spaces and on some geometric questions, Trudy Inst. Mat. Akad. Nauk Ukrain. SSR 11 (1948), 97-112.
[28] P. Kurasov, I. Lelyavin, and S. Naboko, On the essential spectrum of a class of singular matrix differential operators. II. Weyl's limit circles for the Hain-Lüst operator whenever quasi-regularity conditions are not satisfied, Proc. Roy. Soc. Edinburgh Sect. A 138 (2008), no. 1, $109-138$.
[29] P. Kurasov and S. Naboko, On the essential spectrum of a class of singular matrix differential operators. I. Quasiregularity conditions and essential self-adjointness, Math. Phys. Anal. Geom. 5 (2002), no. 3, 243-286.
[30] H. Langer and R. Mennicken, A transformation of right-definite $S$-hermitian systems to canonical systems, Differential Integral Equations 3 (1990), 901-908.
[31] H. Langer, R. Mennicken, and M. Möller, A second order differential operator depending non-linearly on the eigenvalue parameter. In Topics in operator theory: Ernst D. Hellinger Memorial Volume, Oper. Theory Adv. Appl., vol. 48, Birkhäuser, Basel, 1990, pp. 319-332.
[32] M. Lesch and M. M. Malamud, On the deficiency indices and selfadjointness of symmetric Hamiltonian systems, J. Differential Equations 189 (2003), 556-615.
[33] A. E. Lifshits, Magnetohydrodynamics and spectral theory, Kluwer, Dordrecht-Boston-London, 1989.
[34] R. Mennicken and M. Möller, Non-self-adjoint boundary eigenvalue problems, North-Holland Math. Stud., vol. 192, North-Holland Publishing Co., Amsterdam, 2003.
[35] R. Mennicken, S. Naboko, and C. Tretter, Essential spectrum of a system of singular differential operators and the asymptotic Hain-Lüst operator, Proc. Amer. Math. Soc. 130 (2002), no. 6, 1699-1710 (electronic).
[36] M. Möller, The essential spectrum of a system of singular ordinary differential operators of mixed order. III. A strongly singular case, Math. Nachr. 272 (2004), 104-112.
[37] J. von Neumann, Über adjungierte Operatoren, Ann. Math. 33 (1932), 294-310.
[38] H.-D. Niessen, Singuläre S-hermitesche Rand-Eigenwertprobleme, Manuscripta Math. 3 (1970), 35-68.
[39] H.-D. Niessen, Zum verallgemeinerten zweiten Weylschen Satz, Arch. Math. (Basel) 22 (1971), 648-656.
[40] H.-D. Niessen, Greensche Matrix und die Formel von Titchmarsh-Kodaira für singuläre $S$-hermitesche Eigenwertprobleme, J. Reine Angew. Math. 261 (1973), 164-193.
[41] B. C. Orcutt, Canonical differential equations, Dissertation, University of Virginia, 1969.
[42] S. A. Orlov, Nested matrix disks depending analytically on a parameter, and the theorem on invariance of the ranks of the radii of limiting matrix disks, Izv. Akad. Nauk SSSR Ser. Mat. 40 (1967), 593-644.
[43] J. Qi and S. Chen, Essential spectra of singular matrix differential operators of mixed order, J. Differential Equations 250 (2011), no. 12, 42194235.
[44] A. Schneider, Untersuchungen über singuläre reelle S-hermitesche Differentialgleichungssysteme im Normalfall, Math. Z. 107 (1968), $271-296$.
[45] A. Schneider, Geometrische Bedeutung eines Satzes vom Weylschen Typ für S-hermitesche Differentialgleichungssysteme im Normalfall, Arch. Math. 20 (1969), 147-154.
[46] H. de Snoo, Regular Sturm-Liouville problems whose coefficients depend rationally on the eigenvalue parameter, Math. Nachr. 182 (1996), 99-126.
[47] C. Tretter, Spectral theory of block operator matrices and applications, Imperial College Press, London, 2008.
[48] H. Weyl, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen, Math. Ann. 68 (1910), no. 2, 220-269.
[49] H. Weyl, Über das Pick-Nevanlinna'sche Interpolationsproblem und sein infinitesimales Analogon, Ann. Math. (2) 36 (1935), no. 1, $230-254$.

How to cite this article: Hassi S, Möller M, de Snoo H. Limit-point/limit-circle classification for Hain-Lüst type equations. Mathematische Nachrichten. 2018;291:652-668. https://doi.org/10.1002/mana.201600254

